Supersymmetry and Gauge Theory (7CMMS41)

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1 Introduction

Particle Physics is the study of matter at the smallest scales that can be accessed by experiment. Currently energy scales are as high as $100GeV$ which corresponds to distances of $10^{-16}cm$ (recall that the atomic scale is about $10^{-9}cm$ and the nucleus is about $10^{-13}cm$). Our understanding of Nature up to this scale is excellent[1]. Indeed it must be one of the most successful and accurate scientific theories and goes by the least impressive name "The Standard Model of Elementary Particle Physics". This theory is a relativistic quantum theory which postulates that matter is made up of point-sized particles (in so far as it makes sense to talk about particles as opposed to waves). The mathematical framework for such a theory is quantum field theory. There are an infinite list of possible quantum field theories and the Standard Model is one of these, much like a needle in a haystack.

Technically the Standard Model is a relativistic quantum gauge field theory. What does this mean? The word quantum presumably requires no explanation. The term relativistic refers to the fact that it is consistent with special relativity.

What does is mean to be a gauge theory? The precise meaning of this is a substantial part of this course. In a nutshell a gauge theory means that there are symmetries which are 'local'. I hope that you are familiar with a symmetry. This is transformation of the fields that leaves the action invariant. If we denote the fields of the theory by Φ^A then this means that there is an infinitesimal transformation

$$\delta\Phi^A = \epsilon^r T_r{}^A{}_B \Phi^B \ , \tag{1.1}$$

where ϵ^r are constant infinitesimal parameters labeled by r, such that

$$\delta\mathcal{L} = \sum_A \frac{\partial\mathcal{L}}{\partial\Phi^A}\delta\Phi^A = 0 \tag{1.2}$$

(up to a possible total derivative). To be a gauge symmetry means that this is still true when ϵ^r is allowed to become arbitrary functions of spacetime, *i.e.* a different symmetry transformation is allowed at each point in spacetime! This seems remarkable (at least it does to me). But what is more remarkable is that this is how Nature really works.

In fact one can prove that the only renormalization four-dimensional quantum field theories are gauge theories. What does this mean? Well it is sometimes said that quantum field theories only make sense if they are renormalizable. Renormalizable means that all the divergences that arise in calculations can be compensated for by a (divergent) shift of the parameters of the theory (coupling constants etc.). This isn't the modern picture. Following Wilson we now suppose that there is some better, or more fundamental, theory at high energy that doesn't have divergences. As one does to lower energies it is necessary to average out over the higher energy modes above some cut-off Λ. This process is referred to as 'integrating out'. As a result the various

[1]This ignores important issues that arise in large and complex systems such as those that are studied in condensed matter physics.

parameters of the theory 'run' via the renormalization group as a function of the energy scale Λ. Renormalizable interactions remain finite as $\Lambda \to 0$ whereas non-renormalizable interactions have their coupling constants run to zero when $\Lambda \to 0$. Thus at low energies the only interactions that we observe are renormalizable ones (these are sometimes called relevant or marginal where as non-renormalizable interactions are call irrelevant). So whatever the final theory of Nature is it contains gauge symmetries and interactions and only this part of the theory is observable at the low scales that we have observed.

In this language the statement that gravity is non-renormalizable turns into the statement that it can be neglected at low energy as a force that acts between two particles. Of course this interpretation of renormalization presupposes that there is a good theory at high energy (know as a 'UV completion'). Such a theory may be provided by String Theory.

There is currently a great deal of interest focused on the LHC (Large Hadron Collider) in CERN. In a year or two these experiments will probe higher energy scalars and therefore shorter distances. The great hope is that new physics, beyond that predicted by the Standard Model, will be observed. One of the main ideas, in fact probably the most popular, is that supersymmetry will be observed. There are a few reasons for this:

- The Hierarchy problem: The natural scale of the Standard Model is the electroweak scale which is at about $1TeV$ (hence the excitement about the LHC). In a quantum field theory physical parameters, such as the mass of the Higg's Boson, get renormalized by quantum effects. Why then is the Higg's mass not renormalized up to the Planck scale? The masses of Fermions can be protected by invoking a symmetry but there is no such mechanism for scalar fields. To prevent this requires and enormous amount of fine-tuning (parameters in the Standard Model must be fixed to an incredible order or magnitude). However in a supersymmetric model these renormalizations are less severe and fine-tuning is not required (or at least is not as bad).

- Unification: Another key idea about beyond the Standard Model is that all the gauge fields are unified into a single, simple gauge group at some high scale, roughly $10^{15}GeV$. There is some non-trivial evidence for this idea. For example the particle content is just right and also grand unification gives an accurate prediction for the Weinberg angle. Another piece of evidence is the observation that, although the electromagnetic, strong and weak coupling constants differ at low energy, they 'run' with energy and meet at about $10^{15}GeV$. That any two of them should meet is trivial but that all three meet at the same scale is striking. In fact they don't quite meet in the Standard Model but they do in a supersymmetric version.

- Dark Matter: It would appear that most, roughly 70%, of the matter floating around in the universe is not the stuff that makes up the Standard Model. Supersymmetry predicts many other particles other than those observed in the Standard Model, the so so-called superpartners, and the lightest superpartner (LSP) is considered a serious candidate for dark matter.

- Sting Theory: Although there is no empirical evidence for String Theory it is a very compelling framework to consider fundamental interactions. Its not clear that String Theory predicts supersymmetry but it is certainly a central ingredient and, symbiotically, supersymmetry has played a central role in String Theory and its successes. Indeed there is no clear boundary between supersymmetry and String Theory and virtually all research in fundamental particle physics involves both of them (not that this is necessarily a good thing).

If supersymmetry is observed in Nature it will be a great triumph of theoretical physics. Indeed the origin of supersymmetry is in string theory and the two fields have been closely linked since their inception. If not one can always claim that supersymmetry is broken at a higher energy (although in so doing the arguments in favour of supersymmetry listed above will cease to be valid). Nevertheless supersymmetry has been a very fruitful subject of research and has taught us a great deal about mathematics and quantum field theory. For example supersymmetric quantum field theories, especially those with extended supersymmetry, can be exactly solved (in some sense) at the perturbative and non-perturbative levels. Hopefully this course will convince the student that supersymmetry is a beautiful and interesting subject.

2 Gauge Theory

2.1 Electromagnetism

Our first task is to explain the concept of a gauge theory. The classic example of a gauge theory is Maxwell's theory of electromagnetism.

However we will start by considering quantum mechanics and the Schrödinger equation:

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2m}\partial_i\partial^i\Psi + V(x)\Psi \tag{2.3}$$

As you are no doubt familiar in quantum mechanics the overall phase of the the wave function Ψ has no physical interpretation what so ever. Thus the physical state space is actually a quotient $L^2(\mathbf{R}^3)/U(1)$ where $L^2(\mathbf{R}^3)$ is the Hilbert space of square integrable functions on $\mathbf{R}^3$ and $U(1)$ acts by

$$\Psi \to e^{iq\theta}\Psi \tag{2.4}$$

for some constant q.[2] Clearly the Schrödinger equation respects this transformation.

To gauge such a symmetry we suppose that now the parameter θ is allowed to be a function of spacetime: $\theta(x)$. The problem with Schrödinger's equation is that $\partial_\mu(e^{iq\theta}\Psi) \neq e^{iq\theta}\partial_\mu\Psi$ but rather

$$\partial_\mu(e^{iq\theta}\Psi) = e^{iq\theta}(\partial_\mu\Psi + iq\partial_\mu\theta\Psi) \tag{2.5}$$

[2] This quotient is also known as a projective Hilbert space.

To remedy this we introduce a covariant derivative

$$D_\mu = \partial_\mu - iqA_\mu \tag{2.6}$$

where we have also introduced a new field $A_\mu(x)$ that is called a gauge field.[3] Let us suppose that under the transformation (2.4) A_μ transforms as $A_\mu \to A_\mu + \delta A_\mu$. What we want is to choose δA_μ such that $D_\mu\Psi \to e^{iq\theta}D_\mu\Psi$, *i.e.* we want $D_\mu\Psi$ to transform covariantly, which means to say that it transforms in the same was as Ψ. This will be the case if

$$e^{iq\theta}(\partial_\mu\Psi + iq\partial_\mu\theta\Psi - iqA_\mu\Psi - iq\delta A_\mu\Psi) = e^{iq\theta}(\partial_\mu\Psi - qA_\mu\Psi) \tag{2.7}$$

Thus we see that $\delta A_\mu = \partial_\mu\theta$ so that

$$A_\mu \to A_\mu + \partial_\mu\theta \tag{2.8}$$

You will have seen such a transformation before in Maxwell's Theory and indeed we will see that our resulting equations describe a particle in a background electromagnetic field.

To continue now note that we can write a modified Schrödinger equation that is invariant under (2.4):

$$iD_t\Psi = -\frac{1}{2m}D_iD^i\Psi + V(x)\Psi \tag{2.9}$$

If we recall that the Schrödinger equation corresponds to the classical equation $E = \frac{p^2}{2m} + V$ then we see that our new equation comes from the classical equation

$$E + qA_0 = \frac{1}{2m}(p_i - qA_i)(p^i - qA^i) + V \tag{2.10}$$

This is precisely what one expects for a particle with charge q moving in a background electromagnetic field described by the vector potential A_μ (for example this is derived in the String Theory and Brane course).

Next we set the potential to zero and consider a relativistic generalization, namely the Klein-Gordon equation:

$$D_\mu D^\mu\Psi = 0 \tag{2.11}$$

Problem: Show that this equation comes from the action

$$S = -\frac{1}{2}\int d^4x(D_\mu\Psi)^*D^\mu\Psi \tag{2.12}$$

Our next step is to introduce some dynamics for the gauge field A_μ. Our first step is to notice that the combination

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.13}$$

[3] Mathematically D_μ is known as a connection and A_μ is the connection one-form.

is gauge invariant; $F_{\mu\nu} \to F_{\mu\nu}$ under $A_\mu \to A_\mu + \partial_\mu\theta$. It is helpful to see this from a different perspective. An elementary theory in calculus states that $[\partial_\mu, \partial_\nu] = 0$ (indeed this is why $F_{\mu\nu}$ is gauge invariant). However we now find

$$\begin{aligned}
[D_\mu, D_\nu]\Psi &= (\partial_\mu - iqA_\mu)(\partial_\nu - iqA_\nu)\Psi - (\mu \leftrightarrow \nu) \\
&= \partial_\mu\partial_\nu\Psi - iqA_\mu\partial_\nu\Psi - iq\partial_\mu A_\nu\Psi - iqA_\nu\partial_\mu\Psi - q^2A_\mu A_\nu\Psi - (\mu \leftrightarrow \nu) \\
&= -iq(\partial_\mu A_\nu - \partial_\nu A_\mu)\Psi
\end{aligned} \tag{2.14}$$

Thus we can write

$$[D_\mu, D_\nu]\Psi = -iqF_{\mu\nu}\Psi \tag{2.15}$$

From here its easy to see that, under (2.4), the left hand side transforms as $[D_\mu, D_\nu]\Psi \to e^{iq\theta}[D_\mu, D_\nu]\Psi$ while the right hand side transforms as $-iqF_{\mu\nu}\Psi \to -iqF'_{\mu\nu}e^{iq\theta}\Psi$. Since these must be equal we find

$$F'_{\mu\nu} = e^{iq\theta}F_{\mu\nu}e^{-iq\theta} = F_{\mu\nu} \tag{2.16}$$

$F_{\mu\nu}$ is called the field strength of A_μ and is of course familiar from Electromagnetism.

Since $F_{\mu\nu}$ involves a derivative acting on A_μ it can be used to give a kinetic term. Thus we are lead to the action

$$S_{ScalarQED} = -\int d^4x \; \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\Psi)^* D^\mu\Psi \tag{2.17}$$

Problem: Calculate the equations of motion of this Lagrangian to derive Maxwell's equations

$$\partial^\mu F_{\mu\nu} = j_\nu \qquad \partial_{[\mu}F_{\nu\lambda]} = 0 \tag{2.18}$$

What is j_ν in terms of Ψ?

Note that the second equation is trivially true by the definition of $F_{\mu\nu}$ in terms of A_μ. It is called a Bianchi identity.

2.2 Yang-Mills Theory

We can now try to generalize this construction. Before we do so let us repeat what have. We had a complex scalar field ϕ that transforms under a local (*i.e.* spacetime dependent) $U(1)$ group element as[4]

$$\phi \to e^{iq\theta}\phi \tag{2.19}$$

Here q is an arbitrary parameter that we latter identified with the electric charge. More abstractly one sees that the map

$$U_q : \theta \to e^{iq\theta} \tag{2.20}$$

[4]Note that we are now using ϕ instead of Ψ to denote the scalar field. This is more in line with standard practice, which we will follow below, where ψ is used to denote a Fermion.

for any fixed q, provides a representation of $U(1)$. In other words the scalar field ϕ takes values in vector space (here $\mathbf{C}$) that is endowed with representation of $U(1)$. We also introduced a gauge field A_μ that defined a covariant derivative

$$D_\mu\phi = \partial_\mu\phi - iqA_\mu\phi \tag{2.21}$$

We constructed a field strength through the formula

$$[D_\mu, D_\nu] = -iqF_{\mu\nu} \tag{2.22}$$

Clearly we could also introduce many scalar fields, each with a different charge q. The resulting action is

$$S = -\int d^4x\ \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\sum_q (D_\mu\phi_q)^* D^\mu\phi_q \tag{2.23}$$

Mathematically this corresponds to allowing fields with different representations of $U(1)$.

Under a gauge transformation $F_{\mu\nu}$ had a trivial transformation. However given the discussion around (2.16) we see that it can be thought of as

$$F_{\mu\nu} \to e^{iq\theta}F_{\mu\nu}e^{-iq\theta} \tag{2.24}$$

where the two phase factors cancel. This corresponds to the adjoint action of the group

$$U_{ad}(g) : X \to gXg^{-1} \tag{2.25}$$

Because $U(1)$ is Abelian this group action is trivial.

We can now see how to try to generalize this by replacing $U(1)$ by any Lie group G and taking the scalars to take values in a space that carries a representation $U(g)$ of G. Recall that this means that $U(g)$ is a matrix $U(g)^a{}_b$ that acts on a vector space. This in turn means that the scalars will be vectors in some complex vector space (not a vector in spacetime!) that we denote by ϕ^a. Although in some cases, where the group G admits a real representation, they can be real. To make equations clearer we will often drop the $a, b, ...$ indices and matrix multiplication will be understood to apply.

Thus we postulate the gauge symmetry

$$\phi \to U(g)\phi = \phi', \qquad i.e. \qquad \phi^a \to U(g)^a{}_b\phi^b = \phi'^a \tag{2.26}$$

where $g(x)$ is allowed to depend on spacetime. Next we need to construct a covariant derivative. To this end we postulate

$$D_\mu\phi = \partial_\mu\phi - iU(A_\mu)\phi \tag{2.27}$$

and require that $(D_\mu\phi)' = D'_\mu\phi'$. This is equivalent to

$$U(g)(\partial_\mu\phi - iU(A_\mu)\phi) = \partial_\mu(U(g)\phi) - iU(A'_\mu)U(g)\phi \tag{2.28}$$

This implies that

$$U(A'_\mu)U(g) = -i\partial_\mu U(g) + U(g)U(A_\mu) \tag{2.29}$$

and hence

$$U(A'_\mu) = -i\partial_\mu U(g)U(g^{-1}) + U(g)U(A_\mu)U(g^{-1}) \tag{2.30}$$

We need this to apply regardless of the representation that the scalar fields live in. Hence we take

$$A'_\mu = -i\partial_\mu g g^{-1} + gA_\mu g^{-1} \tag{2.31}$$

Problem: Show that (2.31) can be written as

$$A'_\mu = ig\partial_\mu g^{-1} + gA_\mu g^{-1} \tag{2.32}$$

We next need to obtain a field strength. Again we consider

$$\begin{aligned}
[D_\mu, D_\nu]\phi &= (\partial_\mu - iU(A_\mu))(\partial_\nu - iU(A_\nu))\phi - (\mu \leftrightarrow \nu) \\
&= \partial_\mu\partial_\nu\phi - iU(A_\mu)\partial_\nu\phi - i\partial_\mu U(A_\nu)\phi - iU(A_\nu)\partial_\mu\phi - U(A_\mu)U(A_\nu)\phi - (\mu \leftrightarrow \nu) \\
&= -i(\partial_\mu U(A_\nu) - \partial_\nu U(A_\mu) - i[U(A_\mu), U(A_\nu)])\phi \\
&= -iU(F_{\mu\nu})\phi
\end{aligned} \tag{2.33}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{2.34}$$

How does $F_{\mu\nu}$ change under a gauge transformation? Well we have

$$[D'_\mu, D'_\nu]\phi' = -iU(g)[D_\mu, D_\nu]\phi = -iU(g)U(F_{\mu\nu})\phi \tag{2.35}$$

as well as

$$[D'_\mu, D'_\nu]\phi' = -iU(F'_{\mu\nu})\phi' = -iU(F'_{\mu\nu})U(g)\phi \tag{2.36}$$

Since these must be equal we find

$$F'_{\mu\nu} = gF_{\mu\nu}g^{-1} \tag{2.37}$$

Now consider the special case of constant, *i.e.* global symmetry transformations. In this case we can neglect the derivative term and we see that A_μ is in the adjoint representation of the group G. This implies that A_μ and $F_{\mu\nu}$ take values in Lie-algebra $L(G)$ and indeed we see the presence of the commutator.

To construct an action we require that the representation vector space where the representation $U(g)$ acts has an invariant inner-product

$$\langle \phi_1, \phi_2 \rangle = h_{ab}(\phi_1^a)^*\phi_2^b \tag{2.38}$$

for some matrix h_{ab} (which we will always take to simply be the identity matrix). Invariance means that

$$\langle U(g)\phi_1, U(g)\phi_1 \rangle = \langle \phi_1, \phi_2 \rangle \tag{2.39}$$

Note that the inner-product will depend on the representation. This gives rise to a norm which we simply denote by $|\phi|^2 = \langle \phi | \phi \rangle$.

For the gauge field we require that the Lie-algebra $L(G)$ has also has an invariant inner-product. This is sometimes called a trace-form; $\mathrm{Tr}(X, Y)$. Invariance means that $\mathrm{Tr}(gXg^{-1}, gYg^{-1}) = \mathrm{Tr}(X, Y)$.

The classic example is to consider ϕ to be a complex vector in $\mathbf{C}^N$ with $\langle \phi_1, \phi_2 \rangle = \phi_1^\dagger \phi_2$, where † is the ordinary matrix Hermitian conjugate, *i.e.* transpose and complex conjugated. The invariance of the inner-product corresponds to the condition

$$\phi_1^\dagger U(g)^\dagger U(g) \phi_2 = \phi_1^\dagger \phi_2 \tag{2.40}$$

Thus invariance means that $U(g)$ is a unitary matrix: $U(g)^{-1} = U(g)^\dagger$, *i.e.* $G \subseteq U(N)$. Note that G need not be all of $U(N)$. In this case the invariant trace-form on the Lie-algebra $L(G)$ is simple the usual matrix trace when we view A_μ as an $N \times N$ matrix.

Thus we find a gauge invariant action by taking

$$S_{YM} = -\frac{1}{g_{YM}^2} \int d^4x \frac{1}{4} \mathrm{Tr}(F_{\mu\nu}, F^{\mu\nu}) + \frac{1}{2} |D_\mu \phi|^2 \tag{2.41}$$

and again one could include additional fields in different representations of G. This is the celebrated Yang-Mills action (although we haven't yet included Fermions). The case without any scalar (of Fermion) fields is called pure Yang-Mills (or sometimes pure glue in the case of QCD).

Note that we have included an arbitrary constant $1/g_{YM}^2$ in front known as the coupling constant. Don't confuse this g_{YM} which is a real constant, with the $g \in G$ that we have used for a group element. Perhaps it would be best to choose a different variable for the coupling constant however g_{YM} is more or less exclusively used in the literature. We could have introduced separate constants in front of each term however by rescaling the fields we can always put the action in this form. Indeed we can still make the rescaling

$$A_\mu \to g_{YM} A_\mu \qquad \phi \to g_{YM} \phi \tag{2.42}$$

Problem: Show that the result of this rescaling puts the action in the form

$$S_{YM} = -\int d^4x \frac{1}{4} \mathrm{Tr}(\tilde{F}_{\mu\nu}, \tilde{F}^{\mu\nu}) + \frac{1}{2} |\tilde{D}_\mu \phi|^2 \tag{2.43}$$

where $\tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_{YM} [A_\mu, A_\nu]$ and $\tilde{D}_\mu \phi = \partial_\mu \phi - i g_{YM} U(A_\mu) \phi$.

In this form one sees that g_{YM} controls the strength of the non-linear - interacting - terms. More generally we could add a potential to the action, so long as it is gauge invariant. This is the case if $V = V(|\phi|^2)$.

What are the interacting terms? Well we see that if G is non-Abelian then there are terms that involve $\partial^\mu A^\nu [A_\mu, A_\nu]$ and $[A^\mu, A^\nu][A_\mu, A_\nu]$. Thus the gauge fields interact with themselves. This is in distinction to electromagnetism where the photons propagate freely away from sources. There are also interactions $\partial^\mu\phi U(A_\mu)\phi$ and $U(A_\mu)\phi U(A^\mu)\phi$ between the scalars and gauge fields.

Problem: Calculate the equations of motion that result from S_{YM}.

In the physics literature one often doesn't write the Yang-Mills action in this form. Physicists like indices and rather than looking at finite transformations they content themselves with infinitesimal variations. To do this we represent $g = exp(i\theta^r T_r) = 1 + i\theta^r T_r + \ldots$ where T_r are a matrix representation of the Lie-algebra of G, *i.e.* $(T_r)^a{}_b$, $r = 1, ..., \dim(G)$, $a, b, = 1, ..., \dim(U)$. Here θ^r are spacetime dependent parameters which we take to be infinitesimal. The generators T_r satisfy a commutation relation

$$[T_r, T_s] = i f_{rs}{}^t T_t \tag{2.44}$$

This remains true for all representations although the nature and size of the matrices T_r will change from representation to representation. The factor of i is to ensure that the T_r are Hermitian. We can now write

$$\begin{aligned} D_\mu\phi^a &= \partial_\mu\phi^a - iA^r_\mu (T_r)^a{}_b\phi^b \\ F^r_{\mu\nu} &= \partial_\mu A^r_\nu - \partial_\nu A^r_\mu + f_{st}{}^r A^s_\mu A^t_\nu \end{aligned} \tag{2.45}$$

Problem: Derive this second formula by computing

$$[D_\mu, D_\nu]\phi^a = -iU(F_{\mu\nu})^a{}_b\phi^b = -iF^r_{\mu\nu}(T_r)^a{}_b\phi^b \tag{2.46}$$

We now have that

$$\mathrm{Tr}(F_{\mu\nu}, F^{\mu\nu}) = F^r_{\mu\nu}F^{s\mu\nu}\mathrm{Tr}(T^{ad}_r, T^{ad}_s) \tag{2.47}$$

where T^{ad}_r are the representation matrices in the adjoint representation. The adjoint representation of a Lie-algebra acts as

$$ad_T(X) = [T, X] \tag{2.48}$$

The Jacobi identity ensures that this is indeed a representation:

$$\begin{aligned} [ad_T, ad_S](X) &= [T, [S, X]] - [S, [T, X]] \\ &= -[[S, X], T] - [[X, T], S] \\ &= [[T, S], X] \\ &= ad_{[T,S]}(X) \end{aligned} \tag{2.49}$$

To see what the matrix representation is we expand an element $X = X^s T^{ad}_s$

$$[T_r, X] = [T_r, T_s]X^s = i f_{rs}{}^t X^s T_t = (T^{ad}_r)^t{}_s X^s T_t \tag{2.50}$$

Note that here the a, b indices are the r, s indices since the representation space is the Lie-algebra itself. This means

$$(T_r^{ad})^t{}_s = i f_{rs}{}^t \tag{2.51}$$

and hence

$$\mathrm{Tr}(F_{\mu\nu}, F^{\mu\nu}) = F^r_{\mu\nu} F^{s\mu\nu} \kappa_{rs} \qquad \kappa_{rs} = -f_{rp}{}^t f_{st}{}^p \tag{2.52}$$

In general one finds $\kappa_{rs} = C\delta_{rs}$ for a compact Lie-group G, where $C > 0$.

We can now write the Yang-Mills action as

$$S_{YM} = -\frac{1}{g^2_{YM}} \int d^4x \; \frac{1}{4} F^r_{\mu\nu} F^{s\mu\nu} \kappa_{rs} + \frac{1}{2} h_{ab} (D^\mu \phi^a)^* D_\mu \phi^b \tag{2.53}$$

or equivalently

$$S_{YM} = -\int d^4x \; \frac{1}{4} \tilde{F}^r_{\mu\nu} \tilde{F}^{s\mu\nu} \kappa_{rs} + \frac{1}{2} h_{ab} (\tilde{D}^\mu \phi^a)^* \tilde{D}_\mu \phi^b \tag{2.54}$$

Problem: Determine δA^r_μ under and infinitesimal gauge transformation.

3 Fermion: Clifford Algebras and Spinors

We are still not in a position to write down the Standard Model since we haven't discussed Fermions. Since the details are crucial before proceeding it is necessary to review in detail the formalism that is needed to describe spinors and Fermions. We shall now do this. It is helpful to generalize to spacetime with D dimensions. The details of spinors vary slightly from dimension to dimension (although conceptually things are more or less the same). To help highlight the differences between vectors and spinors it is useful to consider a general dimension.

Fermions first appeared with Dirac who thought that the equation of motion for an electron should be first order in derivatives. Hence, for a free electron, where the equation should be linear, it must take the form

$$(\gamma^\mu \partial_\mu - M)\psi = 0 \tag{3.1}$$

Acting on the left with $(\gamma^\mu \partial_\mu + M)$ one finds

$$(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu - M^2)\psi = 0 \tag{3.2}$$

This should be equivalent to the Klein Gordon equation (which is simply the mass-shell condition $E^2 - p^2 - m^2 = 0$)

$$(\partial^2 - m^2)\psi = 0 \tag{3.3}$$

Thus we see that we can take $m = M$ to be the mass and, since $\partial_\mu \partial_\nu \psi = \partial_\nu \partial_\mu \psi$, we also require that

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \tag{3.4}$$

This seemingly innocent condition is in fact quite deep. It first appeared in Mathematics in the geometrical work of Clifford (who was a student at King's). The next step is to find representations of this relation which reveals an 'internal' spin structure to Fermions.

3.1 Clifford Algebras

Introducing Fermions requires that we introduce a set of γ-matrices. These furnish a representation of the Clifford algebra, which is generically taken to be over the complex numbers, whose generators satisfy the relation

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{3.5}$$

Note that we have suppressed the spinor indices α, β. In particular the right hand side is proportional to the identity matrix in spinor space. We denote spinor indices by α, β.... Although we will only be interested in the four-dimensional Clifford algebra in this course it is instructive to consider Clifford algebras in a variety of dimensions. Each dimension has its own features and these often play an important role in the supersymmetric theories that can arise.

One consequence of this relation is that the γ-matrices are traceless (at least for $D > 1$). To see this we evaluate

$$\begin{aligned} 2\eta_{\mu\nu}\mathrm{Tr}(\gamma_\lambda) &= \mathrm{Tr}(\{\gamma_\mu, \gamma_\nu\}\gamma_\lambda) \\ &= \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\lambda + \gamma_\nu\gamma_\mu\gamma_\lambda) \\ &= \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\lambda + \gamma_\mu\gamma_\lambda\gamma_\nu) \\ &= \mathrm{Tr}(\gamma_\mu\{\gamma_\nu, \gamma_\lambda\}) \\ &= 2\eta_{\nu\lambda}\mathrm{Tr}(\gamma_\mu) \end{aligned} \tag{3.6}$$

Choosing $\mu = \nu \neq \lambda$ immediately implies that $\mathrm{Tr}(\gamma_\lambda) = 0$

Theorem: In even dimensions there is only one non-trivial irreducible representation of the Clifford algebra, up to conjugacy, *i.e.* up to a transformation of the form $\gamma_\mu \to U\gamma_\mu U^{-1}$. In particular the (complex) dimension of this representation is $2^{D/2}$, *i.e.* the γ-matrices will be $2^{D/2} \times 2^{D/2}$ complex valued matrices.

Without loss of generality one can choose a representation such that

$$\gamma_0^\dagger = -\gamma_0 \ , \qquad \gamma_i^\dagger = \gamma_i \tag{3.7}$$

which can be written as

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0 \tag{3.8}$$

An even-dimensional Clifford algebra naturally lifts to a Clifford algebra in one dimension higher. In particular one can show that

$$\gamma_{D+1} = c\gamma_0\gamma_1...\gamma_{D-1} \tag{3.9}$$

anti-commutes with all the γ_μ's. Here c is a constant which we can fix, up to sign, by taking $\gamma_{D+1}^2 = 1$. In particular a little calculation shows that

$$\gamma_{D+1}^2 = -(-1)^{D(D-1)/2}c^2 \tag{3.10}$$

Here the first minus sign comes from γ_0^2 whereas the others come from anti-commuting the different γ_μ's through each other. In this way we find that

$$c = \pm i(-1)^{D(D-1)/4} \tag{3.11}$$

Thus we construct a Clifford Algebra in $(D+1)$-dimensions. It follows that the dimension (meaning the range of the spinor indices $\alpha, \beta ...$) of a Clifford algebra in $(D+1)$-dimensions is the same as the dimension of a Clifford algebra in D-dimensions when D is even.

In odd dimensions there are two inequivalent representations. To see this one first truncates down one dimension. This leads to a Clifford algebra in a even dimension which is therefore unique. We can then construct the final γ-matrix using the above procedure. This leads to two choices depending on the choice of sign above. Next we observe that in odd-dimensions γ_{D+1}, defined as the product of all the γ-matrices, commutes with all the γ_μ's. Hence by Shur's lemma it must be proportional to the identity. Under conjugacy one therefore has $\gamma_{D+1} \to U\gamma_{D+1}U^{-1} = \gamma_{D+1}$. The constant of proportionality is determined by the choice of sign we made to construct the final γ-matrix. Since this is unaffected by conjugation we find two representation we constructed are inequivalent.

We can also construct a Clifford algebra in $D+2$ dimensions using the Clifford algebra in D dimensions. Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.12}$$

be the ubiquitous Pauli matrices. If we have the Clifford algebra in D-dimensions given by γ_μ, $\mu = 0, 1, 2, ..., D-1$ then let

$$\begin{aligned} \Gamma_\mu &= 1 \otimes \gamma_\mu \\ \Gamma_D &= \sigma_1 \otimes \gamma_{D+1} \\ \Gamma_{D+1} &= \sigma_3 \otimes \gamma_{D+1} \end{aligned} \tag{3.13}$$

where we have used Γ_μ for $(D+2)$-dimensional γ-matrices. One readily sees that this gives a Clifford algebra in $(D+2)$-dimensions. Note that this gives two algebras corresponding to the two choices of sign for γ_{D+1}. However these two algebras are equivalent under conjugation by $U = \sigma_2 \otimes 1$. This is to be expected from the uniqueness of an even-dimensional Clifford algebra representation.

Having constructed essentially unique γ-matrices for a given dimension there are two special things that can happen. We have already seen that in even dimensions one finds an "extra" Hermitian γ-matrix, γ_{D+1} (so in four dimensions this is the familiar γ_5). Since this is Hermitian it has a basis of eigenvectors with eigenvalues ± 1 which are called the chirality. Indeed since the γ-matrices are traceless half of the eigenvalues are $+1$ and the other half -1. We can therefore write any spinor ψ uniquely as

$$\psi = \psi_+ + \psi_- \tag{3.14}$$

where $\psi_\pm$ has γ_{D+1} eigenvalue ± 1. A spinor with a definite γ_{D+1} eigenvalue is called a Weyl spinor.

The second special case occurs when the γ-matrices can be chosen to be purely real. In which case it is possible to chose the spinors to also be real. A real spinor is called a Majorana spinor.

Either of these two restrictions will cut the number of independent spinor components in half. In some dimensions it is possible to have both Weyl and Majorana spinors simultaneously. These are called Majorana-Weyl spinors. This reduces the number of independent spinor components to a quarter of the original size. Spinors without any such restrictions are called Dirac spinors. Which restrictions are possible in which dimensions comes in a pattern which repeats itself for dimensions D modulo 8.

Let us illustrate this by starting in low dimensions and work our way up. We will give concrete example of γ-matrices but it is important to bare in mind that these are just choices - there are other choices.

3.1.1 D=1

If there is only one dimension, time, then the Clifford algebra is the simple relation $(\gamma_0)^2 = -1$. In other words $\gamma_0 = i$ or one could also have $\gamma_0 = -i$. It is clear that there is no Majorana representation.

3.1.2 D=2

Here the γ-matrices can be taken to be

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \gamma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{3.15}$$

One can easily check that $\gamma_0^2 = -\gamma_1^2 = -1$ and $\gamma_0\gamma_1 = -\gamma_1\gamma_0$.

Here we have a real representation so that we can choose the spinors to also be real. We can also construct $\gamma_3 = c\gamma_0\gamma_1$ and it is also real:

$$\gamma_3 = -\gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.16}$$

Thus we can have Weyl spinors, Majorana spinors and Majorana-Weyl spinors. These will have 2, 2 and 1 real independent components respectively whereas a Dirac spinor will have 2 complex, *i.e.* 4 real, components.

3.1.3 D=3

Here the γ-matrices can be constructed from $D = 2$ and hence a natural choice is

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$
\begin{aligned}
\gamma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\gamma_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{3.17}
$$

(we could also have taken the opposite sign for γ_2). These are just the Pauli matrices (up to a factor of i for γ_0). Since we are in an odd dimension there are no Weyl spinors but we can choose the spinors to be Majorana with only 2 real independent components.

3.1.4 D=4

Following our discussion above a natural choice is

$$
\begin{aligned}
\gamma_0 &= 1 \otimes i\sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\
\gamma_1 &= 1 \otimes \sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
\gamma_2 &= \sigma_1 \otimes \sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
\gamma_3 &= \sigma_3 \otimes \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{3.18}
$$

By construction this is a real basis of γ-matrices. Therefore we can chose to have Majorana, *i.e.* real, spinors.

Since we are in an even dimension we can construct the chirality operator $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Note the factor of i which is required to ensure that $\gamma_5^2 = 1$. Thus in our basis γ_5 is purely imaginary and, since it is Hermitian, it must be anti-symmetric. This means that it cannot be diagonalized over the reals. Of course since it is Hermitian it can be diagonalized over the complex numbers, *i.e.* there is another choice of γ-matrices for which γ_5 is real and diagonal but in this basis the γ_μ cannot all be real.

Thus in four dimensions we can have Majorana spinors or Weyl spinors but not both simultaneously. In many books, especially those that focus on four-dimensions, a Weyl basis of spinors is used. Complex conjugation then acts to flip the chirality. However we prefer to use a Majorana basis whenever possible (in part because it applies to more dimensions).

3.2 Lorentz and Poincare Algebras

We wish to construct relativistic theories which are covariant with respect to the Lorentz and Poincare symmetries. These consists of translations along with the Lorentz transformations (which in turn contain rotations and boosts). In particular the theory is invariant under the infinitesimal transformations

$$x^\mu \to x^\mu + a^\mu + \omega^\mu{}_\nu x^\nu \qquad i.e. \qquad \delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu \tag{3.19}$$

Here a^μ generates translations and $\omega^\mu{}_\nu$ generates Lorentz transformations. The principle of Special relativity requires that the spacetime proper distance $\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$ between two points is invariant under these transformations. Expanding to first order in $\omega^\mu{}_\nu$ tells us that

$$\begin{aligned} \Delta s^2 &\to \eta_{\mu\nu}(\Delta x^\mu + \omega^\mu{}_\lambda \Delta x^\lambda)(\Delta x^\nu + \omega^\nu{}_\rho \Delta x^\rho) \\ &= \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu + \eta_{\mu\nu}\omega^\mu{}_\lambda \Delta x^\lambda \Delta x^\nu + \eta_{\mu\nu}\omega^\nu{}_\rho \Delta x^\mu \Delta x^\rho \\ &= \Delta s^2 + (\omega_{\mu\nu} + \omega_{\nu\mu})\Delta x^\mu \Delta x^\nu \end{aligned} \tag{3.20}$$

where we have lowered the index on $\omega^\mu{}_\nu$. Thus we see that the Lorentz symmetry requires $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

Next we consider the algebra associated to such generators. To this end we want to know what happens if we make two Poincare transformations and compare the difference, *i.e.* we consider $\delta_1\delta_2 x^\mu - \delta_2\delta_1 x^\mu$. First we calculate

$$\delta_1\delta_2 x^\mu = \omega_1^\mu{}_\nu a_2^\nu + \omega_1^\mu{}_\lambda \omega_2^\lambda{}_\nu x^\nu \tag{3.21}$$

from which we see that

$$(\delta_1\delta_2 - \delta_2\delta_1)x^\mu = (\omega_1^\mu{}_\lambda a_2^\lambda - \omega_2^\mu{}_\lambda a_1^\lambda) + (\omega_1^\mu{}_\lambda \omega_2^\lambda{}_\nu - \omega_2^\mu{}_\lambda \omega_1^\lambda{}_\nu)x^\nu \tag{3.22}$$

This corresponds to a new Poincare transformation with

$$a^\mu = \omega_1^\mu{}_\lambda a_2^\lambda - \omega_2^\mu{}_\lambda a_1^\lambda \qquad \omega^\mu{}_\nu = \omega_1^\mu{}_\lambda \omega_2^\lambda{}_\nu - \omega_2^\mu{}_\lambda \omega_1^\lambda{}_\nu \tag{3.23}$$

note that $\omega_{(\mu\nu)} = \frac{1}{2}(\omega_{\mu\nu} + \omega_{\nu\mu}) = 0$ so this is indeed a Poincare transformation.

More abstractly we think of these transformations as being generated by linear operators P_μ and $M_{\mu\nu}$ so that

$$\delta x^\mu = i a^\nu P_\nu(x^\mu) + \frac{i}{2}\omega^{\nu\lambda} M_{\nu\lambda}(x^\mu) \tag{3.24}$$

The factor of $\frac{1}{2}$ arises because of the anti-symmetry (one doesn't want to count the same generator twice). The factors of i are chosen for later convenience to ensure that the generators are Hermitian. These generators can then also be though of as applying on different objects, *e.g.* spacetime fields rather than spacetime points. In other words we have an abstract algebra and its action on x^μ is merely one representation.

This abstract object is the Poincare algebra and it defined by the commutators

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[P_\mu, M_{\nu\lambda}] &= -i\eta_{\mu\nu}P_\lambda + i\eta_{\mu\lambda}P_\nu \\
[M_{\mu\nu}, M_{\lambda\rho}] &= -i\eta_{\nu\lambda}M_{\mu\rho} + i\eta_{\mu\lambda}M_{\nu\rho} - i\eta_{\mu\rho}M_{\nu\lambda} + i\eta_{\nu\rho}M_{\mu\lambda}
\end{aligned} \tag{3.25}$$

which generalizes (3.22).

Problem: Using (3.24) and(3.25) show that (3.22) is indeed reproduced.

The Poincare group has two clear pieces: translations and Lorentz transformations. It is not quite a direct product because of the non-trivial commutator $[P_\mu, M_{\nu\lambda}]$. It is a so-called a semi-direct product. Translations by themselves form an Abelian and non-compact subgroup. On physical grounds one always takes $P_\mu = -i\partial_\mu$. This seems obvious from the physical interpretation of spacetime. Mathematically the reason for this is simply Taylor's theorem for a function $f(x^\mu)$:

$$\begin{aligned}
f(x+a) &= f(x) + \partial_\mu f(x) a^\mu + \ldots \\
&= f(x) + i a^\mu P_\mu f(x) + \ldots
\end{aligned} \tag{3.26}$$

Thus acting by P_μ will generate a infinitessimal translation. Furthermore Taylor's theorem is the statement that finite translations are obtained from exponentiating P_μ:

$$\begin{aligned}
f(x+a) &= e^{ia^\mu P_\mu} f(x) \\
&= f(x) + a^\mu \partial_\mu f(x) + \frac{1}{2!} a^\mu a^\nu \partial_\mu \partial_\nu f(x) + \ldots
\end{aligned} \tag{3.27}$$

However the other part, the Lorentz group, is non-Abelian and admits interesting finite-dimensional representations. For example the Standard Model contains a scalar field $H(x)$ (the Higg's Boson) which carries a trivial representation and also vector fields $A_\mu(x)$ (*e.g.* photons) and spinor fields $\psi_\alpha(x)$ (*e.g.* electrons). A non-trivial representation of the Lorentz group implies that the field carries some kind of index. In the two cases above these are μ and α respectively. The Lorentz generators then act as matrices with two such indices (one lowered and one raised). Different representations mean that there are different choices for these matrices which still satisfies (3.25). For example in the vector representation one can take

$$(M_{\mu\nu})^\lambda{}_\rho = i\eta_{\mu\rho}\delta^\lambda_\nu - i\delta^\lambda_\mu \eta_{\nu\rho} \tag{3.28}$$

Notice the dual role of μ, ν indices as labeling both the particular Lorentz generator as well as it's matrix components. Whereas in the spinor representation we have

$$(M_{\mu\nu})_\alpha{}^\beta = -\frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta = -\frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\mu\gamma_\nu)_\alpha{}^\beta \tag{3.29}$$

Here $(\gamma_\mu)_\alpha^{\ \beta}$ are the Dirac γ-matrices. However in either case it is important to realize that the defining algebraic relations (3.25) are reproduced.

Problem: Verify that these two representation of $M_{\mu\nu}$ do indeed satisfy the Lorentz subalgebra of (3.25).

3.3 Spinors

Having defined Clifford algebras we next need to discuss the properties of spinors in greater detail. We will see later that $M_{\mu\nu} = \frac{i}{2}\gamma_{\mu\nu}$ gives a representation of the Lorentz algebra, known as the spinor representation. A spinor is simply an object that transforms in the spinor representation of the Lorentz group (it is a section of the spinor bundle over spacetime). Hence it carries a spinor index α. From our definitions this means that under a Lorentz transformation generated by $\omega^{\mu\nu}$, a spinor ψ_α transforms as

$$\delta\psi_\alpha = \frac{1}{4}\omega^{\mu\nu}(\gamma_{\mu\nu})_\alpha^{\ \beta}\psi_\beta \tag{3.30}$$

Note that we gives spinors a lower spinor index. As such the γ-matrices naturally come with one upper and one lower index, so that matrix multiplication requires contraction on one upper and one lower index.

Let us pause for a moment to consider a finite Lorentz transformation. To begin with consider an infinitesimal rotation by an angle θ in the (x^1, x^2)-plane,

$$\delta \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \theta \begin{pmatrix} x^0 \\ -x^2 \\ x^1 \\ x^3 \end{pmatrix} \tag{3.31}$$

i.e.

$$\omega^{12} = -\omega_{21} = \theta \qquad M_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.32}$$

A finite rotation is obtained by exponentiating M_{12}:

$$x^\mu \to (e^{\omega^{\lambda\rho}M_{\lambda\rho}})^\mu_{\ \nu}x^\nu \tag{3.33}$$

Since $M_{12}^2 = -1$ one finds that, using the same proof as the famous relation $e^{i\theta} = \cos\theta + i\sin\theta$,

$$e^{\theta M_{12}} = \cos\theta + M_{12}\sin\theta \tag{3.34}$$

In particular we see that if $\theta = 2\pi$ then $e^{2\pi M_{12}} = 1$ as expected.

How does a spinor transform under such a rotation? The infinitesimal transformation generated by ω^{12} is, by definition,

$$\delta\psi = \frac{1}{4}\omega^{\mu\nu}\gamma_{\mu\nu}\psi = \frac{1}{2}\theta\gamma_{12}\psi \tag{3.35}$$

If we exponentiate this we find

$$\psi \to e^{\frac{1}{2}\theta\gamma_{12}}\psi = \cos(\theta/2) + \gamma_{12}\sin(\theta/2) \tag{3.36}$$

We see that now, if $\theta = 2\pi$, then $\psi \to -\psi$. Thus we recover the well known result that under a rotation by 2π a spinor (such as an electron) picks up a minus sign.

Let us now try to contract spinor indices to obtain Lorentz scalars. It follows that the Hermitian conjugate transforms as

$$\delta\psi^\dagger = \frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_\nu^\dagger\gamma_\mu^\dagger = \frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_0\gamma_{\nu\mu}\gamma_0 = \frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_0\gamma_{\mu\nu}\gamma_0 \tag{3.37}$$

Here we have ignored the spinor index. Note that the index structure is ${(\gamma_0\gamma_{\mu\nu}\gamma_0)_\alpha}^\beta$ and therefore it is most natural to write $(\psi^\dagger)^\alpha = \psi^{*\alpha}$ with an upstairs index.

However we would like to contract two spinors to obtain a scalar. One can see that the naive choice

$$\lambda^\dagger\psi = \lambda^{*\alpha}\psi_\alpha \tag{3.38}$$

will not be a Lorentz scalar due to the extra factors of γ_0 that appear in $\delta\lambda^\dagger$ as compared to $\delta\psi$. To remedy this one defines the Dirac conjugate

$$\bar\lambda = \lambda^\dagger\gamma_0 \tag{3.39}$$

In which case on finds that, under a Lorentz transformation,

$$\delta\bar\lambda = -\frac{1}{4}\bar\lambda\omega^{\mu\nu}\gamma_{\mu\nu} \tag{3.40}$$

and hence

$$\begin{aligned} \delta(\bar\lambda\psi) &= \delta\bar\lambda\psi + \bar\lambda\delta\psi \\ &= -\frac{1}{4}\bar\lambda\omega^{\mu\nu}\gamma_{\mu\nu}\psi + \frac{1}{4}\bar\lambda\omega^{\mu\nu}\gamma_{\mu\nu}\psi \\ &= 0 \end{aligned} \tag{3.41}$$

Thus we have found a Lorentz invariant way to contract spinor indices.

Note that from two spinors we can construct other Lorentz covariant objects such as vectors and anti-symmetric tensors:

$$\bar\lambda\gamma_\mu\psi\ , \quad \bar\lambda\gamma_{\mu\nu}\psi\ , \ldots \tag{3.42}$$

Problem: Show that $V_\mu = \bar\lambda\gamma_\mu\psi$ is a Lorentz vector, *i.e.* show that $\delta V_\mu = {\omega_\mu}^\nu V_\nu$ under the transformation (3.30).

So far our discussion applied to general Dirac spinors. In much of this course we will be interested in Majorana spinors where the γ_μ are real. The above discussion is then valid if we replace the Hermitian conjugate $\dagger$ with the transpose T so that

$\gamma_\mu^T = -\gamma_0\gamma_\mu\gamma_0^{-1}$. More generally such a relationship always exists because if $\{\gamma_\mu\}$ is a representation of the Clifford algebra then so is $\{-\gamma_\mu^T\}$. Therefore, since there is a unique representation up to conjugacy, there must exist a matrix C such that $-\gamma_\mu^T = C\gamma_\mu C^{-1}$. C is called the charge conjugation matrix. The point here is that in the Majorana case it is possible to find a representation in which C coincides with Dirac conjugation matrix γ_0.

Problem: Show that, for a general Dirac spinor in any dimension, $\lambda^T C\psi$ is Lorentz invariant, where C is the charge conjugation matrix.

One way to think about charge conjugation is to view the matrix $C^{\alpha\beta}$ as a metric on the spinor indices with inverse $C^{-1}_{\alpha\beta}$. In which case $\psi^\alpha = \psi_\beta C^{\beta\alpha}$.

Finally we note that spinor quantum fields are Fermions in quantum field theory (this is the content of the spin-statistics theorem). This means that spinor components are anti-commuting Grassmann variables

$$\psi_\alpha\psi_\beta = -\psi_\beta\psi_\alpha \tag{3.43}$$

We also need to define how complex conjugation acts. Since ultimately in the quantum field theory the fields are elevated to operators we take the following convention for complex conjugation

$$(\psi_\alpha\psi_\beta)^* = \psi_\beta^*\psi_\alpha^* \tag{3.44}$$

which is analogous to the Hermitian conjugate. This leads to the curious result that, even for Majorana spinors, one has that

$$(\bar\psi\chi)^* = (\psi_\alpha^* C^{\alpha\beta}\chi_\beta)^* = \chi_\beta C^{\alpha\beta}\psi_\alpha = -\psi_\alpha C^{\alpha\beta}\chi_\beta = -\bar\psi\chi \tag{3.45}$$

is pure imaginary!

3.4 QED, QCD and the Standard Model

We are now in a position to give the actions for QED and QCD. We will also describe the Standard Model however it has many 'bells and whistles' that mean that it would take too long for this course to write the full action out. We will simply sketch it.

First consider QED - quantum electrodynamics, the quantum theory of light interacting with electrons. This theory was developed by Feynman, Schwinger and Tomanga (and others) in the 1940's and is very successful. It is simply Maxwell's theory coupled to Fermions (the electrons). As such it is a $U(1)$ gauge theory. The action is

$$S_{QED} = -\int d^4x \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar\psi\gamma^\mu D_\mu\psi + \frac{i}{2}m\bar\psi\psi \tag{3.46}$$

Here ψ is a Dirac Fermion (and $\bar\psi = \psi^\dagger\gamma_0$) that represents the electron field with mass m and $D_\mu\psi = \partial_\mu\psi - ieA_\mu\psi$. Thats it! Quite simple and yet it describes a vast amount of our physical world very accurately. It is successful because the electric charge e that acts as a coupling constant is small. The number relevant for perturbation theory is the so-called fine structure constant $\alpha = e^2/4\pi \sim 1/137$. In addition there is only one interaction: $\bar{\psi}\gamma^\mu A_\mu \psi$ corresponding to an electron emitting a photon.

Problem: Show the the equations of motion of S_{QED} are

$$\begin{aligned} \partial^\mu F_{\mu\nu} &= \frac{e}{2}\bar{\psi}\gamma_\mu\psi \\ \gamma^\mu\partial_\mu\psi + m\psi - ieA_\mu\psi &= 0 \end{aligned} \tag{3.47}$$

Next consider QCD - quantum chromodynamics, this is the quantum theory of quarks. The quarks are taken to be Fermions which are live in the fundamental representation of $SU(3)$, *i.e.* each quark field q_i is a spinor which takes values in $\mathbf{C}^3$. There are 6 quarks (u,d,c,s,t,b) and they are labeled by $i = 1, 2, ..., 6$. Thus if we write all the indices we have $q_{ia\alpha}$ where i is the flavour index, $a = 1, 2, 3$ the colour index and $\alpha = 1, 2, 3, 4$ is the spinor index. There are also gauge fields G^r_μ known as gluons $r = 1, ..., 8 = \dim(SU(3))$. The action is

$$S_{QCD} = -\frac{1}{g^2}\int d^4x \frac{1}{4}F^r_{\mu\nu}F^{s\mu\nu}\kappa_{rs} + \frac{i}{2}\sum_i \delta_{ab}(\bar{q}^a_i\gamma^\mu D_\mu q^b_i + \frac{i}{2}m_i\bar{\psi}^a_i\psi^b_i) \tag{3.48}$$

where m_i are the quark masses and $D_\mu q^b_i = \partial_\mu q^b_i - iG^r_\mu T^b_{r\,a} q^a_i$ with $(T_r)^a{}_b$ a complex, 3×3 basis of $L(SU(3))$. Again the Fermions are Dirac and $\bar{q} = q^\dagger\gamma_0$. QCD is highly non-linear and in addition the relevant coupling constant $\alpha_{QCD} = g^2/4\pi$ is not small. So there are many interactions between quarks and gluons and also gluons with gluons, all of which happen relatively strongly (hence the name of the strong force). It is believed but still not proven that the quarks are confined to colour neutral states so that a single quark cannot be observed. Things are slightly better at higher energies where α_{QCD} gets smaller. This allows one to test QCD against experiment.

Lastly some words about the Standard Model. This includes both QED and QCD. However it also includes another gauge theory based on $SU(2)$. It also has a scalar field known as the Higg's H and additional Fermions called neutrinos. The total gauge group is $SU(3)\times SU(2)\times U(1)$. Here $SU(3)$ is the $SU(3)$ of QCD that only acts on the quarks. All the (left-handed) fields transform under $SU(2)\times U(1)$. There are four gauge fields but three are made massive by a Higg's mechanism (more about this later). These are called $W^\pm$, Z^0 and A_μ. A_μ is the massless photon of QED however it is not the gauge field for the $U(1)$ in $SU(2)\times U(1)$. Instead it corresponds to a linear combination of this $U(1)$ and the T^3 generator of $SU(2)$. $W^\pm$ and Z^0 are massive. The Higgs field has not yet been scene and it one of the main motivations for the experiments at LHC in CERN. The action is rather involved as it contains all sorts of mixings between the various fields and many so-called Yukawa terms of the form $\bar{\psi}H\lambda$ where H is the Higgs and ψ, λ represent Fermions.

Problem: Why are their factors of i in front of the Fermion terms in these actions?

4 Supersymmetry

4.1 Symmetries, A No-go Theorem and How to Avoid It

Quantum field theories are essentially what you get from the marriage of quantum mechanics with special relativity (assuming locality). A central concept of these ideas is the notion of symmetry. And indeed quantum field theories are thought of and classified according to their symmetries.

The most important symmetry is of course the Poincare group of special relativity which we have already discussed. To say that the Poincare algebra is fundamental in particle physics means that everything is assumed fall into some representation of this algebra. The principle of relativity then asserts that the laws of physics are covariant with respect to this algebra.

The Standard Model and other quantum field theories also have other important symmetries. Most notably gauge symmetries that we have discussed above. These symmetries imply that there is an additional Lie-algebra with a commutation relation of the form

$$[T_r, T_s] = i f_{rs}{}^{t} T_t \tag{4.49}$$

where the T_r are Hermitian generators and $f_{rs}{}^{t}$ are the structure constants. This means that every field in the Standard model Lagrangian also carries a representation of this algebra. If this is a non-trivial representation then there is another 'internal' index on the field. For example the quarks are in the fundamental (*i.e.* three-dimensional) representation of $SU(3)$ and hence, since they are spacetime spinors, the field carries the indices $\psi^a_\alpha(x)$.

Finally we recall Noether's theorem which asserts that for every continuous symmetry of a Lagrangian one can construct a conserved charge. Suppose that a Lagrangian $\mathcal{L}(\Phi_A, \partial_\alpha \Phi_A)$, where we denoted the fields by Φ_A, has a symmetry: $\mathcal{L}(\Phi_A) = \mathcal{L}(\Phi_A + \delta\Phi_A)$. This implies that

$$\frac{\partial \mathcal{L}}{\partial \Phi_A}\delta\Phi_A + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\delta\partial_\alpha\Phi_A = 0 \tag{4.50}$$

This allows us to construct a current:

$$J^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\delta\Phi_A \tag{4.51}$$

which is, by the equations of motion,

$$\begin{aligned} \partial_\alpha J^\alpha &= \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\right)\delta\Phi_A + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\partial_\alpha\delta\Phi_A \\ &= \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\right)\delta\Phi_A - \frac{\partial \mathcal{L}}{\partial \Phi_A}\delta\Phi_A \\ &= 0 \end{aligned} \tag{4.52}$$

conserved. This means that the integral over space of J^0 is a constant defines a charge

$$Q = \int_{space} J^0 \tag{4.53}$$

which is conserved

$$\begin{aligned} \frac{dQ}{dt} &= \int_{space} \partial_0 J^0 \\ &= -\int_{space} \partial_i J^i \\ &= 0 \end{aligned}$$

Thus one can think of symmetries and conservations laws as being more or less the same thing.

So the Standard Model of Particle Physics has several symmetries built into it (*e.g.* $SU(3) \times SU(2) \times U(1)$) and this means that the various fields carry representations of various algebras. These algebras split up into those associated to spacetime (Poincare) and those which one might call internal (such as the gauge symmetry algebra). In fact the split is a direct product in that

$$[P_\mu, T_a] = [M_{\mu\nu}, T_a] = 0 \tag{4.54}$$

where T_a refers to any internal generator. Physically this means the conserved charges of these symmetries are Lorentz scalars.

Since the Poincare algebra is so central to our understanding of spacetime it is natural to ask if this direct product is necessarily the case or if there is, in principle, some deeper symmetry that has a non-trivial commutation relation with the Poincare algebra. This question was answered by Coleman and Mandula:

Theorem: In any spacetime dimension greater than two the only interacting quantum field theories have Lie algebra symmetries which are a direct product of the Poincare algebra with an internal symmetry.

In other words the Poincare algebra is apparently as deep as it gets. There are no interacting theories which have conserved charges that are not Lorentz scalars. Intuitively the reasons is that tensor-like charge must be conserved in any interaction and this is simply too restrictive as the charges end up being proportional to (products of) the momenta. Thus one finds that the individual momenta are conserved, rather than the total momentum.

But one shouldn't stop here. A no-go theorem is only as good as its assumptions. This theorem has several assumptions, for example that there are a finite number of massive particles and no massless ones. However the key assumption of the Coleman-Mandula theorem is that the symmetry algebra should be a Lie-algebra. We recall that a Lie-algebra can be thought of as the tangent space at the identity of a continuous group, so that, an infinitessimal group transformation has the form

$$g = 1 + i\epsilon A \tag{4.55}$$

where A is an element of the Lie-algebra and ϵ is an infinitessimal parameter. The Lie-algebra is closed under a bilinear operation, the Lie-bracket,

$$[A, B] = -[B, A] \tag{4.56}$$

subject to the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{4.57}$$

If we relax this assumption then there is something deeper - Supersymmetry. So how do we relax it since Lie-algebras are inevitable whenever you have continuous symmetries and because of Noether's theorem we need a continuous symmetry to give a conserved charge?

The way to proceed is to note that quantum field theories such as the Standard Model contain two types of fields: Fermions and Bosons. These are distinguished by the representation of the field under the Lorentz group. In particular a fundamental theorem in quantum field theory - the spin-statistics theorem - asserts that Bosons must carry representations of the Lorentz group with integer spins and their field operators must commute outside of the light-cone whereas Fermions carry half-odd-integer spins and their field operators are anti-commuting. This means that the fields associated to Fermions are not ordinary (so-called c-number) valued field but rather Grassmann variables that satisfy

$$\psi_1(x)\psi_2(x) = -\psi_2(x)\psi_1(x) \tag{4.58}$$

So a way out of this no-go theorem is to find a symmetry that relates Bosons to Fermions. Such a symmetry will require that the 'infinitessimal' generating parameter is a Grassmann variable and hence will not lead to a Lie-algebra. More precisely the idea is to consider a Grassmann generator (with also carries a spinor index) and which requires a Grassmann valued spinorial parameter. One then is lead to something called a superalgebra, or a $\mathbf{Z}_2$-graded Lie-algebra. This means that the generators can be labeled as either even and odd. The even generators behave just as the generators of a Lie-algebra and obey commutation relations. An even and an odd generator will also obey a commutator relation. However two odd generators will obey an anti-commutation relation. The even-ness or odd-ness of this generalized Lie-bracket is additive modulo two: the commutator of two even generators is even, the anti-commutator of two odd generators is also even, whereas the commutator of an even and an odd generator is odd. Schematically, the structure of a superalgebra takes the form

$$\begin{aligned} [even, even] &\sim even \\ [even, odd] &\sim odd \\ \{odd, odd\} &\sim even \end{aligned} \tag{4.59}$$

In particular one does not consider things that are the sum of an even and an odd generator (at least physicists don't but some Mathematicians might), nor does the commutator of two odd generators, or anti-commutator of two even generators, play any role. Just as in Lie-algebras there is a Jacobi identity. It is a little messy as whether or not one takes a commutator or anti-commutator depends on the even/odd character of the generator. It can be written as

$$(-1)^{ac}[A,[B,C]_\pm]_\pm + (-1)^{ba}[B,[C,A]_\pm]_\pm + (-1)^{cb}[C,[A,B]_\pm]_\pm = 0 \tag{4.60}$$

where $a, b, c \in \mathbf{Z}_2$ are the gradings of the generators A, B, C respectively and $[\ ,\ ]_\pm$ is a commutator or anti-commutator according to the rule (4.59).

There is a large mathematical literature on superalgebras as abstract objects. However we will simply focus on the case most relevant for particle physics. In particular the even elements will be the Poincare generators $P_\mu, M_{\nu\lambda}$ and the odd elements supersymmetries Q_α. The important point here is that the last line in (4.59) takes the form

$$\{Q, Q\} \sim P + M \tag{4.61}$$

(in fact one typically finds only P or M on the right hand side, and in this course just P). Thus supersymmetries are the square-root of Poincare transformations. Thus there is a sensible algebraic structure that is "deeper" that the Poincare group. Surely this is worth of study.

One final comment is in order. Although we have found a symmetry that underlies the Poincare algebra one generally still finds that supersymmetries commute with the other internal symmetries. Thus a refined version of the Coleman-Mandula theorem still seems to apply and states that the symmetry algebra of a non-trivial theory is at most the direct product of the superalgebra and an internal Lie-algebra.[5]

5 Elementary Consequences of Supersymmetry

The exact details of the supersymmetry algebra vary from dimension to dimension, depending on the details of Clifford algebras, however the results below for four-dimensions are qualitatively unchanged. If there are Majorana spinors then the algebra is, in addition to the Poincare algebra relations (3.25),[6]

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= -2(\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \\
[Q_\alpha, P_\mu] &= 0 \\
[Q_\alpha, M_{\mu\nu}] &= \frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta
\end{aligned}$$

[5]Note that one should be careful here, while this statement is true in spirit it is imprecise and in some sense counter examples can be found (*e.g.* in gauged supergavity).

[6]More precisely this is the minimal $N = 1$ super-Poincare algebra. One can have N-extended supersymmetry algebras and centrally extended supersymmetry algebras, which we will come to later. There are also superalgebras based on other Bosonic algebras than the Poincare algebra, *e.g.*, the anti-de-Sitter algebra.

$$ \tag{5.62} $$

The primary relation is the first line. The second line simply states that the Q_α's are invariant under translations and the third line simply states that they are spacetime spinors.

At first sight one might wonder why there is a C^{-1} on the right hand side. The point is that this is used to lower the second spinor index. Furthermore it is clear that the left hand side is symmetric in α and β and therefore the right hand side must also be symmetric. To see that this is the case we observe that, since we have assumed a Majorana basis where $C = -C^T = \gamma_0$,

$$ (\gamma_\mu C^{-1})^T = (C^{-1})^T \gamma_\mu^T = -(C^{-1})^T C \gamma_\mu C^{-1} = \gamma_\mu C^{-1} \tag{5.63} $$

is indeed symmetric.

Let us take the trace of the primary supersymmetry relation

$$ \sum_\alpha \{Q_\alpha, Q_\alpha\} = 8P_0 \tag{5.64} $$

Here we have used the fact that $C^{-1} = \gamma^0$, $\mathrm{Tr}(\gamma_{\mu\nu}) = 0$ and $\mathrm{Tr}(1) = 2^2$. We can identify $P_0 = E$ with the energy and hence we see that

$$ E = \frac{1}{4}\sum_\alpha Q_\alpha^2 \tag{5.65} $$

Since Q_α is Hermitian it follows that the energy is positive definite. Furthermore the only states with $E = 0$ must have $Q_\alpha|0 >= 0$, *i.e.* they must preserve the supersymmetry.

Supersymmetry, like other symmetries in quantum field theory, can be spontaneously broken. This means that the vacuum state $|vacuum >$, *i.e.* the state of lowest energy, does not satisfy $Q_\alpha|vacuum >= 0$. We see that in a supersymmetric theory this will be the case if and only if the vacuum energy is positive.

Next let us consider the representations of supersymmetry. First we observe that since $[P_\mu, Q_\alpha] = 0$ we have $[P^2, Q_\alpha] = 0$. Thus P^2 is a Casmir, that is to say irreducible representations of supersymmetry (*i.e.* of the Q's) all carry the same value of $P^2 = -m^2$. Thus all the particles in a supermultiplet (*i.e.* in a irreducible representation) have the same mass.

Let us first consider a massive supermultiplet. We can therefore go to the rest frame where $P_\mu = (m, 0, 0, 0)$. In this case the algebra becomes

$$ \{Q_\alpha, Q_\beta\} = 2m\delta_{\alpha\beta} \tag{5.66} $$

We can of course assume that $m > 0$ and rescale $\tilde{Q}_\alpha = m^{-1/2} Q_\alpha$ which gives

$$ \{\tilde{Q}_\alpha, \tilde{Q}_\beta\} = 2\delta_{\alpha\beta} \tag{5.67} $$

This is just a Clifford algebra in 4 Euclidean dimensions! As such we know that it has $2^{4/2} = 4$ states. We can construct the analogue of γ_5:

$$ (-1)^F = Q_1 Q_2 Q_3 ... Q_4 \tag{5.68} $$

Since we are in 4 Euclidean dimensions we have that $((-1)^F)^2 = 1$. Again $(-1)^F$ is traceless and Hermitian. Therefore it has 2 eigenvalues equal to +1 and 2 equal to −1. What is the significance of these eigenvalues? Well if $|\pm>$ is a state with $(-1)^F$ eigenvalue ±1 then $Q_\alpha|\pm>$ will satisfy

$$(-1)^F Q_\alpha|\pm>= -Q_\alpha(-1)^F|\pm>= \mp Q_\alpha|\pm> \tag{5.69}$$

Thus acting by Q_α will change the sign of the eigenvalue. However since Q_α is a Fermionic operator it will map Fermions to Bosons and vise-versa. Thus $(-1)^F$ measures whether or not a state is Fermionic or Bosonic. Since it is traceless we see that a supermutliplet contains an equal number of Bosonic and Fermionic states. This is only true on-shell since we have assumed that we are in the rest frame.

Next let us consider massless particles. Here we can go to a frame where $P_\mu = (E, E, 0, 0)$ so that the supersymmetry algebra becomes

$$\{Q_\alpha, Q_\beta\} = 2E(\delta_{\alpha\beta} + (\gamma_{01})_{\alpha\beta}) \tag{5.70}$$

We observe that $\gamma_{01}^2 = 1$ and also that $\mathrm{Tr}(\gamma_{01}) = 0$. Therefore the matrix $1 - \gamma_{01}$ has half its eigenvalues equal to 0 and the others equal to 2. It follows the algebra splits into two pieces:

$$\{Q_{\alpha'}, Q_{\beta'}\} = 4E\delta_{\alpha'\beta'} \qquad \{Q_{\alpha''}, Q_{\beta''}\} = 0 \tag{5.71}$$

where the primed and doubled primed indices only take on 2 values each. Again by rescaling, this time $\tilde{Q}_{\alpha'} = (2E)^{-1/2}Q_{\alpha'}$ we recover a Clifford algebra but in 2 dimensions. Thus there are just 2 states. Again we find that half are Fermions and the other half Bosons.

Finally we note that the condition $[Q_\alpha, M_{\mu\nu}] = \frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta$ implies that states in a supermultiplet will have spins that differ in steps of 1/2. In an irreducible mutliplet there is a unique state $|j_{max}>$ with maximal spin (actually helicity). The remaining states therefore have spins $j_{max} - 1/2, j_{max} - 1, ...$

It should be noted that often these multiplets will not be CPT complete. For example if they are constructed by acting with lowering operators on a highest helicity state then the tend to have more positive helicity states than negative ones. Therefore in order to obtain a CPT invariant theory, as is required by Lorentz invariance, one has to add in a CPT mirror multiplet (for example based on using raising operators on a lowest helicity state).

In higher dimensions the number of states in a supermultiplet grows exponentially quickly. This is essentially because the number of degrees of freedom of a spinor grow exponentially quickly. However the number of degrees of freedom of Bosonic fields (such as scalars and vectors) do not grow so quickly, if at all, when the spacetime dimension is increase. Although one can always keep adding in extra scalar modes to keep the Bose-Fermi degeneracy this becomes increasingly unnatural. In fact one finds that if we only wish to consider theories with spins less than two (*i.e.* do not include gravity) then the highest spacetime dimension for which there exists supersymmetric theories is $D = 10$. If we do allow for gravity then this pushes the limit up to $D = 11$.

We can also consider the Witten index:

$$\mathcal{W} = \mathrm{Tr}_{\mathcal{H}}(-1)^F \tag{5.72}$$

where the trace is over all states in the Hilbert space of the theory. This is not necessarily zero. What we have shown above is that there is a Bose-Fermi degeneracy among states with a non-zero energy. However there need not be such a degeneracy between supersymmetric, *i.e.* zero energy, vacuum state and hence

$$\mathcal{W} = \#\text{ of Bosonic vacua} - \#\text{ of Fermionic vacua} \tag{5.73}$$

By definition this is an integer. Therefore, since it cannot be continuously varied, it cannot receive corrections as the coupling constants are varied (so for example it is unaffected by perturbation theory).

Problem: Show that in four-dimensions, where Q_α is a Majorna spinor, the first line of the supersymmetry algebra (5.62) can be written as

$$\begin{aligned}
\{Q_{W\alpha}, Q_{W\beta}\} &= 0 \\
\{Q^*_{W\alpha}, Q^*_{W\beta}\} &= 0 \\
\{Q_{W\alpha}, Q^*_{W\beta}\} &= -((1+\gamma_5)\gamma_\mu C^{-1})_{\alpha\beta} P_\mu \\
\{Q^*_{W\alpha}, Q_{W\beta}\} &= -((1-\gamma_5)\gamma_\mu C^{-1})_{\alpha\beta} P_\mu
\end{aligned} \tag{5.74}$$

where $Q_{W\alpha}$ is a Weyl spinor and $Q^*_{W\alpha}$ is its complex conjugate. (Hint: Weyl spinors are chiral and are obtained from Majorana spinors Q_M through $Q_W = \frac{1}{2}(1+\gamma_5)Q_M$, $Q^*_W = \frac{1}{2}(1-\gamma_5)Q_M$.)

In Weyl notation one chooses a different basis for the four-dimensional Clifford Algebra. In particular one writes, in terms of block 2×2 matrices,

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{5.75}$$

where σ_i are the Pauli matrices. Note that the charge conjugation matrix, defined by $\gamma_\mu^T = -C\gamma_\mu C^{-1}$, is no longer $C = \gamma_0$. Rather we find

$$C = \gamma_0\gamma_2\gamma_5 \ .$$

Since Weyl spinors only have two independent components one usually introduces a new notation: $a, \dot{a} = 1, 2$ so that a general 4-component Dirac spinor is decomposed in terms of two complex Weyl spinors as

$$\psi_D = \begin{pmatrix} \lambda_a \\ \chi_{\dot{a}} \end{pmatrix} \tag{5.76}$$

i.e. the first two indices are denoted by a and the second two by $\dot{a}$.

Let us define $\sigma^{\mu}_{a\dot{b}} = \frac{1}{2}((1-\gamma_5)\gamma^{\mu}C^{-1})_{a\dot{b}}$, $\bar{\sigma}^{\mu}_{\dot{a}b} = \frac{1}{2}((1+\gamma_5)\gamma^{\mu}C^{-1})_{\dot{a}b}$. In this case the algebra is

$$\begin{aligned}
\{Q_a, Q_b\} &= 0 \\
\{Q^*_{\dot{a}}, Q^*_{\dot{b}}\} &= 0 \\
\{Q_a, Q^*_{\dot{b}}\} &= -2\sigma^{\mu}_{a\dot{b}}P_{\mu} \\
\{Q^*_{\dot{a}}, Q_b\} &= -2\bar{\sigma}^{\mu}_{\dot{a}b}P_{\mu}
\end{aligned} \tag{5.77}$$

Here we have dropped the subscript W since the use of a and $\dot{a}$ indices implies that we are talking about Weyl spinors. This form for the algebra appears in many text books and is also known as the two-component formalism.

Problem: Show that

$$\begin{aligned}
(\sigma_{\mu})_{a\dot{b}} &= \left(\delta_{a\dot{b}}, \sigma^{i}_{a\dot{b}}\right) \\
(\bar{\sigma}_{\mu})_{a\dot{b}} &= \left(\delta_{a\dot{b}}, -\sigma^{i}_{a\dot{b}}\right)
\end{aligned} \tag{5.78}$$

And therefore

$$\gamma_{\mu} = \begin{pmatrix} 0 & \bar{\sigma}_{\mu} \\ -\bar{\sigma}_{\mu} & 0 \end{pmatrix} . \tag{5.79}$$

Recall that we defined two Lorentz invariant contractions of spinors; Dirac: $\psi^{\dagger}\gamma_0\psi$ and Majorana: $\psi^T C\psi$? In the Majorana notation with real spinors these are manifestly the same (but not if ψ isn't real). In two component notation these are

$$\begin{aligned}
\psi^{\dagger}\gamma_0\psi &= \lambda^{\dagger}\chi - \chi^{\dagger}\lambda \\
\psi^T C\psi &= \lambda^T\sigma^2\chi + \chi^T\sigma^2\lambda .
\end{aligned}$$

Finally, what is a Majorana spinor in this notation? Well its one for which the Dirac conjugate and Majorana conjugate coincide:

$$\psi^{\dagger}\gamma_0 = \psi^T C .$$

Taking the transpose leads to $\psi^* = \gamma_2\gamma_5\psi$. In terms of two-component spinors this gives:

$$\lambda^* = -\sigma^2\chi , \qquad \chi^* = \sigma^2\lambda .$$

6 Super-Yang Mills

We can now start to construct a version of Yang-Mills theory that has supersymmetry. Since we must have a gauge field in the adjoint representation we see that supersymmetry will force us to have a Fermion that is also in adjoint representation. We can then add Fermions in other representations provided that we also include scalar superpartners for them.

6.1 Super-Maxwell

We start our first construction of a supersymmetric theory by looking at a very simple theory: Electromagnetism coupled to a Majorana Fermion in the adjoint. Since the adjoint of $U(1)$ is trivial the Fermion is chargeless and we have a free theory!

Why these fields? We just saw that the simplest supermultiplet is massless with 2 real Fermions and 2 real Bosons on-shell. Furthermore since supersymmetries commute with any internal symmetries we see that the Fermions need to be in the same representation of the gauge group as the Bosons. In Maxwell theory the gauge field is in the adjoint so the Fermion must also be in the adjoint.

Let us now check that the number of degrees of freedom is correct. We fix the gauge to Lorentz gauge $\partial^\mu A_\mu = 0$. Maxwell's equation is then just $\partial^2 A_\mu = 0$. However this only partially fixes the gauge since we can also take $A_\mu \to A_\mu + \partial_\mu\theta$ so long as $\partial^2\theta = 0$. This allow us to remove one component of A_μ, say A_3. However imposing $\partial^\mu A_\mu = 0$ provides a further constraint leaving 2 degrees of freedom. In particular in momentum space choosing $p^\mu = (E, 0, 0, E)$ we see that $p^\mu A_\mu = E(A_0 + A_3) = 0$ and hence $A_0 = A_3 = 0$ leaving just A_1 and A_2.

For the Fermion λ we have the Dirac equation $\gamma^\mu\partial_\mu\lambda = 0$. In momentum space this is $p^\mu\gamma_\mu\lambda = 0$. Choosing $p^\mu = (E, 0, 0, E)$ we find

$$E(\gamma_0 + \gamma_3)\lambda = 0 \tag{6.80}$$

For $E \neq 0$ this implies $\gamma_{03}\lambda = \lambda$. Since γ_{03} is traceless and squares to one we see that this projects out 2 of the 4 components of λ.

The action is

$$S_{SuperMaxwell} = -\int d^4x \; \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda \tag{6.81}$$

where $\bar{\lambda} = \lambda^T C$. Not very exciting except that it has the following symmetry

$$\begin{aligned} \delta A_\mu &= i\bar{\epsilon}\gamma_\mu\lambda \\ \delta\lambda &= -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon \end{aligned} \tag{6.82}$$

To see this we first note that, since $C\gamma^\mu$ is symmetric,

$$\begin{aligned} \delta\bar{\lambda}\gamma^\mu\partial_\mu\lambda &= \partial_\mu(\delta\bar{\lambda}\gamma^\mu\lambda) - \partial_\mu\delta\bar{\lambda}\gamma^\mu\lambda \\ &= \partial_\mu(\delta\bar{\lambda}\gamma^\mu\lambda) + \bar{\lambda}\gamma^\mu\partial_\mu\delta\lambda \end{aligned} \tag{6.83}$$

We can drop the total derivative term in the action and find

$$\begin{aligned} \delta S &= -\int d^4x F^{\mu\nu}\partial_\mu\delta A_\nu + i\bar{\lambda}\gamma^\rho\partial_\rho\delta\lambda \\ &= -\int d^4x F^{\mu\nu} i\bar{\epsilon}\gamma_\nu\partial_\mu\lambda - \frac{i}{2}\bar{\lambda}\gamma^\rho\partial_\rho F_{\mu\nu}\gamma^{\mu\nu}\epsilon \end{aligned} \tag{6.84}$$

To continue we note that $C\gamma^\mu$ is symmetric and $\gamma^\rho\gamma^{\mu\nu} = \gamma^{\rho\mu\nu} + \eta^{\rho\mu}\gamma^\nu - \eta^{\rho\nu}\gamma^\mu$. Thus we have

$$\delta S = -\int d^4x - F^{\mu\nu} i\partial_\mu \bar{\lambda}\gamma_\nu\epsilon - \frac{i}{2}\bar{\lambda}\partial_\rho F_{\mu\nu}(\gamma^{\rho\mu\nu} + 2\eta^{\rho\mu}\gamma^\nu)\epsilon \tag{6.85}$$

Now $\gamma^{\rho\mu\nu}\partial_\rho F_{\mu\nu} = \gamma^{\rho\mu\nu}\partial_{[\rho}F_{\mu\nu]} = 0$ so we are left with

$$\begin{aligned} \delta S &= -\int d^4x - F^{\mu\nu} i\bar{\partial}_\mu \lambda\gamma_\nu\epsilon - i\bar{\epsilon}\partial^\mu F_{\mu\nu}\gamma^\nu\lambda \\ &= \int d^4x \partial_\mu (iF^{\mu\nu}\bar{\lambda}\gamma^\nu\epsilon) \\ &= 0 \end{aligned} \tag{6.86}$$

Our next task is to show that these symmetries do indeed close into the supersymmetry algebra. First we compute the closure on the gauge field

$$\begin{aligned} [\delta_1, \delta_2]A_\mu &= i\bar{\epsilon}_2\gamma_\mu(\frac{1}{2}F^{\lambda\rho}\gamma_{\lambda\rho}\epsilon_2) - (1 \leftrightarrow 2) \\ &= i\bar{\epsilon}_2(\frac{1}{2}\gamma_{\mu\lambda\rho} + \eta_{\mu\lambda}\gamma_\rho)F^{\lambda\rho}\epsilon_1 - (1 \leftrightarrow 2) \end{aligned} \tag{6.87}$$

Now consider the spinor contractions in the first term. We note that

$$(C\gamma_{\mu\lambda\rho})^T = -C\gamma_{\rho\lambda\mu}C^{-1}C^T = C\gamma_{\rho\lambda\mu} = -C\gamma_{\mu\lambda\rho} \tag{6.88}$$

Thus $\epsilon_2\gamma_{\mu\lambda\rho}\epsilon_1$ is symmetric under $1 \leftrightarrow 2$ and hence doesn't contribute to the commutator. Hence

$$\begin{aligned} [\delta_1, \delta_2]A_\mu &= -2i\bar{\epsilon}_2\gamma^\nu\epsilon_1 F_{\mu\nu} \\ &= (2i\bar{\epsilon}_2\gamma^\nu\epsilon_1)\partial_\nu A_\mu - \partial_\mu(2i\bar{\epsilon}_2\gamma^\nu\epsilon_1 A_\nu) \end{aligned} \tag{6.89}$$

We recognize the first term as a translation and the second a gauge transformation. Thus the supersymmetry algebra closes correctly on A_μ.

Next we must look at the Fermions. Here we find

$$\begin{aligned} [\delta_1, \delta_2]\lambda &= -2\partial_\mu(\frac{i}{2}\bar{\epsilon}_1\gamma_\nu\lambda)\gamma^{\mu\nu}\epsilon_2 - (1 \leftrightarrow 2) \\ &= -i\gamma^{\mu\nu}(\bar{\epsilon}_1\gamma_\nu\partial_\mu\lambda)\epsilon_2 - (1 \leftrightarrow 2) \end{aligned} \tag{6.90}$$

The problem here is that the spinor index on λ is contracted with $\bar{\epsilon}_1$ on the right hand side and the free spinor index comes from ϵ_2 whereas the left hand side has a free spinor coming from λ. There is a way to rewrite the right hand side using the so-called Fierz identity, valid for any three, anti-commuting, spinors ρ, ψ and χ in four spacetime dimensions,

$$\begin{aligned} (\bar{\rho}\psi)\chi_\alpha &= -\frac{1}{4}(\bar{\rho}\chi)\psi_\alpha - \frac{1}{4}(\bar{\lambda}\gamma_5\chi)\gamma_5\psi_\alpha - \frac{1}{4}(\bar{\rho}\gamma_\mu\chi)(\gamma^\mu\psi)_\alpha \\ &\quad + \frac{1}{4}(\bar{\rho}\gamma_\mu\gamma_5\chi)(\gamma^\mu\gamma_5\psi)_\alpha + \frac{1}{8}(\bar{\rho}\gamma_{\mu\nu}\chi)(\gamma^{\mu\nu}\psi)_\alpha \end{aligned} \tag{6.91}$$

The proof of this identity is given in appendix B and you are strongly encouraged to read it. The point of this identity is that the free spinor index is moved from being on χ on the left hand side to being on ψ on the right hand side.

Returning to the case at hand we can take $\rho = \epsilon_1$, $\chi = \epsilon_2$ and $\psi = \gamma_\nu \partial_\mu \lambda$. This leads to

$$\begin{aligned}[]
[\delta_1, \delta_2]\lambda &= \frac{i}{4}\gamma^{\mu\nu}(\bar\epsilon_1\epsilon_2)\gamma_\nu\partial_\mu\lambda + \frac{i}{4}\gamma^{\mu\nu}(\bar\epsilon_1\gamma_5\epsilon_2)\gamma_5\gamma_\nu\partial_\mu\lambda + \frac{i}{4}\gamma^{\mu\nu}\left(\bar\epsilon_1\gamma_\rho\epsilon_2\right)\gamma^\rho\gamma_\nu\partial_\mu\lambda \\
&\quad -\frac{i}{4}\gamma^{\mu\nu}\left(\bar\epsilon_1\gamma_\rho\gamma_5\epsilon_2\right)\gamma^\rho\gamma_5\gamma_\nu\partial_\mu\lambda - \frac{i}{8}\gamma^{\mu\nu}\left(\bar\epsilon_1\gamma_{\rho\sigma}\epsilon_2\right)\gamma^{\rho\sigma}\gamma_\nu\partial_\mu\lambda - (1\leftrightarrow 2)
\end{aligned} \tag{6.92}$$

Problem: Show that

$$\begin{aligned}
\bar\epsilon_1\epsilon_2 - \bar\epsilon_2\epsilon_1 &= 0 \\
\bar\epsilon_1\gamma_5\epsilon_2 - \bar\epsilon_2\gamma_5\epsilon_1 &= 0 \\
\bar\epsilon_1\gamma_\rho\gamma_5\epsilon_2 - \bar\epsilon_2\gamma_\rho\gamma_5\epsilon_1 &= 0 \\
\bar\epsilon_1\gamma_{\rho\sigma}\epsilon_2 + \bar\epsilon_2\gamma_{\rho\sigma}\epsilon_1 &= 0
\end{aligned} \tag{6.93}$$

Given this we have

$$[\delta_1, \delta_2]\lambda = \frac{i}{2}\left(\bar\epsilon_1\gamma_\rho\epsilon_2\right)\gamma^{\mu\nu}\gamma^\rho\gamma_\nu\partial_\mu\lambda - \frac{i}{4}\left(\bar\epsilon_1\gamma_{\rho\sigma}\epsilon_2\right)\gamma^{\mu\nu}\gamma^{\rho\sigma}\gamma_\nu\partial_\mu\lambda \tag{6.94}$$

Now look at the first term

$$\begin{aligned}
\gamma^{\mu\nu}\gamma^\rho\gamma_\nu &= -\gamma^{\mu\nu}\gamma_\nu\gamma^\rho + 2\gamma^{\mu\rho} \\
&= -3\gamma^\mu\gamma^\rho + 2\gamma^{\mu\rho} \\
&= -3\eta^{\mu\rho} - \gamma^{\mu\rho} \\
&= -4\eta^{\mu\rho} + \gamma^\rho\gamma^\mu
\end{aligned} \tag{6.95}$$

And the second

$$\begin{aligned}
\gamma^{\mu\nu}\gamma^{\rho\sigma}\gamma_\nu &= [\gamma^{\mu\nu}, \gamma^{\rho\sigma}]\gamma_\nu + \gamma^{\rho\sigma}\gamma^{\mu\nu}\gamma_\nu \\
&= 2(\eta^{\nu\rho}\gamma^{\mu\sigma} - \eta^{\mu\rho}\gamma^{\nu\sigma} + \eta^{\mu\sigma}\gamma^{\nu\rho} - \eta^{\nu\sigma}\gamma^{\mu\rho})\gamma_\nu + 3\gamma^{\rho\sigma}\gamma^\mu \\
&= 2\gamma^{\mu\sigma}\gamma^\rho + 6\eta^{\mu\rho}\gamma^\sigma - 6\eta^{\mu\sigma}\gamma^\rho - 2\gamma^{\mu\rho}\gamma^\sigma + 3\gamma^{\rho\sigma}\gamma^\mu \qquad (6.96) \\
&= 2\gamma^{\mu\sigma\rho} + 2\gamma^\mu\eta^{\rho\sigma} + 4\eta^{\mu\rho}\gamma^\sigma - 2\gamma^{\mu\rho\sigma} - 2\gamma^\mu\eta^{\rho\sigma} - 4\eta^{\mu\sigma}\gamma^\rho + 3\gamma^{\rho\sigma}\gamma^\mu \\
&= 4\gamma^{\sigma\rho\mu} + 4\eta^{\mu\rho}\gamma^\sigma - 4\eta^{\mu\sigma}\gamma^\rho + 3\gamma^{\rho\sigma}\gamma^\mu \\
&= 4\gamma^{\sigma\rho}\gamma^\mu + 3\gamma^{\rho\sigma}\gamma^\mu \\
&= -\gamma^{\rho\sigma}\gamma^\mu
\end{aligned} \tag{6.97}$$

In the second line we used a result from the problems that showed $-\frac{i}{2}\gamma_{\mu\nu}$ satisfy the Lorentz algebra.

Putting this altogether we find

$$[\delta_1, \delta_2]\lambda = -2i\,(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\,\partial_\mu\lambda + \frac{i}{2}\,(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\,\gamma_\nu\gamma^\mu\partial_\mu\lambda \\ +\frac{i}{4}\,(\bar{\epsilon}_1\gamma_{\rho\sigma}\epsilon_2)\,\gamma^{\rho\sigma}\gamma^\mu\partial_\mu\lambda \tag{6.98}$$

We recognize the first term as a translation (the same one since $\bar{\epsilon}_1\gamma^\mu\epsilon_2 = -\bar{\epsilon}_2\gamma^\mu\epsilon_1$). Since λ has a trivial gauge transformation we do not expect anything else. But there clearly is stuff. However this extra stuff vanishes if the Fermion is on-shell: $\gamma^\mu\partial_\mu\lambda = 0$. Thus we say that the supersymmetry algebra closes on-shell.

This is good enough for us since this course is classical (indeed it is often good enough in the quantum theory too). In fact we can see that it couldn't have closed off-shell since the degrees of freedom don't match. In particular A_μ has four degrees of freedom but one is removed by a gauge transformation whereas λ_α also has four degrees of freedom but none can be removed by a gauge transformation. On-shell however A_μ has two degrees of freedom and λ also has two.

6.2 Super-Yang-Mills

Our next task is to find an interacting supersymmetric theory. To this end we try to generalize the previous action to an arbitrary Lie group G. In particular we take have a gauge field A_μ and Fermion λ, both of which are in the adjoint representation

$$S_{susyYM} = -\frac{1}{g_{YM}^2}\int d^4x\ \frac{1}{4}\mathrm{tr}(F_{\mu\nu}, F^{\mu\nu}) + \frac{i}{2}\mathrm{Tr}(\bar{\lambda}, \gamma^\mu D_\mu\lambda) \tag{6.99}$$

with $D_\mu\lambda = \partial_\mu\lambda - i[A_\mu, \lambda]$. The natural guess for the supersymmetry transformation is

$$\delta A_\mu = i\bar{\epsilon}\gamma_\mu\lambda \\ \delta\lambda = -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon \tag{6.100}$$

Note that although this looks the same as in the Abelian case above it is in fact rather complicated and interacting. Nevertheless the steps to prove invariance are very similar but more involved.

The first thing to note is that there is a term in δS coming from $\bar{\lambda}\gamma^\mu[\delta A_\mu, \lambda]$ that is cubic in λ. This is the only term that is cubic in λ and hence must vanish:

$$\mathrm{Tr}(\bar{\lambda}, \gamma^\mu[(\bar{\epsilon}\gamma_\mu\lambda), \lambda]) = 0 \tag{6.101}$$

Problem: Using the Fierz identity show that, in four dimensions,

$$\mathrm{Tr}(\bar{\lambda}, \gamma^\mu[(\bar{\epsilon}\gamma_\mu\lambda), \lambda]) = f^{abc}\bar{\lambda}_c\gamma^\mu(\bar{\epsilon}\gamma_\mu\lambda_a)\lambda_b = 0. \tag{6.102}$$

This is a crucial condition, without it we would be sunk. In fact it is only true in a few dimensions ($D = 3, 4, 6, 10$) and hence what is called pure super-Yang Mills (*i.e.* super Yang-Mills with the minimum number of fields) only exists in these dimensions. Super-Yang-Mills theories exist in lower dimensions but they always contain additional fields such as scalars (you can construct them by compactification one of the pure theories). Ultimately the reason for this is that these are the only dimensions where one can match up the number of Bose and Fermi degrees of freedom on-shell.

Okay now we can proceed. We first note that

$$\mathrm{Tr}(\bar{\delta}\lambda, \gamma^\mu D_\mu \lambda) = \partial_\mu \mathrm{Tr}(\bar{\delta}\lambda, \gamma_\mu \lambda) + \mathrm{Tr}(\bar{\lambda}, \gamma^\mu D_\mu \delta\lambda) \tag{6.103}$$

We have see that this is true when $D_\mu = \partial_\mu$ so we now need to check the A_μ term. The left hand side gives

$$\begin{aligned} -i\mathrm{Tr}(\bar{\delta}\lambda, \gamma^\mu [A_\mu, \lambda]) &= i\mathrm{Tr}([A_\mu, \bar{\lambda}], \gamma^\mu \delta\lambda) \\ &= -i\mathrm{Tr}([\bar{\lambda}, A_\mu], \gamma^\mu \delta\lambda) \\ &= -i\mathrm{Tr}(\bar{\lambda}, \gamma^\mu [A_\mu, \delta\lambda]) \end{aligned} \tag{6.104}$$

and this is indeed the right hand side. Note that in the first line we used the fact that $C\gamma^\mu$ is symmetric to interchange the two spinors with a minus sign and in the last line we used the fact that $\mathrm{Tr}([A, B], C) = \mathrm{Tr}(A, [B, C])$

Thus we find that, up to boundary terms,

$$\delta S = -\int d^4x \frac{1}{2}\mathrm{Tr}(F^{\mu\nu}, \delta F_{\mu\nu}) + i\mathrm{Tr}(\bar{\lambda}, \gamma^\rho D_\rho \delta\lambda)$$

Next we need to compute

$$\begin{aligned} \delta F_{\mu\nu} &= \partial_\mu \delta A_\nu - i[A_\mu, \delta A_\nu] - (\mu \leftrightarrow \nu) \\ &= i\bar{\epsilon}\gamma_\nu D_\mu \lambda - (\mu \leftrightarrow \nu) \end{aligned} \tag{6.105}$$

Thus we have

$$\delta S = -\int d^4x i\mathrm{Tr}(F^{\mu\nu}, \bar{\epsilon}\gamma_\nu D_\mu \lambda) - \frac{i}{2}\mathrm{Tr}(\bar{\lambda}, \gamma^\rho D_\rho \gamma_{\mu\nu} F^{\mu\nu} \epsilon) \tag{6.106}$$

Again we can use the identity

$$\gamma_\rho \gamma_{\mu\nu} = \gamma_{\rho\mu\nu} + \eta_{\rho\mu}\gamma_\nu - \eta_{\rho\nu}\gamma_\mu \tag{6.107}$$

so that we find

$$\begin{aligned} \delta S &= -\int d^4x - i\mathrm{Tr}(F^{\mu\nu}, D_\mu \bar{\lambda}\gamma_\nu \epsilon) - i\mathrm{Tr}(\bar{\lambda}, D_\mu F^{\mu\nu}\gamma_\nu \epsilon) \\ &\quad - \frac{i}{2}\mathrm{Tr}(\bar{\lambda}, \gamma^{\rho\mu\nu} D_\rho F_{\mu\nu} \epsilon) \end{aligned} \tag{6.108}$$

The first line adds up to a total derivative and hence can be dropped. This leaves us the the final line. This indeed vanishes because of the so-called Bianchi identity

$$D_{[\mu}F_{\nu\lambda]} = 0 \tag{6.109}$$

Problem: Prove the Bianchi identity $D_{[\mu}F_{\nu\lambda]} = 0$, where $D_\mu F_{\nu\lambda} = \partial_\mu F_{\nu\lambda} - i[A_\mu, F_{\nu\lambda}]$.

Thus we have established that the action is supersymmetric. It is also important to show that the supersymmetry variations close (on-shell). Let us consider the gauge field first. Little changes from the above Abelian calculation and we find

$$\begin{aligned}[\delta_1, \delta_2]A_\mu &= -(2i\bar{\epsilon}_2\gamma^\nu\epsilon_1)F_{\mu\nu} \\ &= 2i(\bar{\epsilon}_2\gamma^\nu\epsilon_1)\partial_\nu A_\mu - D_\mu(2i\bar{\epsilon}_2\gamma^\nu\epsilon_1 A_\nu)\end{aligned} \tag{6.110}$$

Here we have used the fact that $F_{\mu\nu} = D_\mu A_\nu - \partial_\nu A_\mu$.

Problem: Show that the transformations (6.100) close on-shell on the Fermions to Poincare transformations and gauge transformations.

6.3 Super-Yang-Mills with Matter

The Lagrangian we constructed is supersymmetric but it is not particularly relevant for the real world as the Fermions are necessarily in the adjoint. In the real world the Fermions are in the fundamental representation of the gauge group. More generally we are interested in constructing a supersymmetric theory with Fermions in an arbitrary representation R of the gauge group. In order to accommodate this we must also include superpartners for such Fermions. These must also be in the representation R. Furthermore these superpartners should be scalars (any vectors must be gauge Bosons and hence in the adjoint). Such matter multiplets must therefore have two scalars (why?) and are generically called matter multiplets. We won't discuss these Lagrangians in detail here (the complete answer is rather complicated). However we can discuss come features.

Let us denote the fields in the matter multiplet by ψ^a_α and ϕ^a. Here ψ_α is a Fermion and ϕ^a a complex Boson. Both are taken to be in some representation of the gauge group with generators $(T_r)^a{}_b$. A natural guess for the action is

$$S = -\frac{1}{g^2}\int d^4x \frac{1}{4}\kappa_{rs}F^r_{\mu\nu}F^{s\mu\nu} + \frac{i}{2}\kappa_{rs}\bar{\lambda}^r\gamma^\mu D_\mu\lambda^s + \frac{i}{2}\delta_{ab}\bar{\psi}^a\gamma^\mu D_\mu\psi^b + \frac{1}{2}\delta_{ab}D_\mu\phi^{*a}D_\mu\phi^b \tag{6.111}$$

with the supersymmetries

$$\begin{aligned}\delta A^r_\mu &= i\bar{\epsilon}\gamma_\mu\lambda^r \\ \delta\lambda^r &= -\frac{1}{2}F^r_{\mu\nu}\gamma^{\mu\nu}\epsilon \\ \delta\phi^a &= i\bar{\epsilon}\psi^a \\ \delta\psi^a &= \gamma^\mu D_\mu\phi^a\epsilon\end{aligned} \tag{6.112}$$

Note that this is just a guess, we expect that they will be corrected by additional terms.

However one can see that this won't work. There will be a cubic Fermion term that comes from the supervariation of the gauge field in $\bar{\psi}\gamma^\mu D_\mu\psi$ of the form

$$\delta_3 S = \frac{i}{2}\delta_{ab}\bar{\psi}^a\gamma^\mu(\bar{\epsilon}\gamma_\mu\lambda^r)(T_r)^b{}_c\psi^c \tag{6.113}$$

The Feirz transformation won't kill this term off now. To get rid of it we must introduce other terms that give a suitable cubic Fermion term. Such a term must have two Fermions and a Boson so that the supervariation of the Boson gives a term that can cancel $\delta_3 S$. Such a term is

$$S_{Yukawa} = \int d^4x h_{ab}\bar{\psi}^a\lambda^r(T_r)^b{}_c\phi^c + c.c. \tag{6.114}$$

Here $+c.c.$ denotes the complex conjugate since the first term is clearly not real on its own. This can indeed be made to cancel $\delta_3 S$. However we now find additional terms coming from the supervariation of the Fermions in S_{Yukawa}. These in turn need to be canceled by introducing a potential term of the form $V(\phi, \phi^*)$ and then also by a correction to the λ and ψ supervariation. The Lagrangian now takes the form

$$\begin{aligned} S_{matter} &= -\frac{1}{g^2}\int d^4x \frac{1}{4}\kappa_{rs}F^r_{\mu\nu}F^{s\mu\nu} + \frac{i}{2}\kappa_{rs}\bar{\lambda}^r\gamma^\mu D_\mu\lambda^s + \frac{i}{2}\delta_{ab}\bar{\psi}^a\gamma^\mu D_\mu\psi^b + \frac{1}{2}\delta_{ab}D_\mu\phi^{*a}D_\mu\phi^b \\ &\quad + \left(ih_{ab}\bar{\psi}^a\lambda^r(T_r)^b{}_c\phi^c + c.c.\right) - \frac{1}{2}\sum_r |\phi^{*a}(T_r)_{ab}\phi^b - \xi^r|^2 \end{aligned} \tag{6.115}$$

Here the ξ^r are constants known as Fayet-Illiopoulos parameters.

In fact there are even more things that can be added and one can include an entire holomorphic function $W(\phi^a)$ known as the superpotential. This gives additional Yukawa terms of the form

$$S_{WYukawa} = \int d^4x \partial_a\partial_b W(\psi^a)^T C\psi^b + c.c. \tag{6.116}$$

as well as an additional contribution to the potential

$$S_{Wpotential} = -\int d^4x h^{ab}\partial_a W(\partial_b W)^* \tag{6.117}$$

The complete action is then $S = S_{matter} + S_{WYukawa} + S_{Wpotential}$.

The upshot of all this is that matter fields can be incorporated into a supersymmetric theory. However we see that these always lead to Yukawa interactions as well as a potential for the scalars. These theories can also be made to be chiral, *i.e.* have the left and right handed Fermions transform under different representations of the gauge group. These are the sorts of models that are proposed to be supersymmetric extentions of the Standard Model. Indeed there is a so-called minimal extension known as the MSSM.

7 Extended Supersymmety

The supersymmetry algebra that we discussed above is not quite unique. It turns out that one can have more than one supersymmetry generator Q^I_α, with $I = 1, 2, .., N$. The superalgebra is

$$\{Q^I_\alpha, Q^J_\beta\} = -2(\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \delta^{IJ} + Z^{IJ}_{\alpha\beta} \tag{7.118}$$

The remaining commutators in (5.62) are unchanged (except that there is now a superscript I on Q_α). This is just N copies of the minimal supersymmetry algebra except that there is an extra term on the right hand side. This term is called the central charge where the term central refers to the fact that $Z^{IJ}_{\alpha\beta}$ commutes with all the generators. Since it a Lorentz scalar we can write

$$Z^{IJ}_{\alpha\beta} = U^{IJ} C^{-1}_{\alpha\beta} + iV^{IJ}(\gamma_5 C^{-1})_{\alpha\beta} \tag{7.119}$$

with $U^{(IJ)} = V^{(IJ)} = 0$. Thus $Z^{IJ}_{\alpha\beta}$ is anti-symmetric in $I \leftrightarrow J$ and also anti-symmetric under $\alpha \leftrightarrow \beta$. Thus is can only exist if $I, J = 1, 2, .., N$ with $N > 1$ (although there can be non-Lorentz invariant 'central' charges even for $N = 1$).

Without central charges a massive state cannot preserve any of the supersymmetries. To see this suppose that $\bar\epsilon^\alpha_I Q^I_\alpha |susy >= 0$ for some ϵ^I_α (which is commuting) then from the algebra we see that, in the rest frame,

$$0 = \bar\epsilon^\alpha_I < susy|\{Q^I_\alpha, Q^J_\beta\}|susy >= 2M\bar\epsilon^\beta_J < susy|susy > \tag{7.120}$$

where we have used the fact that $\bar\epsilon^\alpha_I Q^I_\alpha$ will either annihilate $|susy>$ from the left or from the right. In the massless case we see that this is possible provided that $\bar\epsilon^I(1 + \gamma_{01}) = 0$ *i.e.* $\gamma_{01}\epsilon = \epsilon$.

However central charges offer a way out of this. Since $Z^{IJ}_{\alpha\beta}$ is Hermitian we can find a *commuting* spinor $\bar\epsilon^\alpha_J$ such that $\bar\epsilon^\alpha_J Z^{IJ}_{\alpha\beta} = z\bar\epsilon^I_\beta$. Then we see that, in the rest frame,

$$\{Q, Q\} = 2Q^2 = 2M|\epsilon|^2 + z|\epsilon|^2 \tag{7.121}$$

where $Q = \bar\epsilon^\alpha_I Q^I_\alpha$ and $|\bar\epsilon|^2 = \delta_{\alpha\beta}\delta^{IJ}\bar\epsilon^\alpha_I\bar\epsilon^\beta_J$. Note that we take $\bar\epsilon^\alpha_I$ to be an ordinary commuting spinor so that $|\bar\epsilon|^2 \geq 0$. Now take the expectation value of this expression

$$< state|Q^2|state >= (M + \frac{1}{2}z)|\bar\epsilon|^2 < state|state > \tag{7.122}$$

The left hand side is positive definite and vanishes iff $Q|state >= \bar\epsilon^\alpha_I Q^I_\alpha|state >= 0$. Therefore we see that $M \geq -\frac{1}{2}z$. Since $Z^{IJ}_{\alpha\beta}$ is traceless there are equal number of positive and negative eigenvalues z hence we learn that

$$M \geq \frac{1}{2}|z| \tag{7.123}$$

with equality iff $Q|state >= \bar\epsilon^\alpha_I Q^I_\alpha|state >= 0$, *i.e.* iff the state preserves some of the supersymmetries.

Let us now consider the effect of turning on central charges on the representations of extended supersymmetry. Generically we find an algebra that is isomorphic to the appropriate Clifford algebra but now in $4N$ dimensions, corresponding to the $4N$ supersymmetry generator. Therefore we find multiplets with 2^{2N} states, half of which will be Bosons and the other half Fermions.

However there are special cases where the eigenvalues of the central charges agree with the mass eigenvalues. In this case there is cancelation when one goes to the rest frame and $\{Q^I_\alpha, Q^J_\beta\}$ will have zero eigenvalues. This is essentially the same as what happened in the case of massless representations. The result is a Clifford algebra in a lower dimension and hence a smaller representation. In particular if n of the central charge eigenvalues Z are degenerate and equal to the mass m then one finds a Clifford algebra in $4N - n$ dimensions and these will have $2^{2N-n/2}$ states. Again half of these will be Bosons and half Fermions. In addition, as before, there will be a highest spin state and the rest of the states can be obtained by acting with the $2^{N-n/4}$ lowering operators.

The massless case then corresponds to $n = 2N$. The central charge terms are traceless so that at most $2N$ of the eigenvalues can be degenerate (and non-vanishing). Therefore the smallest possible short representation has half has many generators, just as in the massless case.

Representations for which the mass is equal to one or more eigenvalues of the the central charge are called short representations and the states are known as BPS states. The important point about them is that since the dimension of the representation is an integer it cannot be altered by varying the parameters of the theory in a continuous fashion. In particular it cannot pick up any quantum corrections. Thus a relation of the form $m = Z$ (that the mass is degenerate with the central charge) is not corrected by quantum effects (although m and hence Z too may well pick up corrections). This turns out to be a very powerful technique in understanding non-perturbative features of supersymmetric quantum theories and M-theory in particular. For example this property is what allows one to calculate the Beckenstein-Hawking entropy of a black hole by counting the number of microstates in string theory. At weak coupling some D-brane states are described by a gauge theory and the degeneracy can be easily counted, whereas at strong coupling such states appear as black hole solutions. If the D-brane is superymmetric then the various states are in short multiplets and hence their counting is unaffected by going to strong coupling.

7.1 $N = 4$ Super-Yang-Mills

How can we construct a theory with additional supersymmetries? Clearly any theory with extended supersymmetry also has just ordinary $N = 1$ supersymmetry. Therefore these theories must be of the same type that we already considered, *i.e.* they consist of a Yang-Mills gauge multiplet coupled to some matter with some potential and Yukawa terms. What is special is that there must matter in the adjoint representation since there will be Fermion and scalar modes that are in the same supermultiplet as the gauge fields (although for $N = 2$ supersymmetry it is also possible to have additional matter which is not in the adjoint but a general representation). Furthermore the various coupling constants that appear in the potential and Yukawa terms must be related.

Let us then give an example of such a theory. In four-dimensions one can have $N = 1, 2, 4$ supersymmetries. We have studied $N = 1$ so let us jump to the other end and consider $N = 4$. $N = 4$ means that there are 4 supersymmetries Q^I_α, $I = 1, 2, 3, 4$. Since $\alpha = 1, 2, 3, 4$ we see that there are a total of 16 Q^I_α's. What is the easiest way to construct such a theory? Well we can start with super-Yang-Mills is a higher dimension where the spinors have more components and 16 is the number of components of a Majorana-Weyl spinor in 10 dimensions (recall that a general Dirac spinor in 10 dimensions has 32 complex components, Majorana reduces this to 32 real components and Weyl then reduces this to 16).

In 10 dimensions the super-Yang-Mills action is

$$S_{susyYM} = -\frac{1}{g^2_{YM}} \int d^{10}x \ \frac{1}{4}\mathrm{tr}(F_{mn}, F^{mn}) + \frac{i}{2}\mathrm{Tr}(\bar{\Lambda}, \Gamma^m D_m \Lambda) \tag{7.124}$$

This is just as before except that the indices $m, n, = 0, ..., 9$ and the spinors Λ and Γ_m are those of 10 dimensions, *i.e.* 32-component. As mentioned we can chose a Majorana basis and also, simultaneously, restrict to Weyl spinors $\Gamma_{11}\Lambda = \Lambda$, where $\Gamma_{11} = \Gamma_0\Gamma_1...\Gamma_9$ is the 10-dimensional analogue of γ_5.

To prove that this is supersymmetric we can follow the same argument that we did for the four-dimensional case. The variations are taken to be

$$\begin{aligned} \delta A_m &= i\bar{\varepsilon}\Gamma_m\Lambda \\ \delta\Lambda &= -\frac{1}{2}F_{mn}\Gamma^{mn}\varepsilon \ . \end{aligned} \tag{7.125}$$

Note that the preserve $\Gamma_{11}\Lambda = \Lambda$ we must also impose

$$\Gamma_{11}\varepsilon = \varepsilon$$

The only time that the dimension of spacetime showed up was in the cubic variation. As mentioned above, in 10-dimensions, we also have that

$$\mathrm{Tr}(\bar{\Lambda}, \Gamma^m[(\bar{\varepsilon}\Gamma_m\Lambda), \Lambda]) = f^{abc}\bar{\Lambda}_c\Gamma^m(\bar{\varepsilon}\Gamma_m\Lambda_a)\Lambda_b = 0. \tag{7.126}$$

provided that $\Gamma_{11}\Lambda = \Lambda$ and $\Gamma_{11}\varepsilon = \varepsilon$.

Problem: Show this. You may assume the Fierz transformation in 10 dimensions is (why?)

$$\begin{aligned} (\bar{\chi}\psi)\lambda = & -\frac{1}{32}\Big[(\bar{\chi}\lambda)\psi + (\bar{\chi}\Gamma_{11}\lambda)\Gamma^{11}\psi + (\bar{\chi}\Gamma_m\lambda)\Gamma^m\psi - \frac{1}{2!}(\bar{\chi}\Gamma_{mn}\lambda)\Gamma^{mn}\psi \\ & -(\bar{\chi}\Gamma_m\Gamma_{11}\lambda)\Gamma^m\Gamma_{11}\psi - \frac{1}{3!}(\bar{\chi}\Gamma_{mnp}\lambda)\Gamma^{mnp}\psi - \frac{1}{2!}(\bar{\chi}\Gamma_{mn}\Gamma_{11}\lambda)\Gamma^{mn}\Gamma_{11}\psi \\ & +\frac{1}{4!}(\bar{\chi}\Gamma_{mnpq}\lambda)\Gamma^{mnpq}\psi + \frac{1}{4!}(\bar{\chi}\Gamma_{mnp}\Gamma_{11}\lambda)\Gamma^{mnp}\Gamma_{11}\psi + \frac{1}{5!}(\bar{\chi}\Gamma_{mnpqr}\lambda)\Gamma^{mnpqr}\psi \\ & +\frac{1}{4!}(\bar{\chi}\Gamma_{mnpq}\Gamma_{11}\lambda)\Gamma^{mnpq}\Gamma_{11}\psi\Big] \end{aligned} \tag{7.127}$$

Thus the 10-dimensional action is supersymmetric. In fact we should also check that the supersymmetry closes on-shell. The calculation for the gauge fields is just as it was in 4-dimensions. For the Fermions we again need to use the Fierz transformation. This introduces several more terms but nevertheless it all works out (this is to be expected as the Lagrangian is invariant, hence what ever the supersymmetry algebra closes into must be a symmetry of the Lagrangian too).

Problem: Show that in ten-dimensions, with $\Gamma_{11}\Lambda = \Lambda$, the transformations close on-shell on Λ.

Our next task is to dimensionally reduce this theory to 4 dimensions. All this means is that we simply imagine that there is no motion along the $x^4, x^5, ..., x^9$ directions. This is related to the idea of compactification except that we don't imagine there is an infinite tower of Kaluza-Klein states. We are just using this as a trick to obtain a theory in 4 dimensions with $N = 4$ supersymmetry.

Let us consider the Bosons. We have the 10-dimensional adjoint-valued gauge vector field A_m. From the 4-dimensional point of view this can be viewed as a vector gauge field A_μ, $\mu = 0, 1, 2, 3$ along with 6 scalar adjoint-valued fields $\phi_A = A_A$, $A = 4, 5, .., 9$. We note that if we assume that there are no derivatives then under a gauge transformation (that only depends on x^μ) we have

$$A'_\mu = -i\partial_\mu g g^{-1} + gA_\mu g \qquad \phi'_A = g\phi g^{-1} \tag{7.128}$$

Thus indeed the components $\phi_A = A_A$ behave as scalar fields from the 4-dimensional point of view. In addition the field strength reduces to

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \\ F_{\mu A} &= \partial_\mu \phi_A - i[A_\mu, \phi_A] = D_\mu \phi_A \\ F_{AB} &= -i[\phi_A, \phi_B] \end{aligned} \tag{7.129}$$

The 10-dimensional kinetic term can be written as

$$\frac{1}{4}\mathrm{Tr}(F_{mn}, F^{mn}) = \frac{1}{4}\mathrm{Tr}(F_{\mu\nu}, F^{\mu\nu}) + \frac{1}{2}\sum_A \mathrm{Tr}(D_\mu\phi_A, D^\mu\phi_A) - \frac{1}{4}\sum_{A,B}\mathrm{Tr}([\phi_A, \phi_B], [\phi_A, \phi_B]) \tag{7.130}$$

Thus the Bosonic part of the action reduces that of a gauge field and six adjoint-valued scalars in 4 dimensions, along with the potential

$$V = -\frac{1}{4}\sum_{A,B}\mathrm{Tr}([\phi_A, \phi_B], [\phi_A, \phi_B]) \ . \tag{7.131}$$

Next we need to look at the Fermions. We can write the Fermionic term as

$$\frac{i}{2}\mathrm{Tr}(\bar{\Lambda}, \Gamma^m D_m \Lambda) = \frac{i}{2}\mathrm{Tr}(\bar{\Lambda}, \Gamma^\mu D_\mu \Lambda) + \frac{1}{2}\mathrm{Tr}(\bar{\Lambda}, \Gamma^A[\phi_A, \Lambda]) \tag{7.132}$$

The second term is a Yukawa-type term in 4 dimensions. The 4-dimensional action of $N = 4$ super-Yang-Mills is

$$
\begin{aligned}
S_{N=4SYM} &= -\frac{1}{g^2}\int d^4x \frac{1}{4}\mathrm{Tr}(F_{\mu\nu}, F^{\mu\nu}) + \frac{1}{2}\sum_A \mathrm{Tr}(D_\mu\phi_A, D^\mu\phi_A) + \frac{i}{2}\mathrm{Tr}(\bar{\Lambda}, \Gamma^\mu D_\mu\Lambda) \\
&\quad + \frac{1}{2}\mathrm{Tr}(\bar{\Lambda}, \Gamma^A[\phi_A, \Lambda]) - \frac{1}{4}\sum_{A,B}\mathrm{Tr}([\phi_A, \phi_B], [\phi_A, \phi_B]) \qquad (7.133)
\end{aligned}
$$

In principle we are done. However it is a good idea to rewrite the Fermion Λ in terms of 4-dimensional spinors.

To reduce the Fermions we decompose the 10-dimensional Clifford algebra in terms of the 4-dimensional γ_μ's as

$$
\begin{aligned}
\Gamma_\mu &= \gamma_\mu \otimes 1 \\
\Gamma_A &= \gamma_5 \otimes \rho_A \qquad (7.134)
\end{aligned}
$$

Here ρ_A are a Euclidean Clifford algebra in 6-dimensions which we take to be pure imaginary so that Γ_m are Majorana (recall that γ_5 is pure imaginary in a four-dimensional Majorana basis). This is indeed possible and is called a pseudo-Majorana representation. For example we could take

$$
\begin{aligned}
\rho_1 &= 1\otimes 1\otimes \sigma_2 \qquad \rho_2 = 1\otimes\sigma_2\otimes\sigma_3 \\
\rho_3 &= \sigma_2\otimes\sigma_3\otimes\sigma_3 \qquad \rho_4 = \sigma_2\otimes\sigma_1\otimes\sigma_3 \qquad (7.135) \\
\rho_5 &= \sigma_3\otimes\sigma_2\otimes\sigma_1 \qquad \rho_6 = \sigma_2\otimes 1\otimes\sigma_1.
\end{aligned}
$$

Since σ_2 is pure imaginary and σ_1 and σ_3 are real we have a pseudo-Majorana representation of 8×8 matrices.

Similarly we decompose spinors as

$$
\begin{aligned}
\Lambda &= \lambda_I \otimes \eta^I \qquad (7.136) \\
\varepsilon &= \epsilon_I \otimes \eta^I
\end{aligned}
$$

where η^I are a basis of spinors in six-dimensions (which are 8-dimensional). However we note that we require Λ and ε to be chiral with respect to $\Gamma_{11} = -i\gamma_5\otimes\rho^1...\rho^6$. Thus the chirality of η^I with respect to $i\rho^1...\rho^6$ needs to be correlated with the chirality of λ_I with respect to γ_5. This projects out half of the six-dimensional spinors and so there are only four independent values of I. To see this we note that since the η_I are a basis we can write

$$-i\rho_1...\rho_6\eta_I = R_I{}^J\eta_J$$

for some pure imaginary 8×8 matrix $R_I{}^J$. Thus the chirality constraint becomes

$$\lambda^I = \gamma_5 R_J{}^I\lambda^J$$

A similar constraint applies to ϵ^I too. This means that there are no longer 8 independent λ^I and ϵ^I but rather just 4. Finally we assume that these are normalized to $(\eta^I)^T\eta^J = \delta^{IJ}$.

We can now compute

$$
\begin{aligned}
\bar{\Lambda}\Gamma^\mu D_\mu\Lambda &= \lambda_I^T \otimes (\eta^I)^T (C\gamma^\mu \otimes 1) D_\mu \lambda_J \otimes \eta_J \\
&= \delta^{IJ}\bar{\lambda}_I \gamma^\mu D_\mu \lambda_J
\end{aligned} \tag{7.137}
$$

and

$$
\begin{aligned}
\bar{\Lambda}\Gamma_A[\phi^A, \Lambda] &= \lambda_I^T \otimes (\eta^I)^T (C\gamma^5 \otimes \rho^A)[\phi^A, \lambda_J \otimes \eta^J] \\
&= (\eta^I)^T \rho_A \eta^J \bar{\lambda}_I \gamma_5 [\phi^A, \lambda_J] \\
&= \rho_A^{IJ} \bar{\lambda}_I \gamma_5 [\phi^A, \lambda_J]
\end{aligned} \tag{7.138}
$$

where $\rho_A^{IJ} = (\eta^I)^T \rho_A \eta^J$ are the chiral-chiral matrix components of ρ_A. Thus the action can be written as

$$
\begin{aligned}
S_{N=4SYM} = & -\frac{1}{g^2}\int d^4x \frac{1}{4}\mathrm{Tr}(F_{\mu\nu}, F^{\mu\nu}) + \frac{1}{2}\sum_A \mathrm{Tr}(D_\mu\phi_A, D^\mu\phi_A) + \frac{i}{2}\delta^{IJ}\mathrm{Tr}(\bar{\lambda}_I, \gamma^\mu D_\mu \lambda_J) \\
& + \frac{1}{2}\rho_A^{IJ}\mathrm{Tr}(\bar{\lambda}_I\gamma_5[\phi^A, \lambda_J]) - \frac{1}{4}\sum_{A,B}\mathrm{Tr}([\phi_A, \phi_B], [\phi_A, \phi_B])
\end{aligned} \tag{7.139}
$$

Note that we have raised and lowed IJ indices freely with δ^{IJ} and δ_{IJ}.

Our last task is to write the supersymmetry transformations in terms of 4-dimensional spinors.

Problem: Show that the ten-dimensional supersymmetry

$$
\begin{aligned}
\delta A_m &= i\bar{\varepsilon}\Gamma_m \Lambda \\
\delta \Lambda &= -\frac{1}{2}F_{mn}\Gamma^{mn}\varepsilon\ .
\end{aligned} \tag{7.140}
$$

becomes

$$
\begin{aligned}
\delta A_\mu &= i\bar{\epsilon}_I \Gamma_\mu \lambda^I \\
\delta\phi_A &= -\bar{\epsilon}_I \gamma_5 \lambda_J \rho_A^{IJ} \\
\delta\lambda^I &= -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon^I - \gamma^\mu\gamma_5 D_\mu \phi^A \rho_A^{IJ}\epsilon_J + \frac{i}{2}[\phi_A, \phi_B]\rho_{AB}^{JI}\epsilon_I\ .
\end{aligned} \tag{7.141}
$$

where $\rho_{AB}^{JI} = (\eta^J)^T \rho_{AB} \eta^I$. Here we see that there are indeed 4 supersymmetry parameters ϵ_I.

Thus we find a theory in 4 dimensions with one vector field (spins $= \pm 1$) 4 Fermions (spin $= \pm 1/2$) and 6 scalars (spin 0). This is what we expect from our previous discussion. In the fixed (massless) momentum frame there are $4 \times 4 = 16$ supersymmetry generators:

$$
\{Q_\alpha^I, Q_\beta^J\} = 2E(1 - \gamma_{01})_{\alpha\beta}\delta^{IJ}
$$

However since γ_{01} can be diagonalized to the form $\mathrm{diag}(1, 1, -1, -1)$ we see that we can find 8 linear combinations of the Q_α^I, which we denote by $Q_{\ddot{\alpha}}$ that satisfy $\{Q_{\ddot{\alpha}}, Q_{\ddot{\beta}}\} = 0$.

These must act trivially $Q_{\ddot{\alpha}}|state\rangle = 0$. Thus there are 8 nontrivial $Q's$ that we denote by $Q_{\dot{\alpha}}$, $\dot{\alpha} = 1, ..., 8$ and satisfy

$$\{Q_{\dot{\alpha}}, Q_{\dot{\beta}}\} = 4E\delta_{\dot{\alpha}\dot{\beta}}$$

If we rewrite these as

$$\mathcal{Q}_1 = Q_1 + iQ_2 \qquad \mathcal{Q}_2 = Q_3 + iQ_4 \qquad \mathcal{Q}_3 = Q_5 + iQ_6 \qquad \mathcal{Q}_4 = Q_7 + iQ_8$$

then the algebra becomes

$$\{\mathcal{Q}_I, \mathcal{Q}_J\} = \{\mathcal{Q}_I^\dagger, \mathcal{Q}_J^\dagger\} = 0 \qquad \{\mathcal{Q}_I, \mathcal{Q}_J^\dagger\} = 4E\delta_{IJ}$$

This is four copies of the algebra of Fermionic harmonic oscillator with creation and annihilation operators. To construct the represention we start with a highest spin state $|s>$. The $\mathcal{Q}_I$ lower the spin by 1/2 whereas the $\mathcal{Q}_I^\dagger$ raise it. Thus $\mathcal{Q}_I^\dagger|s>= 0$. Thus the states and their helicites are obtained by acting on $|s>$ with $\mathcal{Q}_I$:

$$s \qquad |s> \tag{7.142}$$
$$s - 1/2 \qquad \mathcal{Q}_I|s> \tag{7.143}$$
$$s - 1 \qquad \mathcal{Q}_I\mathcal{Q}_J|s> \tag{7.144}$$
$$s - 3/2 \qquad \mathcal{Q}_I\mathcal{Q}_J\mathcal{Q}_K|s> \tag{7.145}$$
$$s - 2 \qquad \mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3\mathcal{Q}_4|s> \tag{7.146}$$

Note that since the $\mathcal{Q}_I$ anticommute these states must be antisymmetric in I, J, K. Thus there are $1, 4, 6, 4, 1$ states in each row respectively leading to $2^4 = 16$ states. Note that we require $|s| \leq 1$ to remain in field theory without gravity. Therefore we see that we must have precisely $s = 1$ in order that the lowest state has $s \geq -1$. We then find a vector (with states $|1>$ and $|-1>$), 6 scalars (states $|0>$) and 4 Fermions (states $|1/2>$).

8 Physical Features of Yang-Mills Theories

8.1 Vacuum Moduli Space

A key feature of supersymmetric theories, especially those with extended supersymmetry, is that the vacua are not isolated but rather one finds a continuous family of connected vacua. This set is called the vacuum moduli space. More specifically it is the set of all zero-energy field configurations, modulo gauge transformations.

As an example we will look at $N = 4$ Super-Yang-Mills. Zero-energy states are invariant under all the supersymmetries and hence also translations. Thus only the scalars ϕ_A can be non-vanishing (but constant). Setting the potential to zero we see that

$$[\phi_A, \phi_B] = 0 \tag{8.147}$$

for all pairs A, B. Note also that this leads to $\delta A_\mu = \delta\phi_A = \delta\lambda_I = 0$ for any ϵ_I. Hence these configurations are invariant under all the supersymmetries.

The solutions to (8.147) are found by taking the ϕ_A to be mutually commuting elements in $L(G)$. This space is known as the Cartan subalgebra.

For example we can consider $G = SU(N)$. If we write an element of $SU(N)$ as $g = e^{iA}$ then $g^{-1} = e^{-iA}$ and $g^\dagger = e^{-iA^\dagger}$. Thus $g^{-1} = g^\dagger$ implies $A^\dagger = A$. Furthermore the restriction to $\det(g) = 1$ gives $\mathrm{tr}(A) = 0$[7]. So $L(SU(N))$ is the space of all Hermitian traceless $N \times N$ complex matrices. To count the number of linearly independent elements of $L(SU(N))$ we can consider diagonal elements which must therefore be real and have the sum of the diagonal entries equal to zero. This gives $N-1$ independent elements. Because $A^\dagger = A$ we see that the lower-triangular elements of A are just the complex conjugates of the upper-triangular elements. Thus there are $N(N-1)/2$ complex off-diagonal elements giving $N(N-1)$ real elements. So we find the total dimension is $N - 1 + N(N-1) = N^2 - 1$.

What is the Cartan subalgebra of $L(SU(N))$? Mutually commuting matrices can be made to be all diagonal by a similarity transform (in this case this corresponds to a gauge transformation). Thus the Cartan subalgebra is spanned by real, traceless diagonal matrices and has dimension is $N-1$.

However we must consider the space of zero-energy states modulo gauge transformations. As we mentioned we can use gauge transformation $\phi_A \to g\phi_A g^{-1}$ to ensure that all the ϕ_A are diagonal. Let us write

$$\phi_A = v_A^R H_R \ , \tag{8.148}$$

where H_R, $R = 1, ..., \mathrm{rank}(G)$ span the Cartan subalgebra. Let us look at gauge transformations that leave this form of ϕ_A invariant. If we exponentiate the Cartan subalgebra then we obtain a subgroup of G generated by elements of the form $g = e^{i\theta^R H_R}$. However this is clearly just the Abelian group $U(1)^{\mathrm{rank}(G)}$ and is adjoint action on ϕ_A is trivial:

$$\phi'_A = g\phi_A g^{-1} = \phi_A \ . \tag{8.149}$$

Acting with a general element of G will not preserve the form (8.148). However there are typically some discrete elements which do preserve (8.148) under the adjoint action. These form a finite subgroup $\mathcal{W}(G)$ of G called the Weyl group (this is not the usual definition of the Weyl group but it is equivalent for our purposes). Thus the vacuum moduli space of $N = 4$, four-dimensional super-Yang-Mills is the space (8.148), *i.e.* $(\mathbf{R}^6)^{\mathrm{rank}(G)}$ modulo the Weyl group $\mathcal{W}(G)$.

Let us look closely at $SU(2)$. $L(SU(2))$ is spanned by the Pauli matrices. Without loss of generality we can take the Cartan subalgebra to be generated by σ_3. Thus (8.148) is simply

$$\phi_A = v_A \sigma_3 \tag{8.150}$$

with v_A parameterizing a point in $\mathbf{R}^6$. However there are further gauge transformations that preserve the form of ϕ_A.

[7] Here we have used the formula $\det e^X = e^{\mathrm{tr}X}$.

Problem: Show that given any element $g = e^{i\theta^R \sigma_R} \in SU(2)$ which preserves $\phi_A = v_A\sigma_3$, *i.e.* $\phi'_A = g v_A \sigma_3 g^\dagger = v'_A\sigma_3$ then $v'_A = \pm v_A$.

Thus the Weyl group of $SU(2)$ is $\mathbf{Z}_2$ and the vacuum moduli space is

$$\mathcal{M}_{N=2} = \mathbf{R}^6/\mathbf{Z_2} \ . \tag{8.151}$$

For $SU(N)$ the Weyl group is the symmetric group on N elements and we find

$$\mathcal{M}_N = \mathbf{R}^{6(N-1)}/S_N \tag{8.152}$$

Thus the vacuum moduli space is somewhat subtle due to global gauge transformations. I say global here because the Weyl group is not obtained by looking at elements of G that are infinitesimally close to the identity.

Clearly one can consider other theories with more general potentials and scalar fields in various representations. The vacuum moduli space will then of course be rather different.

8.2 Global Symmetries and Spontaneous Symmetry Breaking

We have constructed actions which are invariant under arbitrary local symmetries. However it is important to note that gauge symmetries are not really symmetries. They don't relate two physically distinct field configurations. Rather two field configurations that differ by a gauge transformation should be viewed as physically equivalent.

The actions we have constructed clearly also admit global symmetries where the gauge transformation is constant (and hence does not vanish at infinity). However the vacuum need not respect these symmetries. In particular, suppose that we have theory with a potential $V(\phi)$. Suppose that that ϕ take values in some representation of the group G (which in this subsection we just take to be a global symmetry). For example we could have

$$V(\phi) = \lambda(|\phi|^2 - a^2)^2 \tag{8.153}$$

where ϕ is in some representation of the gauge group (symmetry group) and $|\phi|^2$ is a gauge invariant norm. The point of this expression is that although the potential and indeed the whole theory is invariant under global symmetry transformations, any given vacuum solution (satisfying $|\phi| = a$) is not.

Thus the vacuum breaks the symmetry. Furthermore the low energy fluctuations about such a vacuum also break the symmetry. The symmetry is said to be spontaneously broken. Although the full theory has a symmetry this symmetry is broken by the vacuum. There is an important theorem for spontaneously broken symmetry

Goldstone Theorem In a theory with symmetry group G in a vacuum that only preserves a subgroup $H \subset G$ then there will be $\dim(G) - \dim(H)$ massless fields. These massless fields are called Goldstone Bosons.

To prove this we note that the potential is invariant under $\delta\phi^a = i\eta^r (T_r)^a{}_b\phi^b$ and hence

$$0 = \frac{\partial V}{\partial\phi^a}\delta\phi^a = i\eta^r (T_r)^a{}_b \frac{\partial V}{\partial\phi^a}\phi^b \tag{8.154}$$

We can drop the parameter $i\eta^r$ and differentiate again

$$0 = \frac{\partial^2 V}{\partial\phi^a\partial\phi^c}(T_r)^a{}_b\phi^b + \frac{\partial V}{\partial\phi^a}(T_r)^a{}_c \tag{8.155}$$

Now if we expand V about a vacuum state $\phi^a = v^a$ where $\partial V/\partial\phi^a = 0$ then we see that

$$\frac{\partial^2 V}{\partial\phi^a\partial\phi^c}(T_r)^a{}_b v^b = 0 \tag{8.156}$$

Now $\partial^2 V/\partial\phi^a\partial\phi^c = (M^2)_{ac}$ is the mass-matrix of small fluctuations about the vacuum (it is positive if the vacuum is a minimum). Thus

$$\delta\phi^a = iG^r (T_r)^a{}_b v^b \tag{8.157}$$

is a zero mode of the mass matrix for any parameter G^r. Since $r = 1...\dim(G)$ there are in principle $\dim(G)$ such modes. However if the vacuum state is preserved by a subgroup $H \subset G$ then $(T_r)^a{}_b v^b = 0$ for $\dim(H)$ values of r. Thus we find $\dim(G) - \dim(H)$ zero-modes. Each of these corresponds to a massless particle.

For example consider $N = 4$ super-Yang-Mills with gauge group $SU(2)$. However let us forget the fact that it is a gauge theory for the moment. We will reconsider the gauge symmetry in the next section. In particular we could just consider the scalar sector the theory:

$$\mathcal{L} = -\frac{1}{2}\sum_A \mathrm{tr}(\partial_\mu\phi_A\partial^\mu\phi_A) + \frac{1}{4}\sum_{AB}\mathrm{tr}([\phi_A,\phi_B]^2)$$

This theory has a global symmetry $SU(2)\times SO(6)$, where the $SO(6)$ comes from rotating the scalar fields into each other and the $SU(2)$ is what's left of the gauge symmetry without the gauge fields (so the generators cannot depend on spacetime). We saw that the vacuum corresponds to $[v_A, v_B] = 0$ however the diagonal $U(1) \subset SU(2)$ acts trivially on the scalars since they are in the adjoint (in other words the symmetry group of the vacuum is just $SO(5)$ and not $U(1)\times SO(5)$). Furthermore a non-zero vacuum expectation value for v_A breaks $SO(6)$ to $SO(5)$ (for example we assume that only $v_1 \neq 0$). Thus we expect $\dim(SU(2)\times SO(6)) - \dim(SO(5)) = (3 + 15) - (10) = 8$ massless Goldstone Bosons.

Problem: Write the scalar fields as traceless, Hermitian matrices and expand them about the vacuum as

$$\phi_A = \frac{1}{\sqrt{2}}\begin{pmatrix} m_A & 0 \\ 0 & -m_A \end{pmatrix} + \begin{pmatrix} \phi_A^1 & \phi_A^2 + i\phi_A^3 \\ \phi_A^2 - i\phi_A^3 & -\phi_A^1 \end{pmatrix} \tag{8.158}$$

with only $m_1 \neq 0$. Show that there is a mass for ϕ_A^2 and ϕ_A^3, $A \neq 1$ but no mass for ϕ_1^2 and ϕ_1^3 and ϕ_A^1, $A = 1, 2, 3, 4, 5, 6$.

This gives us the 8 massless Goldstone Bosons.

8.3 The Higg's Mechanism

Let us now return to gauge theories and for concreteness consider (the Bosonic part of) $N = 4$ super-Yang-Mills. Here there is an important effect. Notice that in a gauge theory there is no room to add a mass term of the gauge fields as

$$m^2 \mathrm{Tr}(A_\mu, A^\mu) \tag{8.159}$$

is not be gauge invariant. Since the only massless gauge field observed in Nature is the photon of Electromagnetism the role of Yang-Mills theories was not clear for a long time. It wasn't really until is was proven that the only renormalizable four-dimensional quantum field theories as Yang-Mills theories with some matter content that their role was solidified (actually another feature was asymptotic freedom).

But let us look at a generic,*i.e.* $v_A^R \neq 0$, point in the vacuum moduli space of $N = 4$ super-Yang-Mills. The covariant derivative of ϕ_A contains a term, assuming $\partial_\mu \phi_A = 0$;

$$D_\mu \phi_A = -i[A_\mu, \phi] = -iv_A^R[A_\mu, H_R] + \ldots \tag{8.160}$$

where the ellipses denote fluctuations of the ϕ_A away from its vacuum expectation value. Since the action contains a $(D_\mu \phi_A)^2$ term we see that there is an effective mass for A_μ of the form

$$-v_A^R v_A^S A_\mu^s A^{t\mu} \mathrm{Tr}([H_R, T_s], [H_S, T_t]) \tag{8.161}$$

This gives a mass of order v_A^R to the components of A_μ^r that are not in the Cartan subalgebra. Thus in a generic vacuum most of the gauge fields are massive (most because there are always more generators that are not in the Cartan subalgebra than those that are in it).

We've used the word generic here because it could be that some of the v_A are vanishing. This would then lead to massless gauge Bosons and an enhancement of the symmetry group of the vacuum.

This is an example of the famous Higg's mechanism. More generally suppose we have some scalar fields ϕ^a in an arbitrary representation of the gauge group and some gauge invariant potential $V(|\phi|)$. As we have seen the vacuum can break the global symmetry if there is a non-zero vacuum expectation value v^a for ϕ^a. However in a gauge theory there is always a coupling coming from the covariant derivative

$$D_\mu \phi^a = -iA_\mu^r (T_r)^a{}_b v^b + \ldots \tag{8.162}$$

where the ellipsis denotes fluctuations of ϕ^a away from the vacuum value v^a. Again we see that this gives a mass term for the gauge fields:

$$A_\mu^r A^{r\mu} (T_r)^a{}_b (T_r)^a{}_c v^b v^c \tag{8.163}$$

Note that the gauge fields which are massive are those that correspond to non-trivial Goldstone Bosons, *i.e.* to directions where $(T_r)^a{}_b v^b \neq 0$. The gauge symmetries preserved by the vacuum remain massless.

For example in $N = 4$ super-Yang-Mills the gauge fields corresponding to the $U(1)^{\text{rank}(G)} \subset G$ symmetry group of the vacuum are still massless. However in theories with other matter content it is possible to break the gauge group to smaller subgroups.

You might complain that a massive vector field has three and not two degrees of freedom in four-dimensions (the gauge choice can only remove one massive degree of freedom as opposed to two massless ones). So where does this extra degrees of freedom come from to make the vector field massive? The answer is that the gauge fields 'eat' the massless Goldstone mode associated to breaking the symmetry generated by the non-flat directions. Here 'eat' means that one can show that the associated Goldstone mode can be gauged away.

To see this we note that the Goldstone fields $G^r(x)$ are related to the scalars ϕ^a through (see (8.157))

$$\phi^a(x) = v^a + iG^r(x)(T_r)^a{}_b v^b$$

where we have now introduced a spacetime dependence on the parameters G^r. For simplicity let us suppose that G^r is a infinitesimal perturbation about the vacuum solution $\phi^a = v^a$. We can then consider an infinitesimal gauge transformation:

$$\delta\phi^a = i\theta^r(T_r)^a{}_b\phi^b = i\theta^r(T_r)^a{}_b v^b + \dots$$

where the ellipsis denotes higher order terms. Here we see that we can indeed set the gauge parameters $\theta^r = -G^r$ to 'gauge away' the Goldstone fields G^r. Thus in a particular gauge there are no Goldstone Bosons. However, as we have seen, The vector gauge fields associated to the same directions in the Lie algebra have become massive.

8.4 BPS Monopoles

Finally let us return to the issue of BPS states in the particular example of $N = 4$ Super-Yang-Mills. We saw above there there are special states where the mass saturates a bound determined by the central charges. These states are identified by the fact that they preserve a fraction of the supersymmetry. This means that they solve $\delta A_\mu = \delta\phi_A = \delta\lambda_I = 0$ for some ϵ_I.

We look for classical field configurations where $\lambda_I = 0$. It follows that $\delta A_\mu = \delta\phi_A = 0$. Thus we need to solve

$$-\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon^I - \gamma^\mu\gamma_5 D_\mu\phi^A\rho_A^{IJ}\epsilon_J + \frac{i}{2}[\phi_A,\phi_B]\rho_{AB}^{IJ}\epsilon_J = 0 \tag{8.164}$$

We will consider static solutions with $F_{0i} = D_0\phi_A = 0$ and write $F_{ij} = \varepsilon_{ijk}B^k$, in analogy with the magnetic field in electromagnetism. Let us also assume that only one ϕ_A, say ϕ_1 is non-zero so that $[\phi_A,\phi_B] = 0$. This equation reduces to

$$-\frac{1}{2}\epsilon_{ijk}B^k\gamma^{ij}\epsilon^I = \gamma^i\gamma_5 D_i\phi^1\rho_1^{IJ}\epsilon_J \tag{8.165}$$

Next we note that $\gamma^{ij} = i\epsilon^{ijk}\gamma_k\gamma_0\gamma_5$ so that

$$-iB^k\gamma_k\gamma_0\gamma_5\epsilon^I = \gamma^k\gamma_5 D_k\phi^1\rho_1^{IJ}\epsilon_J \tag{8.166}$$

If we set

$$B_k = \pm D_k\phi_1 \tag{8.167}$$

then our equation reduces to

$$\pm i\gamma_0\rho_1^{IJ}\epsilon_J = \epsilon_I \tag{8.168}$$

This is a projection onto a subset of all the ϵ_I. In particular recall that

$$\gamma_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{8.169}$$

Thus is we write

$$\epsilon_I = \begin{pmatrix} \epsilon_{1I} \\ \epsilon_{2I} \\ \epsilon_{3I} \\ \epsilon_{4I} \end{pmatrix} \tag{8.170}$$

then we find

$$\begin{aligned} \epsilon_{1I} &= \pm i\rho_1^{IJ}\epsilon_{2J} \qquad \epsilon_{2I} = \mp i\rho_1^{IJ}\epsilon_{1J} \\ \epsilon_{3I} &= \pm i\rho_1^{IJ}\epsilon_{4J} \qquad \epsilon_{4I} = \mp i\rho_1^{IJ}\epsilon_{3J} \end{aligned} \tag{8.171}$$

These equations determine ϵ_{2I} and ϵ_{4I} in terms of ϵ_{1I} and ϵ_{3I}. Thus half of the supersymmetries are preserved.

There is a second way to arrive at the same equations, without discussing supersymmetry due to Bogomol'nyi. Let us look for static, Bosonic solutions which minimize the energy (since we are assuming no time-dependence this means the Classical equations of motion will be satisfied). We may choice a so-called Coulomb gauge where $A_0 = 0$ and hence $D_0 = 0$. The total energy can be written as (again assuming $E_i = 0$)

$$\begin{aligned} E &= \frac{1}{g_{YM}^2}\int d^3x \frac{1}{4}\mathrm{tr}(F_{ij}F^{ij}) + \frac{1}{2}\mathrm{tr}(D_i\phi_A D^i\phi_A) \\ &= \frac{1}{g_{YM}^2}\int d^3x \frac{1}{2}\mathrm{tr}(B_iB^i) + \frac{1}{2}\mathrm{tr}(D_i\phi_A D^i\phi_A) \\ &= \frac{1}{g_{YM}^2}\int d^3x \Big(\frac{1}{2}\mathrm{tr}((B_i \mp D_i\phi_1)(B^i \mp D_i\phi_1)) + \frac{1}{2}\sum_{A=2}^{6}\mathrm{tr}(D_i\phi_A D^i\phi_A) \\ &\qquad \pm\mathrm{tr}(B^iD_i\phi_1)\Big) \end{aligned} \tag{8.172}$$

We see that the terms on the first line are squares and hence positive definite. The final term is in fact a total derivative:

$$\mathrm{tr}(B^iD_i\phi_1) = \partial_i\mathrm{tr}(B^i\phi_1) \tag{8.173}$$

Problem: Show this!

Since the remaining terms are sums of squares we find a bound on the energy

$$\begin{aligned} E &\geq \pm\frac{1}{g_{YM}^2}\int d^3x \mathrm{tr}(B^i D_i\phi_1) \\ &= \pm\frac{1}{g_{YM}^2}\int d^3x\partial_i \mathrm{tr}(B^i\phi_1) \\ &= \pm\frac{1}{g_{YM}^2}\int_{S^2_\infty} r^2\sin\theta d\theta d\varphi\ \mathrm{tr}(B_r\phi_1) \end{aligned} \tag{8.174}$$

where n^i is a unit normal vector to the boundary at infinity. This is called the Bogomol'nyi bound. Furthermore we see that the bound is saturated whenever

$$B_i = \pm D_i\phi_1 \qquad D_i\phi_A = 0\ (A\neq 1) \tag{8.175}$$

These are precisely the equations we previously derived. We see that they are the minimum energy solutions for a fixed boundary condition. In particular at infinity the fields will behave as

$$\phi_1^a = v_1^a + \ldots \qquad B_r^a = \frac{Q_M^a}{4\pi r^2} + \ldots \tag{8.176}$$

so that the bound becomes

$$E \geq \pm\frac{v_1^a}{4\pi g_{YM}^2}\int_{S^2_\infty}\sin\theta d\theta d\varphi\ Q_M^a = \left|\frac{v_1^a Q_M^a}{g_{YM}^2}\right| \tag{8.177}$$

Note that since $B_i = \pm D_i\phi_1$ the right-hand-side of (8.174) is positive definite and we can therefore put an absolute value sign. This states that the mass of a state is bounded below by its magnetic charge (and is proportional to $1/g_{YM}^2$, *i.e.* is very large at weak coupling).

Note that there is one final condition to be satisfied. Since we have been dealing directly with the field strength and not the gauge fields we must ensure that the Bianchi identity is satisfied:

$$D_{[i}F_{jk]} = 0 \tag{8.178}$$

(recall that we have set $D_0 = 0$). Contracting this with ϵ^{ijk} gives

$$D^iB_i = 0 \tag{8.179}$$

Thus we must have

$$D^iD_i\phi_1 = 0. \tag{8.180}$$

The exact solutions to these equations are rather difficult to describe (although there is a complete analytic solution available in principle). They describe magnetic monopole states in a non-Abelian gauge theory.

We have seen that these states preserve half of the supersymmetries and therefore fall into short multiplets. That is to say there are additional states in the theory which, along with the monopole, fill out a short multiplet. These other solutions can be obtained by acting on the monopole solution with the broken supersymmetries (so they correspond to turning on the Fermions). It is not hard to show that the multiplet is that of a vector multiplet with highest spin 1. Indeed one essentially just repeats the discussion at the end of section 7. However there is one change. Rather than restricting to highest spin 1 because of not wanting to include gravity (this isn't needed if the states are massive) we require that since the theory is CPT invariant the soliton specturm must be CPT self-conjugate. Thus the highest spin of the monopole multiplet is minus the lowest spin. This fixes $s = 1$. Thus the monopole spectrum is the same as the original perturbative spectrum of the field theory (in $N = 4$ Super-Yang-Mills). These states are conjectured to be the strong coupling duals to the perturabative states.

Appendix A: Conventions

In these notes we are generally in 4 spacetime dimensions labeled by x^μ, $\mu = 0, 1, 2, ..., 3$. When we only want to talk about the spatial components we use x^i with $i = 1, ..., 3$. We use the the "mostly plus" convention for the metric:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{8.1}$$

Spinor indices will in general be denoted by $\alpha, \beta = 1, ..., 4$. When we talk about Weyl spinors we will use the spinor indices $a, \dot{a} = 1, 2$. We will briefly talk about more general D dimensions. In this case $\mu = 0, 1, 2, ..., D - 1$ and $\alpha, \beta = 1, ..., 2^{[D/2]}$.

We also assume, according to the spin-statistics theorem, that spinorial quantities and fields are Grassmann variables, *i.e.* anti-commuting. We will typically use Greek symbols for Fermionic Grassmann fields, $\psi, \lambda, ...$ and Roman symbols for Bosonic c-number fields.

Appendix B: The Fierz Transformation

The γ-matrices have several nice properties. Out of them one can construct the additional matrices

$$1, \gamma_\mu, \gamma_\mu\gamma_{D+1}, \gamma_{\mu\nu}, \gamma_{\mu\nu}\gamma_{D+1}, ... \tag{8.1}$$

where $\gamma_{\mu\nu\lambda...}$ is the anti-symmetric product over the given indices with weight one, *e.g.*

$$\gamma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \tag{8.2}$$

Because of the relation $\gamma_0\gamma_1...\gamma_{D-1} \propto \gamma_{D+1}$ not all of these matrices are independent. The list stops when the number of indices is bigger than $D/2$. It is easy to convince yourself that the remaining ones are linearly independent.

Problem: Using the fact that $< M_1, M_2 >= \mathrm{Tr}(M_1^\dagger M_2)$ defines a complex inner product, convince yourself that the set (8.1), where the number of spacetime indices is no bigger than $D/2$, is a basis for the space of $2^{[D]/2} \times 2^{[D]/2}$ matrices ($[D/2]$ is the integer part of $D/2$).

Thus any matrix can be written in terms of γ-matrices. In particular one can express

$$\delta_\alpha^\beta \delta_\gamma^\delta = \sum_{\Gamma\Gamma'} c_{\Gamma\Gamma'} (\gamma_\Gamma)_\gamma^{\ \beta} (\gamma_{\Gamma'})_\alpha^{\ \delta} \tag{8.3}$$

for some constants $c_{\Gamma\Gamma'}$. Here Γ and Γ' are used as a indices that range over all independent γ-matrix products in (8.1).

To proceed one must determine the coefficients $c_{\Gamma\Gamma'}$. To do this we simply multiply (8.3) by $(\gamma_{\Gamma''})_\beta^{\ \gamma}$ which gives

$$(\gamma_{\Gamma''})_\alpha^{\ \delta} = \sum_{\Gamma\Gamma'} c_{\Gamma\Gamma'} \mathrm{Tr}(\gamma_\Gamma \gamma_{\Gamma''}) (\gamma_{\Gamma'})_\alpha^{\ \delta} \tag{8.4}$$

Now we have observed that $\mathrm{Tr}(\gamma_\Gamma \gamma_{\Gamma''}) = 0$ unless $\Gamma = \Gamma''$ so we find

$$(\gamma_{\Gamma''})_\alpha^{\ \delta} = \sum_{\Gamma''\Gamma'} c_{\Gamma''\Gamma'} \mathrm{Tr}(\gamma_{\Gamma''}^2) (\gamma_{\Gamma'})_\alpha^{\ \delta} \tag{8.5}$$

From here we see that $c_{\Gamma''\Gamma'} = 0$ unless $\Gamma' = \Gamma''$ and hence

$$c_{\Gamma\Gamma} = \frac{1}{\mathrm{Tr}(\gamma_\Gamma^2)} = \pm_\Gamma \frac{1}{2^{[D/2]}} \tag{8.6}$$

Here the $\pm_\Gamma$ arises because $\gamma_\Gamma^2 = \pm 1$ and $2^{[D/2]} = \mathrm{Tr}(1)$ is the dimension of the representation of the Clifford algebra.

The point of doing all this is that the index contractions have been swapped and hence one can write

$$\begin{aligned}
(\bar\lambda\psi)\chi_\alpha &= \bar\lambda^\gamma \psi_\delta \chi_\beta \delta_\alpha^\beta \delta_\gamma^\delta \\
&= -\sum_\Gamma c_{\Gamma\Gamma} \bar\lambda^\gamma (\gamma_\Gamma)_\gamma^{\ \beta} \chi_\beta (\gamma_\Gamma)_\alpha^{\ \delta} \psi_\delta \\
&= -\frac{1}{2^{[D/2]}} \sum_\Gamma \pm_\Gamma \left(\bar\lambda \gamma_\Gamma \chi\right) (\gamma_\Gamma \psi)_\alpha
\end{aligned} \tag{8.7}$$

here the minus sign out in front arises because we must interchange the order of ψ and χ which are anti-commuting. This is called a Fierz rearrangement and it has allowed us to move the free spinor index from χ to ψ. Its draw back is that it becomes increasingly complicated as the spacetime dimension D increases, but generally speaking there isn't an alternative so you just have to slog it out.

In particular consider four dimensions. The independent matrices are

$$1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \gamma_{\mu\nu} \tag{8.8}$$

One can see that this is the case by noting that $\gamma_{\mu\nu 5} = \frac{i}{2}\varepsilon_{\mu\nu\lambda\rho}\gamma^{\lambda\rho}$. You can check that the Fierz identity is

$$
\begin{aligned}
(\bar{\lambda}\psi)\chi_\alpha &= -\frac{1}{4}(\bar{\lambda}\chi)\psi_\alpha - \frac{1}{4}(\bar{\lambda}\gamma_5\chi)\gamma_5\psi_\alpha - \frac{1}{4}\left(\bar{\lambda}\gamma_\mu\chi\right)(\gamma^\mu\psi)_\alpha \\
&\quad + \frac{1}{4}\left(\bar{\lambda}\gamma_\mu\gamma_5\chi\right)(\gamma^\mu\gamma_5\psi)_\alpha + \frac{1}{8}\left(\bar{\lambda}\gamma_{\mu\nu}\chi\right)(\gamma^{\mu\nu}\psi)_\alpha
\end{aligned}
\tag{8.9}
$$

note the extra factor of $\frac{1}{2}$ in the last term that is there is ensure that $\gamma_{\mu\nu}$ and $\gamma_{\nu\mu}$ don't contribute twice. We will use this at various points in the course.

Problem: Show that in three dimensions the Fierz rearrangement is

$$
(\bar{\lambda}\psi)\chi_\alpha = -\frac{1}{2}(\bar{\lambda}\chi)\psi_\alpha - \frac{1}{2}\left(\bar{\lambda}\gamma_\mu\chi\right)(\gamma^\mu\psi)_\alpha \tag{8.10}
$$

Using this, show that in the special case that $\lambda = \chi$ one simply has

$$
(\bar{\lambda}\psi)\lambda_\alpha = -\frac{1}{2}(\bar{\lambda}\lambda)\psi_\alpha \tag{8.11}
$$

for Majorana spinors. Convince yourself that this is true by considering the explicit 3D γ-matrices above and letting

$$
\lambda = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}, \qquad \psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{8.12}
$$

What is the Fierz rearrangement in two dimensions (Hint: this last part should take you very little time)?

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