Common foundations for belief revision, belief merging and voting

D. M. Gabbay, G. Pigozzi and O. Rodrigues

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Abstract

In this paper, we consider a number of different ways of reasoning about voting as a problem of conciliating contradictory interests. The mechanisms that do the reconciliation are belief revision and belief merging. By investigating the relationship between different voting strategies and their associated counterparts in revision theory, we find that whereas the counting mechanism of the voting process is more easily done at the meta-level in belief merging, it can be brought to the object level in base revision. In the former case, the counting can be tweaked according to the aggregation procedure used, whereas in base revision, we can only rely on the notion of minimal change and hence the syntactical representation of the voters’ preferences plays a crucial part in the process. This highlights the similarities between the revision approaches on the one hand and voting on the other, but also opens up a number of interesting questions.

1 Introduction

In this paper we explore the connections between voting, belief revision and merging. We recognize the following three scenarios:

- **Voting** studies how individual conflicting preferences can be aggregated into a collective preference.
- **Belief revision** investigates which changes need to be made in order to make an inconsistent or unacceptable logical theory again consistent.
- **Belief merging** defines the aggregation of several knowledge bases which together are possibly inconsistent.

The novelty of our approach is to bring the mechanism for representing preferences to the object level and hence we can analyse it from a entirely logical perspective.

2 Voting: aggregating preferences

Social choice theory [Arr63, Sen70] studies how individual preferences can be aggregated into a collectively preferred alternative. The problem is of not easy solution as
the Condorcet paradox shows: Given a set of individual preferences, we compare each of the alternatives in pairs. For each pair, we determine the winner by majority voting, and the final collective ordering is obtained by a combination of all partial results. Unfortunately, this method can lead to cycles in the collective ordering.

Let \( C = \{c_1, \ldots, c_k\} \) be the set of candidates. There are exactly \(|C| \times (|C| - 1)\) distinct ordered pairs of candidates \((c_i, c_j)\). A binary relation \(<\) is defined on \(C^2\), where \(c_i < c_j\) denotes that candidate \(c_i\) is (strictly) preferred to candidate \(c_j\). The desired properties of preference relations associated to strict linear orders are given below, where the variables \(\{x, y, z\}\) range over elements of \(C\).

1. **Transitivity**
   \[
   \forall x, y, z \left( (x < y \land y < z) \rightarrow x < z \right)
   
   \]

2. **Totality**
   \[
   \forall x, y \left( x \neq y \rightarrow (x < y \lor y < x) \right)
   
   \]

3. **Asymmetry**
   \[
   \forall x, y \left( (x < y) \rightarrow \neg(y < x) \right)
   
   \]

Suppose that there are three possible candidates \(a, b, c\) and three voters \(V_1, V_2\) and \(V_3\), who express their total preferences in the following way: \(V_1 = \{a < b, b < c\}\), \(V_2 = \{b < c, c < a\}\) and \(V_3 = \{c < a, a < b\}\).

According to Condorcet’s method, \(a < b\) has the majority of the voters (\(V_1\) and \(V_3\)), so does \(b < c\) (\(V_1\) and \(V_2\)) and also \(c < a\) (\(V_2\) and \(V_3\)). This leads us to the collective outcome \(a < b, b < c\) and \(c < a\), which together with transitivity (P1) violates (P3). This is the Condorcet paradox.

Unfortunately, K. Arrow showed that this is not a problem specific of pairwise majority comparison. In his famous impossibility theorem, Arrow proved that there exists no aggregation procedure that satisfies few desirable properties.

Let \(X\) be a non-empty set of mutually exclusive social alternatives and \(\leq_i\) be a total, reflexive and transitive preference relation for an individual \(i\) over the states in \(X\) (and \(<_i\) its strict counterpart). Suppose there are \(n\) individuals \(V = \{V_1, \ldots, V_n\}\) in the society. A social welfare function (SWF) is a function that produces a total, reflexive and transitive social preference relation \(\succeq\) from a given \(n\) tuple of individual orderings \(\preceq_1, \ldots, \preceq_n\) (again we use \(<\) to denote \(\preceq\)’s strict counterpart). A tuple of \(n\) rankings one for each individual over the set of alternatives is called a profile and will be denoted by \(V\).

Arrow’s impossibility theorem states that whenever \(|X| > 2\), the following conditions are incompatible:

1. **Universal domain** The social preference function accepts all admissible individual preference relations.
2. **Independence of irrelevant alternatives - IIA** The social preference on any pair of alternatives depends exclusively on the individual preferences over that pair.
3. **Weak Pareto** If, for all \(i, x <_i y\), then \(x < y\).
4. **Non-dictatorship** There is no individual \(i\) such that for each \(\{x, y\} \in X\) and every profile \(\langle<_1, \ldots, <_i, \ldots, <_n\rangle\), \(x <_i y\) implies \(x < y\).

### 3 Aggregating preferences via belief revision

One way of analysing the interaction between belief revision, merging and voting is to express voting principles in a logical framework and then consider what belief revision
and merging would do in specific voting scenarios. We start by considering a logic theory of order and its relation with belief revision.

Consider the language of predicate logic with binary relation \(<\); the constants \(a, b, c\) and the equality symbol \(=\). Assume the axioms \(\forall x(x = a \lor x = b \lor x = c)\) and \(a \neq b \neq c\) (this means \(\neg(a = b) \land \neg(b = c) \land \neg(a = c)\)). Let \(T\) be \(Cn\{a < b, b < c, c < a\}\) and consider an input \(\tau\) to \(T\) saying that \(< is the strictly linear order of the three elements \(a, b, c\), i.e., \(\tau = P1 \land P2 \land P3\).

It can be easily seen that both \(a < c\) and \(c < a\) follow from \(T + \tau\) and this contradicts \(P3\), hence \(T + \tau\) is not consistent. If we want to analyse what aspects of \(T\) are compatible with a strict linear order of \(a, b, c\), we can consider the revision of \(T\) by \(\tau\). This would replace \(T\) with a new consistent theory \(T \circ \tau\) containing \(\tau\), by making minimal changes in \(T\). The most well known belief revision theory is AGM [AGM85], which provides a set of postulates that constrain how the new theory \(T \circ \tau\) is related to \(\tau\) and to \(T\). The new theory \(T \circ \tau\) is closed under logical consequence, i.e., if \(T \circ \tau \vdash A\), then \(A \in T \circ \tau\), but the AGM framework does not give an algorithm for how to find any such \(T \circ \tau\). One algorithm is given below.

Starting with \(T_1 \models \neg \tau = \{T_1, T_2, T_3, \ldots\}\), \(T \circ \tau\) can be constructed from any \(T_i \in T_1 \models \neg \tau\), say \(Cn(T_i \cup \{\tau\})\) (this would give a maxichoice revision of \(T\) by \(\tau\)). We can find such \(T_i \cup \{\tau\}\) by listing all sentences which \(T\) proves as the list \(A_1, \ldots, A_n, \ldots\) and defining a sequence \(S_0 \subseteq S_1 \subseteq S_2, \ldots\) as follows. Let \(S_0 = \{\tau\}\) and for \(n \geq 0\), define define \(S_{n+1}\) in the following way

\[
S_{n+1} = \begin{cases} 
S_n \cup \{A_{n+1}\}, & \text{if this set is consistent} \\
S_n, & \text{otherwise}
\end{cases}
\]

Finally, let \(S = \bigcup_{i \in \mathbb{N}} S_i\).

If we want to look at what we retain from our original \(T\), we see that \(S - \{\tau\} \subseteq T\) is a maximal subtheory of \(T\) consistent with \(\tau\), i.e. \(S - \{\tau\} = T_i\) for some \(i\). Which \(T_i\) we get depends on the way we present \(T\) as a sequence.

Let us now see what happens if we apply these procedures to our concrete example. \(\tau\) says that \(\{a, b, c\}\) is strictly linearly ordered. \(T\) says that \(a < b\) and \(b < c\) and \(c < a\). \(T\) is not consistent with \(\tau\). The maximal subtheories of \(T\) consistent with \(\tau\) include \(T_1 = Cn\{a < b, b < c\}\), \(T_2 = Cn\{b < c, c < a\}\) and \(T_3 = Cn\{a < b, c < a\}\).

When \(\tau\) is added, we obtain three options for revision: \(V_1 = Cn\{a < b, b < c, \tau\}\), \(V_2 = Cn\{b < c, c < a, \tau\}\) and \(V_3 = Cn\{c < a, a < b, \tau\}\).

Note that this logical revision approach is entirely compatible with AGM revision as it uses three basic assumptions:

1. We must replace the inconsistent \(T + \tau\) by a single consistent theory \(T \circ \tau\).
2. The replacement contains \(\tau\) and as much of \(T\) as possible.
3. We are dealing with two valued logic. In other words, preferences have to be represented as yes/no statements.

From the revision point of view, our voting example consists of three candidate options and several voters who express their total preferences regarding these options. When put together, these preferences result in the theory \(T\).
Since we need to make a group decision, we require an aggregation function $H$ based on the preferences of the three voters motivated by some general principles, such that:

$$H(\text{Voter 1, Voter 2, Voter 3}) = \text{Collective ordering}$$

Reasonable conditions on $H$ are, for instance, those given at the end of of Section 2. One such condition is that $H$ does not choose the ordering of always the same individual to be the collective outcome — the non-dictatorship requirement. Another condition is IIA, i.e., the group decision on how two distinct elements $x$ and $y$ relate ($x < y$ or $y < x$) depends only on how the different voters voted on their relationship. Note that whereas the principle of non-dictatorship is a purely meta-level one on the function $H$ and does not make use of the contents of the theories $T_i$, IIA relates to the properties of the order predicate of $T_i$.

Let us now look at our revision example from the voting point of view. The consistent theories $T_1, T_2, T_3$ can stand for voters. The sentence $\tau$ is a statement of the layout of the voting system. It says that the combination of the voters preferences is strictly linearly ordered. The theory $T$ can be obtained back from the voters as the result of majority vote.

We now have a voting interpretation of our revision theory situation. Maxichoice logical revision simply chooses a dictator (though this is not always the case). We can construct a consistent theory $T$ from a number of voters $V_1, V_2, \ldots$ that is incompatible with the voting rules $\tau$, but whose subsequent revision by $\tau$ will not necessarily pick a dictator even if the revision turns out to be maxichoice. This is illustrated below.

Let $V_1 = \{a < b, b < c, a < c\}$, $V_2 = \{c < b, b < a, c < a\}$ and $\tau$ be the voting rules as before. Now take $T = V_1 \cup V_2 = \{a < b, b < c, a < c, c < b, b < a, c < a\}$. $T$ is consistent, since it does not know about the properties of linear orders. If we now enforce these, i.e., revise $T$ by $\tau$, a maxichoice revision would look at $T \parallel \neg \tau$. One of the sets in $T \parallel \neg \tau$ is, for instance, $\{a < c, c < b, a < b\}$ which together with $\tau$ would result in a strict linear order which does not correspond to either $V_1$ or $V_2$.

Let us return to the expectations of voting theory from the point of view of revision theory. Voting theory expects some social outcome satisfying certain conditions. Belief revision tries to find some compromise theory $S_{\text{comp}}$ between all the $T_i \subseteq T$ that are consistent with $\tau$. This is left mostly to the selection function. Maxichoice revision operations look at all $T_i$’s and sets $T \circ \tau$ as $T_i + \tau$ for some $T_i$. In general, this $T_i$ is not constrained at all on containing consequences of all of the voters. Full meet revisions will be too restrictive and comprise only the consequences of the voting system $\tau$ (since $\cap(T_i \neg \tau) = \text{Cn}(\emptyset)$). On the other hand, partial meet revisions would be based on the particular subsets $T_i$ picked by the selection function which again could leave the wishes of some voters out — an unfair prospect. Therefore, an acceptable $S_{\text{comp}}$ from the voting point of view would have to rely on some meta-level principle in the case of AGM revision functions. What can we then expect of the relationship between $S_{\text{comp}}$ and AGM? In summary,

1. If we stick to AGM, certain conditions of the voting system ($\tau$) can be enforced, but we cannot ensure a fair outcome unless we also adopt some meta-level principles.
2. However, the AGM postulates may hold for a desirable $S_{comp}$ even though they do not incorporate themselves any fairness principles from the voting point of view.

4 Preference aggregation via belief merging

Let us now apply a model-based merging operator to the same example. As before, we want to restrict the set of all possible outcomes to the set of linear preference orderings satisfying $P1$, $P2$ and $P3$. That is, we make the set of integrity constraints ($IC$) equal to $\tau$. This results in the six possible preference orderings $K_1$—$K_6$ illustrated in the table below. In order to simplify the presentation, we consider only three propositions $a < b$, $b < c$ and $a < c$. These are sufficient to represent all possible linear orders with the three elements $a$, $b$ and $c$ (note that $(a < b) = 0$ iff $(b < a) = 1$). We use $x < y < z$ as a shorthand for $Cn\{x < y, y < z, \tau\}$.

<table>
<thead>
<tr>
<th>order</th>
<th>$a &lt; b$</th>
<th>$b &lt; c$</th>
<th>$a &lt; c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_2$</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$K_3$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$K_4$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_5$</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$K_6$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Each voter $V_i$ is satisfied exactly by one of these models: $V_1 = \{(1, 1, 1)\} (K_6)$; $V_2 = \{(0, 1, 0)\} (K_2)$; $V_3 = \{(1, 0, 0)\} (K_4)$.

The idea behind the majoritarian merging operator is to select as social outcome that option that has minimal distance to all the voters. Formal definitions will be given in the next section. Here we are interested in capturing the intuition. As distance we use the Hamming distance $d$, which counts the number of propositions on which two models differ. When we calculate the distances between each $V_i$ and the possible social outcomes, we obtain the following ($D^d$ is the sum of the individual Hamming distances):

<table>
<thead>
<tr>
<th></th>
<th>$d(., V_1)$</th>
<th>$d(., V_2)$</th>
<th>$d(., V_3)$</th>
<th>$D^d(., V)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>$K_2$</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$K_3$</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$K_4$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$K_5$</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>$K_6$</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

There are three social orderings with minimal distance to the profile $V = \{V_1, V_2, V_3\}$. These are $K_2$, $K_4$ and $K_6$, which coincide with $V_2 \lor V_3$ and $V_1$ respectively. The result of the belief merging operator is a tie: $\{V_1 \lor V_2 \lor V_3\}$. This means that, although belief merging (with the help of the $IC$) avoids the paradoxical result, this is done at the price of indecision, i.e., the winner will be randomly chosen.
5 Bringing the voting concepts to the object level

We now want to express the ideas presented in Section 2 in the context of a propositional language $L$, and thus we will associate to each pair of candidates $c_i, c_j$ taken from $C$, a propositional variable “$c_i < c_j$” implicitly meaning that candidate $c_i$ is preferred to candidate $c_j$. We use $\mathcal{P}$ to denote the set of all propositional variables of $L$ constructed in this way. Complex formulae of $L$ are defined as usual. We assume the usual semantics for $L$ and use the symbol $W$ to denote the set of all of its valuations.

A set of propositional variables $\Delta$ is a faithful representation of a strict total order $<$ on $C$ if the following conditions are met:

1. $c_i < c_j \in \Delta$ and $c_j < c_m \in \Delta$ implies $c_i < c_m \in \Delta$ – (P1)
2. for every pair $\{c_i, c_j\}$ of distinct elements $c_i, c_j \in C$, either $c_i < c_j \in \Delta$ or $c_j < c_i \in \Delta$, but not both – (P2) and (P3)
3. no other propositional variables appear in $\Delta$ – (MIN)

In general, we would like to consider sets containing any complex formulae, so we need to impose some conditions on these sets as to what constitutes a faithful representation of a strict total order on $C$. We do this by defining the integrity constraint $\tau$ in the language $L$ as follows: $\tau = \tau_1 \land \tau_2$, where $\tau_1 = \bigwedge_{i \neq j \neq m}[(c_i < c_j) \land (c_j < c_m)) \rightarrow (c_i < c_m)]$ and $\tau_2 = \bigwedge_{i \neq j}[(c_i < c_j) \lor (c_j < c_i)] \land [(c_i < c_j) \rightarrow \neg(c_j < c_i)]$.

$\tau_1$ and $\tau_2$ must be constructed for every distinct pair of candidates taken from $C$. The first conjunct of $\tau_2$ can be rewritten as $\neg(c_i < c_j) \rightarrow (c_j < c_i)$, and hence $\tau_2$ can be rewritten as $(c_i < c_j) \iff \neg(c_j < c_i)$. (All proofs are omitted due to page limit constraints.)

Proposition 1. For every strict linear order $\prec$ on $C$, there is a valuation $w$ in $\mod(\tau)$ such that $x' < y$ iff $w \models x < y$.

Proposition 2. For every valuation $w$ in $\mod(\tau)$, there exists a strict linear order $<_w$ such that $w \models x < y$ iff $x <_w y$.

If there are no extra propositional variables in $L$ apart from the ones necessary to represent $<$, then there is a correspondence between $\mod(\tau)$ and the set of all strict linear orders. We assume this is the case and use $w_<$ to denote the valuation associated with a particular strict linear order $<$ and $<_w$ to denote the strict linear order with a particular valuation $w$.

Example 3. Suppose our candidates are represented by the set $C = \{a, b, c\}$. For this configuration, $\tau$ as defined above would be $\tau = \tau_1 \land \tau_2$, where

$$
\tau_1 = 
\begin{align*}
&[(a < b \land b < c) \rightarrow c < a) \land (a < c \land c < b) \rightarrow a < b)\land \\
&(b < c \land c < a) \rightarrow c < b)\land (b < c \land c < a) \rightarrow b < a)\land \\
&(c < a \land a < b) \rightarrow c < b)\land (c < b \land b < a) \rightarrow c < a)
\end{align*}
$$

There is only significance in the uniqueness when we come to consider distances between valuations for pairs of orders. If there is more than one valuation associated with each order $<$, for a pair of orders $<_1$ and $<_2$, we will be interested in the minimum distance between any two valuations associated with them.
\[ \tau_2 = \left[ (a < b \leftrightarrow \neg b < a) \land (b < a \leftrightarrow \neg a < b) \land (a < c \leftrightarrow \neg c < a) \land (c < a \leftrightarrow \neg a < c) \land (b < c \leftrightarrow \neg c < b) \land (c < b \leftrightarrow \neg b < c) \right] \]

Note that \( \text{mod}(\tau) \) corresponds exactly to all possible strict total orders on a set of three elements: \( c < b < a \), \( b < c < a \), \( b < a < c \), \( c < a < b \), \( a < c < b \) or \( a < b < c \).

**Definition 4.** A voter \( V_i \) is a set of propositional formulae.

Voters need not be consistent, nor complete in their preferences. However, some voters have special characteristics:

**Definition 5.** An opinionated voter is a conjunction of propositional formulae.

In other words, an opinionated voter wants all of his preferences to be considered atomically. It will prove useful to relate an opinionated voter with his more flexible counterpart \( V_i \) and so we will denote the former using the symbol \( \overline{V}_i \). Similarly, a voter may have more or less information about his preferences over candidates:

**Definition 6.** A well-informed voter is a voter whose propositional formulae admit exactly one chain in \( C \).

**Definition 7.** Let \( \Delta \) be a set of formulae.
\[
\text{mod}(\Delta) = \{ w \in W \mid w \Vdash \Delta \} 
\]

The Hamming distance can now be formally defined:

**Definition 8.** Let \( w_1 \) and \( w_2 \) be two valuations in \( W \). The distance \( d \) between \( w_1 \) and \( w_2 \) is the number of propositional variables \( p \), for which \( w_1(p) \neq w_2(p) \).

We extend the Hamming distance to cater for distances between a single valuation and a set of valuations:

**Definition 9.** Let \( w \in W \) and \( W \subseteq W \). The pointwise distance between \( w \) and \( W \), in symbols \( d(w, W) \) is defined as
\[
\min \{ d(w, w') \mid w' \in W \}
\]

We now define the model-based majority merging operator, as introduced by [KPP02].

**Definition 10.** Let \( V_1, \ldots, V_n \) be \( n \) voters and \( IC \) a set of integrity constraints. The majoritarian merging of \( V_1, \ldots, V_2 \) with integrity constraints \( IC \), in symbols \( M_{IC}(V_1, \ldots, V_n) \), is defined as
\[
M_{IC}(V_1, \ldots, V_n) = \{ w \in \text{mod}(IC) \mid \sum_{1 \leq i \leq n} d(w, \text{mod}(V_i)) \text{ is minimum} \}
\]

**Definition 11.** The Kemény distance between two profiles \( P_1 \) and \( P_2 \), in symbols \( d_K(P_1, P_2) \), is defined as
\[
d_K(P_1, P_2) = | \{(c_1, c_2) \in C^2 \mid (c_1, c_2) \in (P_1 \cup P_2) - (P_1 \cap P_2)\} |
\]
Proposition 12. For any two profiles $P_1$ and $P_2$

$$d_K(P_1, P_2) = d(w_{P_1}, w_{P_2})$$

The result above was proved in [EM05].

Definition 13. The Kemeny distance between a rank $P$ and a profile $P_1, \ldots, P_n$, in symbols $d_K(P, \langle P_1, \ldots, P_n \rangle)$, is defined as

$$d_K(P, \langle P_1, \ldots, P_n \rangle) = \sum_{1 \leq i \leq n} d_K(P, P_i)$$

A profile $P$ is called a Kemeny consensus of a set of profiles $P_1, \ldots, P_n$, if $d_K(P, \langle P_1, \ldots, P_n \rangle)$ is minimum.

Definition 14. A candidate $c$ is called a Kemeny winner if there exists a Kemeny consensus $P$ in which $c$ is minimum.

We set $IC = \{\tau_1, \tau_2\}$ and represent $M_{IC}$ simply as $M_\tau$.

Proposition 15. Let $V_1, \ldots, V_n$ be $n$ well-informed voters. $w_p \in M_\tau(V_1, \ldots, V_n)$ iff $w_p$ represents a Kemeny consensus $P_w$ of $\langle V_1, \ldots, V_n \rangle$.

Proposition 16. A candidate $x$ is a Kemeny winner for $\langle V_1, \ldots, V_n \rangle$ iff $\exists w \in M_\tau(V_1, \ldots, V_n)$, such that $w \in \text{mod}(\land x \neq y x < y)$.

Note that in the above characterization, the actual election result was mostly calculated in the meta-level. It was mostly the distance values (calculated outside the logical theory) that determined how the votes were counted. To be more general, we need to include the mechanism of counting the votes in the object level itself.

6 Counting via belief revision

In the previous section, we had one preference relation $<$ and the voters expressed their opinion as to how they wished $<$ to be like. $\tau$ imposed further conditions on such relation. However, we now want to move one step further by expressing the machinery that computes the overall $<$ from each of the individual voter’s preference relations.

In order to distinguish the overall social preference relation from that of the individual voters, we will re-formulate the problem by introducing new symbols as follows. We replace $<_M$ for $<$ in $\tau (\tau_1 \land \tau_2)$ and introduce a collection of propositional symbols $V_i = \{c_i < c_j\}$, for each distinct pairs of candidates $c_i, c_j \in C$ and voter $V_i$.

The set of the candidates and the set of the voters do not necessarily coincide [AT05]. When they do the above scenario represents a different aggregation problem [AT05]. Now, in general, we want to represent a voter’s $V_i$ preferences through one or more formulae taken from $V_i$. This could be done either by a set of formulae or by a conjunction of formulae. We normally will not want the representation of a voter’s preference to be a closed theory. In fact, a number of issues need to be considered when choosing the most appropriate representation. Base revision, for instance is syntax-dependent and this has a direct effect on the concept of minimality. Revision is done by comparing consistent subsets of the original belief base. In particular,
a) if we choose the representation of a voter’s preferences to be a single formulae, then by doing base revision, the result will incorporate all or none of that voter’s preferences. This is the opinionated version. As a side effect, votes can be counted in a more coherent way.

b) if we choose the representation of a voter’s preferences to be the set of propositional variables associated with each of her preferences for a candidate \( c_i \) over \( c_j \), it may be possible to retain at least some of these preferences in the revision process. This approach is more flexible.

In addition, if we take a voter’s preference to be associated with a strict linear order, we will then once again arrive at a dictator at the end of the revision.

As we mentioned at the end of the previous section, we now want to simplify the revision/merging mechanism by placing some of the machinery in the object level. So we will now turn to this problem by considering base revision (instead of belief merging), and hence we need to include in the object-level some information about what it means to win the election according to some criteria. Arguably, the simplest voting rule to incorporate is that of majority. We would like to express under what circumstances the majority of voters prefer one candidate to another. This of course depends on the number of voters. In our example with three voters, a candidate \( c_i \) is preferred over candidate \( c_j \) by the majority of voters if and only if any two voters prefer \( c_i \) over \( c_j \).

**Definition 17** (Majority preference).

\[ \Theta_M = \{ c_i <_M c_j \} \leftrightarrow [(c_i <_1 c_j) \land (c_i <_2 c_j)] \lor ((c_i <_1 c_j) \land (c_i <_3 c_j)) \lor ((c_i <_2 c_j) \land (c_i <_3 c_j)) \mid c_i, c_j \in C \text{ and } i \neq j \} \]

It will prove convenient to use the information in the above (finite) set as a formula \( \theta_M = \text{def } \bigwedge \Theta_M \). This will only tell us that a particular candidate \( c_i \) is preferred over another candidate \( c_j \), when one of the disjuncts on the right of the formula is true, i.e., two out of three of the voters. For larger number of voters, one needs to calculate what the majority number \( m \) of voters is and write formulae accordingly for each combination of \( m \) voters — a tedious and repetitive process, but nevertheless easy. In order to be a Condorcet winner, a candidate \( c_i \) needs to be preferred by the majority over every other candidate \( c_j \). Hence, for each \( c_i \in C \): \( CW(c_i) = \land_{i \neq j} c_i <_M c_j \). And there will be such a winner in a voting scenario, if one of the candidates is a Condorcet winner, i.e.,

\[ ECW = \lor_{c_i \in C} CW(c_i) \]

Now that the voting mechanism is expressed in the logic’s language itself, we can consider different combinations of revision mechanisms and constraints and compare the results of the revision. Note that we can also choose to revise the machinery itself, but let us leave this point for later and as such, require that the machinery is preserved
by the revision process. All we need is to revise by the machinery, because of the success postulate. We then need to decide what kind of revision we want.

AGM belief revision itself is independent of the syntax form of the formulae. This is due to the fact that what is revised is in fact a theory. As a result, voters’ preferences contain a lot more information than can be at first realised and one needs to be careful to constrain exactly what is allowed to remain from the original theory. In our case, each propositional symbol in a voter’s preference representation is associated with a vote of that voter for one candidate over another. In doing the revision we want to minimise the loss of these propositions. Therefore, one reasonable way of seeing the process is to consider base revision instead. Proper subsets of a voter’s representation will be associated with the failure to satisfy all of that voter’s original preferences.

There are two ways of calculating the change to the original base. The first one is to consider all maximal subsets of a belief base $K$ that fail to imply a sentence $\alpha$ (we denote the following set by $K_\perp \alpha$): $\{K' \subseteq K | K' \not\vdash \alpha \text{ and for every } K'' \supset K', K'' \vdash \alpha\}$.

Using the above notion, it is possible to define the revision of a belief base $K$ by a formula $\alpha$, in symbols $K \star \alpha$, by picking one element of $K_\perp \neg \alpha$ if any exists and then expand it by $\alpha$. If $K_\perp \neg \alpha$ is empty, we can simply take $\alpha$ itself as the result of the revision:

$$K \star \alpha = \begin{cases} K' \cup \alpha & \text{for some } K' \in K_\perp \neg \alpha \\ \{\alpha\} & \text{if } K_\perp \neg \alpha = \emptyset \end{cases}$$

This is the traditional way of evaluating minimal change to $K$. One could also consider the subsets of $K$ that are consistent with $\alpha$ and have maximum cardinality (i.e., failure of set inclusion between sets is not sufficient to qualify a set as maximally consistent). This can be formalised as follows:

$K_{\perp c} \alpha = \{K' \subseteq K | K' \not\vdash \alpha \text{ and for every } K'' \subseteq K, |K''| > |K'|, K'' \vdash \alpha\}$

In [Kon99, Chapter 9], syntactic fusion operators are investigated. One of the operators defined was a syntactic merging operator that selects a consistent set with the maximum cardinality among all maximally consistent subsets.

The base revision according to this policy can be defined in a similar way.

$$K \star_{c} \alpha = \begin{cases} K' \cup \alpha & \text{for some } K' \in K_{\perp c} \neg \alpha \\ \{\alpha\} & \text{if } K_{\perp c} \neg \alpha = \emptyset \end{cases}$$

Definition 18 (Young winner). A Young winner is a candidate that can be made a Condorcet winner by the least removal of voters.

The first conjecture we want to prove is that:

Proposition 19. Let $B = \bigcup_i V_i$ and $\alpha = \tau \land ECW \land \theta_M$. $B \star_{c} \alpha \vdash CW(c_i)$ iff $c_i$ is a Young winner for $\langle V_1, \ldots, V_n \rangle$.

However, notice that picking a voter as a set of propositional variables as opposed to a conjunction of those symbols may potentially change the result.
Example 20. Consider the scenario described earlier on this paper. \( \tau = \tau_1 \land \tau_2 \) is exactly as in Example 3, except that we replace \( <_M \land \tau \) for \( <. \)

The three voters are represented as the sets \( V_1 = a < b < c = \{a <_1 b, b <_1 c, a <_1 c\} \); \( V_2 = b < c < a = \{b <_2 c, c <_2 a, b <_2 a\} \) and \( V_3 = c < a < b = \{c <_3 a, a <_3 b, c <_3 b\} \). \( \theta_M \) will be the conjunction of formulae such as \( a <_M b \iff [(a <_1 b \land a <_2 b) \lor (a <_1 b \land a <_3 b) \lor (a <_2 b \land a <_3 b)] \) and we will have one conjunct for each distinct pair \( c_i, c_j \in C \). A candidate \( c_i \) is a Condorcet winner if the formula \( CW(c_i) \) is true. For, say candidate \( a \), this is \( CW(a) = a <_M b \land a <_M c \) and there will be a Condorcet winner if the formula above is true for one of the candidates, i.e., \( (a <_M b \land a <_M c) \lor (b <_M a \land b <_M c) \lor (c <_M a \land c <_M b) \). Now what happens when we revise \( V = V_1 \cup V_2 \cup V_3 = \{a <_1 b, b <_1 c, a <_1 c, b <_2 c, c <_2 a, b <_2 a, c <_3 a, a <_3 b, c <_3 b\} \) by \( \tau \land ECW \land \theta_M \)? Since \( V \land \theta_M \vdash a <_M b \land b <_M c \land c <_M a \) and the latter formula is inconsistent with \( \tau \), we need to retract some formulae and, because of the way the revision operation is defined, these can only be removed from \( V \).

The inconsistency here can be solved by breaking the cycle \( a <_M b \land b <_M c \land c <_M a \), i.e., by removing from \( V \) formulae that support the derivation of any of \( a <_M b \) or \( b <_M c \) or \( c <_M a \). These were obtained by the formulae in \( \theta_M \), which defines majority. Therefore, the options are the ones enclosed in a square in the tree below.

```
\[
\begin{center}
\begin{tikzpicture}
\node (root) {$\tau$};
\node (a) at (root -| -0.5) [below] {$a <_M b$};
\node (b) at (root -| 0.5) [below] {$b <_M c$};
\node (c) at (root -| 0) [below] {$c <_M a$};
\draw (root) -- (a);
\draw (root) -- (b);
\draw (root) -- (c);
\node (theta) at ([xshift=-1cm] root) {$\theta_M$};
\node (a1) at ([xshift=-1cm] theta) [below] {$\land$};
\node (a2) at ([xshift=-0.5cm] theta) [below] {$a <_1 b$};
\node (a3) at ([xshift=0.5cm] theta) [below] {$a <_3 b$};
\node (b1) at ([xshift=-1cm] theta) [below] {$b <_1 c$};
\node (b2) at ([xshift=0.5cm] theta) [below] {$b <_2 c$};
\node (c1) at ([xshift=-1cm] theta) [below] {$c <_2 a$};
\node (c2) at ([xshift=0.5cm] theta) [below] {$c <_3 a$};
\end{tikzpicture}
\end{center}
\]
```

There are hence six subsets of \( V \) that are maximally consistent with \( \tau \land ECW \land \theta_M \) (with respect to set inclusion). One just needs to remove any of the formulae in the boxes above from \( V \). Each removal will be associated with a linearization of the candidates \( a, b \) and \( c \) in \( <_M \). In other words, the maximally consistent subsets will all have maximal cardinality.

Note that in this particular example, any of the three candidates can be made a Young winner by removing exactly two of the voters (the other two). The choice of winner will depend on the selection function.

**Definition 21** (Majority relation induced by a profile). Let \( P = \{P_1, \ldots, P_n\} \) be a profile and let the relation \( M_P \) on \( C^2 \) be the set \( M_P = \{(c_i, c_j) \mid P_{[i]} > \frac{n}{2}\} \).

Note that \( M_P \) is simply a relation. It may, for instance, contain cycles, as it happens in the case of the Condorcet paradox. We will then be interested in linearizations over the set of candidates that resemble as much as possible to \( M_P \). Note how the similarity with the concept of minimal change in belief revision and belief merging.

More similarities with other social choice concepts, such as that of a Slater winner, can be drawn but have been omitted due to space limitations.
7 Conclusions

We have considered a number of different ways of reasoning about voting as a problem of conciliating contradictory interests. The machineries that do the reconciliation are belief revision and belief merging. Belief merging relies more on meta-level operations than belief revision does. In a sense, it draws a similar distinction to the models of each belief base in the merging process as update does for each model of the belief set.

Now, this observation comes about from the way we “counted” the votes in belief merging and belief revision. In the former, each voter was represented as a separate belief base and the counting was done in the meta-level, unashamedly taking advantage of the fact that we know where each preference of one voter for one candidate over another comes from because each voter has his unique belief base. The counting can be tweaked depending on the aggregation technique used.

When we try to do the same in belief revision, there is no such distinction of bases and hence the only way we can differentiate between the voters is to enrich the language, such that preferences are kept apart, e.g., $a <_1 b$ and $b <_2 a$ for voters 1 and 2. In addition, we also need to “code” the counting mechanism in the logic, because now we can no longer rely on the sophisticated meta-level aggregation procedure. We now have to represent things such as what it means for the majority of voters to prefer $a$ to $b$.

We have shown that this is possible, at least under some controlled constraints, and hence the next question is, can this be further generalised? Our intuition is that merging could be a special case of belief revision, where the language is rich enough to distinguish the origin of the formula and the operator careful enough to evaluate “change” with respect to different partitions of the belief set. We leave this for future work.

References


