String Theory and Branes(7CCMMS34)

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1 Introduction: Whats Up in Modern High Energy Physics Theory?

Why are you taking this course? Why am I, or anyone else in the Theoretical Physics Group, paid? Well here are some reasons. Particle physics, using the machinery of relativistic quantum field theory, has in some sense produced the most successful scientific theory ever known: the so-called Standard Model of Particle Physics. It is the most successful in the sense that no other theory can claim to describe Nature, to such a high level of accuracy over such a complete range of physical phenomena using such a modest number of assumptions and parameters. It is unreasonably good and was never intended to be so successful. Since its formulation around 1970 there has not been a single experimental result that has produced even the slightest disagreement. Nothing, despite an enormous amount of effort. But there are skeletons in the closet. Let me mention just three.

The first is the following: Where does the Standard Model come from? For example it has quite a few parameters which are only fixed by experimental observation. What fixes these? It postulates a certain spectrum of fundamental particle states but why these? In particular these particle states form three families, each of which is a copy of the others, differing only in their masses. Furthermore only the lightest family seems to have much to do with life in the universe as we know it, so why the repetition? It is somewhat analogous to Mendelev's periodic table of the elements. There is clearly a discernible structure but this wasn't understood until the discovery of quantum mechanics. We are looking for the underlying principle that gives the somewhat bizarre and apparently ad hoc structure of the Standard Model.

The second problem is that, for all its strengths, the Standard Model does not include gravity. For that we must use General Relativity which is a classical theory and as such is incompatible with the rules of quantum mechanics. Observationally this is not a problem since the effect of gravity, at the energy scales which we probe, is smaller by a factor of 10^{-40} than the effects of the subnuclear forces which the Standard Model describes. You can experimentally test this assertion by lifting up a piece of paper with your little finger. You will see that the electromagnetic forces at work in your little finger can easily overcome the gravitational force of the entire earth which acts to pull the paper to the floor.

However this is clearly a problem theoretically. We can't claim to understand the universe physically until we can provide one theory which consistently describes gravity and the subnuclear forces. If we do try to include gravity into QFT then we encounter two problems. The first is that the result is non-renormalizable. This means that we cannot use the methods of QFT as a fundamental principle for gravity. Another problem is that the Standard Model makes a prediction for the vacuum energy density. Once gravity is included this will warp spacetime in the form of the so-called cosmological constant. The problem is that the QFT prediction is off by some 10^{120} orders of magnitude

[1]! This is undoubtably the worst prediction of any scientific theory.

The third problem I want to mention is more technical. Quantum field theories generically only make mathematical sense if they are viewed as a low energy theory. Due to the effects of renormalization the Standard Model cannot be valid up to all energy scales, even if gravity was not a problem. Mathematically we know that there must be something else which will manifest itself at some higher energy scale. All we can say is that such new physics must arise before we reach the quantum gravity scale, which is some 10^{17} orders of magnitude above the energy scales that we have tested to date. To the physicists who developed the Standard Model the surprise is that we have not already seen such new physics many years ago.

With these comments in mind this course will introduce string theory, which, for good or bad, has become the dominant, and arguably only, framework for a complete theory of all known physical phenomena. As such it is in some sense a course to introduce the modern view of particle physics at its most fundamental level. Whether or not String Theory is ultimately relevant to our physical universe is unknown, and indeed may never be known. However it has provided many deep and powerful ideas. Certainly it has had a profound effect upon pure mathematics. But an important feature of String Theory is that it naturally includes gravitational and subnuclear-type forces consistently in a manner consistent with quantum mechanics and relativity (as far as anyone knows). Thus it seems fair to say that there is a mathematical framework which is capable of describing all of the physics that we know to be true. This is no small achievement.

However it is also fair to say that no one actually knows what String Theory really is. In any event this course can only attempt to be a modest introduction. There will be much that we will not have time to discuss.

2 Classical and Quantum Dynamics of Point Particles

2.1 Classical Action

We want to describe a single particle moving in spacetime. For now we simply consider flat D-dimensional Minkowski space

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + ... + (dx^{D-1})^2 \tag{2.1}$$

A particle has no spatial extent but it does trace out a curve - its worldline - in spacetime. Furthermore in the absence of external forces this will be a straight line (geodesic if you know GR). In other words the equation of motion should be that the length of the

[1]This is not quite the correct way to think about it. QFT does not predict the vacuum energy since in the process of renormalization one can add an arbitrary bare vacuum energy to arrange for any physical value of the vacuum energy that we like. The point is that to arrange for the observed value requires an absurd amount of fine-tuning.

worldline is extremized. Thus we take

$$\begin{aligned} S_{pp} &= -m\int ds \\ &= -m\int\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}d\tau \end{aligned} \tag{2.2}$$

where τ parameterizes the points along the worldline and $X^\mu(\tau)$ gives the location of the particle in spacetime, *i.e.* the embedding coordinates of the worldline into spacetime.

Let us note some features of this action. Firstly it is manifestly invariant under spacetime Lorentz transformations $X^\mu \to \Lambda^\mu{}_\nu X^\nu$ where $\Lambda^T\eta\Lambda = \eta$. Secondly it is reparameterization invariant under $\tau \to \tau'(\tau)$ for any invertible change of worldline coordinate

$$d\tau = \frac{d\tau}{d\tau'}d\tau' \ , \qquad \dot{X}^\mu = \frac{dX^\mu}{d\tau} = \frac{d\tau'}{d\tau}\frac{dX^\mu}{d\tau'} \tag{2.3}$$

thus

$$\begin{aligned} S_{pp} &= -m\int\sqrt{-\eta_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}}d\tau \\ &= -m\int\sqrt{-\eta_{\mu\nu}\left(\frac{d\tau'}{d\tau}\right)^2\frac{dX^\mu}{d\tau'}\frac{dX^\nu}{d\tau'}}\frac{d\tau}{d\tau'}d\tau' \\ &= -m\int\sqrt{-\eta_{\mu\nu}\frac{dX^\mu}{d\tau'}\frac{dX^\nu}{d\tau'}}d\tau' \end{aligned} \tag{2.4}$$

Thirdly we can see why the m appears in front and with a minus sign by looking at the non-relativistic limit. In this case we choose a gauge for the worldline reparameterization invariance such that $\tau = X^0$ *i.e.* worldline 'time' is the same as spacetime 'time'. This is known as static gauge. It is a gauge choice since, as we have seen, we are free to take any parameterization we like. The nonrelativistic limit corresponds to assuming that $\dot{X}^i << 1$. In this case we can expand

$$S_{pp} = -m\int\sqrt{1-\delta_{ij}\dot{X}^i\dot{X}^j}d\tau = \int -m + \frac{1}{2}m\delta_{ij}\dot{X}^i\dot{X}^j d\tau + \ldots \tag{2.5}$$

where the ellipses denotes terms with higher powers of the velocities $\dot{X}^i$. The second term is just the familiar kinetic energy $\frac{1}{2}mv^2$. The first term is simply a constant and doesn't affect the equations of motion. However it can be interpreted as a constant potential energy equal to the rest mass of the particle. Thus we see that the m and minus signs are correct.

Moving on let us consider the equations of motion and conservation laws that follow from this action. The equations of motion follow from the usual Euler-Lagrange equations applied to S_{pp}:

$$\frac{d}{d\tau}\left(\frac{\dot{X}^\nu}{\sqrt{-\eta_{\lambda\rho}\dot{X}^\lambda\dot{X}^\rho}}\right) = 0 \tag{2.6}$$

These equations can be understood as conservations laws since the Lagrangian is invariant under constant shifts $X^\mu \to X^\mu + b^\mu$. The associated charge is

$$p^\mu = \frac{m\dot{X}^\mu}{\sqrt{-\eta_{\lambda\rho}\dot{X}^\lambda\dot{X}^\rho}} \tag{2.7}$$

so that indeed the equation of motion is just $\dot{p}^\mu = 0$. Note that I have called this a charge and not a current. In this case it doesn't matter because the Lagrangian theory we are talking about, the worldline theory of the point particle, is in zero spatial dimensions. So I could just as well called p^μ a conserved current with the conserved charge being obtained by integrating the temporal component of p^μ over space. Here there is no space p^μ only has temporal components.

Warning: We are thinking in terms of the worldline theory where the index μ labels the different scalar fields X^μ, it does not label the coordinates of the worldline. In particular p^0 is not the temporal component of p^μ from the worldline point of view. This confusion between worldvolume coordinates and spacetime coordinates arises throughout string theory

If we go to static gauge again, where $\tau = X^0$ and write $v^i = \dot{X}^i$ then we have the equations of motion

$$\frac{d}{d\tau}\frac{v^i}{\sqrt{1-v^2}} = 0 \tag{2.8}$$

and conserved charge

$$p^i = m\frac{v^i}{\sqrt{1-v^2}} \tag{2.9}$$

which is simply the spatial momentum. These are the standard relativistic expressions.

We can solve the equation of motion in terms of the constant of motion p^i by writing

$$\frac{v^i}{\sqrt{1-v^2}} = p^i/m \iff p^2/m^2 = \frac{v^2}{1-v^2} \iff v^2 = \frac{p^2}{p^2+m^2} \tag{2.10}$$

hence

$$X^i(\tau) = X^i(0) + \frac{p^i\tau}{\sqrt{p^2+m^2}} \tag{2.11}$$

and we see that v^i is constant with $v^2 < 1$.

2.2 Electromagnetic field

Next we can consider a particle interacting with an external electromagnetic field. An electromagnetic field is described by a vector potential A_μ and its field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The natural action of a point particle of mass m and charge q in the presence of such an electromagnetic field is

$$S_{pp} = -m\int\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}d\tau + q\int A_\mu(X)\dot{X}^\mu d\tau \tag{2.12}$$

For those who know differential geometry the vector potential is a connection one-form on spacetime and $A_\mu \dot{X}^\mu d\tau$ is simply the pull-back of A_μ to the worldline of the particle.

The equation of motion is now

$$-m\frac{d}{d\tau}\left(\frac{\eta_{\mu\nu}\dot{X}^\nu}{\sqrt{-\eta_{\lambda\rho}\dot{X}^\lambda\dot{X}^\rho}}\right) - q\frac{d}{d\tau}A_\mu + q\partial_\mu A_\nu \dot{X}^\nu = 0 \tag{2.13}$$

which we rewrite as

$$m\frac{d}{d\tau}\left(\frac{\eta_{\mu\nu}\dot{X}^\nu}{\sqrt{-\eta_{\lambda\rho}\dot{X}^\lambda\dot{X}^\rho}}\right) = qF_{\mu\nu}\dot{X}^\nu \tag{2.14}$$

To be more concrete we could choose static gauge again and we find

$$m\frac{d}{d\tau}\left(\frac{v^i}{\sqrt{1-v^2}}\right) = qF_{i0} + qF_{ij}v^j \tag{2.15}$$

Here we see the Lorentz force magnetic law arising as it should from the second term on the right hand side. The first term on the right hand side shows that an electric field provides a driving force.

At this point we should pause to mention a subtlety. In addition to (2.15) there is also the equation of motion for $X^0 = \tau$. However the reparameterization gauge symmetry implies that this equation is automatically satisfied. In particular the X^0 equation of motion is

$$-m\frac{d}{d\tau}\left(\frac{1}{\sqrt{1-v^2}}\right) = qF_{0i}v^i \tag{2.16}$$

Problem: Show that if (2.15) is satisfied then so is (2.16)

Problem: Show that, in static gauge $X^0 = \tau$, the Hamiltonian for a charged particle is

$$H = \sqrt{m^2 + (p^i - qA^i)(p^i - qA^i)} - qA_0 \tag{2.17}$$

2.3 Quantization

Next we'd like to quantize the point particle. This is made difficult by the highly non-linear form of the action. To this end we will consider a new action which is classically equivalent to the old one. In particular consider

$$S_{HT} = -\frac{1}{2}\int d\tau e\left(-\frac{1}{e^2}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} + m^2\right) \tag{2.18}$$

Here we have introduced a new field $e(\tau)$ which is non-dynamical, *i.e.* has no kinetic term. This action is now just quadratic in the fields X^μ. The point of it is that it reproduces the same equations of motion as before. To see this consider the e equation of motion:

$$\frac{1}{e^2}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} + m^2 = 0 \tag{2.19}$$

we can solve this to find $e = m^{-1}\sqrt{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}}$. We now compute the X^μ equation of motion

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(\frac{1}{e}\dot{X}^\mu\right) \\ &= m\frac{d}{d\tau}\left(\frac{\dot{X}^\mu}{\sqrt{-\dot{X}^\lambda \dot{X}^\rho \eta_{\lambda\rho}}}\right) \end{aligned} \tag{2.20}$$

This is exactly what we want. Thus we conclude that S_{HT} is classically equivalent to S_{pp}.

One way to think about this action is that we have introduced a worldline metric $\gamma_{\tau\tau} = -e^2$ and its inverse $\gamma^{\tau\tau} = -1/e^2$ so that infinitesimal distances along the worldline have length

$$ds^2 = \gamma_{\tau\tau} d\tau^2 \tag{2.21}$$

Note that previously we never said that $d\tau$ represented the physical length of a piece of worldline, just that τ labeled points along the worldline.

There is another advantage to this form of the action; we can smoothly set $m^2 = 0$ and describe massless particles, which was impossible with the original form of the action.

Now the action is quadratic in the fields X^μ we calculate the Hamiltonian and quantize more easily. The first step here is to obtain the momentum conjugate to each of the X^μ

$$\begin{aligned} p_\mu &= \frac{\partial L}{\partial \dot{X}^\mu} \\ &= \frac{1}{e}\eta_{\mu\nu}\dot{X}^\nu \end{aligned} \tag{2.22}$$

There is no conjugate momentum to e, it acts as a constraint and we will deal with it later. The Hamiltonian is

$$\begin{aligned} H &= p_\mu \dot{X}^\mu - L \\ &= \frac{e}{2}\left(\eta_{\mu\nu}p^\mu p^\nu + m^2\right) \end{aligned} \tag{2.23}$$

To quantize this system we consider wavefunctions $\Psi(X, \tau)$ and promote X^μ and p_μ to the operators

$$\hat{X}^\mu \Psi = X^\mu \Psi \qquad \hat{p}_\mu \Psi = -i\frac{\partial \Psi}{\partial X^\mu} \tag{2.24}$$

We then arrive at the Schrodinger equation

$$i\frac{\partial \Psi}{\partial \tau} = \frac{e}{2}\left(-\eta^{\mu\nu}\frac{\partial^2 \Psi}{\partial X^\mu \partial X^\nu} - m^2\Psi\right) \tag{2.25}$$

Lastly we must deal with e which we saw has no conjugate momentum. Classically its equation of motion imposes the constraint

$$p^\mu p_\mu + m^2 = 0 \tag{2.26}$$

which is the mass-shell condition for the particle. Quantum mechanically we realize this by restricting our physical wavefunctions to those that satisfy the corresponding constraint

$$-\eta^{\mu\nu}\frac{\partial^2\Psi}{\partial X^\mu \partial X^\nu} + m^2\Psi = 0 \tag{2.27}$$

However this is just the condition that $\hat{H}\Psi = 0$ so that the wavefunctions are τ independent. If you trace back the origin of this time-independence it arises as a consequence of the reparameterization invariance of the worldline theory. It simply states that wavefunctions must also be reparameterization invariant, *i.e.* they can't depend on τ. This is deep issue in quantum gravity. In effect it says that there is no such thing as time in the quantum theory.

This equation should be familiar if you have learnt quantum field theory. In particular if we consider a free scalar field Ψ in D-dimensional spacetime the action is

$$S = -\int d^D x \left(\frac{1}{2}\partial_\mu\Psi^*\partial^\mu\Psi + \frac{1}{2}m^2\Psi^*\Psi\right) \tag{2.28}$$

and the corresponding equation of motion is

$$\partial^2\Psi - m^2\Psi = 0 \tag{2.29}$$

which is the same as our Schrodinger equation (when restricted to the physical Hilbert space).

Thus we see that there is a natural identification of a free scalar field with a quantum point particle. In particular the quantum states of the point particle are in a one-to-one correspondence with the classical solutions of the free spacetime effective action. However one important difference should be stressed. The quantum point particle gave a Schrodinger equation which could be identified with the classical equation of motion for the scalar field. In quantum field theory one performs a second quantization whereby particles are allowed to be created and destroyed. This is beyond the quantization of the point particle that we considered since by default we studied the effective action on the worldline of a single particle: it would have made no sense to create or destroy particles. Thus the second quantized spacetime action provides a more complete physical theory.

Here we also can see that the quantum description of a point particle in one dimension leads to a classical spacetime effective action in D-dimensions. This is an important concept in String theory where the quantum dynamics of the two-dimensional worldvolume theory, with interactions included, leads to interesting and non-trivial spacetime effective actions.

Problem: Find the Schödinger equation, constraint and effective action for a quantized particle in the background of a classical electromagnetic field using the action

$$S_{pp} = -\int \frac{1}{2}e\left(-\frac{1}{e^2}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} + m^2\right) - A_\mu\dot{X}^\mu \tag{2.30}$$

3 Classical and Quantum Dynamics of Strings

3.1 Classical Action

Having studied point particles from their worldline perspective we now turn to our main subject: strings. Our starting point will be the action the worldvolume of a string, which is two-dimensional. The natural starting point is to consider the action

$$S_{string} = \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det(\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu})} \tag{3.1}$$

which is simply the area of the two-dimensional worldvolume that the string sweeps out. Here $\sigma^\alpha, \alpha = 0,1$ labels the spatial and temporal coordinates of the string: τ, σ. Here $\sqrt{\alpha'}$ is a length scale that determines the size of the string.

Again we don't want to work directly with such a highly non-linear action. We saw above that we could change this by coupling to an auxiliary worldvolume metric $\gamma_{\alpha\beta}$.

Problem: Show that by solving the equation of motion for the metric $\gamma_{\alpha\beta}$ on a d-dimensional worldsheet the action

$$S_{HT} = -\frac{1}{2}\int d^d\sigma\sqrt{-\gamma}\left(\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} + m^2(d-2)\right) \tag{3.2}$$

one finds the action

$$S_{NG} = m^{2-d}\int d^d\sigma\sqrt{-\det\left(\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}\right)} \tag{3.3}$$

for the remaining fields X^μ, *i.e.* calculate and solve the $\gamma_{\alpha\beta}$ equation of motion and then substitute the solution back into S_{HT} to obtain S_{NG}. Note that the action S_{HT} is often referred to as the Howe-Tucker form for the action whereas S_{NG} is the Nambu-Goto form. (Hint: You will need to use the fact that $\delta\sqrt{-\gamma}/\delta\gamma^{\alpha\beta} = -\frac{1}{2}\gamma_{\alpha\beta}\sqrt{-\gamma}$). If you have not yet learnt much about metrics just consider the case of $d = 1$ where $\gamma_{\alpha\beta}$ just has a single component $\gamma_{\tau\tau}$.

So we might instead start with

$$S_{string} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} \tag{3.4}$$

where we have taken $d = 2$ in (3.2).

Problem: What transformation law must $\gamma_{\alpha\beta}$ have to ensure that S_{string} is reparameterization invariant? (Hint: Use the fact that

$$\frac{\partial \sigma'^{\gamma}}{\partial \sigma^{\alpha}} \frac{\partial \sigma^{\beta}}{\partial \sigma'^{\gamma}} = \delta_{\alpha}^{\beta} \tag{3.5}$$

why?)

However this case is very special. If we evaluate the $\gamma_{\alpha\beta}$ equation of motion we find

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} = 0 \tag{3.6}$$

Once again we see that $\gamma_{\alpha\beta} = b\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$ for some b. However in this case nothing fixes b, it is arbitrary. This occurs because there is an addition symmetry of the action that is unique to two-dimensions: it is conformally invariant. That means that under a worldvolume conformal transformation

$$\gamma_{\alpha\beta} \to e^{2\varphi}\gamma_{\alpha\beta} \tag{3.7}$$

(here φ is any function of the worldvolume coordinates) the action is invariant.

There are other features that are unique to two-dimensions. The first is that, up to a reparameterization, we can always choose the metric $\gamma_{\alpha\beta} = e^{2\rho}\eta_{\alpha\beta}$. To see this we note that under a reparameterization we have

$$\gamma'_{\alpha\beta} = \frac{\partial \sigma^{\gamma}}{\partial \sigma'^{\alpha}} \frac{\partial \sigma^{\delta}}{\partial \sigma'^{\beta}} \gamma_{\gamma\delta} \tag{3.8}$$

Thus we simply choose our new coordinates to fix $\gamma'_{01} = 0$ and $\gamma'_{00} = -\gamma'_{11}$. This requires that

$$\frac{\partial \sigma^{\gamma}}{\partial \sigma'^{0}} \frac{\partial \sigma^{\delta}}{\partial \sigma'^{1}} \gamma_{\gamma\delta} = 0 \tag{3.9}$$

and

$$\frac{\partial \sigma^{\gamma}}{\partial \sigma'^{1}} \frac{\partial \sigma^{\delta}}{\partial \sigma'^{1}} \gamma_{\gamma\delta} + \frac{\partial \sigma^{\gamma}}{\partial \sigma'^{0}} \frac{\partial \sigma^{\delta}}{\partial \sigma'^{0}} \gamma_{\gamma\delta} = 0 \tag{3.10}$$

These are two (complicated) differential equation for two functions $\sigma^0(\sigma'^0, \sigma'^1)$ and $\sigma^1(\sigma'^0, \sigma'^1)$. Hence there will be a solution (at least locally).

The second feature is that in two-dimensions the Einstein equation

$$R_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}R = 0 \tag{3.11}$$

vanishes identically. The reason for this is that in two-dimensions there is only one independent component for the Riemann tensor: $R_{0101} = -R_{0110} = -R_{1001} = R_{1010}$. Therefore $R_{00} = R_{0101}\gamma^{11}$, $R_{11} = R_{0101}\gamma^{00}$ and $R_{01} = -R_{0101}\gamma^{01}$. Thus we see that

$$\begin{aligned} R &= 2R_{0101}(\gamma^{00}\gamma^{11} - \gamma^{01}\gamma^{01}) \\ &= 2R_{0101}\det(\gamma^{-1}) \\ &= \frac{2}{\det(\gamma)}R_{0101} \end{aligned} \tag{3.12}$$

Now we note that
$$\begin{pmatrix} \gamma^{00} & \gamma^{01} \\ \gamma^{01} & \gamma^{11} \end{pmatrix} = \frac{1}{\det(\gamma)} \begin{pmatrix} \gamma_{11} & -\gamma_{01} \\ -\gamma_{01} & \gamma_{00} \end{pmatrix} \tag{3.13}$$
and the result follows.

Thus Einstein's equation
$$R_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}R = T_{\alpha\beta} \tag{3.14}$$
will imply that $T_{\alpha\beta} = 0$. Hence even if we include two-dimensional gravity the $\gamma_{\alpha\beta}$ equation of motion imposes the constraint that the energy-momentum tensor vanishes
$$T_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial \gamma^{\alpha\beta}} = 0 \tag{3.15}$$

These facts together imply that the worldvolume metric $\gamma_{\alpha\beta}$ actually decouples from the fields X^μ. This conformal invariance of two-dimensional gravity coupled to the embedding coordinates (viewed as scalar fields) will be our fundamental principle. It allows us to simply set $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$. Thus to consider strings propagating in flat spacetime we use the action (known as the Polyakov action)
$$S_{string} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{3.16}$$
subject to the constraint (3.15) which becomes
$$\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} = 0 \tag{3.17}$$

3.2 Spacetime Symmetries and Conserved Charges

We should also pause to outline how the spacetime symmetries lead to conserved currents and hence conserved charges in the worldsheet theory.

First we summarize Noether's theorem. Suppose that a Lagrangian $\mathcal{L}(\Phi_A, \partial_\alpha \Phi_A)$, where we denoted the fields by Φ_A, has a symmetry: $\mathcal{L}(\Phi_A) = \mathcal{L}(\Phi_A + \delta\Phi_A)$. This implies that
$$\frac{\partial \mathcal{L}}{\partial \Phi_A}\delta\Phi_A + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\delta\partial_\alpha\Phi_A = 0 \tag{3.18}$$
This allows us to construct a current:
$$J^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\delta\Phi_A \tag{3.19}$$
which is conserved
$$\begin{aligned} \partial_\alpha J^\alpha &= \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\right)\delta\Phi_A + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\partial_\alpha\delta\Phi_A \\ &= \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\right)\delta\Phi_A - \frac{\partial \mathcal{L}}{\partial \Phi_A}\delta\Phi_A \\ &= 0 \end{aligned} \tag{3.20}$$

by the equation of motion. This means that the integral over space of J^0 is a constant defines a charge

$$Q = \int_{space} \sigma J^0 \tag{3.21}$$

which is conserved

$$\begin{aligned} \frac{dQ}{dt} &= \int_{space} \partial_0 J^0 \\ &= -\int_{space} \partial_i J^i \\ &= 0 \end{aligned}$$

Let us now consider the action

$$S_{string} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{3.22}$$

This has the spacetime Poincare symmetries: translations $\delta X^\mu = a^\mu$ and Lorentz transformations $\delta X^\mu = \Lambda^\mu{}_\nu X^\nu$. In the first case the conserved current is

$$P^\alpha_{a^\mu} = -\frac{1}{2\pi\alpha'} \partial^\alpha X_\mu a^\mu \tag{3.23}$$

The associated conserved charge is just the total momentum along the direction a^μ and in particular there are D independent choices

$$p_\mu = \frac{1}{2\pi\alpha'} \int d\sigma \dot{X}_\mu \tag{3.24}$$

We can also consider the spacetime Lorentz transformations which lead to the conserved currents

$$J^\alpha_\Lambda = -\frac{1}{2\pi\alpha'} \partial^\alpha X_\mu \Lambda^\mu{}_\nu X^\nu \tag{3.25}$$

The independent conserved charges are therefore given by

$$M^\mu{}_\nu = \frac{1}{4\pi\alpha'} \int d\sigma \dot{X}^\mu X_\nu - X^\mu \dot{X}_\nu \tag{3.26}$$

The Poisson brackets of these worldsheet charges will, at least at the classical level, satisfy the algebra Poincare algebra. In the quantum theory they are lifted to operators that commute with the Hamiltonian.

3.3 Quantization

Next we wish to quantize this action. Unlike the point particle this action is a field theory in $(1 + 1)$-dimensions. As such we must use the quantization techniques of quantum field theory rather than simply constructing a Schrodinger equation. There are several ways to do this. The most modern way is the path integral formulation.

However this requires some techniques that are presumably unfamiliar. So here we will use the method of canonical quantization.

Canonical quantization is essentially the Heisenberg picture of quantum mechanics where the fields X^μ and their conjugate momenta P_μ are promoted to operators which satisfy the equal time commutation relations

$$\begin{aligned}
[\hat{X}^\mu(\tau,\sigma),\hat{P}_\nu(\tau,\sigma')] &= i\delta(\sigma-\sigma')\delta^\mu_\nu \\
[\hat{X}^\mu(\tau,\sigma),\hat{X}^\nu(\tau,\sigma')] &= 0 \\
[\hat{P}_\mu(\tau,\sigma),\hat{P}_\nu(\tau,\sigma')] &= 0
\end{aligned} \tag{3.27}$$

as well as the Heisenberg equation

$$\frac{d\hat{X}^\mu}{d\tau} = i[\hat{H},\hat{X}^\mu] \qquad \frac{d\hat{P}_\mu}{d\tau} = i[\hat{H},\hat{P}_\mu] \tag{3.28}$$

In the case at hand we have

$$\hat{L} = \frac{1}{4\pi\alpha'}\int d\sigma \eta_{\mu\nu}\dot{\hat{X}}^\mu\dot{\hat{X}}^\nu - \eta_{\mu\nu}\hat{X}'^\mu\hat{X}'^\nu \tag{3.29}$$

hence

$$\hat{P}_\mu = \frac{1}{2\pi\alpha'}\eta_{\mu\nu}\dot{\hat{X}}^\nu \tag{3.30}$$

and

$$\begin{aligned}
\hat{H} &= \int d\sigma \hat{P}_\mu\dot{\hat{X}}^\mu - \hat{L} \\
&= \int d\sigma 2\pi\alpha'\eta^{\mu\nu}\hat{P}_\mu\hat{P}_\nu - \int d\sigma\frac{1}{4\pi\alpha'}(2\pi\alpha')^2\eta^{\mu\nu}\hat{P}_\mu\hat{P}_\nu + \frac{1}{4\pi\alpha'}\eta_{\mu\nu}\hat{X}'^\mu\hat{X}'^\nu \\
&= \int d\sigma\pi\alpha'\eta^{\mu\nu}\hat{P}_\mu\hat{P}_\nu + \frac{1}{4\pi\alpha'}\eta_{\mu\nu}\hat{X}'^\mu\hat{X}'^\nu
\end{aligned} \tag{3.31}$$

We can now calculate

$$\begin{aligned}
\dot{\hat{X}}^\mu(\sigma) &= i[\hat{H},\hat{X}^\mu(\sigma)] \\
&= \pi\alpha' i\int d\sigma'\eta^{\lambda\nu}[\hat{P}_\lambda(\sigma')\hat{P}_\nu(\sigma'),\hat{X}^\mu(\sigma)] \\
&= 2\pi\alpha' i\int d\sigma'\eta^{\lambda\nu}\hat{P}_\lambda(\sigma')[\hat{P}_\nu(\sigma'),\hat{X}^\mu(\sigma)] \\
&= 2\pi\alpha'\int d\sigma'\eta^{\lambda\nu}\hat{P}_\lambda(\sigma')\delta^\mu_\nu\delta(\sigma-\sigma') \\
&= 2\pi\alpha'\eta^{\mu\nu}\hat{P}_\nu(\sigma)
\end{aligned} \tag{3.32}$$

which we already knew. But also we can now calculate

$$\begin{aligned}
\dot{\hat{P}}_\mu(\sigma) &= i[\hat{H}, \hat{P}_\mu(\sigma)] \\
&= \frac{i}{4\pi\alpha'}\int d\sigma' \eta_{\lambda\nu}[\hat{X}'^\lambda(\sigma')\hat{X}'^\nu(\sigma'), \hat{P}_\mu(\sigma)] \\
&= \frac{i}{2\pi\alpha'}\int d\sigma' \eta_{\lambda\nu}\hat{X}'^\lambda(\sigma')[\hat{X}'^\nu(\sigma'), \hat{P}_\mu(\sigma)] \\
&= \frac{i}{2\pi\alpha'}\int d\sigma' \eta_{\lambda\nu}\hat{X}'^\lambda(\sigma')\frac{\partial}{\partial\sigma'}[\hat{X}^\nu(\sigma'), \hat{P}_\mu(\sigma)] \\
&= -\frac{i}{2\pi\alpha'}\int d\sigma' \eta_{\lambda\nu}\hat{X}''^\lambda(\sigma')[\hat{X}^\nu(\sigma'), \hat{P}_\mu(\sigma)] \\
&= \frac{1}{2\pi\alpha'}\int d\sigma' \eta_{\lambda\nu}\hat{X}''^\lambda(\sigma')\delta^\nu_\mu\delta(\sigma-\sigma') \\
&= \frac{1}{2\pi\alpha'}\eta_{\mu\nu}\hat{X}''^\nu(\sigma)
\end{aligned} \tag{3.33}$$

or equivalently

$$-\ddot{\hat{X}}^\mu + \hat{X}''^\mu = 0 \tag{3.34}$$

Of course this is just the classical equation of motion reinterpreted in the quantum theory as an operator equation. In two-dimensions the solution to this is simply that

$$\hat{X}^\mu = \hat{X}^\mu_L(\tau+\sigma) + \hat{X}^\mu_R(\tau-\sigma) \tag{3.35}$$

i.e. we can split $\hat{X}^\mu$ into a left and right moving part.

To proceed we expand the string in a Fourier series

$$\hat{X}^\mu = x^\mu + w^\mu\sigma + \alpha' p^\mu\tau + \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\left(\frac{a^\mu_n}{n}e^{-in(\tau+\sigma)} + \frac{\tilde{a}^\mu_n}{n}e^{-in(\tau-\sigma)}\right) \tag{3.36}$$

The various factors of n and α' will turn out to be useful later on. We have also included linear terms since $\hat{X}^\mu$ need not be periodic (more on this later). Or if you prefer

$$\begin{aligned}
\hat{X}^\mu_L &= x^\mu_L + \frac{1}{2}(\alpha' p^\mu + w^\mu)(\tau+\sigma) + \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\frac{a^\mu_n}{n}e^{-in(\tau+\sigma)} \\
\hat{X}^\mu_R &= x^\mu_R + \frac{1}{2}(\alpha' p^\mu - w^\mu)(\tau-\sigma) + \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\frac{\tilde{a}^\mu_n}{n}e^{-in(\tau-\sigma)}
\end{aligned} \tag{3.37}$$

Note that we have dropped the hat on the operators a^μ and $\tilde{a}^\mu$ since they will appear frequently. But don't forget that they are operators! Note also that we haven't yet said what n is, *e.g.* whether or not it is an integer, we will be more specific later. The a^μ_n and $\tilde{a}^\mu_n$ have the interpretation as left and right moving oscillators. Just as in quantum mechanics and quantum field theory these will be related to particle creation and annihilation operators.

Since X^μ is an observable we require that it is Hermitian in the quantum theory. This in turn implies that

$$(a_n^\mu)^\dagger = a_{-n}^\mu \ , \qquad (\tilde{a}_n^\mu)^\dagger = \tilde{a}_{-n}^\mu \tag{3.38}$$

and $(x^\mu)^\dagger = x^\mu, (w^\mu)^\dagger = w^\mu, (p^\mu)^\dagger = p^\mu$. In this basis

$$\begin{aligned} \hat{P}^\mu &= \frac{1}{2\pi\alpha'}\dot{\hat{X}}^\mu \\ &= \frac{1}{2\pi\alpha'}\left(\alpha' p^\mu + \sqrt{\frac{\alpha'}{2}}\sum_{-n\neq 0} a_n^\mu e^{-in(\tau+\sigma)} + \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\tilde{a}_n^\mu e^{-in(\tau-\sigma)}\right) \end{aligned} \tag{3.39}$$

We can work out the commutator. First we take $x^\mu = w^\mu = p^\mu = 0$

$$\begin{aligned} [\hat{X}^\mu(\tau,\sigma), \hat{P}_\nu(\tau,\sigma')] &= \frac{i}{4\pi}\sum_n\sum_m \frac{1}{n}e^{-i(n+m)\tau}e^{-i(n\sigma+m\sigma')}[a_n^\mu, a_m^\nu] \\ &+\frac{i}{4\pi}\sum_n\sum_m \frac{1}{n}e^{-i(n+m)\tau}e^{i(n\sigma+m\sigma')}[\tilde{a}_n^\mu, \tilde{a}_m^\nu] \\ &+\frac{i}{4\pi}\sum_n\sum_m \frac{1}{n}e^{-i(n+m)\tau}e^{i(n\sigma-m\sigma')}[\tilde{a}_n^\mu, a_m^\nu] \\ &+\frac{i}{4\pi}\sum_n\sum_m \frac{1}{n}e^{-i(n+m)\tau}e^{-i(n\sigma-m\sigma')}[a_n^\mu, \tilde{a}_m^\nu] \end{aligned} \tag{3.40}$$

In order for the τ-dependent terms to cancel we see that we need the commutators to vanish if $n \neq -m$. The sum now reduces to

$$\begin{aligned} [\hat{X}^\mu(\tau,\sigma), \hat{P}^\nu(\tau,\sigma')] &= \frac{i}{4\pi}\sum_n \frac{1}{n}e^{-in(\sigma-\sigma')}[a_n^\mu, a_{-n}^\nu] \\ &+\frac{i}{4\pi}\sum_n \frac{1}{n}e^{in(\sigma-\sigma')}[\tilde{a}_n^\mu, \tilde{a}_{-n}^\nu] \\ &+\frac{i}{4\pi}\sum_n \frac{1}{n}e^{in(\sigma+\sigma')}[\tilde{a}_n^\mu, a_{-n}^\nu] \\ &+\frac{i}{4\pi}\sum_n \frac{1}{n}e^{-in(\sigma+\sigma')}[a_n^\mu, \tilde{a}_{-n}^\nu] \end{aligned} \tag{3.41}$$

Next translational invariance implies that the $\sigma+\sigma'$ terms vanish and hence

$$[a_n^\mu, \tilde{a}_m^\nu] = 0 \tag{3.42}$$

A slight rearrangement of indices shows that we are left with

$$[\hat{X}^\mu(\tau,\sigma), \hat{P}^\nu(\tau,\sigma')] = \frac{i}{4\pi}\sum_n \frac{1}{n}e^{-in(\sigma-\sigma')}([a_n^\mu, a_{-n}^\nu] + [\tilde{a}_n^\mu, \tilde{a}_{-n}^\nu]) \tag{3.43}$$

In a Fourier basis

$$\delta(\sigma-\sigma') = \frac{1}{2\pi}\sum_n e^{-in(\sigma-\sigma')} \tag{3.44}$$

Note that there is a contribution from $n=0$ here that doesn't come from the oscillators, we'll deal with it in a moment. Therefore we see that we must take

$$[a^\mu_n, a^\nu_m] = n\eta^{\mu\nu}\delta_{n,-m}\ , \qquad [\tilde{a}^\mu_n, \tilde{a}^\nu_m] = n\eta^{\mu\nu}\delta_{n,-m} \tag{3.45}$$

Next it remains to consider the zero-modes (including the $n=0$ contribution in (3.44)).

Problem: Show that if $x^\mu, w^\mu, p^\mu \neq 0$ then we also have

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \tag{3.46}$$

with the other commutators vanishing.

We also have to consider the constraint $\hat{T}_{\alpha\beta}=0$. Its components are

$$\begin{aligned}
\hat{T}_{00} &= \frac{1}{2}\dot{\hat{X}}^\mu\dot{\hat{X}}^\nu\eta_{\mu\nu} + \frac{1}{2}\hat{X}'^\mu\hat{X}'^\nu\eta_{\mu\nu} \\
\hat{T}_{11} &= \frac{1}{2}\hat{X}'^\mu\hat{X}'^\nu\eta_{\mu\nu} + \frac{1}{2}\dot{\hat{X}}^\mu\dot{\hat{X}}^\nu\eta_{\mu\nu} \\
\hat{T}_{01} &= \dot{\hat{X}}^\mu\hat{X}'^\nu\eta_{\mu\nu}
\end{aligned} \tag{3.47}$$

It is helpful to change coordinates to

$$\begin{array}{l} \sigma^+ = \tau+\sigma \\ \sigma^- = \tau-\sigma \end{array} \iff \begin{array}{l} \tau = \frac{\sigma^+ + \sigma^-}{2} \\ \sigma = \frac{\sigma^+ - \sigma^-}{2} \end{array} \tag{3.48}$$

Problem: Show that in these coordinates

$$\begin{aligned}
\hat{T}_{++} &= \partial_+\hat{X}^\mu\partial_+\hat{X}^\nu\eta_{\mu\nu} \\
\hat{T}_{--} &= \partial_-\hat{X}^\mu\partial_-\hat{X}^\nu\eta_{\mu\nu} \\
\hat{T}_{+-} &= T_{-+} = 0
\end{aligned} \tag{3.49}$$

Let us now calculate T_{++} in terms of oscillators. We have

$$\partial_+\hat{X}^\mu = \sqrt{\frac{\alpha'}{2}}\sum_{n=-\infty}^{\infty} a^\mu_n e^{-in(\tau+\sigma)} \tag{3.50}$$

where we have introduced

$$a^\mu_0 = \sqrt{\frac{\alpha'}{2}}p^\mu + \sqrt{\frac{1}{2\alpha'}}w^\mu \tag{3.51}$$

thus

$$\begin{aligned}\hat{T}_{++} &= \frac{\alpha'}{2}\sum_{nm} a^\mu_n a^\nu_m e^{-i(n+m)(\tau+\sigma)}\eta_{\mu\nu} \\ &= \alpha'\sum_n L_n e^{-in(\tau+\sigma)}\end{aligned} \tag{3.52}$$

with

$$L_n = \frac{1}{2}\sum_m a^\mu_{n-m}a^\nu_m\eta_{\mu\nu} \tag{3.53}$$

where again we've dropped a hat on L_n, even though it is an operator. Similarly we find

$$T_{--} = \alpha'\sum_n \tilde{L}_n e^{-in(\tau-\sigma)} \tag{3.54}$$

with

$$\tilde{L}_n = \frac{1}{2}\sum_m \tilde{a}^\mu_{n-m}\tilde{a}^\nu_m\eta_{\mu\nu} \tag{3.55}$$

and

$$\tilde{a}^\mu_0 = \sqrt{\frac{\alpha'}{2}}p^\mu - \sqrt{\frac{2}{\alpha'}}w^\mu \tag{3.56}$$

We can rewrite the commutators (3.45) using (3.38) as

$$[a^\mu_n, a^{\nu\dagger}_n] = n\eta^{\mu\nu} \qquad [\tilde{a}^\mu_n, \tilde{a}^{\nu\dagger}_n] = n\eta^{\mu\nu} \tag{3.57}$$

with $n > 0$. Thus we can think of a^μ_n and $\tilde{a}^\mu_n$ annihilation operators and $a^{\mu\dagger}_n$ and $\tilde{a}^{\mu\dagger}_n$ as creation operators. Following the standard practice of QFT we consider the ground state $|0>$ to be annihilated by a_n and $\tilde{a}_n$:

$$a_n|0>= 0\ , \qquad \tilde{a}_n|0>= 0\ , \qquad n > 0 \tag{3.58}$$

The zero modes also act on the ground state. Since x^μ and p^μ don't commute we can only chose $|0>$ to be an eigenstate of one, we take

$$\hat{p}^\mu|0>= p^\mu|0> \qquad \hat{w}^\mu|0>= w^\mu|0> \tag{3.59}$$

when we want to be precise we label the ground state $|0; p, w>$. You will have to excuse the clumsy notion where I have reintroduce a hat on an operator to distinguish it from its eigenvalue acting on a state. We can now construct a Fock space of multi-particle states by acting on the ground state with the creation operators a^μ_{-n} and $\tilde{a}^\mu_{-n}$. For example

$$a^\mu_{-1}\tilde{a}^\nu_{-1}|0>\ , \qquad a^\mu_{-2}\tilde{a}^\lambda_{-1}\tilde{a}^\rho_{-1}|0>\ , \qquad etc. \tag{3.60}$$

These elements should be familiar from the study of the harmonic oscillator. In a string theory each classical vibrational mode is mapped in the quantum theory to an individual harmonic oscillator with the same frequency.

Note that we really should considering normal ordered operators, where the annihilation operators always appear to the right of the creation operators. For L_n and $\tilde{L}_n$ with $n \neq 0$ there is no ambiguity as a^μ_m and a^ν_{n-m} will commute. However for L_0 and $\tilde{L}_0$ one finds

$$L_0 = \frac{1}{2}a^\mu_0 a^\nu_0 \eta_{\mu\nu} + \sum_{m>0} a^\mu_{-m} a^\nu_m \eta_{\mu\nu} - \frac{1}{2}\sum_{m>0}[a^\mu_{-m}, a^\nu_m]\eta_{\mu\nu} \tag{3.61}$$

The last term is an infinite divergent sum This can be thought of as sum over the zero-point energies of the infinite number of harmonic oscillators. We must renormalize. Clearly $\tilde{L}_0$ has the same problem and this introduces the same sum. Since this is just a number the end result is that we define the normal ordered L_0 and $\tilde{L}_0$ to be

$$\begin{aligned} :L_0: &= \frac{1}{2}a^\mu_0 a^\nu_0 \eta_{\mu\nu} + \alpha' \sum_{m>0} a^\mu_{-m} a^\nu_m \eta_{\mu\nu} \\ :\tilde{L}_0: &= \frac{1}{2}\tilde{a}^\mu_0 \tilde{a}^\nu_0 \eta_{\mu\nu} + \alpha' \sum_{m>0} \tilde{a}^\mu_{-m} \tilde{a}^\nu_m \eta_{\mu\nu} \end{aligned} \tag{3.62}$$

In string theory $:L_n:$ and $:\tilde{L}_n:$ play a central role.

How do we deal with constraints in the quantum theory? We should proceed by reducing to the so-called physical Hilbert space of states which are those states that are animated by $:\hat{T}_{\alpha\beta}:$. However this turns out to be too strong a condition and would remove all states. Instead we impose that the positive frequency components of $:\hat{T}_{\alpha\beta}:$ annihilates any physical state

$$:L_n:|phys>=:\tilde{L}_n:|phys>=0\,, n>0 \qquad (:L_0:-a)|phys>=(:\tilde{L}_0:-a)|phys>=0 \tag{3.63}$$

Here we have introduced a parameter a since $:L_0:$ differs from L_0 by an infinite constant that we must regularize to the finite value a. For historical reasons the parameter a is called the intercept (and α' the slope). However it is not a parameter but rather is fixed by consistency conditions. Indeed it can be calculated by a variety of methods (such as ζ-function regularization or by using the modern BRST approach to quantization). We will see that the correct value is $a = 1$.

This is then sufficient to show that the expectation value of $:\hat{T}_{\alpha\beta}:$ vanishes

$$<phys|:L_n:|phys>=<phys|:\tilde{L}_n:|phys>=0 \qquad \forall n \neq 0 \tag{3.64}$$

since the state on the right is annihilated by the postiche frequency parts where as by taking the Hermitian conjugates one sees that the state on the left is annihilated by the negative frequency part.

It is helpful to calculate the commutator $[:L_m:,:L_n:]$. There will be a similar expression for $[:\tilde{L}_m:,:\tilde{L}_m:]$ and clearly one has $[:L_m:,:\tilde{L}_n:]=0$. To do this we first consider the case without worrying about normal orderings

$$[L_m, L_n] = \frac{1}{4}\sum_{pq}[a^\mu_{m-p}a^\nu_p, a^\lambda_{n-q}a^\rho_q]\eta_{\mu\nu}\eta_{\lambda\rho}$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{pq}\eta_{\mu\nu}\eta_{\lambda\rho}\Big([a^\mu_{m-p}a^\nu_p, a^\lambda_{n-q}]a^\rho_q + a^\lambda_{n-q}[a^\mu_{m-p}a^\nu_p, a^\rho_q]\Big)\\
&= \frac{1}{4}\sum_{pq}\eta_{\mu\nu}\eta_{\lambda\rho}\Big(a^\mu_{m-p}[a^\nu_p, a^\lambda_{n-q}]a^\rho_q + [a^\mu_{m-p}, a^\lambda_{n-q}]a^\nu_p a^\rho_q\\
&\quad +a^\lambda_{n-q}a^\mu_{m-p}[a^\nu_p, a^\rho_q] + a^\lambda_{n-q}[a^\mu_{m-p}, a^\rho_q]a^\nu_p\Big)\\
&= \frac{1}{4}\sum_{p}\eta_{\mu\rho}\Big(pa^\mu_{m-p}a^\rho_{n+p} + (m-p)a^\mu_p a^\rho_{n+m-p}\\
&\quad +pa^\rho_{n+p}a^\mu_{m-p} + (m-p)a^\rho_{n+m-p}a^\mu_p\Big)\\
&= \frac{1}{2}\sum_{p}\eta_{\mu\rho}\Big((p-n)a^\mu_{m+n-p}a^\rho_p \ + (m-p)a^\mu_p a^\rho_{n+m-p}\ \eta_{\mu\rho}\Big)
\end{aligned}
\tag{3.65}
$$

Here we have used the identities

$$
[A, BC] = [A, B]C + B[A, C]\ , \qquad [AB, C] = A[B, C] + [A, C]B \tag{3.66}
$$

and shifted the p=variable in the sum. Thus we find

$$
[L_m, L_n] = (m-n)L_{m+n} \tag{3.67}
$$

This is called the classical Virasoro algebra and is of crucial importance in string theory and conformal field theory in general. Recall that it is the algebra of constraints that arose from the condition $\hat{T}_{\alpha\beta} = 0$ which is the statement of conformal invariance.

In the quantum theory we must consider the issues associated with normal ordering. We saw that this only affected $: L_0 :$. It follows that the only effect this can have on the Virasoro algebra is in terms with an $: L_0 :$. Since the effect on $: L_0 :$ is a shift by an infinite constant it won't appear in the commutator on the left hand side. Thus any new terms can only appear with $: L_0 :$ on the right hand side. Thus the general form is

$$
[: L_m :, : L_n :] = (m-n) : L_{m+n} : +C(n)\delta_{m\ -n} \tag{3.68}
$$

The easiest way to determine the $C(n)$ is to note the following (one can also perform a direct calculation but it is notoriously complicated and messy). First one imposes the Jacobi identity

$$
[: L_k :, [: L_m :, : L_n :]] + [: L_m :, [: L_n :, : L_k :]] + [: L_m :, [: L_n :, : L_k :]] = 0 \tag{3.69}
$$

If we impose that $k + m + n = 0$ with $k, m, n \neq 0$ (so that no pair of them adds up to zero) then this reduces to

$$
(m-n)C(k) + (n-k)C(m) + (k-m)C(n) = 0 \tag{3.70}
$$

If we pick $k = 1$ and $m = -n - 1$ one finds

$$
-(2n+1)C(1) + (n-1)C(-n-1) + (n+2)C(n) = 0 \tag{3.71}
$$

Now we note that $C(-n) = -C(n)$ by definition. Hence we learn that $C(0) = 0$ and

$$C(n+1) = \frac{(n+2)C(n) - (2n+1)C(1)}{n-1} \tag{3.72}$$

This is just a difference equation and given $C(2)$ it will determine $C(n)$ for $n > 1$ (note that it can't determine $C(2)$ given $C(1)$). We can look for a solution to this by considering polynomials. Since it must be odd in n the simplest guess is

$$c(n) = c_1 n^3 + c_2 n \tag{3.73}$$

In this case the right hand side becomes

$$\begin{aligned}
\frac{(n+1)(c_1 n^3 + c_2 n) - (2n+1)(c_1 + c_2)}{n-1} &= \frac{c_1 n^4 + 2c_1 n^3 + c_2 n^2 - 2c_1 n - (c_1 + c_2)}{n-1} \\
&= \frac{(n-1)(c_1 n^3 + 3c_1 n^2 + (3c_1 + c_2)n + c_1 + c_2)}{n-1}
\end{aligned} \tag{3.74}$$

Expanding out the left hand side gives

$$c_1(n+1)^3 + c_2(n+1) = c_1 n^3 + 3c_1 n^2 + (3c_1 + c_2)n + c_1 + c_2 \tag{3.75}$$

and hence they agree.

Note that if we shift L_0 by a constant l then $C(n)$ is shifted by $2nl$ (note that in so doing we'd have to shift a as well). This means that we can change the value of c_2. Therefore we will fix it to be $c_1 = -c_2$. Finally we must calculate c_1. To do this we consider the ground state with no momentum $|0;0,0>$ This state is annihilated by $:L_n:$ for all $n \geq 0$. Thus we have

$$\begin{aligned}
<0,0;0| :L_2::L_{-2}: |0;0,0> &= <0,0;0|[:L_2:,:L_{-2}:]|0;0,0> \\
&= 4 <0,0;0| :L_0: |0;0,0> + 6c_1 <0,0;0|0;0,0> \\
&= 6c_1
\end{aligned} \tag{3.76}$$

where we assume that the ground state has unit norm.

Problem: Show that

$$<0,0;0| :L_2::L_{-2}: |0;0,0> = \frac{D}{2} \tag{3.77}$$

So we deduce that

$$[:L_m:,:L_n:] = (m-n) :L_{m+n}: + \frac{D}{12}(m^3 - m)\delta_{m\ -n} \tag{3.78}$$

Of course there is a similar expression for $[:\tilde{L}_m:,:\tilde{L}_m:]$. This is called the central extension of the Virasoro algebra and D is the central charge which has arisen as a quantum effect. From now on we will always take operators to be normal ordered and we will drop the :: symbol, unless otherwise stated.

Let us return to our Fock space of states. It is built up out of the ground state which we take to have unit norm $< 0|0 >= 1$. One sees that the one-particle state $a^\mu_{-1}|0>$ has norm

$$< 0|a^\mu_1 a^\mu_{-1}|0 >=< 0|[a^\mu_1, a^\mu_{-1}]|0 >= \eta^{\mu\mu} \tag{3.79}$$

where we do not sum over μ. Thus the state $a^0_{-1}|0>$ has negative norm!

Problem: Show that the state $(a^0_{-1} + a^1_{-1})|0>$ has zero norm.

Thus the natural innerproduct on the Fock space is not positive definite because the time-like oscillators come with the wrong sign. This also occurs in other quantum theories such as QED and doesn't necessarily represent any kind of sickness.

There are stranger states still. A physical state $|\chi>$ that satisfies $< \chi|phys >= 0$ for all physical states is called null (or spurious if it only satisfies the $n = 0$ physical state condition). It then follows that a null state has zero norm (as it must be orthogonal to itself).

There are many such states. To construct an example just consider

$$|\chi >= L_{-1}|0; p > \quad \text{with} \quad p^2 = 0 \tag{3.80}$$

Note that the zero-momentum ground state satisfies $L_n|0; 0 >= 0$ and for all $n \geq 0$ and this remains true if for $|0; p>$ if $p^2 = 0$. First we verify that $|\chi>$ is physical. We have For $m \geq 0$

$$\begin{aligned} L_m|\chi > &= L_m L_{-1}|0; p > \\ &= [L_m, L_{-1}]|0; p > \\ &= (m+1)L_{m-1}|0; p > + \frac{D}{12}(m^3 - m)\delta_{m1}|0; p > \end{aligned} \tag{3.81}$$

The last term will vanish automatically whereas the first term can only be non-zero for $m = 0$ (since $L_n|0; p >= 0$ for all $n \geq 0$). Here we find $L_0|\chi >= |\chi>$ which is the physical state condition for $a = 1$ which will turn out to be the case. Next we see that $< \chi|phys >=< 0|L_1|phys >= 0$. Note that we could have used any state instead of $|0; p>$ that was annihilated by L_n for all $n \geq 0$ to construct a null state.

Thus if we calculate some amplitude between two physical states $< phys'|phys >$ we can shift $|phys > \rightarrow |phys > + |\chi>$ where $|\chi>$ is a null state. The new state $|phys > + |\chi>$ is still physical but the amplitude will remain the same - for any other choice of physical state $|phys'>$. Thus we have a stringy gauge symmetry whereby two physical states are equivalent if their difference is a null state. This will turn out to be the origin of Yang-Mills and other gauge symmetries within string theory. And furthermore one can prove a no-ghost theorem which asserts that there are no physical states with negative norm (at least for $a = 1$ and $D = 26$).

3.4 Open Strings

Strings come in two varieties: open and closed. To date we have tried to develop as many formulae and results as possible which apply to both. However now we must make a decision and proceed along slightly different but analogous roots. Open strings have two end points which traditionally arise at $\sigma = 0$ and $\sigma = \pi$. We must be careful to ensure that the correct boundary conditions are imposed. In particular we must choose boundary conditions so that the boundary value problem is well defined. This requires that

$$\eta_{\mu\nu}\delta X^\mu \partial_\sigma X^\nu = 0 \tag{3.82}$$

at $\sigma = 0, \pi$.

Problem: Show this!

There are essentially two boundary conditions that one can impose. The first is Dirichlet: we hold X^μ fixed at the end points so that $\delta X^\mu = 0$. The second is Neumann: we set $\partial_\sigma X^\mu = 0$ at the end points. The first condition implies that somehow the end points of the string are fixed in spacetime, like a flag to a flag pole. At first glance this seems unphysical and we will ignore it for now, although such boundary conditions turn out to be profoundly important. So we will start by considering second boundary condition, which states that no momentum leaks off the ends of the string.

The condition that $\partial_\sigma \hat{X}^\mu(\tau, 0) = 0$ implies that

$$w^\mu = 0 \ , \qquad a_n^\mu = \tilde{a}_n^\mu \tag{3.83}$$

i.e. the left and right oscillators are not independent. If we look at the boundary condition at $\sigma = \pi$ then we determine that

$$\sum_{n\neq 0} a_n^\mu e^{in\tau}\sin(n\pi) = 0 \tag{3.84}$$

Thus n is indeed an integer. The mode expansion is therefore

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + \sqrt{2\alpha'}i\sum_{n\neq 0}\frac{a_n^\mu}{n}e^{in\tau}\cos(n\sigma) \tag{3.85}$$

(Note the slightly redefined value of p^μ as compared to before.)

For the open string the physical states are constrained to satisfy

$$L_n|phys>= 0 \ , n > 0 \qquad \text{and} \qquad (L_0 - 1)|phys>= 0 \tag{3.86}$$

in particular there is only one copy of the constraints required since the $\tilde{L}_n$ constraints will automatically be satisfied. The second condition is the most illuminating as it gives the spacetime mass shell condition. To see this we note that translational invariance

$X^\mu \to X^\mu + x^\mu$ gives rise to the conserved current $\hat{P}^\mu = \frac{1}{2\pi\alpha'}\dot{X}^\mu$. This is a worldsheet current and hence the conserved charge (from the worldsheet point of view) is

$$\begin{aligned}
p^\mu &= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma \dot{X}^\mu \\
&= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma 2p^\mu + \sqrt{2\alpha'}\sum_{n\neq 0} a_n^\mu e^{in\tau}\cos(n\sigma) \\
&= p^\mu
\end{aligned} \tag{3.87}$$

where again we have abused notation and confused the operator $\hat{p}^\mu$ that appears in the mode expansion of X^μ with its eigenvalue p^μ which we have now identified with the conserved charge. In any case we do this because we have shown that p^μ is indeed the spacetime momentum of the string. Note that this also explains why we put in the extra factor of 2 in front of $p^\mu\tau$ in the mode expansion.

Next we let

$$N = \sum_{n>0} \eta_{\mu\nu} a_{-n}^\mu a_n^\nu \tag{3.88}$$

Which is the analogue of the number operator that appears in the Harmonic oscillator. Again this is an operator even though we are being lazy and dropping the hat. It is easy to see that for $m > 0$

$$\begin{aligned}
[N, a_{-m}^\lambda] &= \sum_{n>0} \eta_{\mu\nu} a_{-n}^\mu [a_n^\nu, a_{-m}^\lambda] \\
&= m a_{-m}^\lambda
\end{aligned} \tag{3.89}$$

Thus if $|n>$ is a state with $N|n>= n|n>$ then

$$\begin{aligned}
N a_{-m}^\lambda |n> &= ([N, a_{-m}^\lambda] + a_{-m}^\lambda N)|n> \\
&= (m a_{-m}^\lambda + a_{-m}^\lambda n)|n> \\
&= (m+n) a_{-m}^\lambda |n>
\end{aligned} \tag{3.90}$$

Therefore $a_{-m}^\lambda|n>$ is a state with N-eigenvalue $n+m$. You can think of N as counting the number of oscillator modes in a given state.

With this definition we can write the physical state condition $(L_0 - 1)|phys>= 0$ as

$$(p_\mu p^\mu + \frac{1}{\alpha'}(N-1))|phys>= 0 \tag{3.91}$$

Thus we can identify the spacetime mass-squared of a physical state to be the eigenvalue of

$$M^2 = \frac{1}{\alpha'}(N-1) \tag{3.92}$$

We call the eigenvalue of N the level of the state. In other words the higher oscillator modes give more and more massive states in spacetime. In practice one takes $a^{-1/2}$ to be a very high mass scale so that only the massless modes are physically relevant. Note that the number of states at level n grows exponentially in n as the number of possible oscillations will be of order of the number of partitions of n into smaller integers. This exponentially growing tower of massive modes a unique feature of strings as opposed to point particles.

Of course we must not forget the other physical state condition $L_n|phys >= 0$ for $n > 0$. This constraint will take the form of a gauge fixing condition . Let us consider the lowest lying states.

At level zero we have the vacuum $|0; p >$. We see that the mass-shell condition is

$$p^2 - \alpha'^{-1} = 0 \tag{3.93}$$

The other constraint, $L_n|0; p >= 0$ with $n > 0$, is automatically satisfied. This has a negative mass-squared! Such a mode is called a Tachyon. Tachyons arise in field theory if rather than expanding a scalar field about a minimum of the potential one expands about a maximum. Thus they are interpreted as instabilities. The problem is that no one knows in general whether or not the instability associated to this open string tachyon is ever stabilized. We will simply ignore the tachyon. Our reason for doing this is that it naturally disappears once one includes worldsheet Fermions and considers the superstring theories. However the rest of the physics of Bosonic strings remains useful in the superstring. Hence we continue to study it.

Next consider level 1. Here we have

$$|A_\mu >= A_\mu(p)a^\mu_{-1}|0; p > \tag{3.94}$$

Since these modes have $N = 1$ it follows from the mass shell condition that they are massless (for $a = 1$!), *i.e.* the L_0 constraint implies that $p^2A_\mu = 0$. Note that this depends crucially on the fact that $a = 1$. If $a > 1$ then $|A_\mu >$ would be tachyonic whereas if $a < 1$ $|A_\mu >$ would be massive. In either case there is no known constituent theory of a massive (or tachyonic) vector field.

But we must also check that $L_n|A >= 0$ for $n > 0$. Thus

$$\begin{aligned} L_n|A_\mu > &= \frac{1}{2}A_\mu \sum_m \eta_{\nu\lambda} a^\nu_{n-m} a^\lambda_m a^\mu_{-1}|0; p > \\ &= \frac{1}{2}A_\mu \eta_{\nu\lambda} \sum_{m\le 1} a^\nu_{n-m} a^\lambda_m a^\mu_{-1}|0; p > \\ &= \frac{1}{2}A_\mu \eta_{\nu\lambda} \sum_{n-1\le m\le 1} a^\nu_{n-m} a^\lambda_m a^\mu_{-1}|0; p > \end{aligned} \tag{3.95}$$

In the second line we've noted that if $m > 1$ we can safely commute a^λ_m past a^μ_{-1} where it annihilates the vacuum. In the third line we've observed that if $n - m > 1$ then we can safely commute a^ν_{n-m} through the other two oscillators to annihilate the vacuum (recall that for $n > 0$ a^ν_{n-m} always commutes through a^λ_m). Thus for $n > 1$ we automatically have $L_n|A_\mu >= 0$. For $n = 1$ we find just two terms

$$\begin{aligned} L_1|A> &= \frac{1}{2}A_\mu\eta_{\nu\lambda}(a^\nu_1 a^\lambda_0 a^\mu_{-1} + a^\nu_0 a^\lambda_1 a^\mu_{-1})|0;p> \\ &= A_\mu a^\mu_0|0;p> \\ &= \sqrt{2\alpha'}p^\mu A_\mu|0;p> \end{aligned} \tag{3.96}$$

Thus we see that $|A_\mu>$ is represent a massless vector mode with $p^\mu A_\mu = 0$. In position space this is just $\partial^\mu A_\mu = 0$ and this looks like the Lorentz gauge condition for an electromagnetic potential.

Indeed recall that before we found the null state, with $p^2 = 0$,

$$\begin{aligned} |\Lambda> &= i\Lambda(p)L_{-1}|0;p> \\ &= i\eta_{\mu\nu}\Lambda a^\mu_0 a^\nu_{-1}|0;p> \\ &= i\sqrt{2\alpha'}p_\mu\Lambda a^\mu_{-1}|0;p> \end{aligned} \tag{3.97}$$

provided that $p^2 = 0$. Thus we must identify $A_\mu \equiv A_\mu + i\sqrt{2\alpha'}p_\mu\Lambda$ which in position space is the electromagnetic gauge symmetry $A_\mu \equiv A_\mu + \sqrt{2\alpha'}\partial_\mu\Lambda$. Again this occurs precisely when $a = 1$, otherwise $L_{-1}|0;p>$ is not a null state and their would not be a gauge symmetry.

There is one more thing that can be done. Since and open string has two preferred points, its end points, we can attach discrete labels to the end points so that the ground state, of the open string carries two indices

$$|0;p,ab> \tag{3.98}$$

where $a = 1,..,N$ refers the $\sigma = 0$ end and $b = 1,...,N$ refers to the $\sigma = \pi$ end. It then follows that all the Fock space elements built out of $|0;p,ab>$ will carry these indices. These are called Chan-Paton indices. The level one states now have the form

$$|A^{ab}_\mu >= A^{ab}_\mu a^\mu_{-1}|0;p,ab> \tag{3.99}$$

The null states take the form

$$|\Lambda^{ab} >= i\Lambda^{ab}L_{-1}|0;p,ab> \tag{3.100}$$

and the gauge symmetry is

$$A^{ab}_\mu \equiv A^{ab}_\mu + \sqrt{2\alpha'}\partial_\mu\Lambda^{ab} \tag{3.101}$$

These are the gauge symmetries of a non-Abelian Yang-Mills field with gauge group $U(N)$ (at lowest order in the fields). Thus we see that we can obtain non-Abelian gauge field dynamics from open strings.

3.5 Closed Strings

Let us now consider a closed string, so that $\sigma \sim \sigma + 2\pi$. The resulting "boundary condition" is more simple: we simply demand that $\hat{X}^\mu(\tau, \sigma + 2\pi) = \hat{X}^\mu(\tau, \sigma)$. This is achieved by again taking n to be an integer and $w^\mu = 0$. However we now have two independent sets of left and right moving oscillators. Thus the mode expansion is given by

$$X^\mu = x^\mu + \alpha' p^\mu \tau + \sqrt{\frac{\alpha'}{2}} i \sum_{n \neq 0} \left(\frac{a_n^\mu}{n} e^{-in(\tau+\sigma)} + \frac{\tilde{a}_n^\mu}{n} e^{-in(\tau-\sigma)} \right) \tag{3.102}$$

note the absence of the factor of 2 in front of $p^\mu \tau$. The total momentum of such a string is calculated as before to give

$$\begin{aligned} p^\mu &= \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \dot{X}^\mu \\ &= \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} a_n^\mu e^{-in(\tau+\sigma)} + \tilde{a}_n^\mu e^{-in(\tau-\sigma)} \\ &= p^\mu \end{aligned} \tag{3.103}$$

so again p^μ is the spacetime momentum of the string.

We now have double the constraints:

$$\begin{aligned} (L_0 - 1)|phys> &= (\tilde{L}_0 - 1)|phys> = 0 \\ L_n|phys &= \tilde{L}_n|phys> = 0 \end{aligned} \tag{3.104}$$

with $n > 0$. If we introduce the right-moving number operator $\tilde{N}$

$$\tilde{N} = \sum_{n>0} \eta_{\mu\nu} \tilde{a}_{-n}^\mu \tilde{a}_n^\nu \tag{3.105}$$

then the first conditions can be rewritten as

$$(p_\mu p^\mu + \frac{4}{\alpha'}(N-1))|phys> = 0 \qquad (N - \tilde{N})|phys> = 0 \tag{3.106}$$

where we have recalled that, if $w^\mu = 0$, $a_0^\mu = \tilde{a}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$ and $L_0 = \frac{1}{2}\eta_{\mu\nu} a_0^\mu a_0^\nu + N$, $L_0 = \frac{1}{2}\eta_{\mu\nu} \tilde{a}_0^\mu \tilde{a}_0^\nu + \tilde{N}$. The second condition is called level matching. It simply says that any physical state must be made up out of an equal number of left and right moving oscillators. Again the remaining constraints will give gauge fixing conditions.

Let us consider the lowest modes of the closed string. At level 0 (which means level 0 on both the left and right moving sectors by level matching) we simply have the ground state $|0; p>$. This is automatically annihilated by both L_n and $\tilde{L}_n$ with $n > 0$. For $n = 0$ we find

$$p^2 - \frac{4}{\alpha} = 0 \tag{3.107}$$

Thus we again find a tachyonic ground state. No one knows what to do with this instability. It turns out to be much more serious than the open string tachyon that we saw, which can sometimes be dealt with. Most people today would say that the Bosonic string is inconsistent although this hasn't been demonstrated. However for us the cure is the same as for the open string: in the superstring this mode is projected out. So we continuing by simply ignoring it, as our discussion of the other modes still holds in the superstring.

ext we have level 1. Here the states are of the form

$$|G_{\mu\nu}>= G_{\mu\nu}a^{\mu}_{-1}\tilde{a}^{\nu}_{-1}|0;p> \tag{3.108}$$

Just as for the open string these will be massless, *i.e.* $p^2=0$ (again only if $a=1$). Next we consider the constraints $L_m|G_{\mu\nu}>=\tilde{L}_m|G_{\mu\nu}>=0$ with $m>0$.

Problem: Show that these constraints imply that $p^\mu G_{\mu\nu}=p^\nu G_{\mu\nu}=0$

The matrix $G_{\mu\nu}$ is a spacetime tensor. Under the Lorentz group $SO(1,D-1)$ it will decompose into a symmetric traceless, anti-symmetric and trace part. What this means is that under spacetime Lorentz transformations the tensors

$$\begin{aligned} g_{\mu\nu} &= G_{(\mu\nu)}-\frac{1}{D}\eta^{\lambda\rho}G_{\lambda\rho}\eta_{\mu\nu} \\ b_{\mu\nu} &= G_{[\mu\nu]} \\ \phi &= \eta^{\lambda\rho}G_{\lambda\rho} \end{aligned} \tag{3.109}$$

will transform into themselves

Problem: Show this.

Thus from the spacetime point of view there are three independent modes labeled by $g_{\mu\nu}$, $b_{\mu\nu}$ and ϕ. Just as for the open string there is a gauge symmetry

$$|G_{\mu\nu}>\to|G_{\mu\nu}>+i\xi_\mu L_{-1}\tilde{a}^{\mu}_{-1}|0;p>+i\zeta_\mu\tilde{L}_{-1}a^{\mu}_{-1}|0;p> \tag{3.110}$$

where we have used the fact that $\xi_\mu L_{-1}\tilde{a}^{\mu}_{-1}|0;p>$ and $\zeta_\mu\tilde{L}_{-1}a^{\mu}_{-1}|0;p>$ are null states, provided that $p^2=0$. The proof of this is essentially the same as it was for the open string. We need only ensure that the level matching condition is satisfied, which is clear, and that $\tilde{L}_nL_{-1}\tilde{a}_{-1}|0;p>=L_n\tilde{L}_{-1}a_{-1}|0;p>=0$ for $n>0$. Thus we need only check that

$$L_n\tilde{L}_{-1}a^{\mu}_{-1}|0;p> \;=\; \frac{1}{2}\tilde{L}_{-1}\sum_m\eta_{\lambda\rho}a^{\lambda}_{n+m}a^{\rho}_{-m}a^{\mu}_{-1}|0;p>=0 \tag{3.111}$$

Just as before the $n > 1$ terms will vanish automatically. So we need only check

$$\begin{aligned}
L_1 \tilde{L}_{-1} a^\mu_{-1}|0;p> &= \frac{1}{2}\tilde{L}_{-1}\sum_m \eta_{\lambda\rho} a^\lambda_{1+m} a^\rho_{-m} a^\mu_{-1}|0;p> \\
&= \tilde{L}_{-1}\eta_{\lambda\rho} a^\lambda_0 a^\rho_1 a^\mu_{-1}|0;p> \\
&= \tilde{L}_{-1}\eta_{\lambda\rho} a^\lambda_0 [a^\rho_1, a^\mu_{-1}]|0;p> \\
&= \tilde{L}_{-1} a^\mu_0|0;p> \\
&= \frac{\sqrt{\alpha'}}{2}\tilde{L}_{-1} p^\mu|0;p>
\end{aligned} \tag{3.112}$$

Similarly for $\tilde{L}_n L_{-1}\tilde{a}^\mu_{-1}|0;p>$. Thus we also find that $p^\mu \xi_\mu = p^\mu \zeta_\mu = 0$. This of course is required to preserve the condition $p^\mu G_{\mu\nu} = p^\nu G_{\mu\nu} = 0$.

In terms of $G_{\mu\nu}$ this implies that

$$G_{\mu\nu} \rightarrow G_{\mu\nu} + i\sqrt{\frac{\alpha'}{2}} p_\mu \xi_\nu + i\sqrt{\frac{\alpha'}{2}} p_\nu \zeta_\mu \tag{3.113}$$

or, switching to coordinate space representations and the individual tensor modes, we find

$$\begin{aligned}
g_{\mu\nu} &\rightarrow g_{\mu\nu} + \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\partial_\mu(\xi_\nu + \zeta_\nu) + \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\partial_\nu(\xi_\mu + \zeta_\mu) \\
b_{\mu\nu} &\rightarrow B_{\mu\nu} + \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\partial_\mu(\xi_\nu - \zeta_\nu) - \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\partial_\nu(\xi_\mu - \zeta_\mu) \\
\phi &\rightarrow \phi + 2\sqrt{\frac{\alpha'}{2}}\partial_\mu(\xi^\mu + \zeta^\mu)
\end{aligned} \tag{3.114}$$

If we let $v_\mu = \frac{1}{2}\sqrt{\frac{\alpha'}{2}}(\xi_\mu + \zeta_\mu)$ and $\Lambda_\mu = \frac{1}{2}\sqrt{\frac{\alpha'}{2}}(\xi_\mu - \zeta_\mu)$ and use $\partial^\mu \xi_\mu = p^\mu \zeta_\mu = 0$ then we find

$$\begin{aligned}
g_{\mu\nu} &\rightarrow g_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu \\
b_{\mu\nu} &\rightarrow b_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \\
\phi &\rightarrow \phi
\end{aligned} \tag{3.115}$$

The first term line gives the infinitesimal form of a diffeomorphism, $x^\mu \rightarrow x^\mu - v^\mu$ and thus we can identify $g_{\mu\nu}$ to be a metric tensor. The second line gives a generalization of and electromagnetic gauge transformation. The analogue of the gauge invariant field strength is

$$H_{\lambda\mu\nu} = \partial_\lambda b_{\mu\nu} + \partial_\mu b_{\nu\lambda} + \partial_\nu b_{\lambda\mu} \tag{3.116}$$

Thus the massless field content at level 1 consists of a graviton mode $g_{\mu\nu}$, an anti-symmetric tensor field $b_{\mu\nu}$ and a scalar ϕ, subject to the gauge transformations (3.114). Finally the massless condition $p^2 G_{\mu\nu} = 0$ leads to

$$\begin{aligned} \partial^2 g_{\mu\nu} &= 0 \\ \partial^2 b_{\mu\nu} &= 0 \\ \partial^2 \phi &= 0 \end{aligned} \tag{3.117}$$

The conditions $p^\mu G_{\mu\nu} = p^\nu G_{\mu\nu} = 0$ now reduce to the linearized equations

$$\begin{aligned} \partial^\mu g_{\mu\nu} &= 0 \\ \partial^\mu b_{\mu\nu} &= 0 \\ \partial_\mu \phi &= 0 \end{aligned} \tag{3.118}$$

The first two equations can be viewed as gauge fixing conditions whereas the second states that ϕ is a constant. The fields $g_{\mu\nu}$, $b_{\mu\nu}$ and ϕ are known as the graviton (metric), Kalb-Ramond (b-field) and dilaton respectively.

3.6 Light-cone gauge and $D = 26$!

So far we have quantized a string in flat D-dimensional spacetime. Apart from D we have the parameters a and α'. In fact α' is not a parameter, it is a dimensional quantity - it has the dimensions of length-squared - and simply sets the scale. What is important are unitless quantities such as $p^2\alpha'$. For example small momentum means $p^2\alpha' << 1$.

We are left with D and a but actually these are fixed: quantum consistency demands that $D = 26$ and $a = 1$. We have seen that things would go horribly wrong if $a \neq 1$.

The easiest way to see this is to introduce light-cone gauge. Recall that the action we started with had diffeomorphism symmetry. We used this symmetry to fix $\gamma_{\alpha\beta} = e^{2\rho}\eta_{\alpha\beta}$. However there is still a residual symmetry. In particular in terms of the coordinates $\sigma^\pm$ then under a transformation

$$\sigma'^+ = \sigma'^+(\sigma^+) \qquad \sigma'^- = \sigma'^-(\sigma^-) \tag{3.119}$$

we see that $\gamma'_{\alpha\beta} = e^{2\rho'}\eta_{\alpha\beta}$ with

$$\rho' = \rho + \frac{1}{2}\ln\left(\frac{\partial\sigma^+}{\partial\sigma'^+}\frac{\partial\sigma^-}{\partial\sigma'^-}\right) \tag{3.120}$$

i.e. this preserves the conformal gauge. In terms of the worldsheet coordinates σ, τ we see that

$$\tau' = \frac{1}{2}(\sigma'^+ + \sigma'^-) \tag{3.121}$$

and since $\sigma'^{\pm}$ are arbitrary functions of $\sigma^{\pm}$ we see that any τ that solves the two-dimensional wave equation can be obtained by such a diffeomorphism. Therefore, without loss of generality, we can choose the worldsheet 'time' coordinate τ to be any of the spacetime coordinates (since these solve the two-dimensional wave-equation). Of course there are many choices but the usual one is to define

$$\hat{X}^{+} = \frac{1}{2}(X^{0} + X^{D-1}) \qquad \hat{X}^{-} = \frac{1}{2}(X^{0} - X^{D-1}) \tag{3.122}$$

and then take

$$\hat{X}^{+} = x^{+} + \alpha' p^{+}\tau \tag{3.123}$$

This is called light cone gauge.

Next we evaluate the conformal symmetry constraints (3.15). We observe that in these coordinates the spacetime $\eta_{\mu\nu}$ is

$$\eta_{-+} = \eta_{+-} = -2 \qquad \eta_{ij} = \delta_{ij} \tag{3.124}$$

Thus we find that

$$\begin{aligned} T_{00} = T_{11} &= -2\alpha' p^{+}\dot{X}^{-} + \frac{1}{2}\dot{X}^{i}\dot{X}^{j}\delta_{ij} + \frac{1}{2}X'^{i}X'^{j}\delta_{ij} = 0 \\ T_{01} = T_{10} &= -2\alpha' p^{+}\hat{X}'^{-} + \dot{X}^{i}X'^{j}\delta_{ij} = 0 \end{aligned} \tag{3.125}$$

where $i, j = 1, 2, 3, ..., D-2$. This allows one to explicitly solve for X^{-} in term of the mode expansions for X^{i}.

Problem: Show that with our conventions

$$X^{-} = x^{-} + \alpha' p^{-}\tau + i\left(\sum_{n}\frac{a_{n}^{-}}{n}e^{-in\sigma^{+}} + \frac{\tilde{a}_{n}^{-}}{n}e^{-in\sigma^{-}}\right) \tag{3.126}$$

where

$$a_{n}^{-} = \frac{1}{2p^{+}}\sum_{m} a_{n-m}^{i}a_{m}^{j}\delta_{ij} \tag{3.127}$$

and the massshell constraint is

$$-4\alpha' p^{+}p^{-} + \alpha' p^{i}p^{j}\delta_{ij} + 2(N + \tilde{N}) = 0 \tag{3.128}$$

with

$$N + \tilde{N} = \frac{1}{2}\delta_{ij}\sum_{n\neq 0} a_{n}^{i}a_{-n}^{j} + \tilde{a}_{n}^{i}\tilde{a}_{-n}^{j} \tag{3.129}$$

To continue we note that in the quantum theory there is a normal ordering ambiguity in the definition of a_{0}^{-} and we must include our constant a again into the definition. Hence we must take

$$: a_{0}^{-} := \frac{1}{p^{+}}\left(\sum_{m>0} a_{m}^{i}a_{-m}^{j}\delta_{ij} - a\right) \qquad : \tilde{a}_{0}^{-} := \frac{1}{p^{+}}\left(\sum_{m>0} \tilde{a}_{m}^{i}\tilde{a}_{-m}^{j}\delta_{ij} - a\right) \tag{3.130}$$

This will also show up in the mass shell constraint as

$$-4\alpha' p^+ p^- + \alpha' p^i p^j \delta_{ij} + 2(N + \tilde{N} - 2a) = 0 \tag{3.131}$$

Note that $-4p^+p^- + p^ip^j\delta_{ij} = \eta_{\mu\nu}p^\mu p^\nu$ so this really just tells us that the mass of a state is

$$M^2 = \frac{2}{\alpha'}(N + \tilde{N} - 2a) \tag{3.132}$$

We still have a level matching condition for closed strings

$$N = \tilde{N} \tag{3.133}$$

This arises because we only have one spacetime momentum p^μ (not separate ones for left and right moving modes) and hence $a_0^\mu = \tilde{a}_0^\mu$. This implies that $a_0^- = \tilde{a}_0^-$ and hence $N = \tilde{N}$.

Note that this breaks the $SO(1, D-1)$ symmetry of our flat target space since we choose X^0 and X^{D-1} whereas any pair will do (so long as one is timelike). Thus we will not see a manifest $SO(1, D-1)$ symmetry but just an $SO(D-2)$ symmetry from rotations of the $\hat{X}^i$. However it is important to realize that the $SO(1, D-1)$ symmetry is not really broken, we have merely performed a kind of gauge fixing (recall there was this underlying gauge symmetry of the string spectrum). It is just no longer manifest.

On the other hand the benefit of this procedure is that the physical Hilbert space is manifestly postive definite because we remove the oscillators $a_n^0, \tilde{a}_n^0, a_n^{D-1}, \tilde{a}_n^{D-1}$. This is often a helpful way to determine the physical spectrum of the theory.

For example we can reconsider the low lying states that we constructed above. The ground states are unchanged as they do not involve any oscillators. For the open string we find the $D-2$ states at level one

$$|A_i >= a_{-1}^i |0; p > \tag{3.134}$$

These are the transverse components of a massless gauge field. For the closed string we find, at level one,

$$|G_{ij} >= G_{ij} a_{-1}^i \tilde{a}_{-1}^j |0; p > \tag{3.135}$$

These correspond to the physical components, in a certain gauge, of the metric, Kalb-Ramond field and dilaton Note however that there is no remnant at all of gauge symmetry which is a crucial feature of dynamics

Now formally a is given by

$$\begin{aligned} a &= -\frac{1}{2}\sum_{m=1}^{\infty}[a_m^i, a_{-m}^j]\delta_{ij} \\ &= -\frac{D-2}{2}\sum_{m=1}^{\infty} m \end{aligned} \tag{3.136}$$

This is divergent however it can be regularized in the following manner. We note that

$$a = -\frac{D-2}{2}\zeta(-1) \tag{3.137}$$

where $\zeta(s)$ is the Riemann ζ-function

$$\zeta(s) = \sum_{m=1}^{\infty}\frac{1}{m^s} \tag{3.138}$$

This is analytic for complex s with $\mathrm{Re}(s) > 1$. Thus it can be extended to a holomorphic function of the complex plane, with poles at a discrete number of points. Analytically continuing to $s = -1$ one finds $\zeta(-1) = -1/12$ and hence

$$a = \frac{D-2}{24} \tag{3.139}$$

We have seen that in order to have a sensible theory we must take $a = 1$. Hence we must take $D = 26$.

This is not a very satisfactory derivation of the dimension of spacetime but it is rather useful in other circumstances. In light cone gauge it can be summarized by the statement that a periodic scalar field has a zero point energy of $-1/24$. Sometimes one is interested in anti-periodic scalars $X^\mu(\sigma + 2\pi) = -X^\mu(\sigma)$. The difference is that the n which appear in the mode expansion are half-odd-integers. Following the same logic one finds the ground state energy of such a scalar field is

$$a = -\frac{1}{2}\sum_{m=1,odd}^{\infty}\frac{m}{2} \tag{3.140}$$

Now we could write

$$\begin{aligned}\sum_{m=1}^{\infty} m &= \sum_{m\ odd} m + \sum_{m\ even} m \\ &= \sum_{m\ odd} m + 2\sum_{m=1}^{\infty} m\end{aligned} \tag{3.141}$$

hence

$$\sum_{m\ odd} m = -\sum_{m=1}^{\infty} m = \frac{1}{12} \tag{3.142}$$

and we see that for an anti-periodic scalar

$$a = -\frac{1}{4}\frac{1}{12} = -\frac{1}{48} \tag{3.143}$$

So the zero-point energy of anti-periodic scalar is 1/48.

A more convincing argument is the following. Light cone gauge is just a gauge. Therefore although the manifest spacetime Lorentz symmetry is no longer present there is still an $SO(1, D-1)$ Lorentz symmetry, even though only an $SO(D-2)$ subgroup is manifest in light cone gauge. It is too lengthy a calculation to do here, but one can show that the full $SO(1, D-1)$ Lorentz symmetry, generated by the charges (3.26) is preserved in the quantum theory, *i.e.* once normal ordering is taken into account, if and only if $a = 1$ and $D = 26$. You are urged to read the section 2.3 of Green Schwarz and Witten or section 12.5 of Zwiebach where this is shown more detail.

4 Curved Spacetime and an Effective Action

4.1 Strings in Curved Spacetime

We have considered quantized strings propagating in flat spacetime. This lead to a spectrum of states that included the graviton as well as other modes. More generally a string should be allowed to propagate in a curved background with non-trivial values for the metric and other fields. Our ansatz will be to consider the most general two-dimensional action for the embedding coordinates X^μ coupled to two-dimensional gravity subject to the constraint of conformal invariance. This later condition is required so that the two-dimensional worldvolume metric decouples from the other fields. We will consider only closed strings in this section. We will return to open strings in the next section. The reason for this is that these days one views open strings as description soliton like objects, called Dp-branes, that naturally sit inside the closed string theory.

Before proceeding we note that

$$S_{EH} = \frac{1}{4\pi}\int d^2\sigma\sqrt{-\gamma}R = \chi \tag{4.1}$$

is a topological invariant called the Euler number, *i.e.* the integrand is locally a total derivative. Thus we could add the term S_{EH} to the action and not change the equations of motion.

With this in mind the most general action we can write down for a closed string is

$$S_{closed} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\left(\phi(X)R + \gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X) + \frac{1}{\sqrt{-\gamma}}\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu b_{\mu\nu}(X)\right) \tag{4.2}$$

where ϕ is a scalar, $g_{\mu\nu}$ symmetric and $b_{\mu\nu}$ antisymmetric. These are precisely the correct degrees of freedom to be identified with the massless modes of the string. One can think of this worldsheet theory as two-dimensional quantum gravity coupled to some matter in the form of scalar fields. More generally one can think of and conformal field theory (with central charge equal to 26) as defining the action for a string.

Furthermore this action has the diffeomorphism symmetry $X^\mu \to X'^\mu(X)$

$$\partial_\alpha X'^\mu = \frac{\partial X'^\mu}{\partial X^\nu}\partial_\alpha X^\nu \qquad g'_{\mu\nu} = \frac{\partial X^\lambda}{\partial X'^\mu}\frac{\partial X^\rho}{\partial X'^\nu}g_{\lambda\rho} \qquad b'_{\mu\nu} = \frac{\partial X^\lambda}{\partial X'^\mu}\frac{\partial X^\rho}{\partial X'^\nu}b_{\lambda\rho} \qquad \phi' = \phi \tag{4.3}$$

automatically built in. It also incorporates the b-field gauge symmetry

$$b'_{\mu\nu} = b_{\mu\nu} + \partial_\mu \lambda_\nu - \partial_\mu \lambda_\nu \tag{4.4}$$

however to see this we note that

$$\begin{aligned} \delta S_{closed} &= -\frac{1}{2\pi\alpha'} \int d^2\sigma \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\mu \lambda_\nu \\ &= -\frac{1}{2\pi\alpha'} \int d^2\sigma \partial_\alpha (\epsilon^{\alpha\beta} \partial_\beta X^\nu \lambda_\nu) \\ &= 0 \end{aligned} \tag{4.5}$$

where we used the fact that $\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta X^\nu = 0$ in the second to last line and the fact that the worldsheet is a closed manifold in the last line, *i.e.* the periodic boundary conditions.

Notice something important. If the dilaton ϕ is constant then the first term in the action is a topological invariant, the Euler number. In the path integral formulation the partition function for the full theory is defined by summing over all worldsheet topologies

$$Z = \sum_{g=0}^{\infty} \int D\gamma D X e^{-S} \tag{4.6}$$

Here the path integral is over the worldsheet fields $\gamma_{\alpha\beta}$ and X^μ. Now each genus g worldsheet will appear suppressed by the factor $e^{-\phi\chi_g} = e^{-2\phi(g-1)}$. Thus $g_s = e^\phi$ can be thought of as the string coupling constant which counts which genus surface is contributing to a calculation. In particular for $g_s \to 0$ one can just consider the leading order term where the worldsheet is a sphere.

However if one wants to consider the splitting and joining of strings then one must take $g_s > 0$ and include higher genus surfaces. In particular the first non-trivial string interactions arise when the worldsheet is a torus. To see the analogy with quantum field theory note that a torus can be thought of as the worldvolume of a closed string that has gone around in a loop. Thus it is analogous to 1-loop processes in quantum field theory. Similarly higher genus surfaces incorporate higher loop processes. One of the great features of string theory is that each of these contributions is finite. So this defines a finite perturbative expansion of a quantum theory which includes gravity!

As stated above our general principle is the conformal invariance of the worldsheet theory, which ensures that the worldsheet metric $\gamma_{\alpha\beta}$ decouples. The action we just wrote down is conformal as a classical action. However this will not generically be the case in the quantum theory. Divergences in the quantum theory require regularization and renormalization and these effects will break conformal invariance by introducing an explicit scale: the renormalization group scale. It turns out that conformal invariance is more or less equivalent to finiteness of the quantum field theory. This restriction leads to equations of motions for the spacetime fields $\phi, g_{\mu\nu}$ and $b_{\mu\nu}$ (which from the worldvolume point of view are just fancy coupling constants). It is beyond the scope of this course to show this but the constraints of conformal invariance at the one loop level give equations of motion

$$\begin{aligned}
R_{\mu\nu} &= -\frac{1}{4}H_{\mu\lambda\rho}H_{\nu}{}^{\lambda\rho} + 2D_\mu D_\nu\phi \\
D^\lambda H_{\lambda\mu\nu} &= 2D^\lambda\phi H_{\lambda\mu\nu} \\
4D^2\phi - 4(D\phi)^2 &= R + \frac{1}{12}H^2
\end{aligned} \tag{4.7}$$

where $H_{\mu\nu\lambda} = 3\partial_{[\mu}b_{\nu\lambda]}$. In general there will be corrections to these equations coming from all orders in perturbation theory, *i.e.* higher powers of α'. However such terms will be higher order spacetime derivatives and can typically be safely ignored.

4.2 A Spacetime Effective Action

A string propagating in spacetime has an infinite tower of massive excitations. However all but the lightest (massless) modes will be too heavy to observe in any experiment that we do. Thus in many cases one really just wants to consider the dynamics of the massless modes. This introduces the concept of an effective action. This is a very general concept (ubiquitous in quantum field theory) whereby we introduce an action for the light modes that we are interested in (below some scale M). The action is constructed so that it has all the correct symmetries of the full theory and its equations of motion reproduce the correct scattering amplitudes of the light modes that the full theory predicts. In general effective actions need not be renormalizable and they are not expected to be valid at energy scales above the scale M where the massive modes we've ignored can be excited and can no longer be ignored. Often one says that the massive modes have been integrated out. Meaning that one has performed the path integral over modes with momenta larger than M and is just left with a path integral over the low momentum modes.

In our case we have considered a string propagating in a curved spacetime that can be thought of as a background coming from a non-trivial configuration of its massless modes. In particular in our discussion we implicitly assumed that the massive modes were set to zero. The result was that quantum conformal invariance predicted the equations of motion (4.7). These are the on-shell conditions for a string to propagate in spacetime as derived in the full quantum theory. Note that they pick up an infinite series of α' corrections and also an infinite series of g_s corrections (where we allow the splitting an joining of strings). In other words, at lowest order in α' and g_s these are the equations of motion for the spacetime fields. Furthermore these equations of motion can be derived from the spacetime action

$$S_{effective} = -\frac{1}{2\alpha'^{12}}\int d^{26}x\sqrt{-g}e^{-2\phi}\left(R - 4(\partial\phi)^2 + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right) + \ldots \tag{4.8}$$

Problem: Show that the equations of motion of (4.8) are indeed (4.7). You may need to recall that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ and $g^{\mu\nu}\delta R_{\mu\nu} = D_\mu D_\nu \delta g^{\mu\nu} - g_{\mu\nu}D^2\delta g^{\mu\nu}$.

This is therefore the effective action for the massless modes of a closed string. The ellipsis denotes contributions from higher loops which will contain higher numbers of derivatives and which are suppressed by higher powers of α'. Note that string theory also predicts corrections to the effective action from string loops, that is from higher genus Riemann surfaces. These terms will come with factors of $e^{-2g\phi}$ where $g = 0, -1, -2, ...$ and can be ignored if the string coupling $g_s = e^{\phi}$ is small.

Thus we have derived a gravitational theory from the low energy dynamics of closed strings. As such we expect to see such features as black holes and cosmology. Next we return to the discussion of open strings.

5 D-branes

5.1 D-branes from Dirichlet Boundary Conditions

So far we have primarily considered the effective action of closed strings. We note that once we consider string interactions, where two strings can join or split, then one can no longer consider just open strings. Indeed it is geometrically obvious that two open strings can join up and form a closed string. From the physical point of view closed strings give gravitational modes and since everything gravitates, in particular Yang-Mills fields, this statement is simply the fact that turning on a Yang-Mills field will lead to a non-vanishing energy-momentum tensor and hence warp spacetime.

However it has become clear that even a theory of closed string fields should naturally contain sectors with open strings. In particular it is natural to consider a Dp-brane in closed string theory. This will turn out to have the interpretation as a solitonic state within the closed string theory.

By definition a Dp-brane is a $(p+1)$-dimensional worldvolume in spacetime upon which open strings can end. In practice this means that within closed string theory we include objects where open strings can end, *i.e.* we allow for Dirichlet boundary conditions $\delta X^i = 0$ at $\sigma = 0, \pi$

To be precise consider a Dp-brane parallel to the $x^0, x^1, ..., x^p$ dimensions. This means that it sits at a specific location in the $x^{p+1}, ..., x^{25}$ dimensions, say $(x^{p+1}, ..., x^{25}) = (a^{p+1}, ..., a^{25})$. Thus one imposes Dirichlet boundary conditions on the fields $X^{p+1}, ..., X^{25}$. However the $X^0, ..., X^p$ coordinates are freely allowed to move and hence and are subjected to Neumann boundary conditions, *i.e.* $\partial_\sigma X^\mu = 0$ at $\sigma = 0, \pi$ and $\mu = 0, ..., p$. Thus the mode expansion for these fields is as before:

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + \sqrt{2\alpha'}\, i \sum_{n\neq 0} \frac{a_n^\mu}{n} e^{in\tau} \cos(n\sigma) \tag{5.1}$$

On the other hand for the transverse coordinates to the Dp-brane we have the boundary condition that $X^i = a^i$ at $\sigma = 0, \pi$. Starting from the expansion

$$\hat{X}^i = x^i + w^i \sigma + \alpha' p^i \tau + \sqrt{\frac{\alpha'}{2}}\, i \sum_{n\neq 0} \left(\frac{a_n^i}{n} e^{-in(\tau+\sigma)} + \frac{\tilde{a}_n^i}{n} e^{-in(\tau-\sigma)} \right) \tag{5.2}$$

and setting $\sigma = 0$ we see that $x^i = a^i$ and $p^i = 0$. We also find $a^i_n = -\tilde{a}^i_n$. Next we consider the $\sigma = \pi$ end. Here we find $w^i = 0$ and

$$a^i_n e^{-in\pi} + \tilde{a}^i_n e^{in\pi} = 0 \tag{5.3}$$

Using the fact that $a^i_n = -\tilde{a}^i_n$ we see that we need $\sin n\pi = 0$. Thus once again we see that the n are integers. In summary we find

$$X^i = a^i + \sqrt{2\alpha'} i \sum_{n\neq 0} \frac{a^i_n}{n} e^{in\tau} \sin(n\sigma) \tag{5.4}$$

The main difference with the X^μ coordinates are the lack of a momentum zero mode p^i. This means that the states of these open strings cannot move away from $x^i = a^i$, but they can move parallel to the Dp-brane. We denote the ground state of this open string by $|D; p>$ to distinguish it from the ground state of other open strings or closed strings.

What are the low lying states? Well they are very similar to before. Only now the $SO(1, 25)$ symmetry is broken to $SO(1, p) \times SO(25-p)$. There is still a tachyon $|D; p>$ at level 0 with mass-squared $-1/\alpha'$. At level 1 there are two types of massless states:

$$|A_\mu >= A_\mu a^\mu_{-1}|D; p> \qquad |Y_i >= Y_i a^i_{-1}|D; p> \tag{5.5}$$

where we now have $\mu = 0, 1, ..., p$ and $i = p + 1, ..., 25$.

Note that the gauge symmetry that we saw above is also suitably reduced. In particular the null state that we used above is now

$$\begin{aligned} i\Lambda L_{-1}|0; p> &= \frac{i}{2}\Lambda \sum_m \eta_{\mu\nu} a^\mu_{-1+m} a^\nu_{-m}|0; p> \\ &= i\sqrt{2\alpha'} p_\mu \Lambda a^\mu_{-1}|0; p> \end{aligned} \tag{5.6}$$

The point to note here is that p^μ is only nonvanishing for $\mu = 0, 1, 2, ..., p$. Hence the modes $|Y_i>$ are not subject to a gauge symmetry, however $|A_\mu>$ still plays the role of a gauge Boson. The states $|Y_i>$ have the interpretation as $25 - p$ massless scalar fields. They parameterize fluctuations of the Dp-brane in the transverse coordinates.

The importance of D-branes was not appreciated until 1994. Dp-branes should be thought of as solitonic-like states that appear in the closed string theory. As such they are like p-dimensional hypersurfaces in space, which are constant in time. In general we can consider configurations made up of several types of Dp-branes lying in different planes and intersecting with each other. The rules of string theory tell us that for each pair of brane (or for a brane and itself) we must consider the open string that stretches between the two. Each such string leads to additional particle like degrees of freedom.

We can also consider situations with N Dp-branes all parallel to each other. In this case we must label the end points of the open strings by an index $a = 1, ..., N$ to indicate which Dp-brane they end on. Indeed one sees that this is a geometric origin for the Chan-Paton factors that we discussed about and leads to a $U(N)$ gauge symmetry. The D-brane ground state can therefore be denote by $|D; p, ab>$, with one Chan-Paton index for each end point. It follows that all the states in the Fock space created using the string oscillators will carry ab indices and hence can be thought of as matrix valued.

The modern view on string theory is that one thinks of the bulk, 26-dimensional, dynamics are governed by closed strings, whose massless modes are a graviton, Kalb-Ramond field and dilaton. However in addition there are these soliton like D-brane state. On the worldvolume of these D-branes one finds $U(N)$ gauge vector fields, as well as scalars. It may happen that in some cases the D-branes are spacetime filling, meaning that they

For example a D0-brane is essentially a point particle. The open strings are confined to end at a particular point in space (but not time). One also has D1-branes which are much like strings themselves. A D25-brane is simply the original notion of open strings, but now these are viewed as a state within the closed string theory.

Problem: Determine the mode expansion for an open string that stretches between a D1-brane located at $x^2 = ... = x^{25} = 0$ and a D25-brane, which fills all of spacetime. By considering light cone gauge (along the direction X^0, X^1) describe the lightest physical states, what are their masses?

5.2 The D-brane Effective action

How can we include the open string massless modes in an effective action? Well if we consider a 25-brane, that is a space-filling D-brane then this is essentially just the original definition of open strings and there is a massless vector A_μ. Given the Neumann boundary condition there is a natural coupling of A_μ to the worldsheet of a string through its boundary:

$$S_{open} = S_{closed} + \int_{endpoints} d\tau A_\mu \dot{X}^\mu \tag{5.7}$$

where S_{closed} is the closed string σ-model that we discussed above.

For D-branes one finds a vector field A_μ, living on a $(p+1)$ dimensional subspace, plus the scalars Y_i. The corresponding worldvolume action is

$$S_{Dp} = S_{closed} + \int_{endpoints} d\tau A_\mu \dot{X}^\mu + Y_i X'^i \tag{5.8}$$

Just as for the closed string modes one must impose that this defines a conformal field theory. However rather than being defined on a compact 2-dimensional surface, it is now defined on a 2-dimensional surface with boundary. In other words we have a boundary conformal field theory.

The requirement of conformal invariance will give an infinite series of α' perturbation expansion in general. At lowest order in α' the effective action is given by

$$S_{Dp} \;\; = \;\; -T_p \int d^{p+1}x e^{-\phi} \mathrm{Tr}\Big(1 + \frac{1}{4}(F_{\mu\nu} + (2\pi\alpha')^{-1} b_{\mu\nu})(F^{\mu\nu} + (2\pi\alpha')^{-1} b^{\mu\nu})$$

$$+\frac{1}{2}D_\mu Y_i D^\mu Y^i - \frac{1}{4}\sum_{i,j}[Y^i,Y^j][Y^i,Y^j]\Big) \tag{5.9}$$

where $T_p \sim \alpha'^{-\frac{p+1}{2}}$ is the tension of a Dp-brane *i.e.* the energy per unit p-volume. Here

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \tag{5.10}$$

is the Yang-Mills field strength and

$$D_\mu Y_i = \partial_\mu Y_i + i[A_\mu, Y_i] \tag{5.11}$$

is the gauge covariant derivative. Note that due to the Chan-Paton factors the gauge field A_μ and scalars Y_i carry ab indices and as such are best viewed as matrix-valued, *i.e.* we really mean that $A_\mu = A_\mu^{ab}$ and

$$[A_\mu, A_\nu]^{ab} = \sum_{k=1}^{N} A_\mu^{ac}A_\nu^{cb} - A_\nu^{ac}A_\mu^{cb} \tag{5.12}$$

and similarly for $Y_i = Y_i^{ab}$

Note that the b-field enters here (multiplied by the identity matrix in the Lie-algebra, *i.e.* $b_{\mu\nu}\delta^{ab}$). The reason for this is that although we saw that there was a gauge symmetry of the closed strings

$$b'_{\mu\nu} = b_{\mu\nu} + \partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu \tag{5.13}$$

This symmetry is broken in the presence of Dp-branes. The point is that if the worldsheet has a boundary then

$$\begin{aligned}\delta S_{open} &= -\frac{1}{2\pi\alpha'}\int d^2\sigma \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\mu \lambda_\nu \\ &= -\frac{1}{2\pi\alpha'}\int d^2\sigma \partial_\alpha(\epsilon^{\alpha\beta}\partial_\beta X^\nu \lambda_\nu) \\ &= \frac{1}{2\pi\alpha'}\int_{endpoints} d\tau \partial_0 X^\nu \lambda_\nu\end{aligned} \tag{5.14}$$

This no longer vanishes. However one sees that it can be canceled by a shift

$$A_\mu^{ab} \to A_\mu^{ab} - \frac{1}{2\pi\alpha'}\lambda_\mu\delta^{ab} \tag{5.15}$$

so that

$$2\pi\alpha' F_{\mu\nu} + b_{\mu\nu} \tag{5.16}$$

is gauge invariant. Note that this only affects the part of $F_{\mu\nu}$ proportional to the identity element in $U(N)$.

This action has a generalization of the usual electromagnetic gauge symmetry:

Problem: Show that S_{Dp} is invariant under the gauge transformation

$$A_\mu \to -ig^{-1}\partial_\mu g + g^{-1}A_\mu g \qquad Y_i \to g^{-1}Y_i g \tag{5.17}$$

where g is an element of $U(N)$.

This is called a Yang-Mills of non-Abelian gauge symmetry A very special case is to consider 1×1 matrices, *i.e.* just numbers. Now $U(1)$ is the space of number g such that $g^{-1} = g^*$ so that $g = e^{i\theta}$ for a real (and periodic) θ . Here we simply recover the electromagnetic gauge transformation $A_\mu = A_\mu + \partial_\mu\theta$ *i.e.* electromagnetism is a $U(1)$ gauge theory.

So far we have just taken A_μ and Y_i to be matrix valued. Matrix multiplication arises naturally from the joining of two open strings to form a third. Consider a string starts on the first D-brane and ends on a second one. Its described by a non-vanishing matrix entry M^{12}. Next consider an open string that starts on the second D-brane and ends on a third D-brane. Its matrix entry will be M^{23}. Combing these to open strings produces a longer open string that starts on the first D-brane and ends on the third one, the corresponding matrix is $M^{12}M^{23}$. Following the usual rules of quantum mechanics we should sum over all possible choices of the second, intermediate D-brane. Thus the natural product of two open strings that start on the first D-brane and end on the third is $\sum_a M^{1a}M^{a3}$. This is just matrix multiplication.

However we shouldn't just let the open strings be described by any matrices. Open strings are oriented so that the string that goes from the ath D-brane to the bth D-brane is described by M^{ab} whereas the open string that goes from the bth D-brane to the ath D-brane is described by M^{ba}. These two strings are different so we do not want to say that $M^{ab} = M^{ba}$, *i.e.* $M = M^T$. However we would like a string that starts on the ath D-brane and ends on the bth D-brane to join up with an open string that starts of the bth D-brane and ends on the ath D-brane to create a closed string. This would be described by $M^{ab}M^{ba} = \mathrm{Tr}(M^2)$. If this is to be a closed string it should just be a real number, *i.e.* with no Chan-Paton index. Therefore we must take $M^* = M^T$. In other words we restrict to Hermitian matrices. Note that if M is Hermitian then so is $g^{-1}Mg$ if $g \in U(N)$.

Finally let us consider the potential on the coordinates Y_i

$$V = -T_p \sum_{i,j} \mathrm{Tr}\left([Y_i, Y_j]\right)^2 \tag{5.18}$$

Note that since $Y_i^\dagger = Y_i$ we have $[Y_i, Y_j]^\dagger = -[Y_i, Y_j]$. Thus the potential is minus the sum of the square of an anti-Hermitian matrix. Therefore $V \geq 0$. It follows that the vacuum states of this action correspond to $[Y^i, Y^j] = 0$ for all i, j. Therefore, up to a gauge transformation, we can write

$$Y^i = \mathrm{diag}(a_1^i, ..., a_n^i) \tag{5.19}$$

We interpret the a_a^i as the location of the ath Dp-brane in the x^i direction. Thus even though the open strings that stretch between N parallel D-branes carry N^2 degrees of freedom, the vacuum moduli space just consists of N independent vectors $(a_a^{p+1}, ..., a_a^{25})$ which parameterize the position of the ath D-brane in the transverse space. The other modes will generically be massive as a consequence of the Higg's effect. Geometrically this occurs because the open string that stretches between two separated D-branes must have a non-zero length and hence is massive. However whenever two or more D-brane coincide there is an enhanced symmetry and additional massless fields arise.

Thus in general the total action is

$$S = \frac{1}{2\alpha'^{12}} \int d^{26}x\sqrt{-g}e^{-2\phi}\left(R - 4(\partial\phi)^2 + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right) + \sum_{Dp} S_{Dp} \tag{5.20}$$

where the second term is a sum over all the Dp-branes in the background. We emphasize that this is just the lowest order term in an effective action with otherwise contains contributions from higher powers of α' and $g_s = e^{\phi}$.

In the Abelian case, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, it is known that, up to terms involving two derivatives acting on the fields,

$$S_{Dp} = S_{BDI} = -T_p \int d^{p+1}xe^{-\phi}\sqrt{-\det(g_{\mu\nu} + F_{\mu\nu} + (2\pi\alpha')^{-1}b_{\mu\nu} + \partial_\mu Y^i\partial_\nu Y^j\delta_{ij})} \tag{5.21}$$

This action had been studied well before string theory and is known as the Dirac-Born-Infeld action. It has two interesting special cases. If the gauge fields and dilaton are set to zero and the metric to that of Minkowski space then we obtain

$$S_{Dp} = -T_p \int d^{p+1}x\sqrt{-\det(\eta_{\mu\nu} + \partial_\mu Y^i\partial_\nu Y^j\delta_{ij})} \tag{5.22}$$

Problem: Show that this is a gauge fixed form of the action

$$S_{NG} = -T_p \int d^{p+1}x\sqrt{-\det\left(\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}\right)} \tag{5.23}$$

where we take $X^\mu = x^\mu$ if $\mu = 0, ..., p$ and $X^\mu = Y^i$ if $\mu = i = p+1, ..., 25$.

This describes an extended object in spacetime whose action is simply its volume and indeed the Y^i give the position of the Dp-brane in the transverse space. Thus indeed we should think of a Dp-brane as a p-dimensional extended object in space, that propagates through time.

On the other hand if we set the scalars Y^i to zero (again with $\phi = b_{\mu\nu} = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$) then the effective action is

$$S_{Dp} = S_{BI} = -T_p \int d^{p+1}x\sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})} \tag{5.24}$$

This action is known as the Born-Infeld action. It was proposed many years ago as a non-linear version of electrodynamics.

6 Compactification

6.1 Closed String Spectrum on a Circle

We have seen that string theory lives in 26 dimensions. Although it has all the features we want, such as gauge symmetries and gravitation, this would seem to contradict experiment badly. Well the point is that no one said that these dimensions had to be very big. Indeed the natural scale of string theory, the string scale, is $l_s = \sqrt{\alpha'}$ and this is somewhere slightly larger than the Plank length (depending on g_s). Thus if a dimension was to be small it could be very small indeed. However this in itself is a problem as we cannot use effective field theory if the length scales are of order l_s. This is because the effective theory is the theory of the massless modes and these correspond to low oscillations of the string. The higher modes resolve substringy distances and can typically be ignored on larger scales.

First let us consider the spectrum of closed strings if some dimensions, labeled by X^i, are circles: $X^i \sim X^i + 2\pi R_i$. Returning to our mode expansion we see that X^i need not be single valued but rather

$$X^i(\sigma + 2\pi) = X^i(\sigma) + 2\pi n^i R_i \tag{6.1}$$

for some integer n^i. Such a string is wound around the X^i dimension. Thus we see that in our original mode expansion we can have $w^i = n^i R_i$ for an integer n^i. Furthermore the momentum around a circle must be quantized (so that the wavefunction is single valued) and hence $p^i = m^i/R$. It then follows that

$$a_0^i = \sqrt{\frac{\alpha'}{2}} m^i R_i^{-1} + \sqrt{\frac{1}{2\alpha'}} n^i R_i \qquad \tilde{a}_0^i = \sqrt{\frac{\alpha'}{2}} m^i R_i^{-1} - \sqrt{\frac{1}{2\alpha'}} n^i R_i \tag{6.2}$$

and

$$a_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu \tag{6.3}$$

for the non-compact directions. Now that $w^i \neq 0$ we see that

$$\begin{aligned} L_0 - 1 &= \frac{1}{2} a_0^\mu a_0^\nu \eta_{\mu\nu} + N - 1 \\ &= \frac{\alpha'}{4} p^2 + \frac{1}{2} \sum_i \left(\frac{\alpha'}{2} \left(\frac{m_i}{R_i} \right)^2 + \frac{1}{2\alpha'} \left(n^i R_i \right)^2 + n^i m_i \right) + N - 1 \end{aligned} \tag{6.4}$$

On the other hand we have

$$\begin{aligned} \tilde{L}_0 - 1 &= \frac{1}{2} a_0^\mu a_0^\nu \eta_{\mu\nu} + N - 1 \\ &= \frac{\alpha'}{4} p^2 + \frac{1}{2} \sum_i \left(\frac{\alpha'}{2} \left(\frac{m_i}{R_i} \right)^2 + \frac{1}{2\alpha'} \left(n^i R_i \right)^2 - n^i m_i \right) + \tilde{N} - 1 \end{aligned} \tag{6.5}$$

Level matching is now slightly shifted to

$$(N - \tilde{N} + n^i m_i)|phys >= 0 \tag{6.6}$$

And the spacetime mass shell is $p^2 + M^2 = 0$ with

$$M^2 = \sum_i \left(\left(\frac{m_i}{R_i} \right)^2 + \frac{1}{\alpha'^2} \left(n^i R_i \right)^2 \right) + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{6.7}$$

You should notice an interesting symmetry. The spectrum is invariant under $n_i \leftrightarrow m_i$ $R_i \leftrightarrow \alpha'/R_i$, *i.e.* under the interchange of momentum and winding quantum numbers. In fact this symmetry extends to the full interacting theory and is know as T-duality. It implies that there is a sort of minimum length scale built into string theory as a string on a circle of radius R is equivalent to a string on a circle of radius α'/R. Physically the problem is that for distances smaller than $\sqrt{\alpha'}$ the string behaves more and more like an extended object and cannot resolve smaller distances.

In any case one sees that for each circular dimension there is a double tower of increasingly massive states (in addition to the exponentially growing tower of states). In particular each mode in the stringy tower of states now carries two extra integral charges. These are the momentum and winding numbers about each compact dimension. It follows that all but the zero-modes are massive, with a mass-squared of order α'^{-1}. Therefore at low energy only the zero-modes will be physically relevant.

Note that the momentum modes get heavier as we shrink the radii whereas the winding mode will get lighter. However so long as the string length $\sqrt{\alpha'}$ is small we can ensure that all these extra massive momentum and winding modes are too massive to observe.

First let us make a digression in to the size of the Planck scale - the scale at which quantum gravity is important - relative to the string scale given by the size of the strings. Recall that the string coupling constant is $g_s = e^\phi$ and the string length scale is $\sqrt{\alpha'}$. Thus we see that the spacetime effective behaves as

$$S_{effective} \sim \frac{1}{g_s^2 \alpha'^{12}} \int dx^{26} \sqrt{-g} R \sim \frac{1}{l_P^{24}} \int d^{26} \sqrt{-g} R \tag{6.8}$$

where the right-most expression gives the conventional definition of the Planck length l_p. From this we see that $l_P = g_s^{\frac{1}{12}} \sqrt{\alpha'}$. Therefore at weak string coupling, where string perturbation theory is valid, we have $g_s << 1$ and hence the Planck length is much smaller than the string length. This means that it is sensible to talk about strings without worrying about the effects of quantum gravity.

The Planck length is about $l_P \sim 10^{-19} GeV^{-1}$ therefore we are safe if we take $\sqrt{\alpha'} \sim 10^{-17} GeV^{-1}$ - a hundred times larger. In which case the mass of smallest massive string modes (at oscillator level 2) are $m \sim 1/\sqrt{\alpha'} \sim 10^{17} GeV$. This is well beyond any experimental observation. Next consider compactification so that the massless modes pick up massive momentum and winding partners with mass-squared

$$M^2 = \sum_i \left(\left(\frac{m_i}{R_i} \right)^2 + \frac{1}{\alpha'^2} \left(n^i R_i \right)^2 \right) \tag{6.9}$$

The lightest mode will be the ones with a single unit of momentum and winding so that

$$M \sim \frac{1}{R} \qquad \text{and} \qquad M \sim \frac{R}{\alpha'} \tag{6.10}$$

We wish to make these too heavy to observe in accelerators, say $M >> 10^3 GeV$. In the first case we must take $R << 10^{-3}GeV^{-1}$. In the second case we must have $R >> 10^3\alpha' GeV \sim 10^{-31}GeV^{-1}$. Thus for radii in the range $10^{-31}GeV^{-1} << R << 10^{-3}GeV^{-1}$ all but the massless stringy modes will be too heavy to have observed. This is clearly a very wide range of values for the internal radii.

6.2 Dimensional Reduction of the Effective Action

Now we can consider how to reduce the effective action to lower dimensions by compactification on a torus. In effective we want to throw away the massive momentum modes that one finds (the winding modes where never included in (4.8) in the first place). The idea of extra dimensions is not at all new and first appeared in the work of Kaluza and Klein within the context of gravity.

To illustrate the point let us consider a real free scalar field theory in D dimensions with action

$$S = -\frac{1}{2}\int d^D x \partial_\mu \phi \partial^\mu \phi \tag{6.11}$$

Let us suppose that one of the dimensions, labeled by y is a circle with radius R. We can therefore write ϕ as a Fourier series

$$\phi(x,y) = \sum_n \phi_n(x) e^{-iny/R} \tag{6.12}$$

Since ϕ is real it follows that $\phi_n^* = \phi_{-n}$. Substituting back into the action gives

$$S = -\frac{1}{2}\int d^{D-1}x dy \sum_{nm}\left(\partial_\mu\phi_n\partial^\mu\phi_m - \frac{mn}{R^2}\phi_n\phi_m\right)e^{-i(n+m)y/R} \tag{6.13}$$

where now μ only labels the noncompact directions. Now we have that

$$\int_0^{2\pi R} dy e^{-iny/R} = \begin{cases} 2\pi R & n=0 \\ 0 & n\neq 0 \end{cases} \tag{6.14}$$

Thus we find

$$\begin{aligned} S &= -\pi R \sum_n \int d^{D-1}x \left(\partial_\mu\phi_n\partial^\mu\phi_{-n} + \frac{n^2}{R^2}\phi_n\phi_{-n}\right) \\ &= -\pi R \sum_n \int d^{D-1}x \left(\partial_\mu\phi_n\partial^\mu\phi_n^* + \frac{n^2}{R^2}\phi_n\phi_n^*\right) \end{aligned} \tag{6.15}$$

This is a $(D-1)$ dimensional action with an infinite tower of massive, complex scalar fields with masses

$$m_n^2 = \frac{n^2}{R^2} \tag{6.16}$$

However the zero-mode ϕ_0 is real and massless. This is just as we'd expect from our string discussion. At energies below that set by the extra dimension, *i.e.* at energy scales below $1/R$, only the massless mode is physically relevant. Thus we see that in this case the effective action for the massless field is simply

$$S_0 = -\pi R \int d^{D-1}x \partial_\mu \phi_0 \partial^\mu \phi_0 \tag{6.17}$$

The higher dimensional field theory is said to be compactified. This generally means that only the lowest, massless, modes are retained. For a simply scalar field this reduction does not seem very interesting (unless one keeps the Kaluza-Klein tower of states). However we will see that for tensor fields, such as vector fields, Kalb-Raymond field and gravitons, one does see qualitative changes. However one fact that remains is that, at least for circle reductions, the lightest fields are always independent of the coordinate along the circle.

We need to compactify the string effective action on a circle. Let us denote the 26-dimensional quantities by a hat and 25-dimensional quantities without a hat. We will use μ, ν, λ to denote all 26 and 25 dimensions, which is meant exactly should be clear from whether or not it appears with or without a hat. We have seen that we can simply set $\hat{\phi} = \phi$ to be independent of y. However we also need to consider $\hat{b}_{\mu\nu}$ and the metric $\hat{g}_{\mu\nu}$. We will assume that all fields are independent of y.

First we must reduce the metric. A D-dimensional metric can be written as

$$d\hat{s}^2 = ds^2 + e^{2\rho}(dy + A_\mu dx^\mu)^2 \tag{6.18}$$

or

$$\hat{g}_{\mu\nu} = \begin{pmatrix} g_{\mu\nu} + e^{2\rho} A_\mu A_\nu & e^{2\rho} A_\nu \\ e^{2\rho} A_\mu & e^{2\rho} \end{pmatrix} \tag{6.19}$$

We note that

$$\hat{g}^{\mu\nu} = \begin{pmatrix} g^{\mu\nu} & -A_\nu \\ -A_\mu & e^{-2\rho} + A^2 \end{pmatrix} \tag{6.20}$$

and

$$\sqrt{-\hat{g}} = e^\rho \sqrt{-g} \tag{6.21}$$

Here we are taking all fields to be independent of y. Including such a dependence would lead to an infinite tower of massive Fourier modes which we can ignore at low energy. Note that e^ρ controls the physical size of the compact dimension since the proper circumference is

$$\int_0^{2\pi R} d\hat{s} = \int_0^{2\pi R} \sqrt{\hat{g}_{yy}} dy = 2\pi R e^\rho \tag{6.22}$$

and A_μ determines the angle between the compact dimension and the noncompact dimensions. However the residual diffeomorphism symmetries of the compact dimension lead to an electro-magnetic like gauge symmetry:

Problem: Show that the diffeomorphism symmetry acting on the coordinate y, viz. $y \rightarrow y+\lambda(x)$ appears in the $(D-1)$-dimensional theory as a $U(1)$ gauge transformation $A_\mu \rightarrow A_\mu - \partial_\mu \lambda$

The appearance of this electromagnetic gauge field from gravity in higher dimensions was originally pointed out by Kaluza and Klein why back in the 1920's.

Note that the Bosonic string effective action only depends on the metric and Ricci scalar R. Therefore our next job is to work out the 26-dimensional curvature scalar in terms of the 25-dimensional Ricci scalar, A_μ and ρ. A fairly lengthy calculation (try it) shows that

$$\hat{R} = R - 2(\partial\rho)^2 - 2D^2\rho - \frac{1}{4}e^{2\rho}F^2 \tag{6.23}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and all term on the right hand side are evaluated using the metric $g_{\mu\nu}$.

Putting $\hat{H}_{\mu\nu\lambda} = 0$ for now we find that

$$S_{effective} = -\frac{\pi R}{\alpha'^{13}} \int d^{25}x e^{\rho-2\phi}\left(R - 2(\partial\rho)^2 - 2D^2\rho - \frac{1}{4}e^{2\rho}F^2 - 4(\partial\phi)^2\right) \tag{6.24}$$

Next consider $\hat{b}_{\mu\nu}$. This leads to the 25-dimensional fields

$$\begin{aligned} \hat{b}_{\mu\nu} &= b_{\mu\nu} \\ \hat{b}_{\mu y} = -\hat{b}_{y\mu} &= B_\mu \end{aligned} \tag{6.25}$$

The field string $\hat{H}_{\mu\nu\lambda}$ is therefore

$$\begin{aligned} \hat{H}_{\mu\nu\lambda} &= \partial_\mu b_{\nu\lambda} + \partial_\nu b_{\lambda\mu} + \partial_\lambda b_{\mu\nu} \\ &= H_{\mu\nu\lambda} \\ \hat{H}_{\mu\nu y} &= \partial_\mu \hat{b}_{\nu y} + \partial_\nu \hat{b}_{y\mu} \\ &= \partial_\mu B_\nu - \partial_\nu B_\mu \\ &= G_{\mu\nu} \end{aligned} \tag{6.26}$$

where we have used the fact that $\partial_y = 0$. Thus in 25 dimensions we find a 2-form $B_{\mu\nu}$ and in addition a vector field B_μ with field strength $G_{\mu\nu}$.

To complete the reduction of the effective action we need only evaluate $\hat{H}_{\mu\nu\lambda}\hat{H}^{\mu\nu\lambda}$.

$$\begin{aligned} \hat{H}_{\mu\nu\lambda}\hat{H}^{\mu\nu\lambda} &= g^{\mu\mu'}g^{\nu\nu'}g^{\lambda\lambda'}\hat{H}_{\mu\nu\lambda}\hat{H}_{\mu'\nu'\lambda'} + 3g^{\mu\mu'}g^{\nu\nu'}g^{yy}\hat{H}_{\mu\nu z}\hat{H}_{\mu'\nu' y} \\ &\quad + 6g^{\mu\mu'}g^{y\nu'}g^{\lambda y}\hat{H}_{\mu y\lambda}\hat{H}_{\mu'\nu' y} + 6g^{\mu\mu'}g^{\nu\nu'}g^{\lambda y}\hat{H}_{\mu\nu\lambda}\hat{H}_{\mu'\nu' y} \\ &= H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 3(e^{-2\rho} + A^2)G_{\mu\nu}G^{\mu\nu} - 6G_{\mu\nu}G^{\mu\lambda}A^\nu A_\lambda - 6H_{\mu\nu\lambda}G^{\mu\nu}A^\lambda \end{aligned} \tag{6.27}$$

where it is understood that on the right hand side all metric contractions only involve the 25-dimensional metric $g_{\mu\nu}$ whereas on the left hand side the 26-dimensional metric $\hat{g}_{\mu\nu}$ is used. This expression can be cleaned up slightly by writing

$$K_{\mu\nu\lambda} = H_{\mu\nu\lambda} - 3G_{[\mu\nu}A_{\lambda]} \tag{6.28}$$

so that

$$\hat{H}^2 = K_{\mu\nu\lambda}K^{\mu\nu\lambda} + 3e^{-2\rho}G_{\mu\nu}G^{\mu\nu} \tag{6.29}$$

Thus the compactified effective action is

$$\begin{aligned} S_{effective} &= -\frac{\pi R}{\alpha'^{13}}\int d^{25}x e^{\rho-2\phi}\Big(R - 2(\partial\rho)^2 - 2D^2\rho - \frac{1}{4}e^{2\beta\rho}F^2 - 4(\partial\phi)^2 \\ &\quad + \frac{1}{12}K_{\mu\nu\lambda}K^{\mu\nu\lambda} + \frac{1}{4}e^{-2\rho}G_{\mu\nu}G^{\mu\nu}\Big) \end{aligned} \tag{6.30}$$

note that one can integrate by parts to remove the $D^2\rho$ term from the action. In so doing one sees that there will be cross term $\partial_\mu\rho\partial^\mu\phi$ in the kinetic energy. However by rotating the scalar fields into each other one can write the action without such cross terms. However this is not particularly illuminating and so we won't do it here.

Of course one can repeat this process to compactify on n circles reduce to $26-n$ dimensions. This will lead to more and more massless modes. In fact something remarkable happens. These massless modes organize themselves so that they admit an $O(n,n)$ symmetry. This is essentially a combination of diffeomorhisms of the circles as well as T-dualities. However one could compactify on circles all the way down to three dimensions. Here the only propagating degrees of freedom are scalar fields (the vectors can be dualized into scalars). Remarkably one finds that the dimensionally reduced action in three dimensions has an $O(24,24)$ symmetry, rather than just an $O(23,23)$ symmetry.

Furthermore one need not reduce on circles but general compact manifolds. On such spaces the Laplacian will generically have a discrete spectrum and again one is only interested in the massless modes, with the massive modes have masses of order of the inverse size of the manifold. In addition one must find the massless modes for the 3-form field H. This is related to the topology of the compact manifold.

Problem: Show that, in a background with $g_{\mu\nu} = \eta_{\mu\nu}, b_{\mu\nu} = \phi = 0$, the D$p$-brane effective action (5.9) is the dimensional reduction of the D25-brane effective action

$$S_{D25} = -T_{25}\int d^{26}x \mathrm{Tr}\left(1 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{6.31}$$

to $p+1$ dimensions. For simplicity you may treat the closed string fields as non-dynamical and simply take $g_{\mu\nu} = \eta_{\mu\nu}, b_{\mu\nu} = \phi = 0$.

7 Superstrings, M-theory and the Big Picture

In the final section let us try to give an overview of superstrings and M-theory. We can only provide a superficial treatment here but it is important to note that the superstrings can be described in great detail and fairly rigorously (see Green Schwarz and Witten, or Polchinski). M-theory is still undefined and hence that part of the notes might seem particularly vague. In some sense M-theory is a collection of interrelated results that strongly support the existence of an underlying and well-defined quantum theory.

7.1 On the Worldsheet

The starting point for the superstring is include Fermions ψ^μ on the worldsheet so as to construct a supersymmetric action (see for example the supersymmetry notes)

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\eta^{\alpha\beta} + i\bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi^\nu \eta_{\mu\nu} \tag{7.32}$$

This action is also conformally invariant and in addition has the supersymmetry

$$\delta X^\mu = i\bar{\epsilon}\psi^\mu \ , \qquad \delta\psi^\mu = \gamma^\alpha \partial_\alpha X^\mu \epsilon \tag{7.33}$$

for any constant ϵ.

The mode expansion for the X^μ remains as before with the a_n^μ and $\tilde{a}_n^\mu$ oscillators. When we expand the Fermionic fields we can allow for two types of boundary conditions (let us just consider boundary conditions consistent with a closed string where $\sigma \sim \sigma+2\pi$ and $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi)$):

$$\begin{array}{lrcl} \mathrm{R:} & \psi^\mu(\tau,\sigma+2\pi) & = & \psi^\mu(\tau,\sigma) \\ \mathrm{NS:} & \psi^\mu(\tau,\sigma+2\pi) & = & -\psi^\mu(\tau,\sigma) \end{array} \tag{7.34}$$

these are known as the Ramond and Neveu-Schwarz sectors respectively. Thus we find

$$\begin{array}{lrcl} \mathrm{R:} & \psi^\mu(\tau,\sigma+2\pi) & = & \sum_{n\in\mathbf{Z}} d_n e^{-in\sigma^+} + \tilde{d}_n e^{-in\sigma^-} \\ \mathrm{NS:} & \psi^\mu(\tau,\sigma+2\pi) & = & \sum_{r\in\mathbf{Z}+\frac{1}{2}} b_r e^{-ir\sigma^+} + \tilde{b}_r e^{-ir\sigma^-} \end{array} \tag{7.35}$$

One finds that these satisfy the anti-commutation relations

$$\begin{array}{rcll} \{d_m^\mu, d_n^\nu\} & = & \eta^{\mu\nu}\delta_{m,-n} & \{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r,-s} \\ \{\tilde{d}_m^\mu, \tilde{d}_n^\nu\} & = & \eta^{\mu\nu}\delta_{m,-n} & \{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} = \eta^{\mu\nu}\delta_{r,-s} \end{array} \tag{7.36}$$

with all other anti-commutators vanishing.

Let us compute the intercept a. As before we go to light-cone gauge where we fix two of the coordinates X^μ and their superpartners ψ^μ. We then compute the vacuum energy of the remaining $D-2$ Bosonic and Fermionic oscillators. The result depends on the boundary conditions we use. Noting that the sign of the Fermionic contribution is opposite to that of a Boson one finds

$$\begin{aligned} a_R &= -\frac{D-2}{2}\sum_{n=1}^{\infty} n + \frac{D-2}{2}\sum_{n=1}^{\infty} n \\ &= -\frac{D-2}{2}\Big(-\frac{1}{12}+\frac{1}{12}\Big) \\ &= 0 \end{aligned} \tag{7.37}$$

and

$$\begin{aligned} a_{NS} &= -\frac{D-2}{2}\sum_{n=1}^{\infty} n + \frac{D-2}{2}\sum_{r=1}^{\infty}\Big(r+\frac{1}{2}\Big) \\ &= -\frac{D-2}{2}\sum_{n=1}^{\infty} n + \frac{D-2}{4}\sum_{n=odd}^{\infty} n \\ &= -\frac{D-2}{4}\Big(-\frac{2}{12}-\frac{1}{12}\Big) \\ &= \frac{D-2}{16} \end{aligned} \tag{7.38}$$

The vanishing of a_R is a direct consequence of the fact that there is a Bose-Fermi degeneracy in the R-sector.

Let us now look at the lightest states. There is a different ground state for each sector which we denote by $|R;p>$ and $|NS;p>$ where p^μ labels the spacetime momentum. As before we assume that these states are annihilated by any oscillator with positive frequency.

We see that $|R;p>$ is massless and hence all the higher level states created from it by the action of a creation operator will be massive with a mass of order the string scale. However the Ramond ground state $|R;p>$ is degenerate. In particular we see that there are Fermion zero-modes d_0^μ which satisfy $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$, $\mu,\nu = 1,...,D-2$ in light cone gauge. This is a Clifford algebra and it is known that there is a unique representation and it is $2^{[\frac{D-2}{2}]}$-dimensional. Thus the Ramond ground state is in fact a spinor with $2^{[\frac{D-2}{2}]}$ independent components.

Let us look at the Neveu-Schwarz ground state $|NS,p>$. It is clear that since $a_{NS} > 0$ this state is a tachyon. We can then consider the higher level states (for simplicity we just consider open strings)

$$a_{-1}^\mu|NS,p> \qquad M^2 = 1+\frac{D-2}{16}$$

$$b^\mu_{-\frac{1}{2}}|NS,p> \qquad M^2 = \frac{1}{2}+\frac{D-2}{16}$$

Thus the next lightest state is $b^\mu_{-\frac{1}{2}}|NS,p>$ and its mass-squared is $M^2 = \frac{D-10}{16}$. Thus if $D<10$ then these states are also tachyonic. However as before the magic (that is gauge symmetries from null states) happens when these states are massless, *i.e.* $D=10$. In this case the states $a^\mu_{-1}|NS,p>$ are massive. Thus we take $D=10$ and $a_{NS}=1/2$. Indeed as before this is forced upon us if we want the $SO(1,D-1)$ Lorentz symmetry of spacetime to be preserved in the quantum theory.

Nevertheless we are still left with some bad features. For one the Neveu-Schwarz ground state is still a tachyon. There is also another puzzling feature: $|NS,p>$ is a spacetime scalar and hence it must be a Boson. We can then construct the spacetime vector $b^\mu_{-\frac{1}{2}}|NS,p>$. From the spacetime point of view this state should be a Boson since it transforms under Lorentz transformations as a vector. However it is created from $|NS,p>$ by a Fermionic operator and thus will obey Fermi-statistics. This is contradictory.

The solution to both these problems is to project out the odd states and in particular $|NS,p>$. This is known as the GSO projection. More specifically we declare that $|NS,p>$ is a Fermionic state. Mathematically we introduce the operator $(-1)^F$ which acts as $(-1)^F|NS,p>= -|NS,p>$ and $\{\psi^\mu,(-1)^F\}=0$, $[X^\mu,(-1)^F]=0$. We then project out all Fermionic states, *i.e.* states in the eigenspace $(-1)^F=-1$. Thus $|NS,p>$ and $a^\mu_{-1}|NS,p>$ are removed from the spectrum but the massless states $b^\mu_{-\frac{1}{2}}|NS,p>$ remain.

Let us now consider the Ramond sector states. We already saw that the ground state here is massless but degenerate. Indeed it is a spinor of $SO(8)$, that is to say it can be represented by a vector in the 16-dimensional vector space that furnishes a representation of the Clifford algebra relation $\{d_0^\mu,d_0^\nu\}=\eta^{\mu\nu}$, $\mu,\nu=1,...,8$ (in light-cone gauge). We need to discuss how $(-1)^F$ acts here. There is a natural candidate where we take $(-1)^F=\pm\Gamma_9$, the chirality operator in the 8-dimensional Clifford algebra. Thus after the GSO projection $|R,p>$ is a chiral spinor with 8 independent components. In the Ramond sector we project out Bosonic states with $(-1)^F=1$

In the Ramond sector of the open superstring either choice of sign is equivalent to the other, it is just a convention. Thus for the open superstring the lightest states are massless and consist of a spacetime vector (and hence a Boson) $b^\mu_{-\frac{1}{2}}|NS,p>$ along with a spacetime Fermion $|R,p>$ which can be identified with a chiral spinor. Note that there is a Bose-Fermi degeneracy: we find 8 Bosonic and 8 Fermionic states.

Let us consider closed strings. Here the states are essentially obtained by taking a tensor product of left and right moving modes and hence there are four possibilities:

$$\begin{aligned}
|NS>_L &\otimes |NS>_R\\
|R>_L &\otimes |R>_R\\
|NS>_L &\otimes |R>_R\\
|R>_L &\otimes |NS>_L
\end{aligned}$$

$$(7.39)$$

In this case the relative sign taken in the GSO projection is important. There are two choices: either we chose the same chirality projector for the left and right moving modes or the opposite. This leads to two distinction theories known as the type IIB and type IIA superstring respectively. The states one find are of the form

$$\begin{array}{lcl} |NS>_L & \otimes & |NS>_R \\ |R+>_L & \otimes & |R->_R \\ |NS>_L & \otimes & |R->_R \\ |R+>_L & \otimes & |NS>_L \end{array} \tag{7.40}$$

for type IIA and

$$\begin{array}{lcl} |NS>_L & \otimes & |NS>_R \\ |R+>_L & \otimes & |R+>_R \\ |NS>_L & \otimes & |R+>_R \\ |R+>_L & \otimes & |NS>_L \end{array} \tag{7.41}$$

for type IIB. Here the $\pm$ sign corresponds to the different choice of GSO projector for the left and right moving modes.

The spacetime Bosons come from either the NS-NS or R-R sectors whereas the spacetime Fermions from the NS-R or R-NS sectors. One sees that in the type IIA theory there are Fermionic states with both spacetime chiralities but in the type IIB theory only one chirality appears.

Let us look more closely at the Bosonic states. The NS-NS sector is essentially the same as the spectrum of the Bosonic string the we considered before and contains a graviton, Kalb-Ramond field and a dilaton. This sector is universal to all closed string theories. However we also have R-R fields. These give rise to spacetime n-form field strengths $F_{\mu_1 \ldots \mu_n}$ with n even for type IIA and n odd for type IIB.

We motivated superstrings by considering a worldsheet action that was supersymmetric and hence had an equal number of Bosons and Fermions. However it turns out that, after the GSO projection, these theories also have spacetime supersymmetry, *i.e.* their spectrum of states and interactions is supersymmetric in ten-dimensions.

Furthermore on can introduce Dp-branes into the type II string theories. These are supersymmetric and stable if p is even for type IIA and odd for type IIB. In distinction to the Bosonic string case these stable D-branes are electrically or magnetically charged with respect to the R-R fields $F_(p+2)$. Here one again finds massless $U(N)$ gauge fields but in addition massless Fermions that live on the D-brane and transform under the adjoint representation of $U(N)$. Again one can construct very complicated intersecting configurations of D-branes and hopefully even find the Standard Model of Particle Physics.

This has been a short summary of the type II string theories. There are three other possibilities. For example one can introduce open strings. Since open strings can combine into a closed string this theory must also contain closed strings but the presence of open strings leads to $SO(32)$ gauge fields in spacetime. A more bizarre construction is to exploit the fact the left and right moving modes sectors of the string worldsheet do not talk to each other (in a closed string). Thus one could take the left moving modes of a superstring living in 10 dimensions and tensor them with the right moving modes of a Bosonic string, which live in 26 dimensions. Remarkably this can be made to work and leads to two types of string theories known as the Heterotic strings. These theories contain $E_8 \times E_8$ or $SO(32)$ spacetime gauge fields. The most important point about these other three theories is that they have chiral Fermions that are charged under the gauge group and such fields are known to be an important feature of the Standard Model.

7.2 The Spacetime Effective Action

The superstrings have a spacetime supersymmetry and include gravity. Therefore their low energy effective actions are those of a supergravity. Such theories are so tightly constrained by their symmetries that, at least to lowest order in derivatives, their action is unique and known. In particular the Bosonic section of these theories is given by

$$\begin{aligned} S_{IIA} &= \frac{1}{\alpha'^4}\int d^{10}x\sqrt{-g}\left(e^{-2\phi}(R+4(\partial\phi)^2-\frac{1}{12}H_3^2)-\frac{1}{4}F_2^2-\frac{1}{48}F_4^2\right)+\ldots \\ S_{IIB} &= \frac{1}{\alpha'^4}\int d^{10}x\sqrt{-g}\left(e^{-2\phi}(R+4(\partial\phi)^2-\frac{1}{12}H_3^2)-\frac{1}{2}F_1^2-\frac{1}{12}F_3^2-\frac{1}{240}F_5^2\right)+\ldots \end{aligned}$$

where the ellipsis denotes additional terms (known as Chern-Simons terms) and the subscript $n = 1, 2, 3, 4, 5$ indicates the number of anti-symmetric indices of the field strength $F_n = F_{\mu_1\ldots\mu_n}$. Note that in the S_{IIB} case there is field strength $F_\mu = \partial_\mu a$ which can be thought of as arising from an additional scalar. In addition the equation of motion that arises from S_{IIB} must be supplemented by the constraint that the five-index field strength $F_{\mu_1\mu_2\mu_3\mu_4\mu_5}$ is self-dual:

$$F_{\mu_1\mu_2\mu_3\mu_4\mu_5} = \frac{1}{5!}\sqrt{-g}\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\nu_1\nu_2\nu_3\nu_4\nu_5}F^{\nu_1\nu_2\nu_3\nu_4\nu_5} \tag{7.42}$$

Earlier we saw that when compactified on a circle the Bosonic string admits a new duality known as T-duality. In the superstring case one finds that type IIA string theory on a circle of radius R is equivalent to type IIB string theory on a circle of radius α'/R. However one finds more remarkable dualities. It turns out that the type IIB supergravity has a symmetry $\phi \leftrightarrow -\phi$.[2] From the point of view of the string theory this is suggests a duality between strongly coupled strings with g_s large and weakly coupled stings with g_s small. This self-duality of the type IIB string is known as S-duality.

[2]This is simplifying things if the R-R-scalar a is not zero but a more general statement is true in that case.

What happens in the strong coupling limit, $g_s \to \infty$ of the type IIA superstring? Well is it conjectured that $\sqrt{\alpha'}e^{2\phi/3}$ can be interpreted as the radius of an extra, eleventh, dimension. There is a unique supergravity theory in eleven dimensions and indeed the type IIA string effective action comes from dimensional reduction of this theory on a circle. However there is now a great deal of evidence that the whole of type IIA string theory arises as an expansion of an eleven-dimensional theory about zero-radius (in on of its dimensions). This theory is known as M-theory and is rather poorly understood. However it's existence does seem be justified. The lowest order term is in a derivative expansion is fixed by supersymmetry to be

$$S_M = \frac{1}{\kappa^9}\int d^{11}x\sqrt{-g}(R - \frac{1}{48}G_4^2) + \dots \tag{7.43}$$

where again the ellipsis denotes Chern-Simons and Fermionic terms.

Furthermore it promises to be very powerful as if controls not only the strong coupling limit of the type IIA string but, as a consequence of T-duality and S-duality, as well as some other dualities, the strong coupling limit of all the five known string theories. Thus one no longer thinks of there being five separate string theories but instead one unique theory, M-theory, which contains five different perturbative descriptions depending on what one considers to be a small parameter.

There are no D-branes in M-theory as there is no microscopic description in terms of open strings. However there are supersymmetric 2-brane and 5-brane solutions which are electrically and magnetically charged with respect to the 4-index field G_4. These are known as the M2-brane and M5-brane.

Appendix: Conventions

We work in D-dimensional spacetime with "mostly plus" signature

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & & & & \\ & +1 & & & \\ & & +1 & & \\ & & & ... & \\ & & & & +1 \end{pmatrix} \tag{7.44}$$

We use Greek indices from the middle of the alphabet for D-dimensional spacetime x^μ, $\mu = 0, 1, , 2, ..., D-1$ and Roman ones for space alone x^i, $i = 1, 2, ..., D-1$. We use Greek letters from the beginning of the alphabet for worldvolume coordinates σ^α, $\alpha = 0, 1$ say. Repeated indices are summed over. For a metric $\gamma_{\alpha\beta}$ we use $\gamma = \det(\gamma_{\alpha\beta})$. In two dimensions there is the anti-symmetric ϵ-symbol $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ which is defined to have $\epsilon^{01} = 1$. We use $a, b = 1, ..., N$ to label parallel D-branes, *i.e.* as Chan-Paton indices. We use units where $\hbar = c = 1$.

References

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[2] J. Polchinski, *String Theory, Vols I and II*, CUP, 1998.

[3] M. Green, J. Schwarz and E. Witten, *Superstring Theory Vols I and II*, CUP, 1988.