Supersymmetry (CM439Z/CMMS40)

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1 Introduction

Particle Physics is the study of matter at the smallest scales that can be accessed by experiment. Currently energy scales are as high as $100GeV$ which corresponds to distances of $10^{-16}cm$ (recall that the atomic scale is about $10^{-9}cm$ and the nucleus is about $10^{-13}cm$). Our understanding of Nature up to this scale is excellent[1]. Indeed it must be one of the most successful and accurate scientific theories and goes by the least impressive name "The Standard Model of Elementary Particle Physics". This theory is a relativistic quantum theory which postulates that matter is made up of point-sized particles (in so far as it makes sense to talk about particles as opposed to waves). The mathematical framework for such a theory is quantum field theory. There are an infinite list of possible quantum field theories and the Standard Model is one of these, much like a needle in a haystack.

There is currently a great deal of interest focused on the LHC (Large Hadron Collider) in CERN. In a year or two these experiments will probe higher energy scalars and therefore shorter distances. The great hope is that new physics, beyond that predicted by the Standard Model, will be observed. One of the main ideas, in fact probably the most popular, is that supersymmetry will be observed. There are a few reasons for this:

- The Hierarchy problem: The natural scale of the Standard Model is the electroweak scale which is at about $1TeV$ (hence the excitement about the LHC). In a quantum field theory physical parameters, such as the mass of the Higg's Boson, get renormalized by quantum effects. Why then is the Higg's mass not renormalized up to the Planck scale? To prevent this requires and enormous amount of fine-tuning (parameters in the Standard Model must be fixed to an incredible order or magnitude). However in a supersymmetric model these renormalizations are less severe and fine-tuning is not required (or at least is not as bad).

- Unification: Another key idea about beyond the Standard Model is that all the gauge fields are unified into a single, simple gauge group at some high scale, roughly $10^{15}GeV$. There is some evidence for this idea. For example the particle content is just right and also grand unification gives an accurate prediction for the Weinberg angle. Another piece of evidence is the observation that, although the electromagnetic, strong and weak coupling constants differ at low energy, they 'run' with energy and meet at about $10^{15}GeV$. That any two of them should meet is trivial but that all three meet at the same scale is striking. In fact they don't quite meet in the Standard Model but they do in a supersymmetric version.

- Dark Matter: It would appear that most, roughly 70%, of the matter floating around in the universe is not the stuff that makes up the Standard Model. Supersymmetry predicts many other particles other than those observed in the Standard

[1] This ignores important issues that arise in large and complex systems such as those that are studied in condensed matter physics.

Model, the so so-called superpartners, and the lightest superpartner (LSP) is considered a serious candidate for dark matter.

- Sting Theory: Although there is no empirical evidence for String Theory it is a very compelling framework to consider fundamental interactions. Its not clear that String Theory predicts supersymmetry but it is certainly a central ingredient and, symbiotically, supersymmetry has played a central role in String Theory and its successes. Indeed there is no clear boundary between supersymmetry and String Theory and virtually all research in fundamental particle physics involves both of them (not that this is necessarily a good thing).

If supersymmetry is observed in Nature it will be a great triumph of theoretical physics. Indeed the origin of supersymmetry is in string theory and the two fields have been closely linked since their inception. If not one can always claim that supersymmetry is broken at a higher energy (although in so doing the arguments in favour of supersymmetry listed above will cease to be valid). Nevertheless supersymmetry has been a very fruitful subject of research and has taught us a great deal about mathematics and quantum field theory. For example supersymmetric quantum field theories, especially those with extended supersymmetry, can be exactly solved for at the peturbative and non-perturbative levels. Hopefully this course will convince the student that supersymmetry is a beautiful and interesting subject.

2 Symmetries, A No-go Theorem and How to Avoid It

Quantum field theories are essentially what you get from the marriage of quantum mechanics with special relativity (assuming locality). A central concept of these ideas is the notion of symmetry. And indeed quantum field theories are thought of and classified according to their symmetries.

The most important symmetry is of course the Poincare group of special relativity. This consists of translations along with the Lorentz transformations (which in turn contain rotations and boosts). In particular the theory is invariant under the infinitessimal transformations

$$x^\mu \to x^\mu + a^\mu + \omega^\mu{}_\nu x^\nu \qquad i.e. \qquad \delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu \tag{2.1}$$

Here a^μ generates translations and $\omega^\mu{}_\nu$ generates Lorentz transformations. The principle of Special relativity requires that the spacetime proper distance $\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$ between two points is invariant under these transformations. Expanding to first order in $\omega^\mu{}_\nu$ tells us that

$$\begin{aligned}\Delta s^2 &\to \eta_{\mu\nu}(\Delta x^\mu + \omega^\mu{}_\lambda \Delta x^\lambda)(\Delta x^\nu + \omega^\nu{}_\rho \Delta x^\rho) \\ &= \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu + \eta_{\mu\nu}\omega^\mu{}_\lambda \Delta x^\lambda \Delta x^\nu + \eta_{\mu\nu}\omega^\nu{}_\rho \Delta x^\mu \Delta x^\rho \\ &= \Delta s^2 + (\omega_{\mu\nu} + \omega_{\nu\mu})\Delta x^\mu \Delta x^\nu \end{aligned} \tag{2.2}$$

where we have lowered the index on $\omega^{\mu}{}_{\nu}$. Thus we see that the Lorentz symmetry requires $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

Next we consider the algebra associated to such generators. To this end we want to know what happens if we make two Poincare transformations and compare the difference, *i.e.* we consider $\delta_1\delta_2 x^\mu - \delta_2\delta_1 x^\mu$. First we calculate

$$\delta_2\delta_1 x^\mu = \omega_1^{\mu}{}_{\nu} a_2^\nu + \omega_1^{\mu}{}_{\lambda}\omega_2^{\lambda}{}_{\nu} x^\nu \tag{2.3}$$

from which we see that

$$(\delta_1\delta_2 - \delta_2\delta_1)x^\mu = (\omega_2^{\mu}{}_{\lambda} a_1^\lambda - \omega_1^{\mu}{}_{\lambda} a_2^\lambda) + (\omega_2^{\mu}{}_{\lambda}\omega_1^{\lambda}{}_{\nu} - \omega_1^{\mu}{}_{\lambda}\omega_2^{\lambda}{}_{\nu})x^\nu \tag{2.4}$$

This corresponds to a new Poincare transformation with

$$a^\mu = \omega_2^{\mu}{}_{\lambda} a_1^\lambda - \omega_1^{\mu}{}_{\lambda} a_2^\lambda \qquad \omega^{\mu}{}_{\nu} = \omega_2^{\mu}{}_{\lambda}\omega_1^{\lambda}{}_{\nu} - \omega_1^{\mu}{}_{\lambda}\omega_2^{\lambda}{}_{\nu} \tag{2.5}$$

note that $\omega_{(\mu\nu)} = \frac{1}{2}(\omega_{\mu\nu} + \omega_{\nu\mu}) = 0$ so this is indeed a Poincare transformation.

More abstractly we think of these transformations as being generated by linear operators P_μ and $M_{\mu\nu}$ so that

$$\delta x^\mu = i a^\nu P_\nu(x^\mu) + \frac{i}{2}\omega^{\nu\lambda} M_{\nu\lambda}(x^\mu) \tag{2.6}$$

The factor of $\frac{1}{2}$ arises because of the anti-symmetry (one doesn't want to count the same generator twice). The factors of i are chosen for later convenience to ensure that the generators are Hermitian. These generators can then also be though of as applying on different objects, *e.g.* spacetime fields rather than spacetime points. In other words we have an abstract algebra and its action on x^μ is merely one representation.

This abstract object is the Poincare algebra and it defined by the commutators

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[P_\mu, M_{\nu\lambda}] &= i\eta_{\mu\nu}P_\lambda - i\eta_{\mu\lambda}P_\nu \\
[M_{\mu\nu}, M_{\lambda\rho}] &= i\eta_{\nu\lambda}M_{\mu\rho} - i\eta_{\mu\lambda}M_{\nu\rho} + i\eta_{\mu\rho}M_{\nu\lambda} - i\eta_{\nu\rho}M_{\mu\lambda}
\end{aligned} \tag{2.7}$$

which generalizes (2.4).

Problem: Using (2.6) and(2.7) show that (2.4) is indeed reproduced.

To say that the Poincare algebra is fundamental in particle physics means that everything is assumed fall into some representation of this algebra. The principle of relativity then asserts that the laws of physics are covariant with respect to this algebra.

The Poincare group has two clear pieces: translations and Lorentz transformations. It is not quite a direct product because of the non-trivial commutator $[P_\mu, M_{\nu\lambda}]$. It is a so-called a semi-direct product. Translations by themselves form an Abelian and non-compact subgroup. On physical grounds one always takes $P_\mu = -i\partial_\mu$. This seems obvious from the physical interpretation of spacetime. Mathematically the reason for this is simply Taylor's theorem for a function $f(x^\mu)$:

$$\begin{aligned} f(x+a) &= f(x) + \partial_\mu f(x) a^\mu + \dots \\ &= f(x) + i a^\mu P_\mu f(x) + \dots \end{aligned} \tag{2.8}$$

Thus acting by P_μ will generate a infinitessimal translation. Furthermore Taylor's theorem is the statement that finite translations are obtained from exponentiating P_μ:

$$\begin{aligned} f(x+a) &= e^{ia^\mu P_\mu} f(x) \\ &= f(x) + a^\mu \partial_\mu f(x) + \frac{1}{2!} a^\mu a^\nu \partial_\mu \partial_\nu f(x) + \dots \end{aligned} \tag{2.9}$$

However the other part, the Lorentz group, is non-Abelian and admits interesting finite-dimensional representations. For example the Standard Model contains a scalar field $H(x)$ (the Higg's Boson) which carries a trivial representation and also vector fields $A_\mu(x)$ (*e.g.* photons) and spinor fields $\psi_\alpha(x)$ (*e.g.* electrons). A non-trivial representation of the Lorentz group implies that the field carries some kind of index. In the two cases above these are μ and α respectively. The Lorentz generators then act as matrices with two such indices (one lowered and one raised). Different representations mean that there are different choices for these matrices which still satisfies (2.7). For example in the vector representation one can take

$$(M_{\mu\nu})^\lambda{}_\rho = i\eta_{\mu\rho}\delta_\nu^\lambda - i\delta_\mu^\lambda \eta_{\nu\rho} \tag{2.10}$$

Notice the dual role of μ, ν indices as labeling both the particular Lorentz generator as well as it's matrix components. Whereas in the spinor representation we have

$$(M_{\mu\nu})_\alpha{}^\beta = \frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\mu\gamma_\nu)_\alpha{}^\beta \tag{2.11}$$

Here $(\gamma_\mu)_\alpha{}^\beta$ are the Dirac γ-matrices (more about these soon). However in either case it is important to realize that the defining algebraic relations (2.7) are reproduced.

Problem: Verify that these two representation of $M_{\mu\nu}$ do indeed satisfy the Lorentz subalgebra of (2.7).

The Standard Model and other quantum field theories also have other important symmetries. Most notably gauge symmetries. These symmetries imply that there is an additional Lie-algebra with a commutation relation of the form

$$[T_a, T_b] = i f_{ab}{}^c T_c \tag{2.12}$$

where the T_a are Hermitian generators and $f_{ab}{}^c$ are the structure constants. This means that every field in the Standard model Lagrangian also carries a representation of this algebra. If this is a non-trivial representation then there is another 'internal' index on the field. For example the quarks are in the fundamental (*i.e.* three-dimensional) representation of $SU(3)$ and hence, since they are spacetime spinors, the field carries the indices $\psi^a_\alpha(x)$.

Finally we recall Noether's theorem which asserts that for every continuous symmetry of a Lagrangian one can construct a conserved charge. Suppose that a Lagrangian $\mathcal{L}(\Phi_A, \partial_\alpha\Phi_A)$, where we denoted the fields by Φ_A, has a symmetry: $\mathcal{L}(\Phi_A) = \mathcal{L}(\Phi_A + \delta\Phi_A)$. This implies that

$$\frac{\partial\mathcal{L}}{\partial\Phi_A}\delta\Phi_A + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi_A)}\delta\partial_\alpha\Phi_A = 0 \tag{2.13}$$

This allows us to construct a current:

$$J^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi_A)}\delta\Phi_A \tag{2.14}$$

which is, by the equations of motion,

$$\begin{aligned}\partial_\alpha J^\alpha &= \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi_A)}\right)\delta\Phi_A + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi_A)}\partial_\alpha\delta\Phi_A \\ &= \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi_A)}\right)\delta\Phi_A - \frac{\partial\mathcal{L}}{\partial\Phi_A}\delta\Phi_A \\ &= 0\end{aligned} \tag{2.15}$$

conserved. This means that the integral over space of J^0 is a constant defines a charge

$$Q = \int_{space} J^0 \tag{2.16}$$

which is conserved

$$\begin{aligned}\frac{dQ}{dt} &= \int_{space}\partial_0 J^0 \\ &= -\int_{space}\partial_i J^i \\ &= 0\end{aligned}$$

Thus one can think of symmetries and conservations laws as being more or less the same thing.

So the Standard Model of Particle Physics has several symmetries built into it and this means that the various fields carry representations of various algebras. These algebras split up into those associated to spacetime (Poincare) and those which one might call internal (such as the gauge symmetry algebra). In fact the split is a direct product in that

$$[P_\mu, T_a] = [M_{\mu\nu}, T_a] = 0 \tag{2.17}$$

where T_a refers to any internal generator. Physically this means the conserved charges of these symmetries are Lorentz scalars.

Since the Poincare algebra is so central to our understanding of spacetime it is natural to ask if this direct product is necessarily the case or if there is, in principle, some deeper symmetry that has a non-trivial commutation relation with the Poincare algebra. This question was answered by Coleman and Mandula:

Theorem: In any spacetime dimension greater than two the only interacting quantum field theories have Lie algebra symmetries which are a direct product of the Poincare algebra with an internal symmetry.

In other words the Poincare algebra is apparently as deep as it gets. There are no interacting theories which have conserved charges that are not Lorentz scalars. Intuitively the reasons is that tensor-like charge must be conserved in any interaction and this is simply too restrictive as the charges end up being proportional to (products of) the momenta. Thus one finds that the individual momenta are conserved, rather than the total momentum.

But one shouldn't stop here. A no-go theorem is only as good as its assumptions. This theorem has several assumptions, for example that there are a finite number of massive particles and no massless ones. However the key assumption of the Coleman-Mandula theorem is that the symmetry algebra should be a Lie-algebra. We recall that a Lie-algebra can be thought of as the tangent space at the identity of a continuous group, so that, an infinitessimal group transformation has the form

$$g = 1 + i\epsilon A \tag{2.18}$$

where A is an element of the Lie-algebra and ϵ is an infinitessimal parameter. The Lie-algebra is closed under a bilinear operation, the Lie-bracket,

$$[A, B] = -[B, A] \tag{2.19}$$

subject to the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{2.20}$$

If we relax this assumption then there is something deeper - Supersymmetry. So how do we relax it since Lie-algebras are inevitable whenever you have continuous symmetries and because of Noether's theorem we need a continuous symmetry to give a conserved charge?

The way to proceed is to note that quantum field theories such as the Standard Model contain two types of fields: Fermions and Bosons. These are distinguished by the representation of the field under the Lorentz group. In particular a fundamental theorem in quantum field theory - the spin-statistics theorem - asserts that Bosons must carry representations of the Lorentz group with integer spins and their field operators must commute outside of the light-cone whereas Fermions carry half-odd-integer spins and their field operators are anti-commuting. This means that the fields associated to Fermions are not ordinary (so-called c-number) valued field but rather Grassmann variables that satisfy

$$\psi_1(x)\psi_2(x) = -\psi_2(x)\psi_1(x) \tag{2.21}$$

So a way out of this no-go theorem is to find a symmetry that relates Bosons to Fermions. Such a symmetry will require that the 'infinitessimal' generating parameter is a Grassmann variable and hence will not lead to a Lie-algebra. More precisely the idea is to consider a Grassmann generator (with also carries a spinor index) and which requires a Grassmann valued spinorial parameter. One then is lead to something called a superalgebra, or a $\mathbf{Z}_2$-graded Lie-algebra. This means that the generators can be labeled as either even and odd. The even generators behave just as the generators of a Lie-algebra and obey commutation relations. An even and an odd generator will also obey a commutator relation. However two odd generators will obey an anti-commutation relation. The even-ness or odd-ness of this generalized Lie-bracket is additive modulo two: the commutator of two even generators is even, the anti-commutator of two odd generators is also even, whereas the commutator of an even and an odd generator is odd. Schematically, the structure of a superalgebra takes the form

$$\begin{aligned} [even, even] &\sim even \\ [even, odd] &\sim odd \\ \{odd, odd\} &\sim even \end{aligned} \tag{2.22}$$

In particular one does not consider things that are the sum of an even and an odd generator (at least physicists don't but some Mathematicians might), nor does the commutator of two odd generators, or anti-commutator of two even generators, play any role. Just as in Lie-algebras there is a Jacobi identity. It is a little messy as whether or not one takes a commutator or anti-commutator depends on the even/odd character of the generator. It can be written as

$$(-1)^{ac}[A,[B,C]_\pm]_\pm + (-1)^{ba}[B,[C,A]_\pm]_\pm + (-1)^{cb}[C,[A,B]_\pm]_\pm = 0 \tag{2.23}$$

where $a, b, c \in \mathbf{Z}_2$ are the gradings of the generators A, B, C respectively and $[\ ,\ ]_\pm$ is a commutator or anti-commutator according to the rule (2.22).

There is a large mathematical literature on superalgebras as abstract objects. However we will simply focus on the case most relevant for particle physics. In particular the even elements will be the Poincare generators $P_\mu, M_{\nu\lambda}$ and the odd elements supersymmetries Q_α. The important point here is that the last line in (2.22) takes the form

$$\{Q, Q\} \sim P + M \tag{2.24}$$

(in fact one typically finds only P or M on the right hand side, and in this course just P). Thus supersymmetries are the square-root of Poincare transformations. Thus there is a sensible algebraic structure that is "deeper" that the Poincare group. Surely this is worth of study.

One final comment is in order. Although we have found a symmetry that underlies the Poincare algebra one generally still finds that supersymmetries commute with the other internal symmetries. Thus a refined version of the Coleman-Mandula theorem still seems to apply and states that the symmetry algebra of a non-trivial theory is at most the direct product of the superalgebra and an internal Lie-algebra.[2]

3 Preliminaries: Clifford Algebras and Spinors

Before proceeding it is necessary to review in detail the formalism that is needed to describe spinors and Fermions. These first appeared with Dirac who thought that the equation of motion for an electron should be first order in derivatives. Hence, for a free electron, where the equation should be linear, it must take the form

$$(\gamma^\mu \partial_\mu - M)\psi = 0 \tag{3.1}$$

Acting on the left with $(\gamma^\mu \partial_\mu + M)$ one finds

$$(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu - M^2)\psi = 0 \tag{3.2}$$

This should be equivalent to the Klein Gordon equation (which is simply the mass-shell condition $E^2 - p^2 - m^2 = 0$)

$$(\partial^2 - m^2)\psi = 0 \tag{3.3}$$

Thus we see that we can take $m = M$ to be the mass and, since $\partial_\mu \partial_\nu \psi = \partial_\nu \partial_\mu \psi$, we also require that

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \tag{3.4}$$

This seemingly innocent condition is in fact quite deep. It first appeared in Mathematics in the geometrical work of Clifford (who was a student at King's). The next step is to find representations of this relation which reveals an 'internal' spin structure to Fermions.

3.1 Clifford Algebras

Introducing Fermions requires that we introduce a set of γ-matrices. These furnish a representation of the Clifford algebra, which is generically taken to be over the complex numbers, whose generators satisfy the relation

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{3.5}$$

Note that we have suppressed the spinor indices α, β. In particular the right hand side is proportional to the identity matrix in spinor space. We denote spinor indices by α, β....

[2] Note that one should be careful here, while this statement is true in spirit it is imprecise and in some sense counter examples can be found (*e.g.* in gauged supergavity).

One consequence of this relation is that the γ-matrices are traceless (at least for $D > 1$). To see this we evaluate

$$\begin{aligned}
2\eta_{\mu\nu}\mathrm{Tr}(\gamma_\lambda) &= \mathrm{Tr}(\{\gamma_\mu,\gamma_\nu\}\gamma_\lambda) \\
&= \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\lambda + \gamma_\nu\gamma_\mu\gamma_\lambda) \\
&= \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\lambda + \gamma_\mu\gamma_\lambda\gamma_\nu) \\
&= \mathrm{Tr}(\gamma_\mu\{\gamma_\nu,\gamma_\lambda\}) \\
&= 2\eta_{\nu\lambda}\mathrm{Tr}(\gamma_\mu)
\end{aligned} \tag{3.6}$$

Choosing $\mu = \nu \neq \lambda$ immediately implies that $\mathrm{Tr}(\gamma_\lambda) = 0$

Theorem: In even dimensions there is only one non-trivial irreducible representation of the Clifford algebra, up to conjugacy, *i.e.* up to a transformation of the form $\gamma_\mu \to U\gamma_\mu U^{-1}$. In particular the (complex) dimension of this representation is $2^{D/2}$, *i.e.* the γ-matrices will be $2^{D/2} \times 2^{D/2}$ complex valued matrices.

Without loss of generality one can choose a representation such that

$$\gamma_0^\dagger = -\gamma_0\ , \qquad \gamma_i^\dagger = \gamma_i \tag{3.7}$$

which can be written as

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0 \tag{3.8}$$

An even-dimensional Clifford algebra naturally lifts to a Clifford algebra in one dimension higher. In particular one can show that

$$\gamma_{D+1} = c\gamma_0\gamma_1...\gamma_{D-1} \tag{3.9}$$

anti-commutes with all the γ_μ's. Here c is a constant which we can fix, up to sign, by taking $\gamma_{D+1}^2 = 1$. In particular a little calculation shows that

$$\gamma_{D+1}^2 = -(-1)^{D(D-1)/2}c^2 \tag{3.10}$$

Here the first minus sign comes from γ_0^2 whereas the others come from anti-commuting the different γ_μ's through each other. In this way we find that

$$c = \pm i(-1)^{D(D-1)/4} \tag{3.11}$$

Thus we construct a Clifford Algebra in $(D+1)$-dimensions. It follows that the dimension (meaning the range of the spinor indices α, β...) of a Clifford algebra in $(D+1)$-dimensions is the same as the dimension of a Clifford algebra in D-dimensions when D is even.

In odd dimensions there are two inequivalent representations. To see this one first truncates down one dimension. This leads to a Clifford algebra in a even dimension which is therefore unique. We can then construct the final γ-matrix using the above procedure. This leads to two choices depending on the choice of sign above. Next we observe that in odd-dimensions γ_{D+1}, defined as the product of all the γ-matrices, commutes with all the γ_μ's. Hence by Shur's lemma it must be proportional to the identity. Under conjugacy one therefore has $\gamma_{D+1} \to U\gamma_{D+1}U^{-1} = \gamma_{D+1}$. The constant of proportionality is determined by the choice of sign we made to construct the final γ-matrix. Since this is unaffected by conjugation we find two representation we constructed are inequivalent.

We can also construct a Clifford algebra in $D + 2$ dimensions using the Clifford algebra in D dimensions. Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.12}$$

be the ubiquitous Pauli matrices. If we have the Clifford algebra in D-dimensions given by γ_μ, $\mu = 0, 1, 2, ..., D - 1$ then let

$$\begin{aligned} \Gamma_\mu &= 1 \otimes \gamma_\mu \\ \Gamma_D &= \sigma_1 \otimes \gamma_{D+1} \\ \Gamma_{D+1} &= \sigma_3 \otimes \gamma_{D+1} \end{aligned} \tag{3.13}$$

where we have used Γ_μ for $(D + 2)$-dimensional γ-matrices. One readily sees that this gives a Clifford algebra in $(D + 2)$-dimensions. Note that this gives two algebras corresponding to the two choices of sign for γ_{D+1}. However these two algebras are equivalent under conjugation by $U = \sigma_2 \otimes 1$. This is to be expected from the uniqueness of an even-dimensional Clifford algebra representation.

Having constructed essentially unique γ-matrices for a given dimension there are two special things that can happen. We have already seen that in even dimensions one finds an "extra" Hermitian γ-matrix, γ_{D+1} (so in four dimensions this is the familiar γ_5). Since this is Hermitian it has a basis of eigenvectors with eigenvalues ± 1 which are called the chirality. Indeed since the γ-matrices are traceless half of the eigenvalues are $+1$ and the other half -1. We can therefore write any spinor ψ uniquely as

$$\psi = \psi_+ + \psi_- \tag{3.14}$$

where $\psi_\pm$ has γ_{D+1} eigenvalue ± 1. A spinor with a definite γ_{D+1} eigenvalue is called a Weyl spinor.

The second special case occurs when the γ-matrices can be chosen to be purely real. In which case it is possible to chose the spinors to also be real. A real spinor is called a Majorana spinor.

Either of these two restrictions will cut the number of independent spinor components in half. In some dimensions it is possible to have both Weyl and Majorana spinors simultaneously. These are called Majorana-Weyl spinors. This reduces the number of independent spinor components to a quarter of the original size. Spinors without any such restrictions are called Dirac spinors. Which restrictions are possible in which dimensions comes in a pattern which repeats itself for dimensions D modulo 8.

Let us illustrate this by starting in low dimensions and work our way up. We will give concrete example of γ-matrices but it is important to bare in mind that these are just choices - there are other choices.

3.1.1 D=1

If there is only one dimension, time, then the Clifford algebra is the simple relation $(\gamma_0)^2 = -1$. In other words $\gamma_0 = i$ or one could also have $\gamma_0 = -i$. It is clear that there is no Majorana representation.

3.1.2 D=2

Here the γ-matrices can be taken to be

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \gamma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{3.15}$$

One can easily check that $\gamma_0^2 = -\gamma_1^2 = -1$ and $\gamma_0\gamma_1 = -\gamma_1\gamma_0$.

Here we have a real representation so that we can choose the spinors to also be real. We can also construct $\gamma_3 = c\gamma_0\gamma_1$ and it is also real:

$$\gamma_3 = -\gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.16}$$

Thus we can have Weyl spinors, Majorana spinors and Majorana-Weyl spinors. These will have 2, 2 and 1 real independent components respectively whereas a Dirac spinor will have 2 complex, *i.e.* 4 real, components.

3.1.3 D=3

Here the γ-matrices can be constructed from $D = 2$ and hence a natural choice is

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \gamma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \gamma_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{3.17}$$

(we could also have taken the opposite sign for γ_2). These are just the Pauli matrices (up to a factor of i for γ_0). Since we are in an odd dimension there are no Weyl spinors but we can choose the spinors to be Majorana with only 2 real independent components.

3.1.4 D=4

Following our discussion above a natural choice is

$$
\begin{aligned}
\gamma_0 &= 1 \otimes i\sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\
\gamma_1 &= 1 \otimes \sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
\gamma_2 &= \sigma_1 \otimes \sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
\gamma_3 &= \sigma_3 \otimes \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{3.18}
$$

By construction this is a real basis of γ-matrices. Therefore we can chose to have Majorana, *i.e.* real, spinors.

Since we are in an even dimension we can construct the chirality operator $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Note the factor of i which is required to ensure that $\gamma_5^2 = 1$. Thus in our basis γ_5 is purely imaginary and, since it is Hermitian, it must be anti-symmetric. This means that it cannot be diagonalized over the reals. Of course since it is Hermitian it can be diagonalized over the complex numbers, *i.e.* there is another choice of γ-matrices for which γ_5 is real and diagonal but in this basis the γ_μ cannot all be real.

Thus in four dimensions we can have Majorana spinors or Weyl spinors but not both simultaneously. In many books, especially those that focus on four-dimensions, a Weyl basis of spinors is used. Complex conjugation then acts to flip the chirality. However we prefer to use a Majorana basis whenever possible (in part because it applies to more dimensions).

3.2 Spinors

Having defined Clifford algebras we next need to discuss the properties of spinors in greater detail. As we saw in a problem above, $M_{\mu\nu} = \frac{i}{2}\gamma_{\mu\nu}$ gives a representation of the Lorentz algebra, known as the spinor representation. A spinor is simply an object that transforms in the spinor representation of the Lorentz group (it is a section of the spinor bundle over spacetime). Hence it carries a spinor index α. From our definitions this means that under a Lorentz transformation generated by $\omega^{\mu\nu}$, a spinor ψ_α transforms as

$$\delta\psi_\alpha = -\frac{1}{4}\omega^{\mu\nu}(\gamma_{\mu\nu})_\alpha{}^\beta\psi_\beta \tag{3.19}$$

Note that we gives spinors a lower spinor index. As such the γ-matrices naturally come with one upper and one lower index, so that matrix multiplication requires contraction on one upper and one lower index.

Let us pause for a moment to consider a finite Lorentz transformation. To begin with consider an infinitesimal rotation by an angle θ in the (x^1, x^2)-plane,

$$\delta\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \theta\begin{pmatrix} x^0 \\ -x^2 \\ x^1 \\ x^3 \end{pmatrix} \tag{3.20}$$

i.e.

$$\omega^{12} = -\omega_{21} = \theta \qquad M_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.21}$$

A finite rotation is obtained by exponentiating M_{12}:

$$x^\mu \to (e^{\omega^{\lambda\rho}M_{\lambda\rho}})^\mu{}_\nu x^\nu \tag{3.22}$$

Since $M_{12}^2 = -1$ one finds that, using the same proof as the famous relation $e^{i\theta} = \cos\theta + i\sin\theta$,

$$e^{\theta M_{12}} = \cos\theta + M_{12}\sin\theta \tag{3.23}$$

In particular we see that if $\theta = 2\pi$ then $e^{2\pi M_{12}} = 1$ as expected.

How does a spinor transform under such a rotation? The infinitesimal transformation generated by ω^{12} is, by definition,

$$\delta\psi = -\frac{1}{4}\omega^{\mu\nu}\gamma_{\mu\nu}\psi = -\frac{1}{2}\theta\gamma_{12}\psi \tag{3.24}$$

If we exponentiate this we find

$$\psi \to e^{-\frac{1}{2}\theta\gamma_{12}}\psi = \cos(\theta/2) - \gamma_{12}\sin(\theta/2) \tag{3.25}$$

We see that now, if $\theta = 2\pi$, then $\psi \to -\psi$. Thus we recover the well known result that under a rotation by 2π a spinor (such as an electron) picks up a minus sign.

Let us now try to contract spinor indices to obtain Lorentz scalars. It follows that the Hermitian conjugate transforms as

$$\delta\psi^\dagger = -\frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_\nu^\dagger\gamma_\mu^\dagger = \frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_0\gamma_{\nu\mu}\gamma_0 = -\frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_0\gamma_{\mu\nu}\gamma_0 \tag{3.26}$$

Here we have ignored the spinor index. Note that the index structure is $(\gamma_0\gamma_{\mu\nu}\gamma_0)_\alpha{}^\beta$ and therefore it is most natural to write $(\psi^\dagger)^\alpha = \psi^{*\alpha}$ with an upstairs index.

However we would like to contract two spinors to obtain a scalar. One can see that the naive choice

$$\lambda^\dagger \psi = \lambda^{*\alpha}\psi_\alpha \tag{3.27}$$

will not be a Lorentz scalar due to the extra factors of γ_0 that appear in $\delta\lambda^\dagger$ as compared to $\delta\psi$. To remedy this one defines the Dirac conjugate

$$\bar{\lambda} = \lambda^\dagger \gamma_0 \tag{3.28}$$

In which case on finds that, under a Lorentz transformation,

$$\delta\bar{\lambda} = \frac{1}{4}\bar{\lambda}\omega^{\mu\nu}\gamma_{\mu\nu} \tag{3.29}$$

and hence

$$\begin{aligned} \delta(\bar{\lambda}\psi) &= \delta\bar{\lambda}\psi + \bar{\lambda}\delta\psi \\ &= \frac{1}{4}\bar{\lambda}\omega^{\mu\nu}\gamma_{\mu\nu}\psi - \frac{1}{4}\bar{\lambda}\omega^{\mu\nu}\gamma_{\mu\nu}\psi \\ &= 0 \end{aligned} \tag{3.30}$$

Thus we have found a Lorentz invariant way to contract spinor indices.

Note that from two spinors we can construct other Lorentz covariant objects such as vectors and anti-symmetric tensors:

$$\bar{\lambda}\gamma_\mu\psi \ , \quad \bar{\lambda}\gamma_{\mu\nu}\psi \ , \ldots \tag{3.31}$$

Problem: Show that $V_\mu = \bar{\lambda}\gamma_\mu\psi$ is a Lorentz vector, *i.e.* show that $\delta V_\mu = -\omega_\mu{}^\nu V_\nu$ under the transformation (3.19).

So far our discussion applied to general Dirac spinors. In much of this course we will be interested in Majorana spinors where the γ_μ are real. The above discussion is then valid if we replace the Hermitian conjugate $\dagger$ with the transpose T so that $\gamma_\mu^T = -\gamma_0\gamma_\mu\gamma_0^{-1}$. More generally such a relationship always exists because if $\{\gamma_\mu\}$ is a representation of the Clifford algebra then so is $\{-\gamma_\mu^T\}$. Therefore, since there is a unique representation up to conjugacy, there must exist a matrix C such that $-\gamma_\mu^T = C\gamma_\mu C^{-1}$. C is called the charge conjugation matrix. The point here is that in the Majorana case it is possible to find a representation in which C coincides with Dirac conjugation matrix γ_0.

Problem: Show that, for a general Dirac spinor in any dimension, $\lambda^T C\psi$ is Lorentz invariant, where C is the charge conjugation matrix.

One way to think about charge conjugation is to view the matrix $C^{\alpha\beta}$ as a metric on the spinor indices with inverse $C^{-1}_{\alpha\beta}$. In which case $\psi^\alpha = \psi_\beta C^{\beta\alpha}$.

Finally we note that spinor quantum fields are Fermions in quantum field theory (this is the content of the spin-statistics theorem). This means that spinor components are anti-commuting Grassmann variables

$$\psi_\alpha \psi_\beta = -\psi_\beta \psi_\alpha \tag{3.32}$$

We also need to define how complex conjugation acts. Since ultimately in the quantum field theory the fields are elevated to operators we take the following convention for complex conjugation

$$(\psi_\alpha \psi_\beta)^* = \psi^*_\beta \psi^*_\alpha \tag{3.33}$$

which is analogous to the Hermitian conjugate. This leads to the curious result that, even for Majorana spinors, one has that

$$(\bar{\psi}\chi)^* = (\psi^*_\alpha C^{\alpha\beta}\chi_\beta)^* = \chi_\beta C^{\alpha\beta}\psi_\alpha = -\psi_\alpha C^{\alpha\beta}\chi_\beta = -\bar{\psi}\chi \tag{3.34}$$

is pure imaginary!

3.3 The Fierz Transformation

The γ-matrices have several nice properties. Out of them one can construct the additional matrices

$$1, \gamma_\mu, \gamma_\mu\gamma_{D+1}, \gamma_{\mu\nu}, \gamma_{\mu\nu}\gamma_{D+1}, ... \tag{3.35}$$

where $\gamma_{\mu\nu\lambda...}$ is the anti-symmetric product over the given indices with weight one, *e.g.*

$$\gamma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \tag{3.36}$$

Because of the relation $\gamma_0\gamma_1...\gamma_{D-1} \propto \gamma_{D+1}$ not all of these matrices are independent. The list stops when the number of indices is bigger than $D/2$. It is easy to convince yourself that the remaining ones are linearly independent.

Problem: Using the fact that $< M_1, M_2 >= \mathrm{Tr}(M_1^\dagger M_2)$ defines a complex inner product, convince yourself that the set (3.35), where the number of spacetime indices is no bigger than $D/2$, is a basis for the space of $2^{[D]/2} \times 2^{[D]/2}$ matrices ($[D/2]$ is the integer part of $D/2$).

Thus any matrix can be written in terms of γ-matrices. In particular one can express

$$\delta_\alpha^\beta \delta_\gamma^\delta = \sum_{\Gamma\Gamma'} c_{\Gamma\Gamma'} (\gamma_\Gamma)_\gamma^{\ \beta} (\gamma_{\Gamma'})_\alpha^{\ \delta} \tag{3.37}$$

for some constants $c_{\Gamma\Gamma'}$. Here Γ and Γ' are used as a indices that range over all independent γ-matrix products in (3.35).

To proceed one must determine the coefficients $c_{\Gamma\Gamma'}$. To do this we simply multiply (3.37) by $(\gamma_{\Gamma''})_\beta^{\ \gamma}$ which gives

$$(\gamma_{\Gamma''})_\alpha^{\ \delta} = \sum_{\Gamma\Gamma'} c_{\Gamma\Gamma'} \mathrm{Tr}(\gamma_\Gamma \gamma_{\Gamma''}) (\gamma_{\Gamma'})_\alpha^{\ \delta} \tag{3.38}$$

Now we have observed that $\mathrm{Tr}(\gamma_\Gamma\gamma_{\Gamma''}) = 0$ unless $\Gamma = \Gamma''$ so we find

$$(\gamma_{\Gamma''})_\alpha^{\ \delta} = \sum_{\Gamma''\Gamma'} c_{\Gamma''\Gamma'}\mathrm{Tr}(\gamma_{\Gamma''}^2)(\gamma_{\Gamma'})_\alpha^{\ \delta} \tag{3.39}$$

From here we see that $c_{\Gamma''\Gamma'} = 0$ unless $\Gamma' = \Gamma''$ and hence

$$c_{\Gamma\Gamma} = \frac{1}{\mathrm{Tr}(\gamma_\Gamma^2)} = \pm_\Gamma \frac{1}{2^{[D/2]}} \tag{3.40}$$

Here the $\pm_\Gamma$ arises because $\gamma_\Gamma^2 = \pm 1$ and $2^{[D/2]} = \mathrm{Tr}(1)$ is the dimension of the representation of the Clifford algebra.

The point of doing all this is that the index contractions have been swapped and hence one can write

$$\begin{aligned}(\bar\lambda\psi)\chi_\alpha &= \bar\lambda^\gamma\psi_\delta\chi_\beta\delta_\alpha^\beta\delta_\gamma^\delta \\ &= -\sum_\Gamma c_{\Gamma\Gamma}\bar\lambda^\gamma(\gamma_\Gamma)_\gamma^{\ \beta}\chi_\beta(\gamma_\Gamma)_\alpha^{\ \delta}\psi_\delta \\ &= -\frac{1}{2^{[D/2]}}\sum_\Gamma \pm_\Gamma\left(\bar\lambda\gamma_\Gamma\chi\right)(\gamma_\Gamma\psi)_\alpha \end{aligned} \tag{3.41}$$

here the minus sign out in front arises because we must interchange the order of ψ and χ which are anti-commuting. This is called a Fierz rearrangement and it has allowed us to move the free spinor index from χ to ψ. Its draw back is that it becomes increasingly complicated as the spacetime dimension D increases, but generally speaking there isn't an alternative so you just have to slog it out. This may seem somewhat abstract now but we will need to use it in a concrete example soon enough.

Problem: Show that in three dimensions the Fierz rearrangement is

$$(\bar\lambda\psi)\chi_\alpha = -\frac{1}{2}(\bar\lambda\chi)\psi_\alpha - \frac{1}{2}\left(\bar\lambda\gamma_\mu\chi\right)(\gamma^\mu\psi)_\alpha \tag{3.42}$$

Using this, show that in the special case that $\lambda = \chi$ one simply has

$$(\bar\lambda\psi)\lambda_\alpha = -\frac{1}{2}(\bar\lambda\lambda)\psi_\alpha \tag{3.43}$$

for Majorana spinors. Convince yourself that this is true by considering the explicit 3D γ-matrices above and letting

$$\lambda = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}, \qquad \psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{3.44}$$

What is the Fierz rearrangement in two dimensions (Hint: this last part should take you very little time)?

4 Elementary Consequences of Supersymmetry

The exact details of the supersymmetry algebra vary from dimension to dimension, depending on the details of Clifford algebras, however the results below are essentially unchanged. If there are Majorana spinors then the algebra is, in addition to the Poincare algebra relations (2.7),[3]

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= -2(\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \\ [Q_\alpha, P_\mu] &= 0 \\ [Q_\alpha, M_{\mu\nu}] &= \frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta \end{aligned} \tag{4.45}$$

The primary relation is the first line. The second line simply states that the Q_α's are invariant under translations and the third line simply states that they are spacetime spinors.

At first sight one might wonder why there is a C^{-1} on the right hand side. The point is that this is used to lower the second spinor index. Furthermore it is clear that the left hand side is symmetric in α and β and therefore the right hand side must also be symmetric. To see that this is the case we observe that, since we have assumed a Majorana basis where $C = -C^T = \gamma_0$,

$$(\gamma_\mu C^{-1})^T = (C^{-1})^T \gamma_\mu^T = -(C^{-1})^T C \gamma_\mu C^{-1} = \gamma_\mu C^{-1} \tag{4.46}$$

is indeed symmetric.

Let us take the trace of the primary supersymmetry relation

$$\sum_\alpha \{Q_\alpha, Q_\alpha\} = 2^{[D/2]+1} P_0 \tag{4.47}$$

Here we have used the fact that $C^{-1} = \gamma^0$, $\mathrm{Tr}(\gamma_{\mu\nu}) = 0$ and $\mathrm{Tr}(1) = 2^{[D/2]}$. We can identify $P_0 = E$ with the energy and hence we see that

$$E = \frac{1}{2^{[D/2]+1}} \sum_\alpha Q_\alpha^2 \tag{4.48}$$

Since Q_α is Hermitian it follows that the energy is positive definite. Furthermore the only states with $E = 0$ must have $Q_\alpha|0>= 0$, *i.e.* they must preserve the supersymmetry.

Supersymmetry, like other symmetries in quantum field theory, can be spontaneously broken. This means that the vacuum state $|vacuum>$, *i.e.* the state of lowest energy, does not satisfy $Q_\alpha|vacuum>= 0$. We see that in a supersymmetric theory this will be the case if and only if the vacuum energy is positive.

[3]More precisely this is the minimal $N = 1$ super-Poincare algebra. One can have N-extended supersymmetry algebras and centrally extended supersymmetry algebras, which we will come to later. There are also superalgebras based on other Bosonic algebras than the Poincare algebra, *e.g.*, the anti-de-Sitter algebra.

Next let us consider the representations of supersymmetry. First we observe that since $[P_\mu, Q_\alpha] = 0$ we have $[P^2, Q_\alpha] = 0$. Thus P^2 is a Casmir, that is to say irreducible representations of supersymmetry (*i.e.* of the Q's) all carry the same value of $P^2 = -m^2$. Thus all the particles in a supermultiplet (*i.e.* in a irreducible representation) have the same mass.

Let us first consider a massive supermultiplet. We can therefore go to the rest frame where $P_\mu = (m, 0, 0, ..., 0)$. In this case the algebra becomes

$$\{Q_\alpha, Q_\beta\} = 2m\delta_{\alpha\beta} \tag{4.49}$$

We can of course assume that $m > 0$ and rescale $\tilde{Q}_\alpha = m^{-1/2}Q_\alpha$ which gives

$$\{\tilde{Q}_\alpha, \tilde{Q}_\beta\} = 2\delta_{\alpha\beta} \tag{4.50}$$

This is just a Clifford algebra in $2^{[D/2]}$ Euclidean dimensions! As such we know that it has $2^{2^{[D/2]}/2} = 2^{2^{[D/2]-1}}$ states. We can construct the analogue of γ_{D+1}:

$$(-1)^F = Q_1 Q_2 Q_3 ... Q_{2^{2^{[D/2]-1}}} \tag{4.51}$$

Since we are in $2^{2^{[D/2]-1}}$ Euclidean dimensions we have that $((-1)^F)^2 = 1$[4]. Again $(-1)^F$ is traceless and Hermitian. Therefore it has $2^{2^{[D/2]-1}-1}$ eigenvalues equal to $+1$ and $2^{2^{[D/2]-1}-1}$ equal to -1. What is the significance of these eigenvalues? Well if $|\pm>$ is a state with $(-1)^F$ eigenvalue ± 1 then $Q_\alpha|\pm>$ will satisfy

$$(-1)^F Q_\alpha|\pm> = -Q_\alpha(-1)^F|\pm> = \mp Q_\alpha|\pm> \tag{4.52}$$

Thus acting by Q_α will change the sign of the eigenvalue. However since Q_α is a Fermionic operator it will map Fermions to Bosons and vise-versa. Thus $(-1)^F$ measures whether or not a state is Fermionic or Bosonic. Since it is traceless we see that a supermutliplet contains an equal number of Bosonic and Fermionic states. This is only true on-shell since we have assumed that we are in the rest frame.

Next let us consider massless particles. Here we can go to a frame where $P_\mu = (E, E, 0, 0, ..., 0)$ so that the supersymmetry algebra becomes

$$\{Q_\alpha, Q_\beta\} = 2E(\delta_{\alpha\beta} + (\gamma_{01})_{\alpha\beta}) \tag{4.53}$$

We observe that $\gamma_{01}^2 = 1$ and also that $\mathrm{Tr}(\gamma_{01}) = 0$. Therefore the matrix $1 - \gamma_{01}$ has half its eigenvalues equal to 0 and the others equal to 2. It follows the algebra splits into two pieces:

$$\{Q_{\alpha'}, Q_{\beta'}\} = 4E\delta_{\alpha'\beta'} \qquad \{Q_{\alpha''}, Q_{\beta''}\} = 0 \tag{4.54}$$

where the primed and doubled primed indices only take on $2^{[D/2]-1}$ values each. Again by rescaling, this time $\tilde{Q}_{\alpha'} = (2E)^{-1/2}Q_{\alpha'}$ we recover a Clifford algebra but in $2^{[D/2]-1}$ dimensions. Thus there are just $2^{2^{[D/2]-1}-1}$ states. Again we find that half are Fermions and the other half Bosons.

[4]Actually this fails for $D \leq 2$. We will ignore various curiosities that can occur in low dimensions.

Finally we note that the condition $[Q_\alpha, M_{\mu\nu}] = \frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta$ implies that states in a supermultiplet will have spins that differ in steps of 1/2. In an irreducible mutliplet there is a unique state $|j_{max}>$ with maximal spin (actually helicity). The remaining states therefore have spins $j_{max} - 1/2, j_{max} - 1, \ldots$.

It should be noted that often these multiplets will not be CPT complete. For example if they are constructed by acting with lowering operators on a highest helicity state then the tend to have more positive helicity states than negative ones. Therefore in order to obtain a CPT invariant theory, as is required by Lorentz invariance, one has to add in a CPT mirror multiplet (for example based on using raising operators on a lowest helicity state).

The number of states in a supermultiplet grows exponentially quickly. This is essentially because the number of degrees of freedom of a spinor grow exponentially quickly. However the number of degrees of freedom of Bosonic fields (such as scalars and vectors) do not grow so quickly, if at all, when the spacetime dimension is increase. Although one can always keep adding in extra scalar modes to keep the Bose-Fermi degeneracy this becomes increasingly unnatural. In fact one finds that if we only wish to consider theories with spins less than two (*i.e.* do not include gravity) then the highest spacetime dimension for which there exists supersymmetric theories is $D = 10$. If we do allow for gravity then this pushes the limit up to $D = 11$.

We can also consider the Witten index:

$$\mathcal{W} = \mathrm{Tr}_{\mathcal{H}}(-1)^F \tag{4.55}$$

where the trace is over all states in the Hilbert space of the theory. This is not necessarily zero. What we have shown above is that there is a Bose-Fermi degeneracy among states with a non-zero energy. However there need not be such a degeneracy between supersymmetric, *i.e.* zero energy, vacuum state and hence

$$\mathcal{W} = \#\text{ of Bosonic vacua} - \#\text{ of Fermionic vacua} \tag{4.56}$$

By definition this is an integer. Therefore, since it cannot be continuously varied, it cannot receive corrections as the coupling constants are varied (so for example it is unaffected by perturbation theory).

Problem: Show that in four-dimensions, where Q_α is a Majorna spinor, the first line of the supersymmetry algebra (4.45) can be written as

$$\begin{aligned}
\{Q_{W\alpha}, Q_{W\beta}\} &= 0 \\
\{Q^*_{W\alpha}, Q^*_{W\beta}\} &= 0 \\
\{Q_{W\alpha}, Q^*_{W\beta}\} &= -((1+\gamma_5)\gamma_\mu C^{-1})_{\alpha\beta} P_\mu \\
\{Q^*_{W\alpha}, Q_{W\beta}\} &= -((1-\gamma_5)\gamma_\mu C^{-1})_{\alpha\beta} P_\mu
\end{aligned} \tag{4.57}$$

where $Q_{W\alpha}$ is a Weyl spinor and $Q^*_{W\alpha}$ is its complex conjugate. (Hint: Weyl spinors are chiral and are obtained from Majorana spinors Q_M through $Q_W = \frac{1}{2}(1+\gamma_5)Q_M$, $Q^*_W = \frac{1}{2}(1-\gamma_5)Q_M$.)

In a Weyl basis for the four-dimensional Clifford algebra one chooses to write, in terms of block 2×2 matrices,

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{4.58}$$

where σ_i are the Pauli matrices. Note that the charge conjugation matrix is still $C = \gamma_0$. Since Weyl spinors only have two independent components one usually introduces a new notation: $a, \dot{a} = 1, 2$ so that a general 4-component Dirac spinor is decomposed in terms of two complex Weyl spinors as

$$\psi_D = \begin{pmatrix} \lambda_a \\ \chi_{\dot{a}} \end{pmatrix} \tag{4.59}$$

i.e. the first two indices are denoted by a and the second two by $\dot{a}$. One then defines $\sigma^{\mu}_{a\dot{b}} = \frac{1}{2}((1-\gamma_5)\gamma^{\mu}C^{-1})_{a\dot{b}}$, $\bar{\sigma}^{\mu}_{\dot{a}b} = \frac{1}{2}((1+\gamma_5)\gamma^{\mu}C^{-1})_{\dot{a}b}$. In this case the algebra is

$$\begin{aligned} \{Q_a, Q_b\} &= 0 \\ \{Q^*_{\dot{a}}, Q^*_{\dot{b}}\} &= 0 \\ \{Q_a, Q^*_{\dot{b}}\} &= -2\sigma^{\mu}_{a\dot{b}}P_{\mu} \\ \{Q^*_{\dot{a}}, Q_b\} &= -2\bar{\sigma}^{\mu}_{\dot{a}b}P_{\mu} \end{aligned} \tag{4.60}$$

Here we have dropped the subscript W since the use of a and $\dot{a}$ indices implies that we are talking about Weyl spinors. This form for the algebra appears in many text books and is also known as the two-component formalism.

Problem: Show that

$$\begin{aligned} (\sigma_{\mu})_{a\dot{b}} &= \left(\delta_{a\dot{b}}, \sigma^{i}_{a\dot{b}}\right) \\ (\bar{\sigma}_{\mu})_{a\dot{b}} &= \left(\delta_{a\dot{b}}, -\sigma^{i}_{a\dot{b}}\right) \end{aligned} \tag{4.61}$$

5 The Four-dimensional Wess-Zumino Model

So far we have discussed the supersymmetry algebra and some of its consequences but we have not yet provided an example of an interacting theory, or even a free one, which is supersymmetric! This would be a rather short course if it weren't for the fact that there are many interesting examples. In fact there are supersymmetric versions of just about every theory you could consider. We have already seen that supersymmetry requires a balance of the number of Bosonic and Fermionic degrees of freedom. We will also see that it requires that the various parameters in the interaction Lagrangian are related to each other.

For example in a gauge theory one has gauge Bosons that are always in the adjoint representation of the gauge group and have spin one. Since supersymmetries commute with internal symmetries such as gauge symmetries, it follows that a supersymmetric gauge theory must have Fermions with spin 1/2 in the adjoint representation (non gravitational theories that contain particles with spins greater than 1 are inconsistent). These are generically called gauginos. The gauge theory that underlies the standard model also contains Fermions with spin 1/2 in the fundamental representation of the gauge group, *e.g.* quarks and leptons. Therefore supersymmetric versions of these gauge theories must also have Bosonic spin 0 fields which are in the fundamental representation of the gauge group (if they were spin 1, *i.e.* vector Bosons, renormalizability would require them to be gauge bosons and hence in the adjoint representation). These are called squarks and sleptrons. Finally the standard model contains a Higgs Boson in the fundamental representation of the gauge group. This must therefore come with two Fermionic spin 1/2 Higginos. In addition the Higg's field itself must be complex (rather than a real scalar as is the case in the standard model). Thus a supersymmetric standard model more than doubles the number of particles of the standard model. If such a theory is correct then it must be that these superpartners are too massive to have been detected yet. Thus supersymmetry must be broken. It is these particles the people hope to detect at the LHC.

In this course we will not study supersymmetric gauge theories. It is a beautiful subject and there has been much progress made, especially at the quantum level. However gauge symmetries require more technical details which would detract from the technical details that we wish to deal with. In this course we will content ourselves with models that only incorporate scalar fields and spin 1/2 Fermionic fields (with standard kinetic terms). All of the key points of supersymmetry can be studied in such models.

5.1 The Free Theory

The simplest field content of a four-dimensional supersymmetric model contains a single Majorana Fermion $\psi(x)$ along with some scalar fields. Off-shell a Majorana Fermion has 4 real components (as opposed to a Dirac Fermion with 4 complex components) but these are reduced to 2 on-shell. Thus the field content must also have two real scalars $A(x)$ and $B(x)$. Why are the Fermionic degrees of Freedom reduced on-shell? Consider a free massive Fermion. The equation of motion is

$$\gamma^\mu \partial_\mu \psi - m\psi = 0 \tag{5.62}$$

If we go to the rest frame then this becomes

$$(i\gamma^0 E - m)\psi = 0 \qquad \Longleftrightarrow \qquad \left(i\gamma^0 - \frac{m}{E}\right)\psi = 0 \tag{5.63}$$

Since $i\gamma^0$ is Hermitian with eigenvalues ± 1 we see that $E = \pm m$. Furthermore since $i\gamma^0$ is traceless this equation projects out precisely half of the components of ψ, leaving two independent on-shell degrees of freedom.

We start by considering the free theory. The Lagrangian is

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu A \partial^\mu A - \frac{1}{2}\partial_\mu B \partial^\mu B - \frac{i}{2}\bar{\psi}\gamma^\mu \partial_\mu \psi \tag{5.64}$$

The Bosonic part should require no explanation and leads to the equation of motion $\partial^2 A = \partial^2 B = 0$. Let us review the Fermionic part. First why the factor of i? You should think of it as associated to $P_\mu = -i\partial_\mu$. In particular if we take the complex conjugate we find

$$\begin{aligned}(\frac{i}{2}\bar\psi\gamma^\mu\partial_\mu\psi)^* &= (\frac{i}{2}\psi_\alpha(C\gamma^\mu)^{\alpha\beta}\partial_\mu\psi_\beta)^* \\ &= -\frac{i}{2}\partial_\mu\psi_\beta(C\gamma^\mu)^{\alpha\beta}\psi_\alpha \\ &= \frac{i}{2}\psi_\alpha(C\gamma^\mu)^{\alpha\beta}\partial_\mu\psi_\beta \\ &= \frac{i}{2}\bar\psi\gamma^\mu\partial_\mu\psi \end{aligned} \tag{5.65}$$

An alternative is to note that $C\gamma_\mu$ is real and symmetric and hence we have, continuing from the second line,

$$\begin{aligned}(\frac{i}{2}\bar\psi\gamma^\mu\partial_\mu\psi)^* &= -\frac{i}{2}\partial_\mu\psi_\beta(C\gamma^\mu)^{\beta\alpha}\psi_\alpha \\ &= -\frac{i}{2}\partial_\mu(\psi_\beta(C\gamma_\mu)^{\beta\alpha}\psi_\alpha) + \frac{i}{2}\psi_\beta(C\gamma^\mu)^{\beta\alpha}\partial_\mu\psi_\alpha \\ &= \frac{i}{2}\bar\psi\gamma^\mu\partial_\mu\psi \end{aligned} \tag{5.66}$$

where we have dropped a total derivative, which actually vanishes since $\psi_\alpha\psi_\beta = -\psi_\beta\psi_\alpha$. Thus the Lagrangian is real.

Next let us calculate the equation of motion. Varying with respect to ψ gives

$$\begin{aligned}\delta_\psi\mathcal{L}_0 &= -\frac{i}{2}\delta\psi_\alpha(C\gamma_\mu)^{\alpha\beta}\partial^\mu\psi_\beta - \frac{i}{2}\psi_\alpha(C\gamma_\mu)^{\alpha\beta}\partial^\mu\delta\psi_\beta \\ &= -\frac{i}{2}\delta\psi_\alpha(C\gamma_\mu)^{\alpha\beta}\partial^\mu\psi_\beta + \frac{i}{2}\partial^\mu\psi_\alpha(C\gamma_\mu)^{\alpha\beta}\delta\psi_\beta - \frac{i}{2}\partial^\mu(\psi_\alpha(C\gamma_\mu)^{\alpha\beta}\delta\psi_\beta) \\ &= -i\delta\psi_\alpha(C\gamma_\mu)^{\alpha\beta}\partial^\mu\psi_\beta - \frac{i}{2}\partial^\mu(\psi_\alpha(C\gamma_\mu)^{\alpha\beta}\delta\psi_\beta) \end{aligned} \tag{5.67}$$

Here we have used the fact that $C\gamma_\mu$ is symmetric. Assuming boundary conditions that allow us to drop the total derivative (as one always does) leads to equation of motion

$$\gamma^\mu\partial_\mu\psi = 0 \tag{5.68}$$

Thus everything is as it should be.

This action is supersymmetric. The supersymmetry variation is

$$\begin{aligned}\delta A &= i\bar\epsilon\psi \\ \delta B &= -\bar\epsilon\gamma_5\psi \\ \delta\psi &= \gamma^\mu\partial_\mu A\epsilon + i\gamma^\mu\gamma_5\partial_\mu B\epsilon \end{aligned} \tag{5.69}$$

Problem: Show that these variations preserve the reality of A, B and ψ.

To establish the invariance of the action we calculate, using the last line of (5.67), and dropping a total derivative,

$$\begin{aligned} \delta\mathcal{L}_0 &= -\partial_\mu A\partial^\mu\delta A - \partial_\mu B\partial^\mu\delta B - i\delta\psi_\alpha(C\gamma^\mu)^{\alpha\beta}\partial_\mu\psi_\beta - \frac{i}{2}\partial^\mu(\psi_\alpha C\gamma_\mu\delta\psi) \\ &= -i\bar\epsilon\partial_\mu A\partial^\mu\psi + \bar\epsilon\partial_\mu B\gamma_5\partial^\mu\psi - i(\partial^\nu A\gamma_\nu\epsilon + i\partial^\nu B\gamma_\nu\gamma_5\epsilon)_\alpha(C\gamma^\mu)^{\alpha\beta}\partial_\mu\psi_\beta \end{aligned} \tag{5.70}$$

To proceed we need to deal with

$$\begin{aligned} (\partial^\nu A\gamma_\nu\epsilon + i\partial^\nu B\gamma_\nu\gamma_5\epsilon)_\alpha &= (\partial^\nu A\epsilon\gamma_\nu^T + i\partial^\nu B\epsilon\gamma_5^T\gamma_\nu^T)_\alpha \\ &= (-\partial^\nu A\epsilon C\gamma_\nu C^{-1} + i\partial^\nu B\epsilon\gamma_5 C\gamma_\nu C^{-1})_\alpha \\ &= -\bar\epsilon(\partial^\nu A\gamma_\nu C^{-1} + i\partial^\nu B\gamma_5\gamma_\nu C^{-1})_\alpha \end{aligned} \tag{5.71}$$

Thus we find

$$\delta\mathcal{L}_0 = -i\bar\epsilon\partial_\mu A\epsilon\partial^\mu\psi + \bar\epsilon\partial_\mu B\gamma_5\partial^\mu\psi + i\bar\epsilon(\partial^\nu A + i\partial^\nu B\gamma_5)\gamma_\nu\gamma^\mu\partial_\mu\psi \tag{5.72}$$

Finally we observe that $\gamma_\nu\gamma^\mu = \delta_\nu^\mu + \gamma_\nu{}^\mu$ and hence

$$\begin{aligned} \delta\mathcal{L}_0 &= i\bar\epsilon(\partial_\nu A + i\partial_\nu B\gamma_5)\gamma^{\nu\mu}\partial_\mu\psi \\ &= \partial_\mu\left(i\bar\epsilon(\partial_\nu A + i\partial_\nu B\gamma_5)\gamma^{\nu\mu}\psi\right) \end{aligned} \tag{5.73}$$

which is a total derivative. Therefore we see that we indeed have a symmetry of the action.

However it is important to check the algebra of this symmetry to see that we really have a supersymmetry. To this end we must evaluate $[\delta_1, \delta_2]$ acting on the fields. Note that although we are talking about a supersymmetry, the total variation does not change the spin of the the field and can be written in terms of $\bar\epsilon Q = \bar Q\epsilon$ which is Bosonic operator and hence obeys commutation relations. The anti-commutators arise because

$$\begin{aligned} [\delta_1, \delta_2] &= [\bar\epsilon_1^\alpha Q_\alpha\ , \bar\epsilon_2^\beta Q_\beta] \\ &= \bar\epsilon_1^\alpha\bar\epsilon_2^\beta\{Q_\alpha, Q_\beta\} \end{aligned} \tag{5.74}$$

i.e. due to the anti-commutation of ϵ_1 and ϵ_2. We find, for the Bosons,

$$\begin{aligned} [\delta_1, \delta_2]A &= i\bar\epsilon_2(\gamma^\mu\partial_\mu A\epsilon_1 + i\gamma^\mu\gamma_5\partial_\mu B\epsilon_1) - (1 \leftrightarrow 2) \\ &= i(\bar\epsilon_2\gamma_\mu\epsilon_1 - \bar\epsilon_1\gamma_\mu\epsilon_2)\partial_\mu A - (\bar\epsilon_2\gamma_\mu\gamma_5\epsilon_1 - \bar\epsilon_1\gamma_\mu\gamma_5\epsilon_2)\partial_\mu B \\ [\delta_1, \delta_2]B &= -\bar\epsilon_2\gamma_5(\gamma^\mu\partial_\mu A\epsilon_1 + i\gamma^\mu\gamma_5\partial_\mu B\epsilon_1) - (1 \leftrightarrow 2) \\ &= (\bar\epsilon_2\gamma_\mu\gamma_5\epsilon_1 - \bar\epsilon_1\gamma_\mu\gamma_5\epsilon_2)\partial_\mu A + i(\bar\epsilon_2\gamma_\mu\epsilon_1 - \bar\epsilon_1\gamma_\mu\epsilon_2)\partial_\mu B \end{aligned} \tag{5.75}$$

Next we recall that

$$(C\gamma_\mu)^T = C\gamma_\mu \tag{5.76}$$

and hence

$$(C\gamma_\mu\gamma_5)^T = \gamma_5^T(C\gamma_\mu)^T = -\gamma_5(C\gamma_\mu) = -C\gamma_\mu\gamma_5 \tag{5.77}$$

Since ϵ_1 and ϵ_2 are anti-commuting we see that

$$\bar{\epsilon}_2\gamma_\mu\gamma_5\epsilon_1 - \bar{\epsilon}_1\gamma_\mu\gamma_5\epsilon_2 = 0 \tag{5.78}$$

and hence

$$\begin{aligned} [\delta_1, \delta_2]A &= v_\mu\partial^\mu A \\ [\delta_1, \delta_2]B &= v_\mu\partial^\mu B \end{aligned} \tag{5.79}$$

with

$$\begin{aligned} v_\mu &= i\bar{\epsilon}_2\gamma_\mu\epsilon_1 - i\bar{\epsilon}_1\gamma_\mu\epsilon_2 \\ &= -2i\bar{\epsilon}_1\gamma_\mu\epsilon_2 \end{aligned} \tag{5.80}$$

Thus we see that indeed the anti-commutator of two supersymmetries, acting on the Bosons, gives a translation.

We are left with the Fermion. We find

$$[\delta_1, \delta_2]\psi = i\gamma^\mu(\bar{\epsilon}_1\partial_\mu\psi)\epsilon_2 - i\gamma^\mu\gamma_5(\bar{\epsilon}_1\gamma_5\partial_\mu\psi)\epsilon_2 - (1 \leftrightarrow 2) \tag{5.81}$$

So here's the rub, we need to move spinors around so that the free index rests on ψ. This is why we developed the Fierz transformation. In four dimensions the independent γ-matrix products are

$$1, \qquad \gamma_\mu, \qquad \gamma_5, \qquad \gamma_\mu\gamma_5, \qquad \gamma_{\mu\nu} \tag{5.82}$$

together these provide $1 + 4 + 1 + 4 + 6 = 16 = 4^2$ matrices as required. Therefore the Fierz identity is

$$\begin{aligned} (\bar{\lambda}\rho)\chi_\alpha &= -\frac{1}{4}(\bar{\lambda}\chi)\rho_\alpha - \frac{1}{4}\left(\bar{\lambda}\gamma_\mu\chi\right)(\gamma^\mu\rho)_\alpha - \frac{1}{4}\left(\bar{\lambda}\gamma_5\chi\right)(\gamma_5\rho)_\alpha \\ &\quad + \frac{1}{4}\left(\bar{\lambda}\gamma_\mu\gamma_5\chi\right)(\gamma^\mu\gamma_5\rho)_\alpha + \frac{1}{8}\left(\bar{\lambda}\gamma_{\mu\nu}\chi\right)(\gamma^{\mu\nu}\rho)_\alpha \end{aligned} \tag{5.83}$$

Note the extra factor of $\frac{1}{2}$ in the last term. This compensates for the over counting the arises from the fact that $\gamma_{\mu\nu} = -\gamma_{\nu\mu}$. In our case we need

$$\begin{aligned} (\bar{\epsilon}_1\rho)\epsilon_{2\alpha} - (1 \leftrightarrow 2) &= -\frac{1}{4}(\bar{\epsilon}_1\epsilon_2)\rho_\alpha - \frac{1}{4}(\bar{\epsilon}_1\gamma_\mu\epsilon_2)(\gamma^\mu\rho)_\alpha - \frac{1}{4}(\bar{\epsilon}_1\gamma_5\epsilon_2)(\gamma_5\rho)_\alpha \\ &\quad + \frac{1}{4}(\bar{\epsilon}_1\gamma_\mu\gamma_5\epsilon_2)(\gamma^\mu\gamma_5\rho)_\alpha + \frac{1}{8}(\bar{\epsilon}_1\gamma_{\mu\nu}\epsilon_2)(\gamma^{\mu\nu}\rho)_\alpha - (1 \leftrightarrow 2) \end{aligned} \tag{5.84}$$

where $\rho = \partial_\mu\psi$ in one case and $\rho = \gamma_5\partial_\mu\psi$ in the other. Things simplify because the anti-commutivity of ϵ_1 and ϵ_2 implies that we need only keep terms in the expansion for which $(C\gamma_\Gamma)^T = C\gamma_\Gamma$ and only $\gamma_\Gamma = \gamma_\mu, \gamma_{\mu\nu}$ satisfies this. Thus we find

$$\begin{aligned}(\bar\epsilon_1\rho)\epsilon_{2\alpha} - (1 \leftrightarrow 2) &= -\frac{1}{2}(\bar\epsilon_1\gamma_\mu\epsilon_2)(\gamma^\mu\rho)_\alpha + \frac{1}{4}(\bar\epsilon_1\gamma_{\mu\nu}\epsilon_2)(\gamma^{\mu\nu}\rho)_\alpha \\ &= -\frac{i}{4}v_\nu(\gamma^\nu\rho)_\alpha + \frac{1}{4}\omega_{\nu\lambda}(\gamma^{\nu\lambda}\rho)_\alpha\end{aligned} \tag{5.85}$$

where $\omega_{\nu\lambda} = \bar\epsilon_1\gamma_{\nu\lambda}\epsilon_2$. Substituting this back leads to

$$\begin{aligned}[\delta_1,\delta_2]\psi &= \frac{1}{4}\gamma^\mu v_\nu\gamma^\nu\partial_\mu\psi - \frac{i}{4}\gamma^\mu\omega_{\nu\lambda}\gamma^{\nu\lambda}\partial_\mu\psi \\ &\quad -\frac{1}{4}\gamma^\mu\gamma_5 v_\nu\gamma^\nu\gamma_5\partial_\mu\psi + \frac{i}{4}\gamma^\mu\gamma_5\omega_{\nu\lambda}\gamma^{\nu\lambda}\gamma_5\partial_\mu\psi \\ &= \frac{1}{2}v_\nu\gamma^\mu\gamma^\nu\partial_\mu\psi \\ &= \frac{1}{2}v_\nu(2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu)\partial_\mu\psi \\ &= v_\nu\partial^\nu\psi - \frac{1}{2}v_\nu\gamma^\nu\gamma^\mu\partial_\mu\psi\end{aligned} \tag{5.86}$$

The second term is not correct, *i.e.* it does not correspond to a term in the supersymmetry algebra. However this term vanishes on-shell. Indeed the supersymmetry algebra couldn't have worked off-shell as there is no longer a Bose-Fermi degeneracy.

Thus we see that, on-shell, $[\delta_1,\delta_2] = v^\mu\partial_\mu$ or, writing $\delta = \bar\epsilon Q$,

$$\begin{aligned}\bar\epsilon_1^\alpha\bar\epsilon_2^\beta\{Q_\alpha, Q_\beta\} &= 2\bar\epsilon_1\gamma_\mu\epsilon_2 P^\mu \\ &= 2\bar\epsilon_1^\alpha\epsilon_{2\beta}(\gamma_\mu)_\alpha{}^\beta P^\mu \\ &= 2\bar\epsilon_1^\alpha\bar\epsilon_2^\gamma C^{-1}_{\gamma\beta}(\gamma_\mu)_\alpha{}^\beta P^\mu \\ &= -2\bar\epsilon_1^\alpha\bar\epsilon_2^\beta(\gamma_\mu C^{-1})_{\alpha\beta}P^\mu\end{aligned} \tag{5.87}$$

since $C^T = -C$. Thus we have recovered the formal supersymmetry algebra.

5.2 Interactions

Let us now consider an interacting model. We can start by postulating the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$ with

$$\mathcal{L}_{int} = -V(A,B) - \frac{i}{2}U_1(A,B)\bar\psi\psi + \frac{1}{2}U_2(A,B)\bar\psi\gamma_5\psi \tag{5.88}$$

and the supersymmetry transformations

$$\begin{aligned}\delta A &= i\bar{\epsilon}\psi \\ \delta B &= -\bar{\epsilon}\gamma_5\psi \\ \delta\psi &= \gamma^\mu\partial_\mu A\epsilon + i\gamma^\mu\gamma_5\partial_\mu B\epsilon + W_1(A,B)\epsilon + i\gamma_5 W_2(A,B)\epsilon\end{aligned} \tag{5.89}$$

It should be clear that the reality of ψ is preserved provided that U_1, U_2 and W_1, W_2 are real functions. If U_1, U_2 are linear and V quartic polynomials in A and B then our ansatz is the most general one that is compatible with renormalizability. In particular ψ has mass dimension 3/2 and A, B have mass dimension 1. Therefore ϵ has mass dimension $-1/2$ and it is clear that, without introducing constants with a negative mass dimension, *i.e.* non-renormalizable terms, there can be no modification to δA and δB. However we will not need to assume this to construct an interacting model.

To continue we calculate, ignoring total derivatives and using the last line of (5.67),

$$\begin{aligned}\delta\mathcal{L}_0 &= \delta'\mathcal{L}_0 \\ &= -i\bar{\psi}\gamma^\mu\partial_\mu(W_1\epsilon + i\gamma_5 W_2\epsilon) \\ &= -i\bar{\psi}\gamma^\mu\partial_A W_1\partial_\mu A\epsilon - i\bar{\psi}\gamma^\mu\partial_B W_1\partial_\mu B\epsilon + \bar{\psi}\gamma^\mu\gamma_5\partial_A W_2\partial_\mu A\epsilon + \bar{\psi}\gamma^\mu\gamma_5\partial_B W_2\partial_\mu B\epsilon\end{aligned} \tag{5.90}$$

where $\delta'\psi = W_1(A,B)\epsilon + i\gamma_5 W_2(A,B)\epsilon$ corresponds to the additional terms in $\delta\psi$ that we didn't take into account in the free theory.

Next we consider the variation of $\mathcal{L}_{int}$. Up to terms that are linear in the Fermions we find

$$\begin{aligned}\delta_1\mathcal{L}_{int} &= -i\partial_A V\bar{\psi}\epsilon + \partial_B V\bar{\psi}\gamma_5\epsilon \\ &\quad -i\bar{\psi}U_1(\gamma^\mu\partial_\mu A + i\gamma^\mu\gamma_5\partial_\mu B + W_1\epsilon + i\gamma_5 W_2\epsilon) \\ &\quad +\bar{\psi}U_2\gamma_5(\gamma^\mu\partial_\mu A + i\gamma^\mu\gamma_5\partial_\mu B + W_1\epsilon + i\gamma_5 W_2\epsilon)\end{aligned} \tag{5.91}$$

where in the first line we have used the fact that, for Majorana spinors, $\bar{\epsilon}\psi = \bar{\psi}\epsilon$ and $\bar{\epsilon}\gamma_5\psi = \bar{\psi}\gamma_5\epsilon$.

Let us start with the linear terms. Note that we must have $\delta'\mathcal{L}_0 + \delta_1\mathcal{L}_{int} = 0$ since the remaining terms are cubic in ψ and these must vanish separately. We can simply arrange for the various linear terms to cancel which leads to the equations

$$\begin{aligned}0 &= -\partial_A V - U_1W_1 + U_2W_2 \\ 0 &= \partial_B V + U_1W_2 + U_2W_1 \\ 0 &= -\partial_A W_1 - U_1 \\ 0 &= -\partial_B W_1 - U_2 \\ 0 &= \partial_A W_2 - U_2 \\ 0 &= \partial_B W_2 + U_1\end{aligned} \tag{5.92}$$

The middle two equations tell us that

$$U_1 = -\partial_A W_1 \qquad U_2 = -\partial_B W_1 \tag{5.93}$$

whereas the last two equations imply $U_1 = -\partial_B W_2$, $U_2 = \partial_A W_2$. This is only consistent if

$$\partial_A W_1 = \partial_B W_2 \qquad \partial_B W_1 = -\partial_A W_2 \tag{5.94}$$

This should be familiar as the Cauchy-Riemann equations. If we write $Z = A + iB$ then we see that $W = W_1 + iW_2$ is a holomorphic function of Z.

The first two equations now become

$$\partial_A V = \partial_A W_1 W_1 + \partial_A W_2 W_2 \qquad \partial_B V = \partial_B W_2 W_2 + \partial_B W_1 W_1 \tag{5.95}$$

which integrates to give

$$V = \frac{1}{2}(W_1^2 + W_2^2) \tag{5.96}$$

where we have, without loss of generality, set the integration constant to zero.

Lastly we must consider the cubic terms

$$\delta_3 \mathcal{L}_{int} = \frac{1}{2}(\bar\psi\psi)\partial_A U_1 \bar\epsilon\psi + \frac{i}{2}(\bar\psi\psi)\partial_B U_1 \bar\epsilon\gamma_5\psi + \frac{i}{2}(\bar\psi\gamma_5\psi)\partial_A U_2 \bar\epsilon\psi - \frac{1}{2}(\bar\psi\gamma_5\psi)\partial_B U_2 \bar\epsilon\gamma_5\psi \tag{5.97}$$

Again we need to use the Fierz rearrangement (5.83) but this time with we require that $(C\gamma_\Gamma)^T = -(C\gamma_\Gamma)^T$ and so

$$\begin{aligned}
(\bar\psi\psi)\psi &= -\frac{1}{4}(\bar\psi\psi)\psi - \frac{1}{4}(\bar\psi\gamma_5\psi)\gamma_5\psi + \frac{1}{4}(\bar\psi\gamma_\mu\gamma_5\psi)\gamma^\mu\gamma_5\psi \\
(\bar\psi\psi)\gamma_5\psi &= -\frac{1}{4}(\bar\psi\gamma_5\psi)\psi - \frac{1}{4}(\bar\psi\psi)\gamma_5\psi - \frac{1}{4}(\bar\psi\gamma_\mu\gamma_5\psi)\gamma^\mu\psi \\
(\bar\psi\gamma_5\psi)\psi &= -\frac{1}{4}(\bar\psi\psi)\gamma_5\psi - \frac{1}{4}(\bar\psi\gamma_5\psi)\psi + \frac{1}{4}(\bar\psi\gamma_\mu\gamma_5\psi)\gamma^\mu\psi \\
(\bar\psi\gamma_5\psi)\gamma_5\psi &= -\frac{1}{4}(\bar\psi\gamma_5\psi)\gamma_5\psi - \frac{1}{4}(\bar\psi\psi)\psi - \frac{1}{4}(\bar\psi\gamma_\mu\gamma_5\psi)\gamma^\mu\gamma_5\psi
\end{aligned} \tag{5.98}$$

Substituting the fourth line into the first leads to

$$\begin{aligned}
(\bar\psi\psi)\psi &= -\frac{1}{5}(\bar\psi\gamma_5\psi)\gamma_5\psi + \frac{1}{5}(\bar\psi\gamma_\mu\gamma_5\psi)\gamma^\mu\gamma_5\psi \\
&= -\frac{1}{5}\left(-\frac{1}{5}(\bar\psi\psi)\psi + \frac{1}{5}(\bar\psi\gamma_\mu\gamma_5\psi)\gamma^\mu\gamma_5\psi\right) + \frac{1}{5}(\bar\psi\gamma_\mu\gamma_5\psi)\gamma^\mu\gamma_5\psi
\end{aligned} \tag{5.99}$$

hence

$$(\bar\psi\psi)\psi = \frac{1}{4}(\bar\psi\gamma_\mu\gamma_5\psi)\gamma^\mu\gamma_5\psi \tag{5.100}$$

Substituting this into the fourth equation gives

$$(\bar{\psi}\gamma_5\psi)\gamma_5\psi = -\frac{1}{4}(\bar{\psi}\gamma_\mu\gamma_5\psi)\gamma^\mu\gamma_5\psi \tag{5.101}$$

So in particular $(\bar{\psi}\gamma_5\psi)\gamma_5\psi = -(\bar{\psi}\psi)\psi$. Similarly (the calculation is formally the same) one finds $(\bar{\psi}\gamma_5\psi)\psi = -(\bar{\psi}\psi)\gamma_5\psi$.

So all this, once substituted into the condition $\delta_3\mathcal{L}_{int} = 0$, implies

$$\partial_A U_1 = -\partial_B U_2 \qquad \partial_A U_2 = \partial_B U_1 \tag{5.102}$$

but these equations are automatically true as a consequence of the previous constraint that W is a holomorphic function of Z. Well that was a lot of work for nothing! Except that we have now completely established the supersymmetry of the following action

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi \\ &\quad -\frac{1}{2}(W_1^2 + W_2^2) + \frac{i}{2}\bar{\psi}\partial_A W_1\psi - \frac{1}{2}\bar{\psi}\gamma_5\partial_B W_1\psi\end{aligned} \tag{5.103}$$

under the transformations (5.89), provided that $W_1 + iW_2$ is a holomorphic function of $A + iB$. It is remarkable that a holomorphic structure has emerged, seemingly out of nowhere. This fact is at the heart of why supersymmetry is so powerful in quantum field theory. The holomorphic function $\mathcal{W}$ defined, up to an irrelevant constant, by

$$\frac{\partial\mathcal{W}}{\partial Z} = W \tag{5.104}$$

is called the superpotential and plays very important role in four-dimensional supersymmetric theories. We see that each choice of W determines all the interactions in the supersymmetric Wess-Zumino model.

Note that since W is holomorphic its image is the entire complex plane, with the exception of some isolated points. Therefore, even if W never vanishes it must come arbitrarily close to zero. Thus although it is possible to spontaneously break supersymmetry, for example by takeing $W = 1/Z$, there will always be, in some sense, a supersymmetric vacuum at infinity which the system will eventually roll towards (perhaps by tunneling out of a non-supersymmetric vacuum with positive energy).

Problem: Examine the closure of (5.89) to reproduce the supersymmetry algebra on-shell. In so doing deduce the Fermion equation of motion as well as the conditions

$$\partial_A W_1 = \partial_B W_2 \qquad \partial_B W_1 = -\partial_A W_2 \tag{5.105}$$

6 Extended Supersymmetry

6.1 The Extended Superalgebra and it's Representation

Up to this point we have been considering the so-called $N = 1$ supersymmetry algebra. This means that there is a single supersymmetry generator Q_α. However it is possible to have multiple supersymmetry generators Q^I_α where $I = 1, ..., N$. The superalgebra now takes the form

$$\{Q^I_\alpha, Q^J_\beta\} = -2\delta^{IJ}(\gamma^\mu C^{-1})_{\alpha\beta}P_\mu \tag{6.106}$$

in addition to all the terms involving the Poincare generators. This is just N direct copies of the minimal $N = 1$ superalgebra and is called the N-extended super-Poincare algebra. Again, technically this form for the algebra is specific to Majorana Spinors but in spirit it applies more generally.

Let us repeat the analysis above to determine the representations of extended supersymmetry. We start with the massive case with no central charges. Going to the rest frame we find

$$\{Q^I_\alpha, Q^J_\beta\} = 2\delta^{IJ}\delta_{\alpha\beta}m \tag{6.107}$$

Rescaling by $m^{\frac{1}{2}}$ will lead to a Clifford algebra in $2^{[D]/2}N$ dimensions. Thus the representations have the same structure as in the $N = 1$ case only much larger and are called long representations.

Similarly for the massless case one finds

$$\{Q^I_\alpha, Q^J_\beta\} = 2\delta^{IJ}E(1 + \gamma_{01})_{\alpha\beta} \tag{6.108}$$

which will give a Clifford algebra in $\frac{1}{2}2^{[D]/2}N$ dimensions.

For example in four dimensions and $N = 2$ supersymmetry for massive states one finds a Clifford algebra in 8 dimensions and hence $2^4 = 16$ states, 8 of which are Bosons and the other 8 Fermions (again this is on shell). In the massless case one finds a Clifford algebra in 4 dimensions and hence $2^2 = 4$ states. Again these states may have to be supplemented by their CPT conjugates.

6.2 The Wess-Zumino Model in Two-Dimensions

Extended supersymmetry and central charges can occur in all spacetime dimensions up to ten. However since we wish to keep things simple in this course and only consider supersymmetric theories with scalars and Fermions (and cannonical kinetic terms) we must go to lower dimensions. Why? Well it turns out that demanding that the Wess-Zumino model in four dimensions admits $N = 2$ supersymmetry implies that there are no interactions, at most one can add a mass term. There are however interesting and important theories, so-called σ-models, where the kinetic terms are non-standard and interactions can be introduced. The study of these beautiful models would make for a course in itself.

To start we can simply dimensionally reduce the four-dimensional Wess-Zumino model to a lower dimension, say $D=2$ (this is the most popular dimension that is less than four). What do we mean by this? It is clear that the four-dimensional Poincare algebra also contains a two-dimesional Poincare subalgebra. By restricting states that carry no momentum along $x^2$and x^3 one breaks the four-dimensional Poincare algebra to such a two-dimesional Poincare subalgebra. In other words we simply assume that A, B and ψ are functions of x^0 and x^1 but not x^2 and x^3. This is dimensional reduction. Clearly this will not affect supersymmetry. However Majorana spinors in four dimensions have four degrees of freedom whereas those in two dimensions have two. Thus a single spinor in four dimensions reduces to two spinors in two dimensions. In other words if we dimensionally reduce our Wess-Zumino Lagrangian to two-dimensions it will be invariant under two supersymmetries.

The reduced Lagrangian is simply

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi \\
&\quad -\frac{1}{2}(W_1^2+W_2^2) + \frac{i}{2}\bar{\psi}\partial_A W_1\psi - \frac{1}{2}\bar{\psi}\gamma_5\partial_B W_1\psi
\end{aligned} \tag{6.109}$$

where now $\mu = 1,2$ only, *i.e.* all we have done is set $\partial_2 = \partial_3 = 0$! However this is not the best way to write things because the spinor indices are $\alpha,\beta = 1,2,3,4$ whereas in two dimensions spinor indicies should be $\alpha,\beta = 1,2$, in other words we do not have irreducible representations of the Clifford algebra. To this end we write (*cf* (3.18))

$$\begin{aligned}
\gamma_\mu &= 1\otimes\gamma_\mu \\
\gamma_5 &= \sigma_2\otimes\sigma_3 \\
\epsilon &= \eta_I\otimes\epsilon^I \\
\psi &= \eta_I\otimes\psi^I
\end{aligned} \tag{6.110}$$

Note that we have also denoted the two-dimensional γ-matrices by γ_μ *i.e.* $\gamma_0 = i\sigma^2$, $\gamma_1 = \sigma_1$. It should be clear from the context what we mean. In addition η_I, $I = 1,2$ are a basis of (commuting) spinors in two Euclidean dimensions. Explicitly we will take

$$\eta_1 = \begin{pmatrix}1\\0\end{pmatrix} \qquad \eta_2 = \begin{pmatrix}0\\1\end{pmatrix} \tag{6.111}$$

We now have

$$\bar{\psi} = \eta_J^T\otimes(\psi^J)^T\gamma_0 = \eta_J^T\otimes\bar{\psi}^J \qquad \bar{\psi}\gamma_5 = \eta_J^T\sigma_2\otimes(\psi^J)^T\rho_0\sigma_3 = \eta_J^T\sigma_2\otimes\bar{\psi}^J\gamma_3 \tag{6.112}$$

where on the right hand side of these equations we have take a bar to denote the Dirac conjugate in two dimensions and replaced σ_3 with the two-dimensional chirality operator

γ_3 (again not to be confused with γ_3 of the four-dimensional Clifford algebra. Thus we find

$$\begin{aligned} \bar{\psi}\psi &= (\eta_I^T\eta_J)\bar{\psi}^I\psi^J \\ &= \delta_{IJ}\bar{\psi}^I\psi^J \\ \bar{\psi}\gamma_5\psi &= (\eta_I^T\sigma_2\eta_J)\bar{\psi}^I\gamma_3\psi^J \\ &= -i\varepsilon_{IJ}\bar{\psi}^I\gamma_3\psi^J \end{aligned} \tag{6.113}$$

so that the Lagrangian can be written in terms of two-dimensional spinors as

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{i}{2}\delta_{IJ}\bar{\psi}^I\gamma^\mu\partial_\mu\psi^J \\ &\quad -\frac{1}{2}(W_1^2 + W_2^2) + \frac{i}{2}\delta_{IJ}\bar{\psi}^I\partial_A W_1\psi^J + \frac{i}{2}\varepsilon_{IJ}\bar{\psi}^I\gamma_3\partial_B W_1\psi^J \end{aligned} \tag{6.114}$$

Problem: Show that the four-dimensional supersymmetries

$$\begin{aligned} \delta A &= i\bar{\epsilon}\psi \\ \delta B &= -\bar{\epsilon}\gamma_5\psi \\ \delta\psi &= \gamma^\mu\partial_\mu A\epsilon + i\gamma^\mu\gamma_5\partial_\mu B\epsilon + W_1\epsilon + \gamma_5 W_2\epsilon \end{aligned} \tag{6.115}$$

become

$$\begin{aligned} \delta A &= i\delta_{IJ}\bar{\epsilon}^I\psi^J \\ \delta B &= i\epsilon_{IJ}\bar{\epsilon}^I\gamma_3\psi^J \\ \delta\psi^I &= \gamma^\mu\epsilon^I\partial_\mu A + \varepsilon^I{}_J\gamma^\mu\gamma_3\epsilon^J\partial_\mu B + W_1\epsilon^I + \varepsilon^I{}_J W_2\gamma_3\epsilon^J \end{aligned} \tag{6.116}$$

in two dimensions, where now the γ-matices are those of two dimensions. Note that we freely raise and lower I, J indices according to convenience.

From the two dimensional point of view this Lagrangian has $N = 2$ supersymmetry. By construction the supersymmetry algebra of this Lagrangian is the same the four-dimensional one

$$[\delta_1, \delta_2] = 2\bar{\epsilon}_1\gamma_\mu\epsilon_2 P^\mu \tag{6.117}$$

Noting that, in terms of two-dimensional spinors,

$$[\delta_1, \delta_2] = \bar{\epsilon}^\alpha_{I1}\bar{\epsilon}^\alpha_{J2}\{Q^I_\alpha, Q^J_\beta\} \tag{6.118}$$

and

$$2\bar{\epsilon}_1\gamma_\mu\epsilon_2 P^\mu = 2(\eta_I^T\eta_J)\bar{\epsilon}_1^I\gamma_\mu\epsilon_2^J P^\mu = 2\delta^{IJ}\bar{\epsilon}_1^I\gamma_\mu\epsilon_2^J P^\mu \tag{6.119}$$

we see that

$$
\begin{aligned}
\bar{\epsilon}^{\alpha}_{I1}\bar{\epsilon}^{\alpha}_{J2}\{Q^I_\alpha, Q^J_\beta\} &= 2\delta^{IJ}\bar{\epsilon}^I_1\gamma_\mu\epsilon^J_2 P^\mu \\
&= 2\delta^{IJ}\bar{\epsilon}^{\alpha}_{1I}\epsilon_{2J\beta}(\gamma_\mu)_\alpha{}^\beta P^\mu \\
&= 2\delta^{IJ}\bar{\epsilon}^{\alpha}_{1I}\bar{\epsilon}^{\gamma}_{2J}C^{-1}_{\gamma\beta}(\gamma_\mu)_\alpha{}^\beta P^\mu \\
&= -2\delta^{IJ}\bar{\epsilon}^{\alpha}_{I1}\bar{\epsilon}^{\beta}_{2J}(\gamma_\mu C^{-1})_{\alpha\beta}P^\mu
\end{aligned}
\tag{6.120}
$$

which is indeed the $N = 2$ superalgebra.

Note that we find two Bosons and two Fermions in agreement with the counting of states above $2^{[2]/2\cdot 2} = 4$. This is not the most minimal supersymmetric model in two dimensions. Since Majorana spinors have two real components in two dimension the minimal $N = 1$ representation would just have $2^{[2]/2} = 2$ states: a Fermion and a Boson.

Problem: Construct a minimal $N = 1$ supersymmetric model in two dimensions by truncating the $N = 2$ two-dimensional Wess-Zumino model. In particular show that it is consistent to set $W_2 = B = \psi^2 = 0$ and still preserve the supersymmetry generated by ϵ^1 (don't worry about the condition that W is holomorphic, this condition arises by demanding $N = 2$ supersymmetry in two dimensions).

7 Central Extensions and BPS Soliton States

7.1 Central Charges

Extended (and in some cases just $N = 1$) superalgebras naturally admit additional generators $Z^{IJ}_{\alpha\beta}$ such that

$$
\{Q^I_\alpha, Q^J_\beta\} = -2\delta^{IJ}(\gamma^\mu C^{-1})_{\alpha\beta}P_\mu + Z^{IJ}_{\alpha\beta} \tag{7.121}
$$

As always, technically this form for the algebra is specific to Majorana Spinors but in spirit it applies more generally. The second term is known as a central charge. It must be symmetric under the simultaneous transformation $\alpha \leftrightarrow \beta$ and $I \leftrightarrow J$. The central charges are also traceless. Therefore it splits up into a piece that is symmetric in both indices and a piece that is anti-symmetric in both indices:

$$
Z^{IJ}_{\alpha\beta} = U^{(IJ)}_{(\alpha\beta)} + iV^{[IJ]}_{[\alpha\beta]} \tag{7.122}
$$

The word 'central' refers to the fact that the (Bosonic) generators $U^{IJ}_{\alpha\beta}$ and $V^{IJ}_{\alpha\beta}$ commute with all other generators

$$
[U^{IJ}_{\alpha\beta}, \text{anything}] = 0 \qquad [V^{IJ}_{\alpha\beta}, \text{anything}] = 0 \tag{7.123}
$$

Actually this too can be generalized, for example in ten and eleven-dimensional supergravity one finds 'central' charges that break some of the Poincare symmetry. These are carried by p-branes which are extended objects in spacetime with p spatial dimensions. They preserve half of the supersymmetry and a subalgebra of the full Poincare algebra that generates Poincare symmetries in the $(p+1)$ dimensions of their worldvolumes.

Without central charges a massive state cannot preserve any of the supersymmetries. To see this suppose that $\bar{\epsilon}^{\alpha}_I Q^I_{\alpha}|susy>=0$ for some ϵ^I_{α} (which is commuting) then from the algebra we see that, in the rest frame,

$$0=\bar{\epsilon}^{\alpha}_I < susy|\{Q^I_{\alpha},Q^J_{\beta}\}|susy>=2M\bar{\epsilon}^{\beta}_J < susy|susy> \tag{7.124}$$

where we have used the fact that $\bar{\epsilon}^{\alpha}_I Q^I_{\alpha}$ will either annihilate $|susy>$ from the left or from the right. In the massless case we see that this is possible provided that $\bar{\epsilon}^I(1+\gamma_{01})=0$ *i.e.* $\gamma_{01}\epsilon=\epsilon$.

However central charges offer a way out of this. Since $Z^{IJ}_{\alpha\beta}$ is Hermitian we can find a *commuting* spinor $\bar{\epsilon}^{\alpha}_J$ such that $\bar{\epsilon}^{\alpha}_J Z^{IJ}_{\alpha\beta}=z\bar{\epsilon}^I_{\beta}$. Then we see that, in the rest frame,

$$\{Q,Q\}=2QQ^2=2M|\epsilon|^2+z|\epsilon|^2 \tag{7.125}$$

where $Q=\bar{\epsilon}^{\alpha}_I Q^I_{\alpha}$ and $|\bar{\epsilon}|^2=\delta_{\alpha\beta}\delta^{IJ}\bar{\epsilon}^{\alpha}_I\bar{\epsilon}^{\beta}_J$. Note that we take $\bar{\epsilon}^{\alpha}_I$ to be an ordinary commuting spinor so that $|\bar{\epsilon}|^2\geq 0$. Now take the expectation value of this expression

$$< state|Q^2|state>=(M+\frac{1}{2}z)|\bar{\epsilon}|^2 < state|state> \tag{7.126}$$

The left hand side is positive definite and vanishes iff $Q|state>=\bar{\epsilon}^{\alpha}_I Q^I_{\alpha}|state>=0$. Therefore we see that $M\geq -\frac{1}{2}z$. Since $Z^{IJ}_{\alpha\beta}$ is traceless there are equal number of positive and negative eigenvalues z hence we learn that

$$M\geq\frac{1}{2}|z| \tag{7.127}$$

with equality iff $Q|state>=\bar{\epsilon}^{\alpha}_I Q^I_{\alpha}|state>=0$, *i.e.* iff the state preserves some of the supersymmetries.

Let us now consider the effect of turning on central charges on the representations of extended supersymmetry. Generically there is no effect, one will still find an algebra that is isomorphic to the appropriate Clifford algebra in $2^{[D/2]}N$ dimensions with $2^{\frac{1}{2}2^{[D/2]}N}$ states. However there are special cases where the eigenvalues of the central charges agree with the mass eigenvalues. In this case there is cancelation when one goes to the rest frame and $\{Q^I_{\alpha},Q^J_{\beta}\}$ will have zero eigenvalues. This is effectively the same as what happened in the case of massless representations. The result is a Clifford algebra in a lower dimension and hence a smaller representation. In particular if n of the central charge eigenvalues Z are degenerate and equal to the mass m then one finds a Clifford algebra in $2^{[D]/2}N-n$ dimensions. The massless case then corresponds to $n=2^{[D]/2}N/2$. The central charge terms are traceless so that at most $2^{[D]/2}N/2$ of the eigenvalues can be degenerate (and non-vanishing). Therefore the massless case is the smallest possible short representation.

Representations for which the mass is equal to one or more eigenvalues of the the central charge are called short representations and the states are known as BPS states. The important point about them is that since the dimension of the representation is an integer it cannot be altered by varying the parameters of the theory in a continuous fashion. In particular it cannot pick up any quantum corrections. Thus a relation of the form $m = Z$ (that the mass is degenerate with the central charge) is not corrected by quantum effects (although m and hence Z too may well pick up corrections). This turns out to be a very powerful technique in understanding non-perturbative features of supersymmetric quantum theories and M-theory in particular. For example this property is what allows one to calculate the Beckenstein-Hawking entropy of a black hole by counting the number of microstates in string theory. At weak coupling some D-brane states are described by a gauge theory and the degeneracy can be easily counted, whereas at strong coupling such states appear as black hole solutions. If the D-brane is superymmetric then the various states are in short multiplets and hence their counting is unaffected by going to strong coupling.

7.2 BPS States

There is in fact another way to construct such states (which historically came first). Let us look for static Bosonic solutions of the four-dimensional Wess-Zumino model. For the simplest such solutions A and B will only depend on one spatial dimension, say x^3. This is called a domain wall solution because there will be a translational invariance along two spatial dimensions, x^1 and x^2, as well as in time x^0, *i.e.* the solution will look like a wall in spacetime. Thus these solutions will also trivially be solutions of the two dimensional $N = 2$ Wess-Zumino model as well.

In this case we wish to minimize the energy per unit area

$$\mathcal{E} = \frac{1}{2}\int dx^3 (A'^2 + B'^2 + W_1^2 + W_2^2) \qquad (7.128)$$

where a prime denotes differentiation with respect to x^3. It is clear from this expression that the lowest energy solution is obtained by taking A and B to be constant such that $W_1 = W_2 = 0$. This of course is the vacuum (provided that it exists). Note that it preserves all the supersymmetry since

$$\delta A = \delta B = \delta\psi = 0 \qquad (7.129)$$

for any ϵ.

More interesting solutions can be found through a simple and famous trick due to Bogomoln'yi. We rewrite the energy density as

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\int dx^3 \Big((A' - \cos\theta W_1 + \sin\theta W_2)^2 + (B' + \sin\theta W_1 + \cos\theta W_2)^2 \\ &\quad +2A'(\cos\theta W_1 - \sin\theta W_2) - 2B'(\sin\theta W_1 + \cos\theta W_2) \Big) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\int dx^3\Big((A' - \cos\theta W_1 + \sin\theta W_2)^2 + (B' + \sin\theta W_1 + \cos\theta W_2)^2 \\
&\qquad +2\mathrm{Re}((\cos\theta + i\sin\theta)(W_1 + iW_2)(A' + iB'))\Big) \\
&= \frac{1}{2}\int dx^3\Big((A' - \cos\theta W_1 + \sin\theta W_2)^2 + (B' + \sin\theta W_1 + \cos\theta W_2)^2 \\
&\qquad +2\mathrm{Re}(e^{i\theta}\partial\mathcal{W}Z')\Big)
\end{aligned} \tag{7.130}$$

Here θ is a constant angle and we have introduced the holomorphic superpotential $\mathcal{W}$ such that $\partial\mathcal{W} = W_1 + iW_2$. We see that the last term is a total derivative

$$\int dx^3\mathrm{Re}(e^{i\theta}\partial W Z') = \int dx^3\mathrm{Re}(e^{i\theta}W)' = \mathrm{Re}(e^{i\theta}W)\ |_{-\infty}^{\infty} \tag{7.131}$$

Therefore we can write the energy density as

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}\int dx\Big((A' - \cos\theta W_1 + \sin\theta W_2)^2 + (B' + \sin\theta W_1 + \cos\theta W_2)^2\Big) \\
&\quad +\mathrm{Re}(e^{i\theta}\mathcal{W})\ |_{-\infty}^{\infty}
\end{aligned} \tag{7.132}$$

which is of the form of a sum of squares plus a boundary term. Since the squared terms are always positive we have the Bogomoln'yi bound

$$\mathcal{E} \geq \mathrm{Re}(e^{i\theta}\mathcal{W})\ |_{-\infty}^{\infty} \tag{7.133}$$

Furthermore the last term only depends on the boundary conditions it is independent of the behaviour of the fields in the bulk of spacetime. Therefore the minimum energy configuration must come from setting

$$\begin{aligned}
A' &= \cos\theta W_1 - \sin\theta W_2 \\
B' &= -\sin\theta W_1 - \cos\theta W_2
\end{aligned} \tag{7.134}$$

These are a pair of first order differential equations. We can rewrite them as

$$\begin{aligned}
(A + iB)' &= \cos\theta W_1 - \sin\theta W_2 - i(\sin\theta W_1 + \cos\theta W_2) \\
&= (\cos\theta - i\sin\theta)(W_1 - iW_2) \\
i.e. \qquad Z' &= e^{-i\theta}\bar{\partial}\bar{\mathcal{W}}
\end{aligned} \tag{7.135}$$

the bound on the energy is then saturated

$$\mathcal{E} = \mathrm{Re}(e^{i\theta}\mathcal{W})\ |_{-\infty}^{\infty} \tag{7.136}$$

What we have shown is that these equations correspond to the lowest energy state with fixed boundary conditions. Therefore a solution to these equations must be a solution to the equations of motion. However, while a static solution to the equations of motion implies that the energy is a local extremum, these first order, Bogomoln'yi equations imply that, for a fixed boundary condition, the solution is the absolute minimum energy configuration and is therefore totally stable. In general there will be higher energy solutions to the equations of motion with the same boundary conditions and these can be thought of as excited states of the domain wall.

To obtain a finite energy solution it is necessary for the fields to become constant as $x^3 \to \pm\infty$. This implies that the superpotential $\mathcal{W}$ must interpolate between two critical points. In the trivial case where one finds the same vacuum on both sides the boundary term contribution to the energy vanishes and we recover the vacuum solution where Z is constant everywhere. However if there are multiple vacua then there will be domain wall solutions where Z interpolates between two distinct vacua.

Problem: Show that these first order equations imply a solution to the full second order equations of motion.

Problem: Consider the case where $\mathcal{W} = \frac{1}{3}Z^3 - m^2 Z$, where m is real. What are the minima of the potential? Explicitly construct the BPS domain wall solution for case $\theta = 0$, describe its basic features and calculate its energy density. What happens if $\theta = \pi$?

It is no accident that we were able to use Bogomoln'yi's argument in our supersymmetric theory. Recall that it was supersymmetry that told us that the potential could be written as the absolute value squared of a holomorphic function, which was required for the Bogomoln'yi argument. But there is a deeper reason. Let us look at the supersymmetry variations about this domain wall solution. The Bosonic variations automatically vanish as $\psi = 0$. Thus we are left with

$$\begin{aligned} \delta\psi &= \Big(\gamma_3 A' + i\gamma_3\gamma_5 B' + W_1 + i\gamma_5 W_2\Big)\epsilon \\ &= \Big(\gamma_3(\cos\theta W_1 - \sin\theta W_2) + i\gamma_3\gamma_5(-\sin\theta W_1 - \cos\theta W_2) + W_1 + i\gamma_5 W_2\Big)\epsilon \\ &= W_1\Big(\gamma_3\cos\theta - i\gamma_3\gamma_5\sin\theta + 1\Big)\epsilon + W_2\Big(-i\gamma_3\gamma_5\cos\theta - \gamma_3\sin\theta + i\gamma_5\Big)\epsilon \\ &= W_1\gamma_3\Big(\cos\theta - i\gamma_5\sin\theta + \gamma_3\Big)\epsilon - iW_2\gamma_3\gamma_5\Big(\cos\theta - i\gamma_5\sin\theta + \gamma_3\Big)\epsilon \\ &= \Big(W_1\gamma_3 - iW_2\gamma_3\gamma_5\Big)\Big(\cos\theta - i\gamma_5\sin\theta + \gamma_3\Big)\epsilon \end{aligned} \tag{7.137}$$

Thus we see that $\delta\psi = 0$ for supersymmetries such that

$$\Big(\cos\theta - i\gamma_5\sin\theta + \gamma_3\Big)\epsilon = 0 \qquad \Longleftrightarrow \qquad \gamma_3 e^{-i\gamma_5\theta}\epsilon = -\epsilon \tag{7.138}$$

Since $(\gamma_3 e^{-i\gamma_5\theta})^2 = \gamma_3 e^{-i\gamma_5\theta}e^{i\gamma_5\theta}\gamma_3 = 1$ and $\gamma_3 e^{-i\gamma_5\theta}$ is traceless we see that half of the four supersymmetries are preserved by the domain wall.

7.3 The Domain Wall Effective Action

Thus we have indeed found a supersymmetric domain wall solution. In fact there is have a multiplet of domain walls which will fill out a representation the the three-dimensional supersymmetry algebra. It is easy, in principle, to construct these solutions, one simply acts on the solution with the broken supersymmetry generators *i.e.* consider a spinor η such that $\gamma_3 e^{-i\gamma_5\theta}\eta = \eta$.

Explicitly, acting with such a spinor on the Domain wall solution gives (note that we include a factor of $\frac{1}{2}$ for later convenience)

$$\begin{aligned}\psi_0 &= \frac{1}{2}\Big(W_1\gamma_3 - iW_2\gamma_3\gamma_5\Big)\Big(\cos\theta - i\gamma_5\sin\theta + \gamma_3\Big)\eta \\ &= \Big(W_1\gamma_3 - iW_2\gamma_3\gamma_5\Big)\gamma_3\eta \\ &= \Big(W_1 + iW_2\gamma_5\Big)\eta\end{aligned} \tag{7.139}$$

To lowest order in η the Bosonic fields are unchanged. This is called a Fermion zero-mode because it carries no energy. When quantizing the Fermion field in the background this zero-mode leads to a degenerate spectrum of domain walls. By construction this solves the Fermion equation of motion in the domain wall background.

Problem: Explicitly show that ψ_0 solves the Fermion equation of motion in the domain wall background.

There is another zero-mode, this time Bosonic. Since the theory we start with is Lorentz invariant it follows that if $A(x^3)$ and $B(x^3)$ are solutions to the Bogomol'nyi equation then so are $A(a^3 - a)$ and $B(x^3 - a)$ for any a. In other words we can translate the domain wall along the x^3 direction and furthermore this clearly won't change the total energy, *i.e.* the tension, of the domain wall. In general a soliton can have several zero-modes which arise as integration constants when solving the Bogomol'nyi equation. This leads to a moduli space of solutions.

Since the energy of a soliton is unaffected by the value of the moduli it is clear that by adding in an arbitrarily small amount of energy the moduli can be allowed to change slowly in time. Thus they represent massless degrees of freedom of the soliton. In contrast a massive mode always requires that you put in a minimum of energy, equal to the mass, before you can cause the mode to change.

We can calculate the effective action of these moduli by allowing them to depend on the remaining coordinates (x^0, x^1, x^2) of the domain wall and substituting this back into the effective action. For example consider the Bosonic modulus a. Let us suppose that $\partial_\mu a \neq 0$ where now $\mu = 0, 1, 2$. It follows that $\partial_\mu A(x^3 - a) = -A'\partial_\mu a$ and $\partial_\mu B(x^3 - a) = -B'\partial_\mu a$ and hence the Bosonic part of the action is

$$S = -\int d^3x dx^3 \frac{1}{2}(A'^2 + B'^2)\partial_\mu a\partial^\mu a + \frac{1}{2}(A'^2 + B'^2) + \frac{1}{2}(W_1^2 + W_2^2) \tag{7.140}$$

where now $\mu = 0, 1, 2$ and we have used the Bogomoln'yi equation to set $A'^2 + B'^2 = W_1^2 + W_2^2$. Next we note that

$$\frac{1}{2}(A'^2 + B'^2) = \frac{1}{4}(A'^2 + B'^2 + W_1^2 + W_2^2) \tag{7.141}$$

and hence

$$\begin{aligned}\frac{1}{2}\int dx^3(A'^2+B'^2) &= \frac{1}{4}\int dx^3(A'^2+B'^2+W_1^2+W_2^2)\\ &= \frac{1}{2}\mathcal{E}\end{aligned} \tag{7.142}$$

Thus we have derived a three-dimensional effective action for the position a of the domain wall:

$$S_B = -\mathcal{E}\int d^3x\frac{1}{2}\partial_\mu a\partial^\mu a + 1 \tag{7.143}$$

Next we consider the the Fermion zero mode:

$$\psi_0 = \Big(W_1 + iW_2\gamma_5\Big)\eta \tag{7.144}$$

and allow η to vary slowly as a function of x^0, x^1, x^2. Now we see that, for $\mu = 0, 1, 2$,

$$\begin{aligned}\partial_\mu\psi_0 &= \Big(W_1 + iW_2\gamma_5\Big)\partial_\mu\eta + \Big(\partial_\mu W_1 + i\partial_\mu W_2\gamma_5\Big)\eta\\ \psi_0' &= \Big(W_1' + iW_2'\gamma_5\Big)\eta\end{aligned} \tag{7.145}$$

where, for example, $\partial_\mu W_1 = -\partial_A W_1 A'\partial_\mu a - \partial_B W_1 B'\partial_\mu a$.

We note that since $C\gamma_\mu$ is symmetric $\bar\eta\gamma_\mu\eta = 0$ and also, for $\mu = 0, 1, 2$, and using the supersymmetry condition $\gamma_3 e^{-i\gamma_5\theta}\eta = \eta$

$$\begin{aligned}\bar\eta\gamma^\mu\gamma_5\eta &= \eta^T\gamma_0\gamma^\mu\gamma_5\gamma_3 e^{-i\gamma_5\theta}\eta\\ &= -\eta^T\gamma_3 e^{-i\gamma_5\theta}\gamma_0\gamma^\mu\gamma_5\eta\\ &= -\eta^T e^{i\gamma_5\theta}\gamma_3\gamma_0\gamma^\mu\gamma_5\eta\\ &= -(\gamma_3 e^{-i\gamma_5\theta}\eta)^T\gamma_0\gamma^\mu\gamma_5\eta\\ &= -\bar\eta\gamma^\mu\gamma_5\eta\\ &= 0\end{aligned} \tag{7.146}$$

Next we write

$$\bar\psi_0\gamma^\mu = \eta^T(W_1 - i\gamma_5 W_2)\gamma_0\gamma^\mu = \bar\eta\gamma^\mu(W_1 - i\gamma_5 W_2) \tag{7.147}$$

Thus looking at the terms involving derivatives ∂_μ $\mu = 0, 1, 2$ we simply find

$$\bar\psi_0\gamma^\mu\partial_\mu\psi_0 = (W_1^2 + W_2^2)\bar\eta\gamma^\mu\partial_\mu\eta \tag{7.148}$$

Thus the Fermion kinetic term is

$$\begin{aligned}S_F &= -\frac{i}{2}\int d^3x dx^3(W_1^2+W_2^2)\bar\eta\gamma^\mu\partial_\mu\eta\\ &= -\frac{i}{2}\mathcal{E}\int d^3x\bar\eta\gamma^\mu\partial_\mu\eta\end{aligned} \tag{7.149}$$

It remains to calculate $\bar{\psi}_0\psi_0$, $\bar{\psi}_0\gamma_5\psi_0$ and $\bar{\psi}_0\gamma_3\psi_0'$. These terms must vanish since we have seen that ψ_0 is a solution to the equations of motion if η is constant. But it is good to check.

Problem: Explicitly calculate $\bar{\psi}_0\psi_0$, $\bar{\psi}_0\gamma_5\psi_0$ and $\bar{\psi}_0\gamma_3\psi_0'$, where $\psi_0 = (W_1 + iW_2\gamma_5)\eta$ and show that

$$-\frac{i}{2}\bar{\psi}_0\gamma_3\psi_0' + \frac{i}{2}\partial_A W_1\bar{\psi}_0\psi_0 - \frac{1}{2}\partial_A W_2\bar{\psi}_0\gamma_5\psi_0 = 0 \tag{7.150}$$

(HINT: use the property $\gamma_3\eta = (\cos\theta - i\gamma_5\sin\theta)\eta$ to deduce that $\cos\theta\bar{\eta}\eta = i\sin\theta\bar{\eta}\gamma_5\eta$.) If this problem seems too hard try just doing it for $\theta = 0$.

The total effective action is therefore

$$S_{eff} = S_B + S_F = -\int d^3x \left(\frac{1}{2}\partial_\mu a\partial^\mu a + \frac{i}{2}\bar{\eta}\gamma^\mu\partial_\mu\eta + 1\right) \tag{7.151}$$

This is just a free theory. Note that η is a 4-dimensional Marjorana spinor that satisfies a constraint: $\gamma_3 e^{-i\gamma_5\theta}\eta = \eta$. Thus it actually only has 2-independent components. This is the right number for a minimal Majorana spinor in three-dimension. Therefore one can easily see that this action is supersymmetric with two supercharges.

7.4 A Central Charge in the Wess-Zumino Model

Thus we have a solution which preserves a fraction, half, of the supersymmetries. How could this happen? Don't all representations of the four-dimensional supersymmetry algebra have four, no more and no less, generators? Well yes but the point here is that the Poincare symmetry has also been broken, we've lost translational invariance along x^3, as well as boosts and rotations involving x^3. What has been preserved is a superalgebra in three spacetime dimensions. In a sense this algebra lives on the domain wall itself. Since Majorana spinors in three dimensions have only two components there is no contradiction. this is known as the paritial breaking of rigid supersymmetry.

More troubling is the apparent fact that the $N = 1$ algebra does not have any central charges and hence no state can be invariant under a fraction of the supersymmetries (at least in the quantum theory). Well actually there is a central charge but it is rather subtle to construct it. We will do so below for completeness but some details may be a little tricky. To see the subtlety we first observe that the Lagrangian isn't exactly invariant under supersymmetry but rather

$$\delta_\alpha\mathcal{L} = \partial_\mu\Delta^\mu_\alpha \tag{7.152}$$

i.e. it is invariant up to a total derivative. This means that the conserved current is slightly shifted from the usual Noether expression

$$\begin{aligned}\tilde{j}^\mu_\alpha &= j^\mu_\alpha - \Delta^\mu_\alpha \\ &= \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^M}\delta_\alpha\phi^M - \Delta^\mu_\alpha\end{aligned} \tag{7.153}$$

Here we are using ϕ^M as a general symbol to denote all fields. The reason for this is that invariance of the Lagrangian implies

$$\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^M}\partial_\mu\delta_\alpha\phi^M + \frac{\delta\mathcal{L}}{\delta\phi^M}\delta_\alpha\phi^M = \partial_\mu\Delta^\mu_\alpha \tag{7.154}$$

so that, on-shell,

$$\begin{aligned}\partial_\mu\tilde{j}^\mu_\alpha &= \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^M}\right)\delta_\alpha\phi^M + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^M}\partial_\mu\delta_\alpha\phi^M - \partial_\mu\Delta^\mu_\alpha \\ &= \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^M}\right)\delta_\alpha\phi^M - \frac{\delta\mathcal{L}}{\delta\phi^M}\delta_\alpha\phi^M \\ &= 0\end{aligned} \tag{7.155}$$

Thus the conserved charges are modified. This means that the calculation of the commutator of two transformations acting on the fields is not quite the same as calculating the algebra of conserved charges. To date we have passively been assume the two to be the same.

The main reason we haven't noticed this yet is because so far we could avoid this problem by choosing boundary conditions which ensure that the total derivative does not contribute. In the language of differential forms $\Delta = \frac{1}{3!}\epsilon_{\mu\nu\lambda\rho}\Delta^\rho dx^\mu \wedge dx^\nu \wedge dx^\lambda$ is a closed three form. However if there are topologically non-trivial boundary conditions then Δ is not exact and its integral over space will not vanish.

To proceed we need to use the Poisson bracket formalism of classical mechanics that is suitable generalized to field theory (an infinite dimensional system) and Fermions. We will not be too detailed here and simply proceed in analogy with the finite dimensional Bosonic case. The main relation is that for any function f on the phase space one has, on-shell,

$$\frac{df}{dt} = \{f, H\}_{P.B.} \tag{7.156}$$

where $\{\ ,\ \}_{P.B.}$ is Poisson bracket (which has the same properties as the commutator, such as anti-symmetry and the Jacobi identity). A 'conserved charge' Q is just a constant of motion and hence

$$\frac{dQ}{dt} = \{Q, H\}_{P.B.} = 0 \tag{7.157}$$

where H is the Hamiltonian. The symmetry transformation on a function f is generated from the conserved charge by

$$\delta f = \{Q, f\}_{P.B.} \tag{7.158}$$

so that $\delta H = 0$. Two conserved charges always produce a third:

$$\begin{aligned}\frac{d}{dt}\{Q_1, Q_2\}_{P.B.} &= \{\{Q_1, Q_2\}_{P.B.}, H\}_{P.B.} \\ &= -\{\{Q_2, H\}_{P.B}, Q_1\}_{P.B.} - \{\{H, Q_1\}_{P.B.}, Q_2\}_{P.B.} \\ &= 0\end{aligned} \tag{7.159}$$

Thus there is an algebra of conserved charges

$$\{Q_m, Q_n\}_{P.B.} = c_{mn}{}^p Q_p \tag{7.160}$$

where we have used m, n, p to label the conserved charges. In the quantum theory the Poisson bracket is replaced by i times the commutator (or just the anti-commutator for Fermi fields) so that the algebra of charges becomes a familiar (super) Lie-algebra.

Therefore if we shift $\tilde{Q}_m = Q_m - \Delta_m$ we find

$$\begin{aligned}\{\tilde{Q}_m, \tilde{Q}_n\}_{P.B.} &= \{Q_m, Q_n\}_{P.B.} - \{Q_m, \Delta_n\}_{P.B.} - \{\Delta_m, Q_n\}_{P.B.} + \{\Delta_m, \Delta_n\}_{P.B.} \\ &= c_{mn}{}^p Q_p - \{Q_m, \Delta_n\}_{P.B.} + \{Q_n, \Delta_m\}_{P.B.} + \{\Delta_m, \Delta_n\}_{P.B.} \\ &= c_{mn}{}^p \tilde{Q}_p - \delta_m \Delta_n + \delta_n \Delta_m + c_{mn}{}^p \Delta_p + \{\Delta_m, \Delta_n\}_{P.B.}\end{aligned} \tag{7.161}$$

where $\delta_m \Delta_n = \{Q_m, \Delta_n\}_{P.B.}$. Returning to the Wess-Zumino model we need to generalize the previous discussion. There are now infinitely many degrees of freeom (one for each point in space), the charge Q_m is replaced by the charge density $j^0_m(t, x^i)$ and the m index is an anti-commuting spinor index corresponding to supersymmetry. Since

$$\{Q_\alpha, Q_\beta\} = -2(\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \tag{7.162}$$

we have

$$c_{\alpha\beta}{}^\gamma = 0 \qquad c_{\alpha\beta}{}^\rho = -2(\gamma^\rho C^{-1})_{\alpha\beta} \tag{7.163}$$

From the calculations on the invariance of the Wess-Zumino Lagrangian we see that

$$\delta\mathcal{L} = -\frac{i}{2}\partial_\mu(\bar{\psi}\gamma^\mu\delta\psi) \tag{7.164}$$

and hence

$$\Delta^\mu = \bar{\epsilon}^\alpha \Delta^\mu_\alpha = -\frac{i}{2}\bar{\psi}\gamma^\mu\delta\psi \tag{7.165}$$

We also find that translational invariance is only preserved up to a boundary term. In particular under $x^\mu \to x^\mu + v^\mu$ we find $\delta A = v^\nu\partial_\nu A$, $\delta B = v^\nu\partial_\nu B$ and $\delta\psi = v^\nu\partial_\nu\psi$ so that

$$\Delta^\mu = v^\nu \Delta^\mu_\nu = v^\nu T^\mu_{\ \nu} \tag{7.166}$$

where $T_{\mu\nu}$ is the energy momentum tensor.

This is a rather intricate calculation and we will proceed by only considering terms that are independent of the Fermions, in particular we drop the $\{\Delta_m, \Delta_n\}_{P.B.}$ term. Thus putting these pieces together we find

$$\begin{aligned}\{\tilde{j}^0_\alpha(t, x^i), \tilde{j}^0_\beta(t, x'^i)\}_{P.B.} &= -2\delta^3(x^i - x'^i)(\gamma^\nu C^{-1})_{\alpha\beta}\tilde{j}^0_{\ \nu} - 2\delta^3(x^i - x'^i)(\gamma^\nu C^{-1})_{\alpha\beta} T^0_{\ \nu} \\ &\quad -i\delta^3(x^i - x'^i)(\delta_\alpha \Delta^0_\beta + \delta_\beta \Delta^0_\alpha)\end{aligned} \tag{7.167}$$

Note the factor of i that arise because of our conventions for the generators (*e.g.* translations are generated by $\delta\phi^M = iv^\mu P_\mu \phi^M$).

So we must calculate $\delta_\alpha \Delta^0_\beta + \delta_\beta \Delta^0_\alpha$ to this end we observe that, writing $\delta_1 = \bar\epsilon^\alpha_1 \delta_\alpha$, $\delta_2 = \bar\epsilon^\beta_2 \delta_\beta$ and $\Delta^\mu_1 = -\frac{i}{2}\bar\psi\gamma^\mu\delta_1\psi$,

$$\begin{aligned} \delta_2\Delta^\mu_1 &= \bar\epsilon^\beta_2\delta_\beta(\bar\epsilon^\alpha_1\Delta^\mu_\alpha) \\ &= -\bar\epsilon^\beta_2\bar\epsilon^\alpha_1\delta_\beta\Delta^\mu_\alpha \\ &= -\frac{i}{2}\delta_2\bar\psi\gamma^\mu\delta_1\psi - \frac{i}{2}\bar\psi^T\delta_2\delta_1\psi \end{aligned} \tag{7.168}$$

Therefore

$$\begin{aligned} \delta_2\Delta^\mu_1 - \delta_1\Delta^\mu_2 &= -\bar\epsilon^\beta_2\bar\epsilon^\alpha_1(\delta_\beta\Delta^\mu_\alpha + \delta_\alpha\Delta^\mu_\beta) \\ &= -i\delta_2\bar\psi\gamma^\mu\delta_1\psi - \frac{i}{2}\bar\psi\gamma^\mu[\delta_2,\delta_1]\psi \end{aligned} \tag{7.169}$$

and in particular

$$\bar\epsilon^\beta_2\bar\epsilon^\alpha_1(\delta_\beta\Delta^0_\alpha + \delta_\alpha\Delta^0_\beta) = i\delta_2\psi^T\delta_1\psi \tag{7.170}$$

where we've dropped the last term since it contains Fermions.

Problem: Show that, if the Fermions vanish, then

$$\delta_2\psi^T\delta_1\psi = 2(\gamma^\nu C^{-1})_{\alpha\beta}T^0{}_\nu - 2(\partial_i\mathcal{T}\gamma^i)_{\alpha\beta} \tag{7.171}$$

where

$$\mathcal{T} = \mathrm{Re}\mathcal{W} + i\gamma_5\mathrm{Im}\mathcal{W} \tag{7.172}$$

Thus it follows that

$$-i\delta_\beta\Delta^0_\alpha - i\delta_\alpha\Delta^0_\beta = 2(\gamma^\nu C^{-1})_{\alpha\beta}T^0{}_\nu - 2(\partial_i\mathcal{T}\gamma^i)_{\alpha\beta} \tag{7.173}$$

Here we see that the first term cancels the $T^0{}_\nu$ contribution in (7.167). Therefore we find

$$\{\tilde{j}^0_\alpha(t,x^i),\tilde{j}^0_\beta(t,x'^i)\}_{P.B.} = -2\delta^3(x^i - x'^i)(\gamma^\nu C^{-1})_{\alpha\beta}\tilde{j}^0{}_\nu - 2\delta^3(x^i-x'^i)(\partial_i\mathcal{T}\gamma^i)_{\alpha\beta} \tag{7.174}$$

If we had considered terms involving the Fermions then one simply finds that the $\bar\psi\gamma^\mu[\delta_2,\delta_1]\psi$ and $\{\Delta^0_\alpha,\Delta^0_\beta\}_{P.B.}$ terms that we dropped cancel the free and interacting Fermionic terms in $T^0{}_\nu$ respectively.

To find the charge algebra we now integrate this over d^3x an d^3x' to find

$$\{Q_\alpha,Q_\beta\} = -2(\gamma^\mu C^{-1})_{\alpha\beta}P_\mu - 2\int d^3x\partial_i(\mathcal{T}\gamma^i)_{\alpha\beta} \tag{7.175}$$

We finally find the elusive central charge

$$Z_{\alpha\beta} = -2\int d^3x\partial_i(\mathcal{T}\gamma^i)_{\alpha\beta} \tag{7.176}$$

This is a topological charge - *i.e.* it is only non-vanishing if the fields interpolate between distinct vacuum states. Note that it is not quite central as it does not commute with the Lorentz generators, but this is common. This non-Lorentz invariance is reflected in the fact that the states that carry this charge, *e.g.* our domain wall, necessarily break some of the Lorentz symmetry.

If we consider our domain walls in the rest frame then the existence of a supersymmetry implies

$$\begin{aligned}
\bar{\epsilon}(\mathcal{E} - \gamma_3\mathrm{Re}\mathcal{W}\mid_{-\infty}^{\infty} - i\gamma_5\gamma_3\mathrm{Im}\mathcal{W}\mid_{-\infty}^{\infty}) &= 0\\
(\mathcal{E} - \gamma_3\mathrm{Re}\mathcal{W}\mid_{-\infty}^{\infty} + i\gamma_3\gamma_5\mathrm{Im}\mathcal{W}\mid_{-\infty}^{\infty})C^T\epsilon &= 0\\
(\mathcal{E} + \gamma_3\mathrm{Re}\mathcal{W}\mid_{-\infty}^{\infty} + i\gamma_3\gamma_5\mathrm{Im}\mathcal{W}\mid_{-\infty}^{\infty})\epsilon &= 0
\end{aligned} \tag{7.177}$$

We note that

$$\left(\gamma_3\mathrm{Re}\mathcal{W}\mid_{-\infty}^{\infty} + i\gamma_3\gamma_5\mathrm{Im}\mathcal{W}\mid_{-\infty}^{\infty}\right)^2 = (\mathrm{Re}\mathcal{W}\mid_{-\infty}^{\infty})^2 + (\mathrm{Im}\mathcal{W}\mid_{-\infty}^{\infty})^2 = \left|\mathcal{W}\mid_{-\infty}^{\infty}\right|^2 \tag{7.178}$$

and since $\gamma_3\mathrm{Re}\mathcal{W}\mid_{-\infty}^{\infty} + i\gamma_3\gamma_5\mathrm{Im}\mathcal{W}\mid_{-\infty}^{\infty}$ is Hermitian its eigenvalues are $\pm\left|\mathcal{W}\mid_{-\infty}^{\infty}\right|$. Since $\mathcal{E}$ is real and positive we must have

$$\mathcal{E} = \left|\mathcal{W}\mid_{-\infty}^{\infty}\right| \tag{7.179}$$

Finally if we write $\mathcal{W}\mid_{-\infty}^{\infty} = \left|\mathcal{W}\mid_{-\infty}^{\infty}\right| e^{-i\theta}$ then we recover the energy density (7.136) and (7.177) becomes

$$(1 + \gamma_3\cos\theta - i\gamma_3\gamma_5\sin\theta)\epsilon = 0 \tag{7.180}$$

which agrees with (7.138).

8 Off-shell Supersymmetry, Auxiliary Fields and Superspace

8.1 Off-Shell Supersymmetry and Auxiliary Fields

Let us make some comments on on-shell versus off-shell supersymmetry. In this course we are primarily interested in classical field theory. In this case there is little difference since everything is always on-shell. Of course the action must be invariant off-shell (by definition the action is invariant under arbitrary variations on-shell). However in a quantum theory it is important for the algebra to close off-shell. In some cases this can be made to happen by introducing additional Bosonic fields which have no on-shell degrees of freedom but which restore the Bose-Fermi degeneracy off-shell. These are called auxiliary fields. However this is not always possible. These days supersymmetric theories are often viewed as the low energy limits of string theory and as such one does not require them to have algebra's that close off-shell. The quantization of the theory is not achieved by the usual field theory methods. Rather one uses string world sheet techniques. The supersymmetric field theory then arises as a low energy effective theory that reproduces the quantum calculations of the underlying string theory.

To see how auxiliary fields work we can again consider the Wess-Zumino model and introduce two new real scalars F and G. So now there are four Bosonic and four Fermionic degrees of freedom off-shell. F and G must have no on-shell degrees of freedom and so one postulates the action

$$\begin{aligned} \mathcal{L}_{off-shell} &= -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B + \frac{1}{2}F^2 + \frac{1}{2}G^2 - \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi \\ &\quad -\frac{i}{2}U_1(A,B)\bar{\psi}\psi + \frac{1}{2}U_2(A,B)\bar{\psi}\gamma_5\psi - W_1(A,B)F - W_2(A,B)G \end{aligned} \tag{8.181}$$

Here we see that the Fermionic part of the action is unchanged. However the Bosonic part includes quadratic terms in F and G. The point about quadratic terms is that their equation of motion is linear (and in the quantum theory the path-integral is a Gaussian and hence can be performed exactly). In particular the equations of motion for F and G are

$$F = W_1(A,B) \qquad G = W_2(A,B) \tag{8.182}$$

This implies that they do not carry any degrees of freedom. Once one knows the behavior of A and B the solutions for F and G are fixed with no ambiguity. We can now substitute these expressions for F and G directly into the equations of motion for A and B. This will lead to equations of motion that involve only A, B and ψ. In fact these equations of motion will come from a Lagrangian which is obtained by substituting (8.182) into (8.181):

$$\begin{aligned} \mathcal{L}_{on-shell} &= -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}U_1\bar{\psi}\psi \\ &\quad +\frac{1}{2}U_2\bar{\psi}\gamma_5\psi - \frac{1}{2}(W_1^2 + W_2^2) \end{aligned} \tag{8.183}$$

This has a similar form to the Wess-Zumino action that we found above. The fact that we have found such a form without yet imposing supersymmetry is because this whole structure can be placed in a manifestly supersymmetric formalism called superspace.

Problem: Show that the equations of motion arising from $\mathcal{L}_{on-shell}$ are precisely what one gets by substituting (8.182) into the equations of motion obtained from $\mathcal{L}_{off-shell}$.

Our last step is to establish a supersymmetry for (8.181). Since F and G have mass dimension 2 the supersymmetry transformations take the form

$$\begin{aligned}
\delta A &= i\bar{\epsilon}\psi \\
\delta B &= -\bar{\epsilon}\gamma_5\psi \\
\delta F &= i\bar{\epsilon}\gamma^\mu\partial_\mu\psi \\
\delta G &= \bar{\epsilon}\gamma^\mu\gamma_5\partial_\mu\psi \\
\delta\psi &= \gamma^\mu\partial_\mu A\epsilon + i\gamma^\mu\gamma_5\partial_\mu B\epsilon + F\epsilon + i\gamma_5 G\epsilon
\end{aligned} \tag{8.184}$$

The last line follows from our discussion above since, on-shell, we find (8.182) and hence $\delta\psi$ agrees with (5.89).

So let us verify that these transformations generate a supersymmetry algebra that closes off-shell. Proceeding as before we find

$$\begin{aligned}
[\delta_1,\delta_2]A &= i\bar{\epsilon}_2(\gamma^\mu\partial_\mu A\epsilon_1 + i\gamma^\mu\gamma_5\partial_\mu B\epsilon_1 + F\epsilon + i\gamma_5 G\epsilon_1) - (1\leftrightarrow 2) \\
&= i(\bar{\epsilon}_2\gamma^\mu\epsilon_1 - \bar{\epsilon}_1\gamma^\mu\epsilon_2)\partial_\mu A \\
[\delta_1,\delta_2]B &= -\bar{\epsilon}_2\gamma_5(\gamma^\mu\partial_\mu A\epsilon_1 + i\gamma^\mu\gamma_5\partial_\mu B\epsilon_1 + F\epsilon + i\gamma_5 G\epsilon_1) - (1\leftrightarrow 2) \\
&= i(\bar{\epsilon}_2\gamma^\mu\epsilon_1 - \bar{\epsilon}_1\gamma^\mu\epsilon_2)\partial_\mu B
\end{aligned} \tag{8.185}$$

where we dropped the $\bar{\epsilon}_2\gamma^\mu\gamma_5\epsilon_1$, $\bar{\epsilon}_2\epsilon_1$ and $\bar{\epsilon}_2\gamma_5\epsilon_1$ terms since these are symmetric under $(1\leftrightarrow 2)$.

Next we consider the auxiliary fields

$$\begin{aligned}
[\delta_1,\delta_2]F &= i\bar{\epsilon}_2\gamma^\mu\partial_\mu(\gamma^\nu\partial_\nu A\epsilon_1 + i\gamma^\nu\gamma_5\partial_\nu B\epsilon_1 + F\epsilon_1 + i\gamma_5 G\epsilon_1) - (1\leftrightarrow 2) \\
&= i\bar{\epsilon}_2\epsilon_1\partial^2 A - \bar{\epsilon}_2\gamma_5\epsilon_1\partial^2 B + i\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu F - \bar{\epsilon}_2\gamma^\mu\gamma_5\epsilon_1\partial_\mu G) - (1\leftrightarrow 2) \\
&= i(\bar{\epsilon}_2\gamma^\mu\epsilon_1 - \bar{\epsilon}_1\gamma^\mu\epsilon_2)\partial_\mu F \\
[\delta_1,\delta_2]G &= \bar{\epsilon}_2\gamma^\mu\gamma_5\partial_\mu(\gamma^\nu\partial_\nu A\epsilon_1 + i\gamma^\nu\gamma_5\partial_\nu B\epsilon_1 + F\epsilon_1 + i\gamma_5 G\epsilon_1) - (1\leftrightarrow 2) \\
&= -\bar{\epsilon}_2\gamma_5\epsilon_1\partial^2 A - i\bar{\epsilon}_2\epsilon_1\partial^2 B + \bar{\epsilon}_2\gamma^\mu\gamma_5\epsilon_1\partial_\mu F + i\bar{\epsilon}_2\gamma^\mu\gamma_5\epsilon_1\partial_\mu G) - (1\leftrightarrow 2) \\
&= i(\bar{\epsilon}_2\gamma^\mu\epsilon_1 - \bar{\epsilon}_1\gamma^\mu\epsilon_2)\partial_\mu G
\end{aligned} \tag{8.186}$$

where, as always, we've dropped the $\bar{\epsilon}_2\gamma^\mu\gamma_5\epsilon_1$, $\bar{\epsilon}_2\epsilon_1$ and $\bar{\epsilon}_2\gamma_5\epsilon_1$ terms since these are symmetric under $(1\leftrightarrow 2)$.

Lastly we have

$$\begin{aligned}
[\delta_1,\delta_2]\psi &= \gamma^\rho\partial_\rho(i\bar{\epsilon}_1\psi)\epsilon_2 - i\gamma^\rho\gamma_5\partial_\rho(\bar{\epsilon}_1\gamma_5\psi)\epsilon_2 + (i\bar{\epsilon}_1\gamma^\rho\partial_\rho\psi)\epsilon_2 + i\gamma_5(\bar{\epsilon}_1\gamma^\rho\gamma_5\partial_\rho\psi)\epsilon_2 \\
&\quad -(1\leftrightarrow 2)
\end{aligned} \tag{8.187}$$

and again we'll need our good friend Fierz.

$$(\bar{\epsilon}_1\partial_\rho\psi)\epsilon_2 - (1\leftrightarrow 2) = -\frac{1}{2}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma_\mu\partial_\rho\psi + \frac{1}{4}(\bar{\epsilon}_1\gamma^{\mu\nu}\epsilon_2)\gamma_{\mu\nu}\partial_\rho\psi$$

$$\begin{aligned}
(\bar{\epsilon}_1\gamma_5\partial_\rho\psi)\epsilon_2 - (1\leftrightarrow 2) &= -\frac{1}{2}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma_\mu\gamma_5\partial_\rho\psi + \frac{1}{4}(\bar{\epsilon}_1\gamma^{\mu\nu}\epsilon_2)\gamma_{\mu\nu}\gamma_5\partial_\rho\psi \\
(\bar{\epsilon}_1\gamma^\rho\partial_\rho\psi)\epsilon_2 - (1\leftrightarrow 2) &= -\frac{1}{2}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma_\mu\gamma^\rho\partial_\rho\psi + \frac{1}{4}(\bar{\epsilon}_1\gamma^{\mu\nu}\epsilon_2)\gamma_{\mu\nu}\gamma^\rho\partial_\rho\psi \\
(\bar{\epsilon}_1\gamma^\rho\gamma_5\partial_\rho\psi)\epsilon_2 - (1\leftrightarrow 2) &= -\frac{1}{2}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma_\mu\gamma^\rho\gamma_5\partial_\rho\psi + \frac{1}{4}(\bar{\epsilon}_1\gamma^{\mu\nu}\epsilon_2)\gamma_{\mu\nu}\gamma^\rho\gamma_5\partial_\rho\psi
\end{aligned} \tag{8.188}$$

Putting these all together leads to

$$\begin{aligned}
[\delta_1,\delta_2]\psi &= -\frac{i}{2}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma^\rho\gamma_\mu\partial_\rho\psi + \frac{i}{4}(\bar{\epsilon}_1\gamma^{\mu\nu}\epsilon_2)\gamma^\rho\gamma_{\mu\nu}\partial_\rho\psi \\
&\quad -\frac{i}{2}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma^\rho\gamma_\mu\partial_\rho\psi - \frac{i}{4}(\bar{\epsilon}_1\gamma^{\mu\nu}\epsilon_2)\gamma^\rho\gamma_{\mu\nu}\partial_\rho\psi \\
&\quad -\frac{i}{2}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma_\mu\gamma^\rho\partial_\rho\psi + \frac{i}{4}(\bar{\epsilon}_1\gamma^{\mu\nu}\epsilon_2)\gamma_{\mu\nu}\gamma^\rho\partial_\rho\psi \\
&\quad -\frac{i}{2}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\gamma_\mu\gamma^\rho\partial_\rho\psi - \frac{i}{4}(\bar{\epsilon}_1\gamma^{\mu\nu}\epsilon_2)\gamma_{\mu\nu}\gamma^\rho\partial_\rho\psi \\
&= -2i(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\partial_\mu\psi \\
&= i(\bar{\epsilon}_2\gamma^\mu\epsilon_1 - \bar{\epsilon}_1\gamma^\mu\epsilon_2)\partial_\mu\psi
\end{aligned} \tag{8.189}$$

which completes the proof.

Problem: Show that $\mathcal{L}_{off-shell}$ is invariant under the supersymmetry transformations (8.184).

8.2 Superspace in Two Dimensions

Since the Poincare algebra has a clear geometrical origin in terms of the symmetries of spacetime, it is natural to seek such a geometrical interpretation of supersymmetry. This can be done as follows. In addition to the usual coordinates x^μ of spacetime we suppose that there are additional anti-commuting spinorial coordinates θ^α. Superspace is then described by coordinates $z^M = (x^\mu, \bar{\theta}^\alpha)$. x^μ and θ_α are called the even and odd coordinates of superspace. A superfield is a function on superspace with suitable covariance properties. Any such function can be expanded in a polynomial about $\theta = 0$

$$\mathcal{A}(z^M) = A(x) + \bar{\theta}^\alpha\psi_\alpha + ... + \theta^1 ... F \tag{8.190}$$

The key point is that this Taylor expansion eventually stops as a consequence of the anti-commuting nature of θ_α.

We will illustrate how this works in the simplest case relevant to us, namely two-dimensions. Here θ_α has two components which we take to be real thus we can write

$$\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\bar{\theta}\theta \tag{8.191}$$

where

$$\begin{aligned}\bar{\theta}\theta &= (\theta_1, \theta_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \\ &= \theta_1\theta_2 - \theta_2\theta_1 \\ &= 2\theta_1\theta_2 \end{aligned} \tag{8.192}$$

Thus a real superfield can be written in terms of components as

$$\mathcal{A}(z) = A(x) + i\bar{\theta}\psi + \frac{i}{2}\bar{\theta}\theta F \tag{8.193}$$

This is already suggestive of the correct field content that was required for off-shell supersymmetry, namely we have two real Bosons A and F and one two-component Fermion ψ.

If we perform a translation along the odd coordinates of superspace $\bar{\theta} \to \bar{\theta} + \bar{\epsilon}$ then in order to generate a translation in the even part of spacetime we take $x^\mu \to x^\mu - i\bar{\epsilon}\gamma^\mu\theta$. Why shift x^μ too? Well we want to reproduce supersymmetries which are the 'square root' of translations. As such we see that

$$\begin{aligned}[\delta_1, \delta_2]\theta &= 0 \\ [\delta_1, \delta_2]x^\mu &= -i\bar{\epsilon}_2\gamma^\mu\epsilon_1 - (1 \leftrightarrow 2) \\ &= v^\mu \end{aligned} \tag{8.194}$$

where $v^\mu = -2i\bar{\epsilon}_2\gamma^\mu\epsilon_1$ generates a simple translation in x^μ.

Under such a transformation we see that

$$\delta\mathcal{A} = \bar{\epsilon}Q\mathcal{A} \tag{8.195}$$

where

$$Q = \frac{\partial}{\partial\bar{\theta}} - i\gamma^\mu\theta\partial_\mu \tag{8.196}$$

Thus we see that we can define a superfield more generally to be any function $\mathcal{S}$ of superspace such that, under the transformation $\theta \to \theta + \epsilon$, $x^\mu \to x^\mu - i\bar{\epsilon}\gamma^\mu\theta$, transforms as $\delta\mathcal{S} = \bar{\epsilon}Q\mathcal{S}$.

It will prove useful to introduce superderivatives. To this end consider the operator

$$D = \frac{\partial}{\partial\bar{\theta}} + i\gamma^\mu\theta\partial_\mu \tag{8.197}$$

One can see that

$$\begin{aligned}\{D_\alpha, Q_\beta\} &= \left\{\frac{\partial}{\partial\bar{\theta}^\alpha} + i(\gamma^\mu\theta)_\alpha\partial_\mu \, , \frac{\partial}{\partial\bar{\theta}^\beta} - i(\gamma^\nu\theta)_\beta\partial_\nu\right\} \\ &= -i\frac{\partial}{\partial\bar{\theta}^\alpha}(\gamma^\nu\theta)_\beta\partial_\nu + i\frac{\partial}{\partial\bar{\theta}^\beta}(\gamma^\mu\theta)_\alpha\partial_\mu \end{aligned} \tag{8.198}$$

where we used the fact that $\{\frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\theta^\beta}\} = 0$ and

$$\{(\gamma^\mu\theta)_\alpha\partial_\mu, (\gamma^\nu\theta)_\beta\partial_\nu\} = \{(\gamma^\mu\theta)_\alpha, (\gamma^\nu\theta)_\beta\}\partial_\mu\partial_\nu = 0 \tag{8.199}$$

Continuing the calculation we have

$$\frac{\partial}{\partial\bar{\theta}^\alpha}(\gamma^\nu\theta)_\beta = \frac{\partial\theta_\gamma}{\partial\bar{\theta}^\alpha}\frac{\partial}{\partial\theta_\gamma}(\gamma^\nu_\beta{}^\delta\theta_\delta) = C_{\alpha\gamma}\gamma^\nu_\beta{}^\gamma \tag{8.200}$$

We recognize this as $(\gamma^\nu(C^{-1})^T)_{\alpha\beta} = -(\gamma^\nu C^{-1})_{\alpha\beta}$. Since this is symmetric in $\alpha \leftrightarrow \beta$ we find

$$\{D_\alpha, Q_\beta\} = 0 \tag{8.201}$$

It also follows, by a simple change in sign, that

$$\begin{aligned}\{Q_\alpha, Q_\beta\} &= \left\{\frac{\partial}{\partial\bar{\theta}^\alpha} - i(\gamma^\mu\theta)_\alpha\partial_\mu \ , \frac{\partial}{\partial\bar{\theta}^\beta} - i(\gamma^\nu\theta)_\beta\partial_\nu\right\} \\ &= 2i(\gamma^\mu C^{-1})_{\alpha\beta}\partial_\mu \\ &= -2(\gamma^\mu C^{-1})_{\alpha\beta}P_\mu\end{aligned} \tag{8.202}$$

Thus $\bar{\epsilon}Q$ does indeed generate supersymmetry when acting on superfields.

Problem: Show that if $\mathcal{A} = A + i\bar{\theta}\psi + \frac{i}{2}\bar{\theta}\theta F$ then the supersymmetry variations of the component fields are

$$\begin{aligned}\delta A &= i\bar{\epsilon}\psi \\ \delta\psi &= \gamma^\mu\partial_\mu A\epsilon + F\epsilon \\ \delta F &= i\bar{\epsilon}\gamma^\mu\partial_\mu\psi\end{aligned} \tag{8.203}$$

Next it is important to observe that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are two superfields then so are $D\mathcal{S}_1$, $\bar{D}\mathcal{S}_1$ and $\mathcal{S}_1\mathcal{S}_2$. We also see that under supersymmetry $\delta\mathcal{S} = \bar{\epsilon}Q\mathcal{S}$ the highest component of $\delta\mathcal{S}$ always transforms into a total derivative

$$i\bar{\epsilon}Q\mathcal{S}\mid_{\bar{\theta}\theta} = \bar{\epsilon}\gamma^\mu\theta\partial_\mu(\mathcal{S}\mid_{\bar{\theta}}) \tag{8.204}$$

Therefore we can construct an invariant Lagrangian, up to a total derivative, by taking

$$\mathcal{L} = \mathcal{S}\mid_{\bar{\theta}\theta} \tag{8.205}$$

for any choice of superfield $\mathcal{S}$. A natural choice therefore is to take

$$\mathcal{S} = -\frac{1}{2}\bar{D}\mathcal{A}D\mathcal{A} + 2i\mathcal{W}(\mathcal{A}) \tag{8.206}$$

where $\mathcal{W}$ is an arbitrary function of $\mathcal{A}$

Problem: Show that the Lagrangian constructed from this choice of $\mathcal{S}$ is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A \partial^\mu A - \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{1}{2}F^2 + \frac{i}{2}\bar{\psi}\partial^2\mathcal{W}\psi - F\partial\mathcal{W} \tag{8.207}$$

Here we see that F plays the role of an auxiliary field with no on-shell degrees of freedom. To proceed we simply integrate out F through its equation of motion, $F = \partial\mathcal{W}$, to find

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A \partial^\mu A - \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{2}(\partial\mathcal{W})^2 + \frac{i}{2}\bar{\psi}\partial^2\mathcal{W}\psi \tag{8.208}$$

This is precisely the two-dimensional $N = 1$ supersymmetric Lagrangian that we found in one of the problems, with $\mathcal{W}$ an arbitrary real function of A.

One can also define a notion of integration over a Grassmann coordinate θ_α. We take the integration to be a linear functional that takes real values. It is then defined in general by the condition

$$\int d\theta_\alpha = 0 \qquad \int d\theta_\alpha\theta_\alpha = 1 \tag{8.209}$$

where there is no sum on α in the second equation. It follows that if we define

$$\int d^2\theta = -\frac{1}{4}\int d\bar{\theta}d\theta = \frac{1}{2}\int d\theta_2 d\theta_1 \tag{8.210}$$

then

$$\int d^2\theta\mathcal{S} = \mathcal{S}\mid_{\bar{\theta}\theta} \tag{8.211}$$

Thus taking the highest component of the superfield can also be thought of (and usually is) as integration over the odd coordinates of superspace.

One can also consider superfields in other dimensions, most notably four. In four dimensions it is useful to use two-component Weyl spinor notation (recall that we briefly discussed this before in a problem). The general idea remains the same however one finds that the most general superfield contains many more component fields. This leads to a reducible representation of supersymmetry. Thus one usually imposes some kind of constraint on the superfield. For example there are so-called chiral superfields (such that $D_{\dot{a}}\mathcal{S} = 0$) which contain two real scalars, a Weyl spinor and two auxiliary fields. This describes the field content required for the four-dimensional off-shell supersymmetric Wess-Zumino model and a similar superspace construction leads directly to the off-shell Lagrangian (8.181). In these cases the superpotential term is actually expressed as an integral only over a chiral half of superspace. This leads to various quantum properties such as non-renormalization theorems that have been so useful in the study of supersymmetric quantum field theories.

9 The Super-Point Particle

As a final application of supersymmetry let us supersymmetrise the following one-dimensional action:

$$S_{pp} = \frac{1}{2}\int dt\ e^{-1}\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - m^2 e \tag{9.212}$$

where $X^\mu(t)$ are D scalar fields $\mu = 0, 1..., D-1$, a dot denotes differentiation with respect to t and $\eta_{\mu\nu}$ is the Minkowski Metric in D-dimensions. You may recognize this as the worldline action of a point particle (with $\gamma_{tt} = -e^2$). By eliminating e by its equation of motion one finds the more familiar worldline action

$$S_{pp} = -m\int dt\sqrt{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}} \tag{9.213}$$

If you haven't done this calculation in Point Particles and Strings course do it now! In that course we also saw how the quantization of this action lead to the Klein-Gordon equation arising as the constraint equation for the wavefunction $\Psi(X^\mu)$. Therefore the point particle that is described by S_{pp} is a spacetime Boson. So how could we consider a spacetime Fermion such as an electron from a worldline action? Well the answer is supersymmetry (on the worldline).

In one-dimension the spinor indices can only take one value $\alpha, \beta ... = 1$ and the γ-matrices are just numbers. So we can more or less forget about them[5]. We also take our Fermions to a real Grassmann variable. We need to introduce Fermionic superpartners, λ^μ, for X^μ as well as a superpartner χ for e (even though e is not a dynamical field).

For simplicity we will take the particle mass $m = 0$. The most general Lagrangian that contains these fields, ensures that χ is non-dynamical and preserves spacetime translations $\delta X^\mu = a^\mu$ is

$$S_{spp} = \frac{1}{2}\int dt\ e^{-1}\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - i\lambda^\mu \dot{\lambda}^\nu \eta_{\mu\nu} + 2ie^{-1}\lambda^\mu \dot{X}^\nu \eta_{\mu\nu}\chi \tag{9.214}$$

This Lagrangian has the supersymmetry

$$\begin{aligned}
\delta X^\mu &= i\epsilon\lambda^\mu \\
\delta\lambda^\mu &= e^{-1}\dot{X}^\mu\epsilon + ie^{-1}\lambda^\mu\chi\epsilon \\
\delta\chi &= \dot{\epsilon} \\
\delta e &= 2i\epsilon\chi
\end{aligned} \tag{9.215}$$

Notice that we have allowed ϵ to become t-dependent. Thus the supersymmetry is now local and we are in fact considering one-dimensional supergravity. Indeed the Lagrangian is invariant under the local reparameterization $t \to t'(t)$ provided that we take

$$e' = \frac{dt}{dt'}e\ , \qquad \chi' = \frac{dt}{dt'}\chi\ , \qquad X'^\mu = X^\mu\ , \qquad \lambda'^\mu = \lambda^\mu \tag{9.216}$$

[5]Strickly speaking we have $\gamma^0 = C = i$ and there is no Majorana condition but we will proceed by dropping these factors of i and take our spinors to be real. This is essentially a pseudo-Majorana representation.

To see this symmetry we simply calculate, dropping total derivatives,

$$\begin{aligned}
\delta L &= e^{-1}(\dot{X}_\mu + i\lambda_\mu\chi)\delta\dot{X}^\mu - \frac{1}{2e^2}(\dot{X}^\mu\dot{X}_\mu + 2i\lambda^\mu\dot{X}_\mu\chi)\delta e \\
&\quad -i\delta\lambda^\mu(\dot{\lambda}_\mu - \dot{X}_\mu\chi) + ie^{-1}\lambda^\mu\dot{X}_\mu\delta\chi \\
&= ie^{-1}(\dot{X}_\mu + i\lambda_\mu\chi)(\dot{\epsilon}\lambda^\mu + \epsilon\dot{\lambda}^\mu) - \frac{i}{e^2}(\dot{X}^\mu\dot{X}_\mu + 2i\lambda^\mu\dot{X}_\mu\chi)\epsilon\chi \\
&\quad -i(e^{-1}\dot{X}^\mu\epsilon + ie^{-1}\lambda^\mu\chi\epsilon)(\dot{\lambda}_\mu - \dot{X}_\mu\chi) + ie^{-1}\lambda^\mu\dot{X}_\mu\dot{\epsilon} \\
&= ie^{-1}\dot{X}_\mu\dot{\epsilon}\lambda^\mu + ie^{-1}\dot{X}_\mu\epsilon\dot{\lambda}^\mu - e^{-1}\lambda_\mu\chi\epsilon\dot{\lambda}^\mu - ie^{-2}\dot{X}^\mu\dot{X}_\mu\epsilon\chi \\
&\quad -ie^{-1}\dot{X}_\mu\epsilon\dot{\lambda}^\mu + ie^{-1}\dot{X}^\mu\dot{X}_\mu\epsilon\chi + e^{-1}\lambda^\mu\chi\epsilon\dot{\lambda}_\mu + ie^{-1}\lambda^\mu\dot{X}_\mu\dot{\epsilon} \\
&= 0
\end{aligned} \tag{9.217}$$

Next we construct the conjugate momenta:

$$\begin{aligned}
P^\mu &= e^{-1}\dot{X}^\mu + ie^{-1}\lambda^\mu\chi \\
\Pi^\mu &= -\frac{i}{2}\lambda^\mu
\end{aligned} \tag{9.218}$$

and the Hamiltonian

$$\begin{aligned}
H &= P_\mu\dot{X}^\mu + \Pi_\mu\dot{\lambda}^\mu - L \\
&= P_\mu(eP^\mu - i\lambda^\mu\chi) + \Pi_\mu\dot{\lambda}^\mu \\
&\quad -\frac{1}{2}\Big((eP^\mu - i\lambda^\mu\chi)(eP_\mu - i\lambda_\mu\chi) + 2\Pi_\mu\dot{\lambda}^\mu + \frac{2i}{e}\lambda^\mu(eP_\mu - i\lambda_\mu\chi)\Big) \\
&= \frac{e}{2}P_\mu P^\mu - i\lambda^\mu P_\mu\chi
\end{aligned} \tag{9.219}$$

where we have used the fact that $\lambda^\mu\lambda_\mu = 0$.

To quantize this system we consider wavefunctions $\Psi(X^\mu)$ and promote X^μ and P_μ to operators

$$\begin{aligned}
X^\mu &\rightarrow \hat{X}^\mu : \qquad \hat{X}^\mu\Psi = X^\mu\Psi \\
P_\mu &\rightarrow \hat{P}_\mu : \qquad \hat{P}_\mu\Psi = -i\frac{\partial\Psi}{\partial X^\mu}
\end{aligned} \tag{9.220}$$

which satisfy $[\hat{X}^\mu, \hat{P}_\nu] = i\delta^\mu_\nu$. Next we must promote the Fermions λ^μ to (non-dynamical) operators $\hat{\lambda}^\mu$ which satisfy

$$\{\hat{\lambda}^\mu, \hat{\Pi}^\nu\} = i\eta^{\mu\nu} \tag{9.221}$$

However this is equivalent to $\{\hat{\lambda}^\mu, \hat{\lambda}^\nu\} = -2\eta^{\mu\nu}$ which is just the Clifford algebra in D dimensions. Thus we can take

$$\hat{\lambda}^\mu = i\gamma^\mu \tag{9.222}$$

As we have seen this requires introducing a $2^{[D/2]}$ dimensional vector space, the spinor space, in order to realise γ_μ as a matrix. Therefore Ψ must also have a spinor index α and

$$\hat{\lambda}^\mu \Psi = i\gamma^\mu \Psi \tag{9.223}$$

Next we impose the constraints. In the quantum theory this is done by restricting the physical wavefunctions to be those that are annihilated by the constraints. We have two constraints coming from the e and χ equations of motion:

$$P^2 = 0 \qquad \lambda^\mu P_\mu = 0 \tag{9.224}$$

In the quantum theory these therefore lead to the constraints

$$-\eta^{\mu\nu}\frac{\partial^2\Psi}{\partial X^\mu \partial X^\nu} = 0 \qquad \gamma^\mu \frac{\partial \Psi}{\partial X^\mu} = 0 \tag{9.225}$$

Note that, as a consequence of the Clifford relation, the second equation implies the first. We see that the superalgebra in one dimension is just $\hat{H} = \hat{Q}^2$ and hence the condition $\hat{Q} = 0$ implies $\hat{H} = 0$.

The second equation is nothing but the massless Dirac equation in spacetime. Thus the particle we have quantized is a Fermion. Finally we solve the Schrödinger equation which, on the physical Hilbert space, is just

$$i\frac{\partial \Psi}{\partial t} = \hat{H}\Psi = 0 \tag{9.226}$$

and hence we learn that Ψ is independent of t. Since Fermions exist in spacetime (they come pouring out of your tv sets) we see that supersymmetry, indeed supergravity, is indeed realised on their worldvolumes.

Appendix: Conventions

In these notes we will in general be in D-spacetime dimensions labeled by x^μ, $\mu = 0, 1, 2, ..., D-1$. When we only want to talk about the spatial components we use x^i with $i = 1, ..., D-1$. We use the the "mostly plus" convention for the metric:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & ... & \\ & & & & 1 \end{pmatrix} \tag{9.1}$$

Spinor indices will in general be denoted by $\alpha, \beta, ...$ and their range will depend on the dimension of spacetime, *i.e.* on the dimension of the representation of the Clifford algebra. When we talk about four-dimensional Weyl spinors we will use the spinor indices $a, \dot{a} = 1, 2$. A capitol index $I, J = 1, ..., N$ will be used to denote extended supersymmetries and a capitol index M is used to describe superspace.

We also assume, according to the spin-statistics theorem, that spinorial quantities and fields are Grassmann variables, *i.e.* anti-commuting. We will typically use Greek symbols for Fermionic Grassmann fields, $\psi, \lambda, ...$ and Roman symbols for Bosonic c-number fields.

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