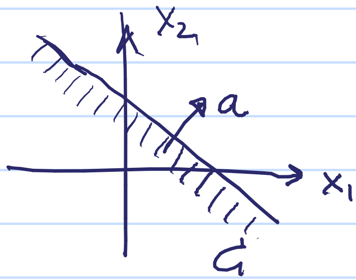
1) It is not convex due to the nonlinear equality constraint.
It can be made convex by changing the constraint to

$$x_1^2+x_2^2 \leq 2 \qquad (*)$$

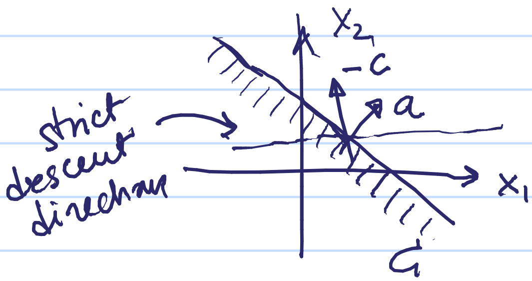
This is because the objective function is decreasing with x_1 and x_2 and hence the optimal point must satisfy $(*)$ with equality.

2) a.



$\nabla f_0(x) = c$

- $p^* = -\infty$ unless $-c$ is parallel to a. In fact, otherwise we get



- If $c + \lambda a = 0$ for some $\lambda > 0$, then $p^* = c^T x = -\lambda a^T x = -\lambda b$ and $X_{opt} = \{x \mid a^T x = b\}$

b. KKT conditions

$$\begin{cases} c + \lambda a = 0 \\ \lambda \geq 0 \\ \lambda(a^T x - b) = 0 \\ a^T x \leq b \end{cases}$$

If $c = \lambda a$ for some $\lambda > 0$, then any value of x such that $a^T x = b$ satisfies the KKT conditions

3) minimize $-\sum_{i=1}^{n} \ln x_i$ strictly convex problem

s.t. $x \in G$

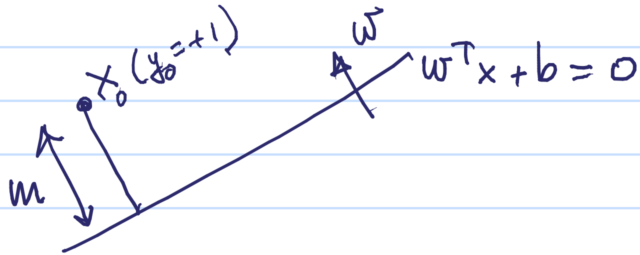
Necessary and sufficient conditions:

$$x^* \in X_{opt} \iff -\sum_{i=1}^{n} \frac{1}{x_i^*}(x_i - x_i^*) \geq 0 \quad \text{for all } x \in G$$

$$\iff \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_i^*} \leq 1 \quad \text{for all } x \in G$$

All other feasible points x must be such that no rate can be proportionally increased, i.e., $\frac{x_i}{x_i^*} > 1$, without making another rate proportionally smaller.

4.



To evaluate the margin m, we first write the half-space $w^T x + b = 0$ as

$$w^T(x - x_0) + w^T x_0 + b = 0$$

$$\iff w^T(x - x_0) = -w^T x_0 - b$$

Now, consider x_0 as the origin of the Cartesian axes. We know that the distance from the origin of a hyperplane $c^T x = d$ is $|d|/\|c\|_2$, so we get

$$m = \frac{|w^T x_0 + b|}{\|w\|_2} = \frac{y_0(w^T x_0 + b)}{\|w\|_2}$$

5. a. It is a strictly convex problem since $-\log x$ is a strictly convex function for $x > 0$.

Weierstrass is not satisfied since G is not closed.

Slater's constraint qualification holds since the constraints are linear

$\Rightarrow$ if a solution x^* exists, it is unique and there is also a corresponding vector of dual variables λ^*.

b. $$\frac{x_i}{a_i}\left(-\frac{a_i}{x_i^2}\right)+\lambda_i+\lambda=0 \Longleftrightarrow -\frac{1}{x_i}+\lambda_i+\lambda=0$$

$$\lambda_i,\lambda \geq 0 \qquad i=1,\dots,n$$

$$\lambda_i(x_i-a)=0$$

$$\lambda\left(\sum_{i=1}^{n} x_i - D\right)=0$$

$$0<x_i\leq a_i \quad , i=1,\dots,n$$

Note that we have not included $x_i \geq 0$ since we must have $x_i>0$ by the implicit constraints.

c. $$x_i=\frac{1}{\underbrace{\lambda_i+\lambda}_{\leq a_i}}=\begin{cases}\frac{1}{\lambda} & \text{if } \frac{1}{\lambda}\leq a_i \rightarrow \lambda_i=0\\ a_i & \text{else} \rightarrow \lambda_i=\frac{1}{a_i}-\lambda\end{cases}$$

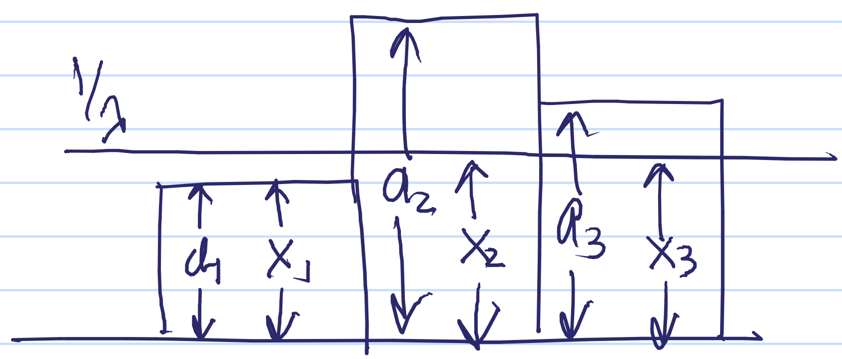
$$=\min\left(a_i,\frac{1}{\lambda}\right)$$

where λ is selected so that

$$\sum_{i=1}^{n}\min\left(a_i,\frac{1}{\lambda}\right) \cancel{\leq} D$$

$=$

as otherwise $\lambda=0$



6) We clearly have

$$p^* = 0$$

$$X_{opt} = \{x \mid a^T x + b = 0\} \text{ (hyperplane)}$$

a. minimize t

s.t. $a^T x + b \leq t$

$-a^T x - b \leq t$

b. $\mathcal{L}(x, t, \nu_1, \nu_2) = t + \nu_1 (a^T x + b - t) + \nu_2 (-a^T x - b - t)$

$= t(1 - \nu_1 - \nu_2) + (\nu_1 a - \nu_2 a)^T x + \nu_1 b - \nu_2 b$

$$g(\nu_1, \nu_2) = \inf_{\substack{x \in \mathbb{R}^n, \\ t \in \mathbb{R}}} \mathcal{L}(x, t, \nu_1, \nu_2)$$

$$= \begin{cases} 0 & \text{if } \nu_1 + \nu_2 = 1 \text{ and } \nu_1 - \nu_2 = 0 \Rightarrow \nu_1 = \nu_2 = 1/2 \\ -\infty & \text{otherwise} \end{cases}$$

$g(\nu_1, \nu_2) = 0 \qquad \mathrm{dom}\, g = \{(1/2, 1/2)\}$

c. We clearly have strong duality as expected due to the fact that the problem is convex and Slater's conditions are satisfied.