Strings, Branes and Quantum Gravity

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1 Introduction: Why String Theory?

Why are you taking this course? Why am I, or anyone else in the Theoretical Physics Group, paid? What are we doing and why do we do it? Well here are some reasons.

The so-called Standard Model of Particle Physics is the most successful scientific theory of Nature in the sense that no other theory has such a high level of accuracy over such a complete range of physical phenomena using such a modest number of assumptions and parameters. It is unreasonably good and was never intended to be so successful. Since its formulation around 1970 there has not been a single experimental result that has produced even the slightest disagreement. Nothing, despite an enormous amount of effort. But there are skeletons in the closet. Let me mention just three.

The first is the following: Where does the Standard Model come from? For example it has quite a few parameters which are only fixed by experimental observation. What fixes these? It postulates a certain spectrum of fundamental particle states but why these? In particular these particle states form three families, each of which is a copy of the others, differing only in their masses. Furthermore only the lightest family seems to have much to do with life in the universe as we know it, so why the repetition? It is somewhat analogous to Mendelev's periodic table of the elements. There is clearly a discernible structure but this wasn't understood until the discovery of quantum mechanics. We are looking for the underlying principle that gives the somewhat bizarre and apparently ad hoc structure of the Standard Model.

The second problem is that, for all its strengths, the Standard Model does not include gravity. For that we must use General Relativity which is a classical theory and as such is incompatible with the rules of quantum mechanics. Observationally this is not a problem since the effect of gravity, at the energy scales which we probe, is smaller by a factor of 10^{-40} than the effects of the subnuclear forces which the Standard Model describes. You can experimentally test this assertion by lifting up a piece of paper with your little finger. You will see that the electromagnetic forces at work in your little finger can easily overcome the gravitational force of the entire earth which acts to pull the paper to the floor.

However this is clearly a problem theoretically. We can't claim to understand the universe physically until we can provide one theory which consistently describes gravity and the subnuclear forces. If we do try to include gravity into QFT then we encounter problems. A serious one is that the result is non-renormalizable, apparently producing an infinite series of divergences which must be subtracted by inventing an infinite series of new interactions, thereby removing any predictive power. Thus we cannot use the methods of QFT as a fundamental principle for gravity. Furthermore there is enormous evidence for Dark Matter, which constitutes most of the mass in the Universe, but which is not at all described by the Standard model.

The third problem I want to mention is more technical. Quantum field theories generically only make mathematical sense if they are viewed as a low energy theory. Due to the effects of renormalization the Standard Model cannot be valid up to all energy scales, even if gravity was not a problem. Mathematically we know that there

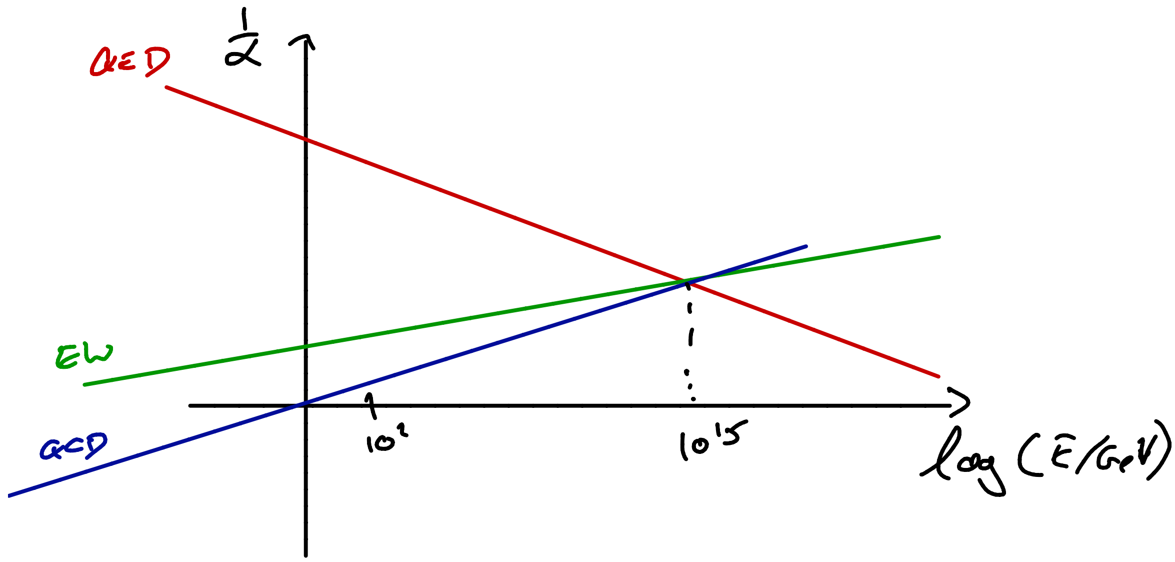

Figure 1.1: The running of the couplings

must be something else which will manifest itself at some higher energy scale. All we can say is that such new physics must arise before we reach the quantum gravity scale, which is some 10^{17} orders of magnitude above the energy scales that we have tested to date. To the physicists who developed the Standard Model the surprise is that we have not already seen such new physics many years ago. And we are all hoping to see it soon at the LHC.

1.1 The Standard Model and Grand Unification

Let us take a slight detour into the modern field of the Standard Model of particle physics. Physicists view this not as a fundamental description of Nature in terms of the known particles but rather as an effective description which is only valid up to some energy scale where new physics will appear. The interactions of the particles in the Standard Model are described by a gauge theory based on $SU(3) \times SU(2) \times U_Y(1)$. The subscript Y indicates that $U_Y(1)$ is not the same as the $U(1)$ of electromagnetism which is a linear combination of $U_Y(1)$ and the Cartan subalgebra of $SU(2)$. (Don't worry if these are just words to you.)

The first piece of evidence that there maybe something more and something simpler is that the coupling constants 'run' with energy and all appear to meet at around $10^{15}GeV$, thats 10^{15} times the mass of the proton. If we plot the inverse couplings as a function of the Log of the energy scale then they appear as straight lines (to first order): see figure 1.1.

That two straight lines intersect is normal, but for all three to intersect is suspicious.

It suggests that they all original from the same force. This is the idea behind a so-called Grand Unified Theory.

Note also that the coupling constant for electromagnetism eventually diverges (corresponding to passing through zero in the above plot). This means that the theory becomes infinitely strongly coupled. As I mentioned one must view the Standard Model as an effective description. Electromagnetism (and also the Higgs') cannot be viewed as working to all energies due to such divergences (known as Landau poles). On the other hand QCD and the electroweak theory can (their couplings get small at high energy and the theory becomes free and well defined there). So something must replace this theory at high energy, even without worrying about gravity. This is often phrased as 'finding a UV completion' which simply means embedding the theory into a theory that could, at least in principle, be valid up to arbitrarily high scales, even if gravity was never included.

The known Fermions, matter particles, of the Standard Model are often written as:

$$
\begin{pmatrix} u \\ d \end{pmatrix} \qquad \begin{pmatrix} c \\ s \end{pmatrix} \qquad \begin{pmatrix} t \\ b \end{pmatrix}
$$
$$
\begin{pmatrix} \bar{\nu}_e \\ e \end{pmatrix} \qquad \begin{pmatrix} \bar{\nu}_\mu \\ \mu \end{pmatrix} \qquad \begin{pmatrix} \bar{\nu}_\tau \\ \tau \end{pmatrix}
$$

These come in three generations which essentially only differ in that they have increasing mass. You and I and everything we see is made of just the first generation. The rest only exist for a short fleeting moment in some accelerator or Cosmic reaction.

In addition to these Fermionic particles there are 12 vector gauge Bosons: 8 gluons for $SU(3)$, W^+, W^-, Z^0 for $SU(2)$ and the photon γ for $U(1)$. These are associated to the adjoint representations of the Standard Model gauge group $SU(3) \times SU(2) \times U_Y(1)$ And last but not least the scalar Higgs h, which is a doublet of $SU(2) \times U_Y(1)$.

Let us look more closely at the first generation (the others will behave in the same way). From a particle physicists point of view we actually think in terms of the following particles:

$$
\begin{aligned}
q_L &= \begin{pmatrix} u_L \\ d_L \end{pmatrix} \in (\mathbf{3}, \mathbf{2}) \ , \qquad u_R \in (\bar{\mathbf{3}}, \mathbf{1}) \ , \qquad d_R \in (\bar{\mathbf{3}}, \mathbf{1}) \\
l_L &= \begin{pmatrix} \bar{\nu}_L \\ e_L \end{pmatrix} \in (\mathbf{1}, \mathbf{2}) \ , \qquad e_R \in (\mathbf{1}, \mathbf{1}) \ , \qquad \nu_R \in (\mathbf{1}, \mathbf{1})?
\end{aligned}
$$

Here the R/L refers to their chirality, *i.e.* their eigenvalue under γ_5. Furthermore a **3** means that the field is a complex triplet that is acted by $SU(3)$ where as a **2** means that it is a complex doublet acted on by $SU(2)$. A **1** indicates that it isn't acted on at all by $SU(3)$ or $SU(2)$. I've put a question mark next to the right-handed neutrino as we believe they have a mass and hence a ν_R but it isn't fully resolved yet. The particles all also have a $U(1)$ hypercharge charge but we won't bother with that here. Indeed we will be a little cavalier about the group theory in the interests of time.

Note that only the left-handed fields interact with the weak force and hence only they carry a non-trivial representation of $SU(2)$ (as well as the Higgs'). Thus for example e_L is a different particle to e_R.

We don't know what lies beyond these particles in the Standard Model. We expect at some higher scale new particles and new physics will emerge, even before we reach the Planck scale. What is remarkable is that the chiral fermions actually fit together into representations of a bigger group $SU(5)$:[2]

$$\begin{aligned} d_R^c \oplus l_L &= (\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}) = \bar{\mathbf{5}} \\ q_L \oplus u_R^c \oplus e_R^c &= (\mathbf{3}, \mathbf{2}) \oplus (\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}) = \mathbf{10} \\ \nu_R &= (\mathbf{1}, \mathbf{1}) = \mathbf{1} \end{aligned} \tag{1.1}$$

Here we have had to take charge conjugates to get the chiralities to fit but thats okay. $\bar{\mathbf{5}}$ is just a complex five-dimensional vector which is the anti-fundamental representation of $SU(5)$ and $\mathbf{10}$ is simply an anti-symmetric 5×5 matrix which is another representation of $SU(5)$.

The Bosons can also be placed into representations of $SU(5)$ but one needs to add new particles. For example the gauge bosons fit into the adjoint of $SU(5)$ but only the block diagonal parts, roughly:

$$\begin{pmatrix} & & & X & X \\ & Gluons & & X & X \\ & & & X & X \\ X & X & X & Z^0 & W^- \\ X & X & X & W^+ & A \end{pmatrix} \tag{1.2}$$

Here the X's are so-called X-Bosons which have yet to be seen. It is assumed that they have been made massive by an additional Higgs' mechanism which breaks $SU(5)$ to $SU(3) \times SU(2) \times U(1)$, just as the discovered Higgs' breaks $SU(2) \times U(1) \to U(1)$. Thus one would start with a Higgs' H:

$$H = \begin{pmatrix} h' \\ h \end{pmatrix} \in \mathbf{5} \tag{1.3}$$

so that the breaking pattern is

$$SU(5) \xrightarrow{\text{h'}} SU(3) \times SU(2) \times U(1) \xrightarrow{\text{h}} U(1) \tag{1.4}$$

The h' Higgs' is also not seen but as with the X-Bosons they are expected to have a mass around the GUT scale of $10^{15} GeV$.

[2] For aficionados $\bar{\mathbf{2}} = \mathbf{2}$.

1.2 Plan

With these comments in mind this course will introduce string theory, which, for good or bad, has become the dominant, and arguably only, framework for a complete theory of all known physical phenomena. As such it is in some sense a course to introduce the modern view of particle physics at its most fundamental level. Whether or not String Theory is ultimately relevant to our physical universe is unknown, and indeed may never be known. However it has provided many deep and powerful ideas. Certainly it has had a profound effect upon pure mathematics. But an important feature of String Theory is that it naturally includes gravitational and subnuclear-type forces consistently in a manner consistent with quantum mechanics and relativity (as far as anyone knows). Thus it seems fair to say that there is a mathematical framework which is capable of describing all of the physics that we know to be true. This is no small achievement.

However it is also fair to say that no one actually knows what String Theory really is. In any event this course can only attempt to be a modest introduction that is aimed at students with no previous knowledge of String Theory. There will be much that we will not have time to discuss

We will first discuss the Bosonic string in some detail. Although this theory is unphysical in several ways (it has a tachyon and no Fermions) it is simpler to study than the superstring but has all the main ideas built-in. We will aim to show how this leads to a theory of gravity. By including open strings we find D-branes and other p-branes, objects with p extended spatial dimensions. These lead to Yang-Mills gauge theories such as those found in the Standard Model of Particle Physics.

2 Classical and Quantum Dynamics of Point Particles

2.1 Classical Action

We want to describe a single particle moving in spacetime. For now we simply consider flat D-dimensional Minkowski space

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + ... + (dx^{D-1})^2 \tag{2.1}$$

A particle has no spatial extent but it does trace out a curve - its worldline - in spacetime. Furthermore in the absence of external forces this will be a straight line (geodesic if you know GR). In other words the equation of motion should be that the length of the worldline is extremized. Thus we take

$$\begin{aligned} S_{pp} &= -m\int ds \\ &= -m\int \sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}d\tau \end{aligned} \tag{2.2}$$

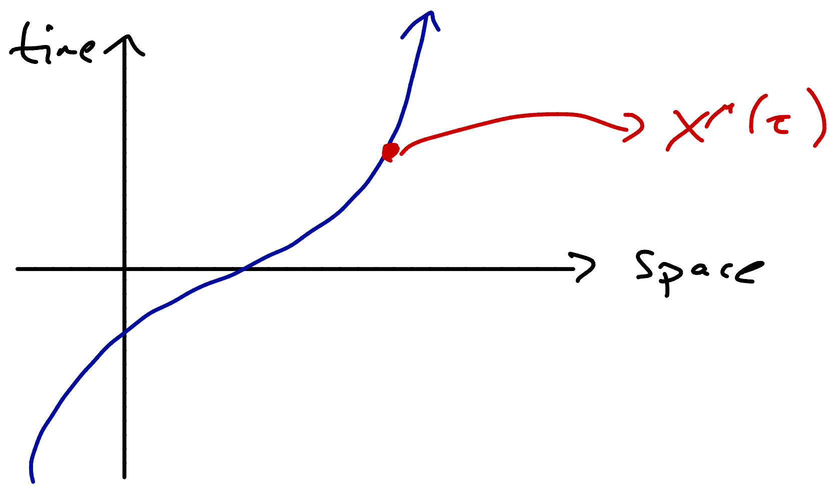

Figure 2.1: A point particle worldline

where τ parameterizes the points along the worldline and $X^\mu(\tau)$ gives the location of the particle in spacetime, *i.e.* the embedding coordinates of the worldline into spacetime.

Let us note some features of this action. Firstly it is manifestly invariant under spacetime Lorentz transformations $X^\mu \to \Lambda^\mu{}_\nu X^\nu$ where $\Lambda^T \eta \Lambda = \eta$. Secondly it is reparameterization invariant under $\tau \to \tau'(\tau)$ for any invertible change of worldline coordinate

$$d\tau = \frac{d\tau}{d\tau'} d\tau' \ , \qquad \dot{X}^\mu = \frac{dX^\mu}{d\tau} = \frac{d\tau'}{d\tau}\frac{dX^\mu}{d\tau'} \tag{2.3}$$

thus

$$\begin{aligned} S_{pp} &= -m \int \sqrt{-\eta_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}} d\tau \\ &= -m \int \sqrt{-\eta_{\mu\nu}\left(\frac{d\tau'}{d\tau}\right)^2 \frac{dX^\mu}{d\tau'}\frac{dX^\nu}{d\tau'}} \frac{d\tau}{d\tau'} d\tau' \\ &= -m \int \sqrt{-\eta_{\mu\nu}\frac{dX^\mu}{d\tau'}\frac{dX^\nu}{d\tau'}} d\tau' \end{aligned} \tag{2.4}$$

Thirdly we can see why the m appears in front and with a minus sign by looking at the non-relativistic limit. In this case we choose a gauge for the worldline reparameterization invariance such that $\tau = X^0$ *i.e.* worldline 'time' is the same as spacetime 'time'. This is known as static gauge. It is a gauge choice since, as we have seen, we are free to take any parameterization we like. The nonrelativistic limit corresponds to assuming that $\dot{X}^i << 1$. In this case we can expand

$$S_{pp} = -m \int \sqrt{1 - \delta_{ij}\dot{X}^i\dot{X}^j} d\tau = \int -m + \frac{1}{2} m \delta_{ij} \dot{X}^i \dot{X}^j d\tau + \dots \tag{2.5}$$

where the ellipses denotes terms with higher powers of the velocities $\dot{X}^i$. The second term is just the familiar kinetic energy $\frac{1}{2}mv^2$. The first term is simply a constant and doesn't affect the equations of motion. However it can be interpreted as a constant potential energy equal to the rest mass of the particle. Thus we see that the m and minus signs are correct.

Moving on let us consider the equations of motion and conservation laws that follow from this action. The equations of motion follow from the usual Euler-Lagrange equations applied to S_{pp}:

$$\frac{d}{d\tau}\left(\frac{\dot{X}^\nu}{\sqrt{-\eta_{\lambda\rho}\dot{X}^\lambda\dot{X}^\rho}}\right) = 0 \tag{2.6}$$

These equations can be understood as conservations laws since the Lagrangian is invariant under constant shifts $X^\mu \to X^\mu + b^\mu$. The associated charge is

$$p^\mu = \frac{m\dot{X}^\mu}{\sqrt{-\eta_{\lambda\rho}\dot{X}^\lambda\dot{X}^\rho}} \tag{2.7}$$

so that indeed the equation of motion is just $\dot{p}^\mu = 0$. Note that I have called this a charge and not a current. In this case it doesn't matter because the Lagrangian theory we are talking about, the worldline theory of the point particle, is in zero spatial dimensions. So I could just as well called p^μ a conserved current with the conserved charge being obtained by integrating the temporal component of p^μ over space. Here there is no space p^μ only has temporal components.

Warning: We are thinking in terms of the worldline theory where the index μ labels the different scalar fields X^μ, it does not label the coordinates of the worldline. In particular p^0 is not the temporal component of p^μ from the worldline point of view. This confusion between worldvolume coordinates and spacetime coordinates arises throughout string theory

If we go to static gauge again, where $\tau = X^0$ and write $v^i = \dot{X}^i$ then we have the equations of motion

$$\frac{d}{d\tau}\frac{v^i}{\sqrt{1-v^2}} = 0 \tag{2.8}$$

and conserved charge

$$p^i = m\frac{v^i}{\sqrt{1-v^2}} \tag{2.9}$$

which is simply the spatial momentum. These are the standard relativistic expressions.

We can solve the equation of motion in terms of the constant of motion p^i by writing

$$\frac{v^i}{\sqrt{1-v^2}} = p^i/m \iff p^2/m^2 = \frac{v^2}{1-v^2} \iff v^2 = \frac{p^2}{p^2+m^2} \tag{2.10}$$

hence

$$X^i(\tau) = X^i(0) + \frac{p^i\tau}{\sqrt{p^2+m^2}} \tag{2.11}$$

and we see that v^i is constant with $v^2\langle 1$.

Next we can consider a particle interacting with an external electromagnetic field. An electromagnetic field is described by a vector potential A_μ and its field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The natural action of a point particle of mass m and charge q in the presence of such an electromagnetic field is

$$S_{pp} = -m\int\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}d\tau + q\int A_\mu(X)\dot{X}^\mu d\tau \tag{2.12}$$

For those who know differential geometry the vector potential is a connection one-form on spacetime and $A_\mu\dot{X}^\mu d\tau$ is simply the pull-back of A_μ to the worldline of the particle.

The equation of motion is now

$$-m\frac{d}{d\tau}\left(\frac{-\eta_{\mu\nu}\dot{X}^\nu}{\sqrt{-\eta_{\lambda\rho}\dot{X}^\lambda\dot{X}^\rho}}\right) + q\frac{d}{d\tau}A_\mu - q\partial_\mu A_\nu\dot{X}^\nu = 0 \tag{2.13}$$

which we rewrite as

$$m\frac{d}{d\tau}\left(\frac{\eta_{\mu\nu}\dot{X}^\nu}{\sqrt{-\eta_{\lambda\rho}\dot{X}^\lambda\dot{X}^\rho}}\right) = qF_{\mu\nu}\dot{X}^\nu \tag{2.14}$$

To be more concrete we could choose static gauge again and we find (for $\mu = i$)

$$m\frac{d}{d\tau}\left(\frac{v^i}{\sqrt{1-v^2}}\right) = qF_{i0} + qF_{ij}v^j \tag{2.15}$$

Here we see the Lorentz force magnetic law arising as it should from the second term on the right hand side. The first term on the right hand side shows that an electric field provides a driving force.

At this point we should pause to mention a subtlety. In addition to (2.15) there is also the equation of motion for $X^0 = \tau$. However the reparameterization gauge symmetry implies that this equation is automatically satisfied. In particular the X^0 equation of motion is

$$-m\frac{d}{d\tau}\left(\frac{1}{\sqrt{1-v^2}}\right) = qF_{0i}v^i \tag{2.16}$$

2.2 Quantization

Next we'd like to quantize the point particle. This is made difficult by the highly non-linear form of the action. To this end we will consider a new action which is classically equivalent to the old one. In particular consider

$$S_{HT} = -\frac{1}{2}\int d\tau e\left(-\frac{1}{e^2}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} + m^2\right) \tag{2.17}$$

Here we have introduced a new field $e(\tau)$ which is non-dynamical, *i.e.* has no kinetic term. This action is now just quadratic in the fields X^μ. The point of it is that it reproduces the same equations of motion as before. To see this consider the e equation of motion:

$$\frac{1}{e^2}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} + m^2 = 0 \tag{2.18}$$

we can solve this to find $e = m^{-1}\sqrt{-\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu}}$. We now compute the X^μ equation of motion

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(\frac{1}{e}\dot{X}^\mu\right) \\ &= m\frac{d}{d\tau}\left(\frac{\dot{X}^\mu}{\sqrt{-\dot{X}^\lambda\dot{X}^\rho\eta_{\lambda\rho}}}\right) \end{aligned} \tag{2.19}$$

This is exactly what we want. Thus we conclude that S_{HT} is classically equivalent to S_{pp}.

One way to think about this action is that we have introduced a worldline metric $\gamma_{\tau\tau} = -e^2$ and its inverse $\gamma^{\tau\tau} = -1/e^2$ so that infinitesimal distances along the worldline have length

$$ds^2 = \gamma_{\tau\tau}d\tau^2 \tag{2.20}$$

Note that previously we never said that $d\tau$ represented the physical length of a piece of worldline, just that τ labeled points along the worldline. In particular the action takes the form

$$S_{HT} = -\frac{1}{2}\int d\tau\sqrt{-\gamma}\left(\gamma^{\tau\tau}\partial_\tau X^\mu\partial_\tau X^\nu\eta_{\mu\nu} + m^2\right) \tag{2.21}$$

which looks like D scalar fields on a space with metric $\gamma_{\tau\tau}$.

There is another advantage to this form of the action; we can smoothly set $m^2 = 0$ and describe massless particles, which was impossible with the original form of the action.

Now the action is quadratic in the fields X^μ we calculate the Hamiltonian and quantize more easily. The first step here is to obtain the momentum conjugate to each of the X^μ

$$\begin{aligned} p_\mu &= \frac{\partial L}{\partial\dot{X}^\mu} \\ &= \frac{1}{e}\eta_{\mu\nu}\dot{X}^\nu \end{aligned} \tag{2.22}$$

There is no conjugate momentum to e, it acts as a constraint and we will deal with it later. The Hamiltonian is

$$\begin{aligned} H &= p_\mu\dot{X}^\mu - L \\ &= \frac{e}{2}\left(\eta_{\mu\nu}p^\mu p^\nu + m^2\right) \end{aligned} \tag{2.23}$$

To quantize this system we consider wavefunctions $\Psi(X,\tau)$ and promote X^μ and p_μ to the operators

$$\hat{X}^\mu \Psi = X^\mu \Psi \qquad \hat{p}_\mu \Psi = -i\frac{\partial \Psi}{\partial X^\mu} \tag{2.24}$$

We then arrive at the Schrodinger equation

$$i\frac{\partial \Psi}{\partial \tau} = \frac{e}{2}\left(-\eta^{\mu\nu}\frac{\partial^2 \Psi}{\partial X^\mu \partial X^\nu} - m^2\Psi\right) \tag{2.25}$$

Lastly we must deal with e which we saw has no conjugate momentum. Classically its equation of motion imposes the constraint

$$p^\mu p_\mu + m^2 = 0 \tag{2.26}$$

which is the mass-shell condition for the particle. Quantum mechanically we realize this by restricting our physical wavefunctions to those that satisfy the corresponding constraint

$$-\eta^{\mu\nu}\frac{\partial^2 \Psi}{\partial X^\mu \partial X^\nu} + m^2\Psi = 0 \tag{2.27}$$

However this is just the condition that $\hat{H}\Psi = 0$ so that the wavefunctions are τ independent. If you trace back the origin of this time-independence it arises as a consequence of the reparameterization invariance of the worldline theory. It simply states that wavefunctions must also be reparameterization invariant, *i.e.* they can't depend on τ. This is deep issue in quantum gravity. In effect it says that there is no such thing as time in the quantum theory.

This equation should be familiar if you have learnt quantum field theory. In particular if we consider a free scalar field Ψ in D-dimensional spacetime the action is

$$S = -\int d^D x \left(\frac{1}{2}\partial_\mu \Psi^T \partial^\mu \Psi + \frac{1}{2}m^2 \Psi^T \Psi\right) \tag{2.28}$$

and the corresponding equation of motion is

$$\partial^2 \Psi - m^2 \Psi = 0 \tag{2.29}$$

which is the same as our Schrodinger equation (when restricted to the physical Hilbert space).

Thus we see that there is a natural identification of a free scalar field with a quantum point particle. In particular the quantum states of the point particle are in a one-to-one correspondence with the classical solutions of the free spacetime effective action. However one important difference should be stressed. The quantum point particle gave a Schrodinger equation which could be identified with the classical equation of motion for the scalar field. In quantum field theory one performs a second quantization whereby particles are allowed to be created and destroyed. This is beyond the quantization of the point particle that we considered since by default we studied the effective action on the

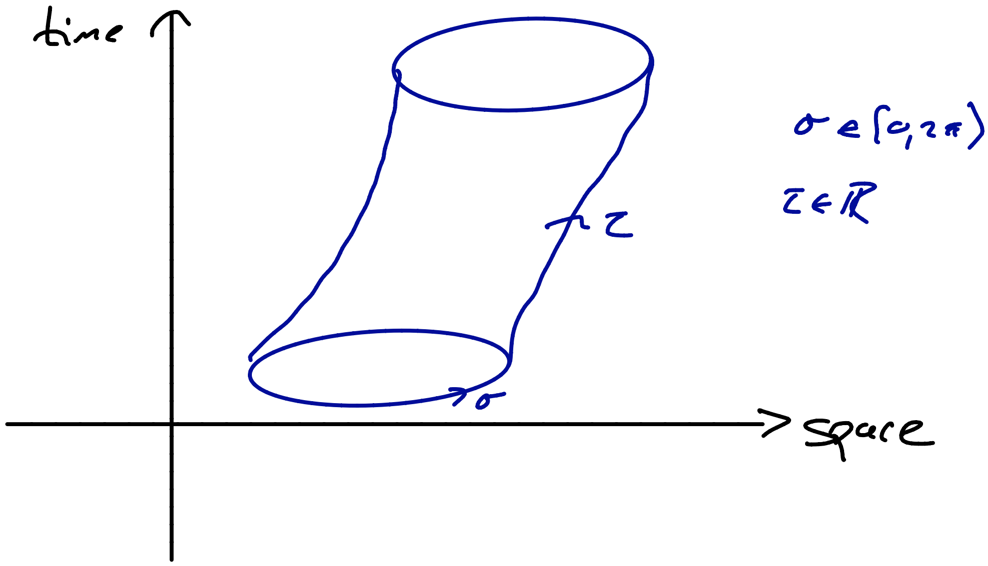

Figure 3.1: A closed string worldsheet

worldline of a single particle: it would have made no sense to create or destroy particles. Thus the second quantized spacetime action provides a more complete physical theory. In short quantizing a point particle leads to a spacetime Klein-Gordon field but not much more. To include interactions and the creation and annihilation of particles we are required to add additional terms into the spacetime action, corresponding to local interactions of particles. But the short distance singularities that these cause means that we shouldn't view this picture as fundamental. And also we lose predictive control as a wide variety of interactions are possible with few guiding principles.

Here we also can see that the quantum description of a point particle in one dimension leads to a classical spacetime effective action in D-dimensions. This is an important concept in String theory where the quantum dynamics of the two-dimensional worldvolume theory, with interactions included, leads to interesting and non-trivial spacetime effective actions.

3 Classical and Quantum Dynamics of Closed Strings

3.1 Classical Action

Having studied point particles from their worldline perspective we now turn to our main subject: strings. Our starting point will be the action the worldvolume of a string, which is two-dimensional. The natural starting point is to consider the action

$$S_{string} = \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det(\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu})} \tag{3.1}$$

which is simply the area of the two-dimensional worldvolume that the string sweeps out. Here $\sigma^\alpha, \alpha = 0,1$ labels the spatial and temporal coordinates of the string: $\tau = \sigma^0, \sigma = \sigma^1$. By convention we take $\sigma \in [0, 2\pi)$. Here $\sqrt{\alpha'}$ is a length scale that determines the size of the string. Or alternatively it's tension: the energy contained in a length L of string $E = L/2\pi\alpha'$.

This action is invariant under spacetime Lorentz transformations:

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu \tag{3.2}$$

as $\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\mu \eta_{\rho\sigma}$. Since we are considering a string in Minkowski space we also have translational symmetries:

$$X^\mu \to X^\mu + a^\mu \tag{3.3}$$

where a^μ is constant.

It is also invariant under worldsheet reparameterization invariance:

$$\sigma^\alpha \to \sigma'^\alpha(\sigma^\beta) \tag{3.4}$$

To see this we note that under such a change of coordinates

$$\partial_\alpha X^\mu = \frac{\partial\sigma'^\beta}{\partial\sigma^\alpha}\frac{\partial X^\mu}{\partial\sigma'^\beta} = \frac{\partial\sigma'^\beta}{\partial\sigma^\alpha}\partial'_\beta X^\mu \tag{3.5}$$

so that, in a condensed matrix notation,

$$\begin{aligned}\sqrt{-\det(\partial X^\mu \partial X^\nu \eta_{\mu\nu})} &= \sqrt{-\det\left(\left(\frac{\partial\sigma'}{\partial\sigma}\right)\partial' X^\mu \left(\frac{\partial\sigma'}{\partial\sigma}\right)\partial' X^\nu \eta_{\mu\nu}\right)} \\ &= \left|\det\left(\frac{\partial\sigma'}{\partial\sigma}\right)\right| \sqrt{-\det(\partial' X^\mu \partial' X^\nu \eta_{\mu\nu})}\ .\end{aligned} \tag{3.6}$$

Whereas on the other hand we have

$$\int d^2\sigma = \int d^2\sigma' \left|\det\left(\frac{\partial\sigma}{\partial\sigma'}\right)\right| \tag{3.7}$$

and the factors of $|\det(\partial\sigma/\partial\sigma')|$ and $|\det(\partial\sigma'/\partial\sigma)|$ will cancel.

Again we don't want to work directly with such a highly non-linear action. We saw above that we could change this by coupling to an auxiliary worldvolume metric. Thus we instead introduce a worldsheet metric $\gamma_{\alpha\beta}$ and consider

$$S_{string} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\ . \tag{3.8}$$

where $\gamma = \det(\gamma_{\alpha\beta}$ and $\gamma^{\alpha\beta}$ is the matrix inverse of $\gamma_{\alpha\beta}$. This action leads to the same classical equation of motion for X^μ and is reparameterization invariant (see the problem sets).

Problem: Show that by solving the equation of motion for the metric $\gamma_{\alpha\beta}$ on a d-dimensional worldvolume the action

$$S_{HT} = -\frac{1}{2}\int d^d\sigma\sqrt{-\gamma}\left(\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - m^2(d-2)\right)$$

one finds the equations of motion of the action

$$S_{NG} = m^{2-d}\int d^d\sigma\sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right)}$$

for the remaining fields X^μ, *i.e.* calculate and solve the $\gamma_{\alpha\beta}$ equation of motion and then substitute the solution back into S_{HT} to obtain S_{NG}. Note that the action S_{HT} is often referred to as the Howe-Tucker form for the action whereas S_{NG} is the Nambu-Goto form. (Hint: You will need to use the fact that $\delta\sqrt{-\gamma}/\delta\gamma^{\alpha\beta} = -\frac{1}{2}\gamma_{\alpha\beta}\sqrt{-\gamma}$). If you have not yet learnt much about metrics just consider the caseb of $d=1$ where $\gamma_{\alpha\beta}$ just has a single component $\gamma_{\tau\tau}$.

There is also an equation of motion arising from $\gamma_{\alpha\beta}$:

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} = 0 \tag{3.9}$$

Here one needs to use the fact that

$$\frac{\delta\sqrt{-\gamma}}{\delta\gamma^{\alpha\beta}} = -\frac{1}{2}\frac{1}{\sqrt{-\gamma}}\gamma_{\alpha\beta}\ . \tag{3.10}$$

However this two-dimensional case is very special. First, in addition to reparameterizations, the action is conformally invariant. That means that under a worldvolume Weyl transformation of the metric

$$\gamma_{\alpha\beta} \to e^{2\varphi}\gamma_{\alpha\beta} \tag{3.11}$$

(here φ is any function of the worldvolume coordinates) the action is invariant. In particular we see that the inverse metric transforms as $\gamma^{\alpha\beta} \to e^{-2\varphi}\gamma^{\alpha\beta}$ and the determinant as $\sqrt{-\gamma} \to e^{2\varphi}\sqrt{-\gamma}$ so that the combination $\sqrt{-\gamma}\gamma^{\alpha\beta}$ is invariant.

Secondly, using reparameterization, we can always choose the metric to take the form $\gamma_{\alpha\beta} = e^{2\rho}\eta_{\alpha\beta}$. To see this we note that under a reparameterization we have

$$\gamma'_{\alpha\beta} = \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\delta}{\partial\sigma'^\beta}\gamma_{\gamma\delta} \tag{3.12}$$

Thus we simply choose our new coordinates to fix $\gamma'_{01} = 0$ and $\gamma'_{00} = -\gamma'_{11}$. This requires that

$$\frac{\partial\sigma^\gamma}{\partial\sigma'^0}\frac{\partial\sigma^\delta}{\partial\sigma'^1}\gamma_{\gamma\delta} = 0 \tag{3.13}$$

and

$$\frac{\partial\sigma^\gamma}{\partial\sigma'^1}\frac{\partial\sigma^\delta}{\partial\sigma'^1}\gamma_{\gamma\delta} + \frac{\partial\sigma^\gamma}{\partial\sigma'^0}\frac{\partial\sigma^\delta}{\partial\sigma'^0}\gamma_{\gamma\delta} = 0 \tag{3.14}$$

These are two (complicated) differential equation for two functions $\sigma^0(\sigma'^0, \sigma'^1)$ and $\sigma^1(\sigma'^0, \sigma'^1)$. Hence there will be a solution (at least locally).

These facts together imply that the worldvolume metric $\gamma_{\alpha\beta}$ actually decouples from the fields X^μ. This conformal invariance of two-dimensional gravity coupled to the embedding coordinates (viewed as scalar fields) will be our fundamental principle. It allows us to simply set $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$. Thus to consider strings propagating in flat spacetime we use the action

$$S_{string} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{3.15}$$

subject to the constraint (3.23) which becomes

$$\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} = 0 \tag{3.16}$$

We finish with some comments about gravity in two-dimensions. We have included a dynamical metric in our string action but it does not have a kinetic term and so imposes the constraint $T_{\alpha\beta} = 0$. Why don't we try to include a derivative term for $\gamma_{\alpha\beta}$? The reason is that in two-dimensions the Einstein equation is trivial as

$$R_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta} R = 0 \tag{3.17}$$

vanishes identically. The reason for this is that in two-dimensions there is only one independent component for the Riemann tensor: $R_{0101} = -R_{0110} = -R_{1001} = R_{1010}$. Therefore

$$\begin{aligned} R_{\alpha\beta} &= R_{\alpha\gamma\beta\delta}\gamma^{\gamma\delta} \\ &= \begin{pmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{pmatrix} \\ &= R_{0101} \begin{pmatrix} \gamma^{11} & -\gamma^{01} \\ -\gamma^{01} & \gamma^{00} \end{pmatrix} \end{aligned} \tag{3.18}$$

Now we note that

$$\begin{pmatrix} \gamma^{00} & \gamma^{01} \\ \gamma^{01} & \gamma^{11} \end{pmatrix} = \frac{1}{\det(\gamma)} \begin{pmatrix} \gamma_{11} & -\gamma_{01} \\ -\gamma_{01} & \gamma_{00} \end{pmatrix} \tag{3.19}$$

and hence

$$R_{\alpha\beta} = \frac{1}{\det(\gamma)} R_{0101}\gamma_{\alpha\beta} \tag{3.20}$$

On the other hand we have

$$\begin{aligned} R &= 2R_{0101}(\gamma^{00}\gamma^{11} - \gamma^{01}\gamma^{01}) \\ &= 2R_{0101}\det(\gamma^{-1}) \\ &= \frac{2}{\det(\gamma)} R_{0101} \end{aligned} \tag{3.21}$$

and the result follows.

Thus Einstein's equation

$$R_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}R = T_{\alpha\beta} \tag{3.22}$$

will imply that $T_{\alpha\beta} = 0$. Hence even if we include two-dimensional gravity the $\gamma_{\alpha\beta}$ equation of motion imposes the constraint that the energy-momentum tensor vanishes

$$T_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial \gamma^{\alpha\beta}} = 0 \tag{3.23}$$

In particular the only term one could add which is reparameterization invariant and conformal inavariant is the usual Einstein-Hilbert term

$$S_{EH} = \frac{1}{4\pi}\int d^2\sigma\sqrt{-\gamma}R(\gamma)\ . \tag{3.24}$$

But in two dimensions this a total derivative. Evaluating is on a curved spacetime leads to a topological invariant known as the Euler-number. Thus the decoupling of the metric from the matter degrees of freedom is a fundamental requirement of our theory.

3.2 Aside: Spacetime Symmetries and Conserved Charges

We should also pause to outline how the spacetime symmetries lead to conserved currents and hence conserved charges in the worldsheet theory.

First we summarize Noether's theorem. Suppose that a Lagrangian $\mathcal{L}(\Phi_A, \partial_\alpha\Phi_A)$, where we denoted the fields by Φ_A, has a symmetry: $\mathcal{L}(\Phi_A) = \mathcal{L}(\Phi_A + \delta\Phi_A)$. This implies that

$$\frac{\partial\mathcal{L}}{\partial\Phi_A}\delta\Phi_A + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi_A)}\delta\partial_\alpha\Phi_A = 0 \tag{3.25}$$

This allows us to construct a current:

$$J^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi_A)}\delta\Phi_A \tag{3.26}$$

which is conserved

$$\begin{aligned} \partial_\alpha J^\alpha &= \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi_A)}\right)\delta\Phi_A + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi_A)}\partial_\alpha\delta\Phi_A \\ &= \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi_A)}\right)\delta\Phi_A - \frac{\partial\mathcal{L}}{\partial\Phi_A}\delta\Phi_A \\ &= 0 \end{aligned} \tag{3.27}$$

by the equation of motion. This means that the integral over space of J^0 is a constant defines a charge

$$Q = \int_{space} J^0 \tag{3.28}$$

which is conserved

$$\begin{aligned}\frac{dQ}{dt} &= \int_{space} \partial_0 J^0 \\ &= -\int_{space} \partial_i J^i \\ &= 0\end{aligned}$$

Let us now consider the action

$$S_{string} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{3.29}$$

This has the spacetime Poincare symmetries: translations $\delta X^\mu = a^\mu$ and Lorentz transformations $\delta X^\mu = \Lambda^\mu{}_\nu X^\nu$. In the first case the conserved current is

$$P^\alpha(a^\mu) = -\frac{1}{2\pi\alpha'}\partial^\alpha X_\mu a^\mu \tag{3.30}$$

The associated conserved charge is just the total momentum along the direction a^μ and in particular there are D independent choices

$$p_\mu = \frac{1}{2\pi\alpha'}\int d\sigma \dot{X}_\mu \tag{3.31}$$

We can also consider the spacetime Lorentz transformations which lead to the conserved currents

$$J^\alpha_\Lambda = -\frac{1}{2\pi\alpha'}\partial^\alpha X_\mu \Lambda^\mu{}_\nu X^\nu \tag{3.32}$$

The independent conserved charges are therefore given by

$$M^\mu{}_\nu = \frac{1}{4\pi\alpha'}\int d\sigma \dot{X}^\mu X_\nu - X^\mu \dot{X}_\nu \tag{3.33}$$

The Poisson brackets of these worldsheet charges will, at least at the classical level, satisfy the algebra Poincare algebra. In the quantum theory they are lifted to operators that commute with the Hamiltonian.

3.3 Quantization

Next we wish to quantize this action. Unlike the point particle this action is a field theory in $(1+1)$-dimensions. As such we must use the quantization techniques of quantum field theory rather than simply constructing a Schrodinger equation. There are several ways to do this. The most modern way is the path integral formulation. However this requires some techniques that are presumably unfamiliar. So here we will use the method of canonical quantization.

Canonical quantization is essentially the Heisenberg picture of quantum mechanics where the fields X^μ and their conjugate momenta P_μ are promoted to operators which satisfy the equal time commutation relations

$$\begin{aligned}
[\hat{X}^\mu(\tau,\sigma),\hat{P}_\nu(\tau,\sigma')] &= i\delta(\sigma-\sigma')\delta^\mu_\nu \\
[\hat{X}^\mu(\tau,\sigma),\hat{X}^\nu(\tau,\sigma')] &= 0 \\
[\hat{P}_\mu(\tau,\sigma),\hat{P}_\nu(\tau,\sigma')] &= 0
\end{aligned} \tag{3.34}$$

as well as the Heisenberg equation

$$\frac{d\hat{X}^\mu}{d\tau} = i[\hat{H},\hat{X}^\mu] \qquad \frac{d\hat{P}_\mu}{d\tau} = i[\hat{H},\hat{P}_\mu] \tag{3.35}$$

In the case at hand we have

$$\hat{L} = \frac{1}{4\pi\alpha'}\int d\sigma \eta_{\mu\nu}\dot{\hat{X}}^\mu\dot{\hat{X}}^\nu - \eta_{\mu\nu}\hat{X}'^\mu\hat{X}'^\nu \tag{3.36}$$

hence

$$\hat{P}_\mu = \frac{1}{2\pi\alpha'}\eta_{\mu\nu}\dot{\hat{X}}^\nu \tag{3.37}$$

and

$$\begin{aligned}
\hat{H} &= \int d\sigma \hat{P}_\mu\dot{\hat{X}}^\mu - \hat{L} \\
&= \int d\sigma 2\pi\alpha'\eta^{\mu\nu}\hat{P}_\mu\hat{P}_\nu - \int d\sigma\frac{1}{4\pi\alpha'}(2\pi\alpha')^2\eta^{\mu\nu}\hat{P}_\mu\hat{P}_\nu + \frac{1}{4\pi\alpha'}\eta_{\mu\nu}\hat{X}'^\mu\hat{X}'^\nu \\
&= \int d\sigma\pi\alpha'\eta^{\mu\nu}\hat{P}_\mu\hat{P}_\nu + \frac{1}{4\pi\alpha'}\eta_{\mu\nu}\hat{X}'^\mu\hat{X}'^\nu
\end{aligned} \tag{3.38}$$

We can now calculate

$$\begin{aligned}
\dot{\hat{X}}^\mu(\sigma) &= i[\hat{H},\hat{X}^\mu(\sigma)] \\
&= \pi\alpha' i\int d\sigma'\eta^{\lambda\nu}[\hat{P}_\lambda(\sigma')\hat{P}_\nu(\sigma'),\hat{X}^\mu(\sigma)] \\
&= 2\pi\alpha' i\int d\sigma'\eta^{\lambda\nu}\hat{P}_\lambda(\sigma')[\hat{P}_\nu(\sigma'),\hat{X}^\mu(\sigma)] \\
&= 2\pi\alpha'\int d\sigma'\eta^{\lambda\nu}\hat{P}_\lambda(\sigma')\delta^\mu_\nu\delta(\sigma-\sigma') \\
&= 2\pi\alpha'\eta^{\mu\nu}\hat{P}_\nu(\sigma)
\end{aligned} \tag{3.39}$$

which we already knew. But also we can now calculate

$$
\begin{aligned}
\dot{\hat{P}}_\mu(\sigma) &= i[\hat{H},\hat{P}_\mu(\sigma)] \\
&= \frac{i}{4\pi\alpha'}\int d\sigma'\eta_{\lambda\nu}[\hat{X}'^\lambda(\sigma')\hat{X}'^\nu(\sigma'),\hat{P}_\mu(\sigma)] \\
&= \frac{i}{2\pi\alpha'}\int d\sigma'\eta_{\lambda\nu}\hat{X}'^\lambda(\sigma')[\hat{X}'^\nu(\sigma'),\hat{P}_\mu(\sigma)] \\
&= \frac{i}{2\pi\alpha'}\int d\sigma'\eta_{\lambda\nu}\hat{X}'^\lambda(\sigma')\frac{\partial}{\partial\sigma'}[\hat{X}^\nu(\sigma'),\hat{P}_\mu(\sigma)] \\
&= -\frac{i}{2\pi\alpha'}\int d\sigma'\eta_{\lambda\nu}\hat{X}''^\lambda(\sigma')[\hat{X}^\nu(\sigma'),\hat{P}_\mu(\sigma)] \\
&= \frac{1}{2\pi\alpha'}\int d\sigma'\eta_{\lambda\nu}\hat{X}''^\lambda(\sigma')\delta^\nu_\mu\delta(\sigma-\sigma') \\
&= \frac{1}{2\pi\alpha'}\eta_{\mu\nu}\hat{X}''^\nu(\sigma)
\end{aligned}
\tag{3.40}
$$

or equivalently

$$
-\ddot{\hat{X}}^\mu + \hat{X}''^\mu = 0 \tag{3.41}
$$

Of course this is just the classical equation of motion reinterpreted in the quantum theory as an operator equation. In two-dimensions the solution to this is simply that

$$
\hat{X}^\mu = \hat{X}^\mu_L(\tau+\sigma) + \hat{X}^\mu_R(\tau-\sigma) \tag{3.42}
$$

i.e. we can split $\hat{X}^\mu$ into a left and right moving part.

To proceed we expand the string in a Fourier series

$$
\hat{X}^\mu = \hat{x}^\mu + \hat{w}^\mu\sigma + \alpha'\hat{p}^\mu\tau + \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\left(\frac{a^\mu_n}{n}e^{-in(\tau+\sigma)} + \frac{\tilde{a}^\mu_n}{n}e^{-in(\tau-\sigma)}\right) \tag{3.43}
$$

Note that we have dropped the hat on the operators a^μ and $\tilde{a}^\mu$ since they will appear frequently. But don't forget that they are operators! The various factors of n and α' will turn out to be useful later on. We have also included linear terms since $\hat{X}^\mu$ need not be periodic (more on this later). The factor of α' in front of $\hat{p}^\mu$ is there so that the total momentum of such a string is

$$
\begin{aligned}
\frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\dot{\hat{X}}^\mu &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\left(\alpha' p^\mu + \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}a^\mu_n e^{-in(\tau+\sigma)} + \tilde{a}^\mu_n e^{-in(\tau-\sigma)}\right) \\
&= \hat{p}^\mu
\end{aligned}
\tag{3.44}
$$

So p^μ agrees with the spacetime momentum of the string calculated using the Noether current.

Or if you prefer

$$
\begin{aligned}
\hat{X}^\mu_L &= \hat{x}^\mu_L + \frac{1}{2}(\alpha'\hat{p}^\mu + \hat{w}^\mu)(\tau+\sigma) + \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\frac{a^\mu_n}{n}e^{-in(\tau+\sigma)} \\
\hat{X}^\mu_R &= \hat{x}^\mu_R + \frac{1}{2}(\alpha'\hat{p}^\mu - \hat{w}^\mu)(\tau-\sigma) + \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\frac{\tilde{a}^\mu_n}{n}e^{-in(\tau-\sigma)}
\end{aligned}
\tag{3.45}
$$

Note also that we haven't yet said what n is, *e.g.* whether or not it is an integer, we will be more specific later. The a^μ_n and $\tilde{a}^\mu_n$ have the interpretation as left and right moving oscillators. Just as in quantum mechanics and quantum field theory these will be related to particle creation and annihilation operators.

Since X^μ is an observable we require that it is Hermitian in the quantum theory. This in turn implies that

$$
(a^\mu_n)^\dagger = a^\mu_{-n}\ , \qquad (\tilde{a}^\mu_n)^\dagger = \tilde{a}^\mu_{-n} \tag{3.46}
$$

and $(\hat{x}^\mu)^\dagger = \hat{x}^\mu, (\hat{w}^\mu)^\dagger = \hat{w}^\mu, (\hat{p}^\mu)^\dagger = \hat{p}^\mu$. In this basis

$$
\begin{aligned}
\hat{P}^\mu &= \frac{1}{2\pi\alpha'}\dot{\hat{X}}^\mu \\
&= \frac{1}{2\pi\alpha'}\left(\alpha' p^\mu + \sqrt{\frac{\alpha'}{2}}\sum_{-n\neq 0}a^\mu_n e^{-in(\tau+\sigma)} + \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\tilde{a}^\mu_n e^{-in(\tau-\sigma)}\right)
\end{aligned}
\tag{3.47}
$$

We can work out the commutator. First we take $x^\mu = w^\mu = p^\mu = 0$

$$
\begin{aligned}
[\hat{X}^\mu(\tau,\sigma), \hat{P}_\nu(\tau,\sigma')] &= \frac{i}{4\pi}\sum_n\sum_m \frac{1}{n}e^{-i(n+m)\tau}e^{-i(n\sigma+m\sigma')}[a^\mu_n, a^\nu_m] \\
&\quad + \frac{i}{4\pi}\sum_n\sum_m \frac{1}{n}e^{-i(n+m)\tau}e^{i(n\sigma+m\sigma')}[\tilde{a}^\mu_n, \tilde{a}^\nu_m] \\
&\quad + \frac{i}{4\pi}\sum_n\sum_m \frac{1}{n}e^{-i(n+m)\tau}e^{i(n\sigma-m\sigma')}[\tilde{a}^\mu_n, a^\nu_m] \\
&\quad + \frac{i}{4\pi}\sum_n\sum_m \frac{1}{n}e^{-i(n+m)\tau}e^{-i(n\sigma-m\sigma')}[a^\mu_n, \tilde{a}^\nu_m]
\end{aligned}
\tag{3.48}
$$

In order for the τ-dependent terms to cancel we see that we need the commutators to vanish if $n \neq -m$. The sum now reduces to

$$\begin{aligned} [\hat{X}^\mu(\tau,\sigma), \hat{P}^\nu(\tau,\sigma')] &= \frac{i}{4\pi}\sum_n \frac{1}{n} e^{-in(\sigma-\sigma')}[a^\mu_n, a^\nu_{-n}] \\ &+\frac{i}{4\pi}\sum_n \frac{1}{n} e^{in(\sigma-\sigma')}[\tilde{a}^\mu_n, \tilde{a}^\nu_{-n}] \\ &+\frac{i}{4\pi}\sum_n \frac{1}{n} e^{in(\sigma+\sigma')}[\tilde{a}^\mu_n, a^\nu_{-n}] \\ &+\frac{i}{4\pi}\sum_n \frac{1}{n} e^{-in(\sigma+\sigma')}[a^\mu_n, \tilde{a}^\nu_{-n}] \end{aligned} \tag{3.49}$$

Next translational invariance implies that the $\sigma + \sigma'$ terms vanish and hence

$$[a^\mu_n, \tilde{a}^\nu_m] = 0 \tag{3.50}$$

A slight rearrangement of indices shows that we are left with

$$[\hat{X}^\mu(\tau,\sigma), \hat{P}^\nu(\tau,\sigma')] = \frac{i}{4\pi}\sum_n \frac{1}{n} e^{-in(\sigma-\sigma')}([a^\mu_n, a^\nu_{-n}] + [\tilde{a}^\mu_n, \tilde{a}^\nu_{-n}]) \tag{3.51}$$

In a Fourier basis

$$\delta(\sigma-\sigma') = \frac{1}{2\pi}\sum_n e^{-in(\sigma-\sigma')} \tag{3.52}$$

Note that there is a contribution from $n = 0$ here that doesn't come from the oscillators, we'll deal with it in a moment. Therefore we see that we must take

$$[a^\mu_n, a^\nu_m] = n\eta^{\mu\nu}\delta_{n,-m}\ , \qquad [\tilde{a}^\mu_n, \tilde{a}^\nu_m] = n\eta^{\mu\nu}\delta_{n,-m} \tag{3.53}$$

Next it remains to consider the zero-modes (including the $n = 0$ contribution in (3.52)).

Problem: Show that the zero mode operators $\hat{x}^\mu, \hat{w}^\mu, \hat{p}^\mu \neq 0$ satisfy

$$[\hat{x}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu} \tag{3.54}$$

with the other commutators vanishing.

3.4 Classical Constraints

We also have to consider the constraint $\hat{T}_{\alpha\beta} = 0$. These are known as the Virasoro constraints and lie at the heart of string theory. The components are

$$\begin{aligned} \hat{T}_{00} &= \frac{1}{2}\dot{\hat{X}}^\mu \dot{\hat{X}}^\nu \eta_{\mu\nu} + \frac{1}{2}\hat{X}'^\mu \hat{X}'^\nu \eta_{\mu\nu} \\ \hat{T}_{11} &= \frac{1}{2}\hat{X}'^\mu \hat{X}'^\nu \eta_{\mu\nu} + \frac{1}{2}\dot{\hat{X}}^\mu \dot{\hat{X}}^\nu \eta_{\mu\nu} \\ \hat{T}_{01} &= \dot{\hat{X}}^\mu \hat{X}'^\nu \eta_{\mu\nu} \end{aligned} \tag{3.55}$$

It is helpful to change coordinates to

$$\begin{array}{c} \sigma^+ = \tau + \sigma \\ \sigma^- = \tau - \sigma \end{array} \Longleftrightarrow \begin{array}{c} \tau = \frac{\sigma^+ + \sigma^-}{2} \\ \sigma = \frac{\sigma^+ - \sigma^-}{2} \end{array} \tag{3.56}$$

Problem: Show that in these coordinates

$$\begin{aligned} \hat{T}_{++} &= \partial_+ \hat{X}^\mu \partial_+ \hat{X}^\nu \eta_{\mu\nu} \\ \hat{T}_{--} &= \partial_- \hat{X}^\mu \partial_- \hat{X}^\nu \eta_{\mu\nu} \\ \hat{T}_{+-} &= T_{-+} = 0 \end{aligned} \tag{3.57}$$

Let us now calculate T_{++} in terms of oscillators. We have

$$\partial_+ \hat{X}^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} a_n^\mu e^{-in(\tau+\sigma)} \tag{3.58}$$

where we have introduced

$$a_0^\mu = \sqrt{\frac{\alpha'}{2}} \hat{p}^\mu + \sqrt{\frac{1}{2\alpha'}} \hat{w}^\mu \tag{3.59}$$

thus

$$\begin{aligned} \hat{T}_{++} &= \frac{\alpha'}{2} \sum_{nm} a_n^\mu a_m^\nu e^{-i(n+m)(\tau+\sigma)} \eta_{\mu\nu} \\ &= \alpha' \sum_n L_n e^{-in(\tau+\sigma)} \end{aligned} \tag{3.60}$$

with

$$L_n = \frac{1}{2} \sum_m a_{n-m}^\mu a_m^\nu \eta_{\mu\nu} \tag{3.61}$$

where again we've dropped a hat on L_n, even though it is an operator but it too will appear frequently. Similarly we find

$$\hat{T}_{--} = \alpha' \sum_n \tilde{L}_n e^{-in(\tau-\sigma)} \tag{3.62}$$

with

$$\tilde{L}_n = \frac{1}{2} \sum_m \tilde{a}_{n-m}^\mu \tilde{a}_m^\nu \eta_{\mu\nu} \tag{3.63}$$

and

$$\tilde{a}_0^\mu = \sqrt{\frac{\alpha'}{2}} \hat{p}^\mu - \sqrt{\frac{1}{2\alpha'}} \hat{w}^\mu \tag{3.64}$$

It is instructive to compute the commutator of two components or

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{4}\sum_{pq}[a^\mu_{m-p}a^\nu_p, a^\lambda_{n-q}a^\rho_q]\eta_{\mu\nu}\eta_{\lambda\rho} \\
&= \frac{1}{4}\sum_{pq}\eta_{\mu\nu}\eta_{\lambda\rho}\Big([a^\mu_{m-p}a^\nu_p, a^\lambda_{n-q}]a^\rho_q + a^\lambda_{n-q}[a^\mu_{m-p}a^\nu_p, a^\rho_q]\Big) \\
&= \frac{1}{4}\sum_{pq}\eta_{\mu\nu}\eta_{\lambda\rho}\Big(a^\mu_{m-p}[a^\nu_p, a^\lambda_{n-q}]a^\rho_q + [a^\mu_{m-p}, a^\lambda_{n-q}]a^\nu_p a^\rho_q \\
&\quad +a^\lambda_{n-q}a^\mu_{m-p}[a^\nu_p, a^\rho_q] + a^\lambda_{n-q}[a^\mu_{m-p}, a^\rho_q]a^\nu_p\Big) \\
&= \frac{1}{4}\sum_{p}\eta_{\mu\rho}\Big(pa^\mu_{m-p}a^\rho_{n+p} + (m-p)a^\mu_p a^\rho_{n+m-p} \\
&\quad +pa^\rho_{n+p}a^\mu_{m-p} + (m-p)a^\rho_{n+m-p}a^\mu_p\Big) \\
&= \frac{1}{2}\sum_{p}\eta_{\mu\rho}\Big((p-n)a^\mu_{m+n-p}a^\rho_p \ + (m-p)a^\mu_p a^\rho_{n+m-p}\ \eta_{\mu\rho}\Big)
\end{aligned} \tag{3.65}$$

Here we have used the identities

$$[A, BC] = [A, B]C + B[A, C]\ , \qquad [AB, C] = A[B, C] + [A, C]B \tag{3.66}$$

and shifted the p=variable in the sum. Thus we find

$$[L_m, L_n] = (m-n)L_{m+n} \tag{3.67}$$

This is called the Witt algebra. Similarly we find

$$[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} \tag{3.68}$$

and also $[L_m, \tilde{L}_n] = 0$, *i.e.* two commuting copies of the Witt algebra associated to the left and right moving modes.

3.5 The Virasoro Algebra

How do we impose the constraints in the quantum theory? One issue is that, as defined, L_n and $\tilde{L}_n$ are ambiguous due to the failure of the oscilators modes to commute. Recall that we found the commutation relations

$$[a^\mu_n, {a^\nu_n}^\dagger] = n\eta^{\mu\nu} \qquad [\tilde{a}^\mu_n, {\tilde{a}^\nu_n}^\dagger] = n\eta^{\mu\nu} \qquad n \geq 0\ , \tag{3.69}$$

where $a^{\mu\dagger} = a^\mu_{-n}$, $\tilde{a}^{\mu\dagger} = \tilde{a}^\mu_{-n}$. As is usual in quantum field theory we consider the vacuum to be annhilated by a^μ_n and $\tilde{a}^\mu_n$:

$$a^\mu_n|0\rangle = \tilde{a}^\mu_n|0\rangle = 0 \qquad n > 0 \tag{3.70}$$

we then use $a_n^{\nu\dagger}$ and $\tilde{a^\mu}^\dagger$ to create elements of the Fock space (more on this later). Note that the zero-mode oscillators are neither creation or annhilation operators. As such we take the groundstate to be eigenstates of a_0^μ and $\tilde{a}_0^\mu$ or equivalently p^μ, w^μ. Thus when necessary we will denote the vacuum by $|0; p, w\rangle$.

Therefore we consider normal ordered operators, $: L_n :$ and $: \tilde{L}_n :$, where the annihilation operators always appear to the right of the creation operators. For $n > 0$ there is no problem with the definition as no pair of oscilators appears with $m + n = 0$ and hence

$$\begin{aligned} : L_n : &= : \frac{1}{2} \sum_m a^\mu_{n-m} a^\nu_m \eta_{\mu\nu} : \\ &= \frac{1}{2} a^\nu_0 a^\mu_n \eta_{\mu\nu} + \frac{1}{2} \sum_{m>0} a^\mu_{n-m} a^\nu_m \eta_{\mu\nu} + \frac{1}{2} \sum_{m<0} a^\nu_m a^\mu_{n-m} \eta_{\mu\nu} \\ &= L_n \end{aligned} \tag{3.71}$$

and similarly for $: \tilde{L}_n :$. We find the same for the negative Fourier modes ($n > 0$):

$$\begin{aligned} : L_{-n} : &= : \frac{1}{2} \sum_m a^\mu_{-n-m} a^\nu_m \eta_{\mu\nu} : \\ &= \frac{1}{2} a^\mu_{-n} a^\nu_0 \eta_{\mu\nu} + \frac{1}{2} \sum_{m>0} a^\mu_{-n-m} a^\nu_m \eta_{\mu\nu} + \frac{1}{2} \sum_{m<0} a^\nu_m a^\mu_{-n-m} \eta_{\mu\nu} \\ &= L_{-n} \end{aligned} \tag{3.72}$$

and similarly for $: \tilde{L}_{-n} :$. Note that one has

$$: L_n^\dagger :=: L_{-n} : \qquad : \tilde{L}_n^\dagger :=: \tilde{L}_{-n} : \tag{3.73}$$

However for L_0 and $\tilde{L}_0$ one finds

$$\begin{aligned} : L_0 : &= : \frac{1}{2} \sum_m a^\mu_{-m} a^\nu_m \eta_{\mu\nu} : \\ &= \frac{1}{2} a^\mu_0 a^\nu_0 \eta_{\mu\nu} + \frac{1}{2} \sum_{m>0} a^\mu_{-m} a^\nu_m \eta_{\mu\nu} + \frac{1}{2} \sum_{m<0} a^\nu_m a^\mu_{-m} \eta_{\mu\nu} \\ &= \frac{1}{2} a^\mu_0 a^\nu_0 \eta_{\mu\nu} + \sum_{m>0} a^\mu_{-m} a^\nu_m \eta_{\mu\nu} \\ : \tilde{L}_0 : &= \frac{1}{2} \tilde{a}^\mu_0 \tilde{a}^\nu_0 \eta_{\mu\nu} + \sum_{m>0} \tilde{a}^\mu_{-m} \tilde{a}^\nu_m \eta_{\mu\nu} \end{aligned} \tag{3.74}$$

However in this case

$$\begin{aligned} L_0 &= \frac{1}{2}\sum_m a^\mu_{-m} a^\nu_m \eta_{\mu\nu} \\ &= \frac{1}{2}a^\mu_0 a^\nu_0\eta_{\mu\nu} + \frac{1}{2}\sum_{m>0} a^\mu_{-m} a^\nu_m \eta_{\mu\nu} + \frac{1}{2}\sum_{m<0} a^\mu_{-m} a^\nu_m \eta_{\mu\nu} \\ &= \frac{1}{2}a^\mu_0 a^\nu_0\eta_{\mu\nu} + \frac{1}{2}\sum_{m>0} a^\mu_{-m} a^\nu_m \eta_{\mu\nu} + \frac{1}{2}\sum_{m>0} a^\mu_{m} a^\nu_{-m} \eta_{\mu\nu} \\ &= \frac{1}{2}a^\mu_0 a^\nu_0\eta_{\mu\nu} + \sum_{m>0} a^\mu_{-m} a^\nu_m \eta_{\mu\nu} + \frac{1}{2}\sum_{m>0} [a^\mu_{m}, a^\nu_{-m}] \eta_{\mu\nu} \\ &= \,: L_0 : + \frac{1}{2}\sum_{m>0} [a^\mu_{m}, a^\nu_{-m}] \eta_{\mu\nu} \end{aligned} \tag{3.75}$$

The last term is an infinite divergent sum which can be thought of as sum over the zero-point energies of the infinite number of harmonic oscillators. We must renormalize by subtracting off this divergence. Clearly $\tilde{L}_0$ has the same problem and this introduces the same sum. Thus we have, at some formal level,

$$L_0 =: L_0 : -a \qquad \tilde{L}_0 =: \tilde{L}_0 : -a \tag{3.76}$$

where

$$a = \frac{1}{2}\sum_{m>0}[a^\mu_m, a^\nu_{-m}]\eta_{\mu\nu} \ . \tag{3.77}$$

Assuming that a is real then $: L_0^\dagger :=: Ł_0 :$, $\tilde{L}_0^\dagger :=: \tilde{L}_0 :$. We will be more precise (but still rather unrigorous) about this later in the section on light cone gauge.

Thus we want to consider the algebra satisfied by $: L_n :$ (and clearly $: \tilde{L}_n :$ will satisfy a commuting copy). It follows that the only effect this can have on the algebra is in terms with an $: L_0 :$. Furthermore since the effect on $: L_0 :$ is a shift by an infinite constant (times the identity operator) it won't appear in the commutator on the left hand side. Thus any new terms can only appear with $: L_0 :$ on the right hand side and hence the general form is

$$[: L_m :, : L_n :] = (m-n) : L_{m+n} : + C(n)\delta_{m\ -n} \tag{3.78}$$

The easiest way to determine the $C(n)$ is to note the following (one can also perform a direct calculation but it is notoriously complicated and messy). First one imposes the Jacobi identity

$$[: L_k :, [: L_m :, : L_n :]] + [: L_m :, [: L_n :, : L_k :]] + [: L_m :, [: L_n :, : L_k :]] = 0 \tag{3.79}$$

If we impose that $k+m+n=0$ with $k,m,n \neq 0$ (so that no pair of them adds up to zero) then this reduces to

$$(m-n)C(k) + (n-k)C(m) + (k-m)C(n) = 0 \tag{3.80}$$

If we pick $k = 1$ and hence $m = -n - 1$ we find

$$-(2n+1)C(1) + (n-1)C(-n-1) + (n+2)C(n) = 0 \qquad (3.81)$$

Now we note that $C(-n) = -C(n)$ by definition. Hence we learn that $C(0) = 0$ and

$$C(n+1) = \frac{(n+2)C(n) - (2n+1)C(1)}{n-1} \qquad (3.82)$$

This is just a difference equation and given $C(2)$ it will determine $C(n)$ for $n > 1$ (note that it can't determine $C(2)$ given $C(1)$). We can look for a solution to this by considering polynomials. Since it must be odd in n the simplest guess is

$$C(n) = c_1 n^3 + c_2 n \qquad (3.83)$$

In this case the right hand side becomes

$$\begin{aligned} \frac{(n+2)(c_1 n^3 + c_2 n) - (2n+1)(c_1 + c_2)}{n-1} &= \frac{c_1 n^4 + 2c_1 n^3 + c_2 n^2 - 2c_1 n - (c_1 + c_2)}{n-1} \\ &= \frac{(n-1)(c_1 n^3 + 3c_1 n^2 + (3c_1 + c_2)n + c_1 + c_2)}{n-1} \end{aligned} \qquad (3.84)$$

Expanding out the left hand side gives

$$c_1(n+1)^3 + c_2(n+1) = c_1 n^3 + 3c_1 n^2 + (3c_1 + c_2)n + c_1 + c_2 \qquad (3.85)$$

and hence they agree.

Finally we must calculate c_1 and c_2. To do this we consider the ground state with no momentum or winding $|0;0,0\rangle$. This state is annihilated by $:L_n:$ for all $n \geq 0$ (why?). Thus we have

$$\begin{aligned} \langle 0,0;0| :L_n::L_{-n}: |0;0,0\rangle &= \langle 0,0;0|[:L_n:, :L_{-n}:]|0;0,0\rangle \\ &= 2n\langle 0,0;0| :L_0: |0;0,0\rangle + (c_1 n^3 + c_2 n)\langle 0,0;0|0;0,0\rangle \\ &= c_1 n^3 + c_2 n \end{aligned} \qquad (3.86)$$

where we assume that the ground state has unit norm.

Problem: Show using the original definition in terms of oscillators that

$$\begin{aligned} \langle 0,0;0| :L_1::L_{-1}: |0;0,0\rangle &= 0 \\ \langle 0,0;0| :L_2::L_{-2}: |0;0,0\rangle &= \frac{D}{2} \end{aligned} \qquad (3.87)$$

This tells us that

$$\begin{aligned} c_1 + c_2 &= 0 \\ 8c_1 + 2c_2 &= \frac{D}{2} \end{aligned} \qquad (3.88)$$

From which we learn that $c_1 = -c_2 = D/12$ and hence

$$[: L_m :, : L_n :] = (m-n) : L_{m+n} : + \frac{D}{12}(m^3 - m)\delta_{m\ -n} \tag{3.89}$$

Of course there is a similar expression for $[: \tilde{L}_m :, : \tilde{L}_m :]$ and $[: L_m :, : \tilde{L}_m :] = 0$. This is called the central extension of the Witt algebra. Meaning that there is an extra term on the right hand side which commutes with all the generators. However it much better known as the Virasoro algebra and D is called the central charge which has arisen as a quantum effect. The Virasoro algebra plays a central role in conformal field theories with a central charge c that can be different from D, *e.g.* it need not be an integer. From now on we will always take operators to be normal ordered and we will drop the :: symbol, unless otherwise stated.

3.6 Fock Space and Physical Hilbert Space

As we have mentioned the vacuum state is taken to be annihilated by all positive moded oscilators a_n^μ, $\tilde{a}_n^\mu$:

$$a_n^\mu|0\rangle = 0 \ , \qquad \tilde{a}_n^\mu|0\rangle = 0 \ , \qquad n > 0 \tag{3.90}$$

The zero modes also act on the ground state. Since $\hat{x}^\mu$ and $\hat{p}^\mu$ don't commute we can only chose $|0\rangle$ to be an eigenstate of one of them, we take

$$\hat{p}^\mu|0\rangle = p^\mu|0\rangle \qquad \hat{w}^\mu|0\rangle = w^\mu|0\rangle \tag{3.91}$$

when we want to be precise we label the ground state by $|0; p, w\rangle$. The Fock space is constructed from the vacuum $|0; p, w\rangle$ by acting with arbitrary numbers of creation operators:

$$|Fock\ state\rangle = a^{\mu_1}{}^\dagger_{n_1} \dots a^{\mu_L}{}^\dagger_{n_L} a^{\tilde{\mu}_1}{}^\dagger_{m_1} \dots a^{\tilde{\mu}_R}{}^\dagger_{m_R}|0; p, w\rangle \ . \tag{3.92}$$

These elements should be familiar from the study of the harmonic oscillator and free QFT. In a string theory each classical vibrational mode is mapped in the quantum theory to an individual harmonic oscillator with the same frequency.

Let look more closely at our Fock space of states. It is built up out of the ground state which we take to have unit norm $\langle 0|0\rangle = 1$. We sees that the one-particle state $a_{-1}^\mu|0\rangle$ has norm

$$\langle 0|a_1^\mu a_{-1}^\mu|0\rangle = \langle 0|[a_1^\mu, a_{-1}^\mu]|0\rangle = \eta^{\mu\mu} \tag{3.93}$$

where we do not sum over μ. Thus the state $a_{-1}^0|0\rangle$ has negative norm!

Problem: Show that the state $(a_{-1}^0 + a_{-1}^1)|0\rangle$ has zero norm.

Thus the natural innerproduct on the Fock space is not positive definite because the time-like oscillators come with the wrong sign. This also occurs in other quantum theories such as QED and doesn't necessarily represent any kind of sickness. Rather it is the indicator of a redundancy in the form of a gauge symmetry.

However we still need to impose the constraints. We proceed by reducing to the so-called physical Hilbert space of states. At first we might define this as those states in the Fock space that are annihilated by $\hat{T}_{\alpha\beta}$ and hence by L_n and $\tilde{L}_n$. However this turns out to be too strong a condition and would remove all states. Instead we define the physical state space to be those states which are annihilated by the positive frequency components of $\hat{T}_{\alpha\beta}$. In particular we impose[3]

$$L_n|phys\rangle = \tilde{L}_n|phys\rangle = 0 \ , n > 0 \qquad (L_0 - a)|phys\rangle = (\tilde{L}_0 - a)|phys\rangle = 0 \tag{3.94}$$

Here we have introduced a parameter a due to the renormalization of L_0. For historical reasons the parameter a is sometimes called the intercept and α' the slope. However it is not a parameter but rather is fixed by consistency conditions. Indeed it can be calculated by a variety of methods (such as using the modern BRST approach to quantization). We will see that the correct value is $a = 1$.

This is then sufficient to show that the expectation value of $\hat{T}_{\alpha\beta}$ vanishes

$$\langle phys|L_n|phys\rangle = \langle phys|\tilde{L}_n|phys\rangle = 0 \qquad \forall n \neq 0 \tag{3.95}$$

since the state on the right is annihilated by the positive frequency parts where as by taking the Hermitian conjugates one sees that the state on the left is annihilated by the negative frequency part.

There are stranger states still. A physical state $|\chi\rangle$ that satisfies $\langle\chi|phys\rangle = 0$ for all physical states is called null. It then follows that a null state has zero norm (as it must be orthogonal to itself).

There can be many such states. To construct an example just consider

$$|\chi\rangle = L_{-1}|0;p\rangle \ \text{ with } \ p^2 = 0 \tag{3.96}$$

Note that the zero-momentum ground state satisfies $L_n|0;0\rangle = 0$ and for all $n \geq 0$ and this remains true if for $|0;p\rangle$ if $p^2 = 0$. First we verify that $|\chi\rangle$ is physical. We have, for $m \geq 0$

$$\begin{aligned} L_m|\chi\rangle &= L_m L_{-1}|0;p\rangle \\ &= [L_m, L_{-1}]|0;p\rangle \\ &= (m+1)L_{m-1}|0;p\rangle + \frac{D}{12}(m^3 - m)\delta_{m1}|0;p\rangle \end{aligned} \tag{3.97}$$

The last term will vanish automatically whereas the first term can only be non-zero for $m = 0$ (since $L_n|0;p\rangle = 0$ for all $n \geq 0$). Here we find $L_0|\chi\rangle = |\chi\rangle$ which is the physical state condition for $a = 1$ which will turn out to be the case. Next we see that $\langle\chi|phys\rangle = \langle 0|L_1|phys\rangle = 0$. Note that we could have used any state instead of $|0;p\rangle$ that was annihilated by L_n for all $n \geq 0$ to construct a null state.

[3] Recall that all operators are understood to be normal ordered.

Of course we need to consider states in both the left and right sectors:

Problem: Show that $L_{-1}\tilde{a}^\mu_{-1}|0;p\rangle$ and $\tilde{L}_{-1}a^\mu_{-1}|0;p\rangle$ are null states if $p^2=0$.

Thus if we calculate some amplitude between two physical states $\langle phys'|phys\rangle$ we can shift $|phys\rangle \to |phys\rangle + |null\rangle$ where $|null\rangle$ is a null state. The new state $|phys\rangle + |null\rangle$ is still physical but the amplitude will remain the same - for any other choice of physical state $|phys'\rangle$. Thus we have a stringy gauge symmetry whereby two physical states are equivalent if their difference is a null state:

$$|phys\rangle \cong |phys\rangle + |null\rangle \tag{3.98}$$

for any null state $|null\rangle$. This will turn out to be the origin of spacetime diffeomorphisms and other gauge symmetries within string theory. And furthermore one can prove a no-ghost theorem which asserts that the Hilbert space of physical states is positive definite (provided that $a \leq 1$ and $D \leq 26$).

3.7 Spectrum

Let us now look in detail at the spectrum of states in the physical Hilbert Space of a closed string. We will assume that $a=1$. The constraints are

$$\begin{aligned} (L_0-1)|phys\rangle &= (\tilde{L}_0-1)|phys\rangle = 0 \\ L_n|phys &= \tilde{L}_n|phys\rangle = 0 \end{aligned} \tag{3.99}$$

with $n>0$. Typically the most telling constraints arise from L_0 and $\tilde{L}_0$. We will see that these give rise to spacetime Klein-Gordon equations, from which we can read of the mass of the corresponding particle. We will see the appearance of a gauge symmetry from the null states. The remaining constraints give gauge fixing conditions.

Recall that (we will assume $w^\mu=0$)

$$\begin{aligned} L_0 &= \frac{\alpha'}{4}p^\mu p^\nu \eta_{\mu\nu} + \sum_{n>0}\alpha^\mu_{-n}\alpha^\nu_n\eta_{\mu\nu} \\ \tilde{L}_0 &= \frac{\alpha'}{4}p^\mu p^\nu \eta_{\mu\nu} + \sum_{n>0}\tilde{\alpha}^\mu_{-n}\tilde{\alpha}^\nu_n\eta_{\mu\nu} \end{aligned} \tag{3.100}$$

Let us introduce the left and right-moving number operators N, $\tilde{N}$

$$N = \sum_{n>0}\eta_{\mu\nu}a^\mu_{-n}a^\nu_n \qquad \tilde{N} = \sum_{n>0}\eta_{\mu\nu}\tilde{a}^\mu_{-n}\tilde{a}^\nu_n \tag{3.101}$$

It is not hard to see that for a Fock state of the form (3.92) we have

$$\begin{aligned} N|Fock\ state\rangle &= (n_1+\ldots+n_L)|Fock\ state\rangle \\ \tilde{N}|Fock\ state\rangle &= (m_1+\ldots+m_R)|Fock\ state\rangle \end{aligned} \tag{3.102}$$

i.e. they count the total number of oscilators weighted by their mode number. Then the first conditions can be rewritten as

$$(p_\mu p^\mu + \frac{2}{\alpha'}(N + \tilde{N} - 2))|phys\rangle = 0 \qquad (N - \tilde{N})|phys\rangle = 0 \tag{3.103}$$

The first equation is just a spacetime Klein-Gordon equation. Fourier transformed back to coordinate space it becomes:

$$-\partial_\mu \partial^\mu + M^2 = 0 \qquad M^2 = \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{3.104}$$

The second condition, $N = \tilde{N}$, is called level matching. It arises as both the left and right moving sectors have the same spacetime momentum. It simply says that any physical state must be made up out of an equal number of left and right moving oscillators. It is more or less the only time the two sectors talk to each other.

Let us consider the lowest modes of the closed string. At level 0 (which means level 0 on both the left and right moving sectors by level matching) we simply have the ground state $|0;p\rangle$. This is automatically annihilated by both L_n and $\tilde{L}_n$ with $n > 0$. For $n = 0$ we find

$$p^2 - \frac{4}{\alpha} = 0 \tag{3.105}$$

Since $p^2 = -E^2 + \vec{p} \cdot \vec{p}$ we see that

$$E^2 = \vec{p} \cdot \vec{p} - 4/\alpha' \tag{3.106}$$

i.e. a tachyon! No one knows what to do with this. In the very least it represents an instability: one finds tachyons in ordinary field theory if one expands about a maximum rather than a minimum of the potential. Most people today would say that the Bosonic string is inconsistent although this hasn't been demonstrated. The cure arises by considering superstrings where it is projected out of the physical spectrum. So we continuing by simply ignoring it, as our discussion of the other modes still holds in the superstring.

Next we have level 1. Here the states are of the form

$$|G_{\mu\nu}\rangle = G_{\mu\nu} a^\mu_{-1} \tilde{a}^\nu_{-1} |0;p\rangle \tag{3.107}$$

Just as for the open string these will be massless, *i.e.* $p^2 = 0$ (again only if $a = 1$). Next we consider the constraints $L_m|G_{\mu\nu}\rangle = \tilde{L}_m|G_{\mu\nu}\rangle = 0$ with $m > 0$.

Problem: Show that these constraints imply that $p^\mu G_{\mu\nu} = p^\nu G_{\mu\nu} = 0$

The matrix $G_{\mu\nu}$ is a spacetime tensor. Under the Lorentz group $SO(1, D-1)$ it will decompose into a symmetric traceless, anti-symmetric and trace part.

Problem: Show that under spacetime Lorentz transformations the tensors

$$\begin{aligned} h_{\mu\nu} &= G_{(\mu\nu)} - \frac{1}{D}\eta^{\lambda\rho}G_{\lambda\rho}\eta_{\mu\nu} \\ b_{\mu\nu} &= G_{[\mu\nu]} \\ \phi &= \eta^{\lambda\rho}G_{\lambda\rho} \end{aligned} \tag{3.108}$$

will transform into themselves.

Thus from the spacetime point of view there are three independent dynamical modes labeled by $h_{\mu\nu}$, $b_{\mu\nu}$ and ϕ.

3.8 Spacetime Diffeomorphisms and Gauge Symmetry

Next we should see what identifications the null states give us. In particular we have that fact that $\xi_\mu L_{-1}\tilde{a}^\mu_{-1}|0;p\rangle$ and $\zeta_\mu \tilde{L}_{-1}a^\mu_{-1}|0;p\rangle$ are null states, provided that $p^2 = 0$ (see a previous problem). Expanding out

$$\begin{aligned} L_{-1}|0;p\rangle &= \frac{1}{2}\sum_m \eta_{\lambda\rho}a^\lambda_{-1+m}a^\rho_{-m}|0;p\rangle \\ &= \frac{1}{2}\eta_{\lambda\rho}\left(a^\lambda_{-1}a^\rho_0 + a^\lambda_0 a^\rho_{-1}\right)|0;p\rangle \\ &= \eta_{\lambda\rho}a^\lambda_0 a^\rho_{-1}|0;p\rangle \\ &= \sqrt{\frac{\alpha'}{2}}p_\lambda a^\lambda_{-1}|0;p\rangle \end{aligned} \tag{3.109}$$

where all the other contributions in the infinite sum annihilate the vacuum. Similarly

$$\tilde{L}_{-1}|0;p\rangle = \sqrt{\frac{\alpha'}{2}}p_\lambda \tilde{a}^\lambda_{-1}|0;p\rangle \tag{3.110}$$

Thus

$$|G_{\mu\nu}\rangle \cong |G_{\mu\nu}\rangle + i\sqrt{\frac{\alpha'}{2}}\xi_\mu p_\nu a^\nu_{-1}\tilde{a}^\mu_{-1}|0;p\rangle + i\sqrt{\frac{\alpha'}{2}}\zeta_\mu p_\nu a^\mu_{-1}\tilde{a}^\nu_{-1}|0;p\rangle \tag{3.111}$$

In terms of $G_{\mu\nu}$ this implies that

$$G_{\mu\nu} \cong G_{\mu\nu} + i\sqrt{\frac{\alpha'}{2}}p_\mu\xi_\nu + i\sqrt{\frac{\alpha'}{2}}p_\nu\zeta_\mu \tag{3.112}$$

or, switching to coordinate space representations and the individual tensor modes, we find

$$
\begin{aligned}
h_{\mu\nu} &\cong h_{\mu\nu} + \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\partial_\mu(\xi_\nu + \zeta_\nu) + \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\partial_\nu(\xi_\mu + \zeta_\mu) \\
b_{\mu\nu} &\cong B_{\mu\nu} + \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\partial_\mu(\xi_\nu - \zeta_\nu) - \frac{1}{2}\sqrt{\frac{\alpha'}{2}}\partial_\nu(\xi_\mu - \zeta_\mu) \\
\phi &\cong \phi + 2\sqrt{\frac{\alpha'}{2}}\partial_\mu(\xi^\mu + \zeta^\mu)
\end{aligned}
\tag{3.113}
$$

If we let $v_\mu = \frac{1}{2}\sqrt{\frac{\alpha'}{2}}(\xi_\mu + \zeta_\mu)$ and $\Lambda_\mu = \frac{1}{2}\sqrt{\frac{\alpha'}{2}}(\xi_\mu - \zeta_\mu)$ and use $\partial^\mu \xi_\mu = p^\mu \zeta_\mu = 0$ then we find

$$
\begin{aligned}
h_{\mu\nu} &\cong h_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu \\
b_{\mu\nu} &\cong b_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \\
\phi &\cong \phi
\end{aligned}
\tag{3.114}
$$

We should think of these fields are providing dynamical fluctuations away from the flat space configuration. In particular the first term line gives the infinitesimal form of a diffeomorphism, $x^\mu \to x^\mu - v^\mu$ and thus we can identify

$$
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.115}
$$

to be a metric tensor. The second line gives a generalization of and electromagnetic gauge transformation of a new field $b_{\mu\nu}$. The analogue of the gauge invariant field strength is

$$
H_{\lambda\mu\nu} = \partial_\lambda b_{\mu\nu} + \partial_\mu b_{\nu\lambda} + \partial_\nu b_{\lambda\mu} \tag{3.116}
$$

Thus the massless field content at level 1 consists of a graviton mode $h_{\mu\nu}$, an anti-symmetric tensor field $b_{\mu\nu}$ and a scalar ϕ, subject to the gauge transformations (3.113). Finally the massless condition $p^2 G_{\mu\nu} = 0$ leads to

$$
\begin{aligned}
\partial^2 h_{\mu\nu} &= 0 \\
\partial^2 b_{\mu\nu} &= 0 \\
\partial^2 \phi &= 0
\end{aligned}
\tag{3.117}
$$

Note that

$$
G_{\mu\nu} = b_{\mu\nu} + h_{\mu\nu} + \frac{1}{D}\eta_{\mu\nu}\phi \tag{3.118}
$$

and hence $p^\mu G_{\mu\nu} = p^\nu G_{\mu\nu} = 0$ become

$$
\begin{aligned}
\partial^\mu b_{\mu\nu} + \partial^\mu h_{\mu\nu} + \frac{1}{D}\partial_\nu \phi &= 0 \\
\partial^\nu b_{\mu\nu} + \partial^\nu h_{\mu\nu} + \frac{1}{D}\partial_\mu \phi &= 0
\end{aligned}
\tag{3.119}
$$

or equivalently

$$\begin{aligned}\partial^\mu h_{\mu\nu} &= -\frac{1}{D}\partial_\nu\phi \\ \partial^\mu b_{\mu\nu} &= 0\end{aligned} \tag{3.120}$$

These can be viewed as gauge fixing conditions. The fields $h_{\mu\nu}$, $b_{\mu\nu}$ and ϕ are known as the graviton (metric), Kalb-Ramond (b-field) and dilaton respectively.

It is clear that as we go to higher and higher levels we obtain towers of massive particle states. For example at level 2 we will find four possible tensor structures:

$$a^\mu_{-1}a^\nu_{-1}\tilde{a}^\lambda_{-1}\tilde{a}^\rho_{-1}|0\rangle \qquad a^\mu_{-2}\tilde{a}^\lambda_{-1}\tilde{a}^\rho_{-1}|0\rangle \qquad a^\mu_{-1}a^\nu_{-1}\tilde{a}^\lambda_{-2}|0\rangle \qquad a^\mu_{-2}\tilde{a}^\rho_{-2}|0\rangle \tag{3.121}$$

These will be massive fields with

$$M^2 = 4/\alpha' \ . \tag{3.122}$$

It turns our that the growth in the number of independent states grows exponentially with the level number: roughly speaking (without worrying about constraints) it is the number of ways to write the level number N as a sum over D smaller intergers. This is a signature of string theory: at low energy where only the massless modes are excited the string is more or less rigid and behaves like a particle. But at high energies it becomes increasingly floppy and softer. Once we look at strings scattering above the string scale there are a large number of possible modes that can absorb the shock and redistribute the energy. This is one reason behind its good UV behaviour.

It is worth noting here what would happen if $a \neq 1$. In this case the mass formula would be (taking into account level matching):

$$M^2 = \frac{4}{\alpha'}(N-a) \ . \tag{3.123}$$

For $a < 0$ the groundstate is a massive scalar field and then all other states would also be massive. For $a = 0$ the groundstate would be a single massless scalar field and then all other states would be massive. If $0 < a < 1$ then the ground state is a tachyon and then all other states are massive. Lastly if $a > 1$ then the groundstate and at least the level one states are tachyonic, possibly more levels if $a > 2$ *etc.*. Again, for any a, there is always an exponentially large tower of increasingly massive states.

3.9 Strings in Curved Spacetime and an Effective Action

We have considered quantized strings propagating in flat spacetime. This lead to a spectrum of states that included the graviton as well as other modes. More generally a string should be allowed to propagate in a curved background with non-trivial values for the metric and other fields. Our ansatz will be to consider the most general two-dimensional action for the embedding coordinates X^μ coupled to two-dimensional gravity subject to the constraint of conformal invariance. This later condition is required so that the two-dimensional worldvolume metric decouples from the other fields. We will consider only closed strings in this section. The reason for this is that these days one views open strings as description soliton like objects, called Dp-branes, that naturally sit inside the closed string theory.

Before proceeding we note that

$$S_{EH} = \frac{1}{4\pi}\int d^2\sigma\sqrt{-\gamma}R = \chi \tag{3.124}$$

is a topological invariant called the Euler number, *i.e.* the integrand is locally a total derivative. Thus we could add the term S_{EH} to the action and not change the equations of motion.

With this in mind the most general action we can write down for a closed string is

$$S_{closed} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\alpha'\sqrt{-\gamma}\phi(X)R+\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X)+\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu b_{\mu\nu}(X) \tag{3.125}$$

where ϕ is a scalar, $g_{\mu\nu}$ symmetric and $b_{\mu\nu}$ antisymmetric. These are precisely the correct degrees of freedom to be identified with the massless modes of the string. One can think of this worldsheet theory as two-dimensional quantum gravity coupled to some matter in the form of scalar fields. More generally one can think of and conformal field theory (with central charge equal to 26) as defining the action for a string.

Furthermore this action has the diffeomorphism symmetry $X^\mu \to X'^\mu(X)$

$$\partial_\alpha X'^\mu = \frac{\partial X'^\mu}{\partial X^\nu}\partial_\alpha X^\nu \qquad g'_{\mu\nu} = \frac{\partial X^\lambda}{\partial X'^\mu}\frac{\partial X^\rho}{\partial X'^\nu}g_{\lambda\rho} \qquad b'_{\mu\nu} = \frac{\partial X^\lambda}{\partial X'^\mu}\frac{\partial X^\rho}{\partial X'^\nu}b_{\lambda\rho} \qquad \phi' = \phi\ . \tag{3.126}$$

Problem: Show that S_{closed} is invariant under both worldsheet and spacetime diffeomorphisms.

It also incorporates the b-field gauge symmetry

$$b'_{\mu\nu} = b_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\mu\Lambda_\nu \tag{3.127}$$

however to see this we note that

$$\begin{aligned} \delta S_{closed} &= -\frac{1}{2\pi\alpha'}\int d^2\sigma\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\partial_\mu\Lambda_\nu \\ &= -\frac{1}{2\pi\alpha'}\int d^2\sigma\partial_\alpha(\epsilon^{\alpha\beta}\partial_\beta X^\nu\Lambda_\nu) \\ &= 0 \end{aligned} \tag{3.128}$$

where we used the fact that $\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta X^\nu = 0$ in the second to last line and the fact that the worldsheet is a closed manifold in the last line, *i.e.* the periodic boundary conditions $X^\mu(\sigma + 2\pi) = X^\mu(\sigma)$.

As stated above our general principle is the conformal invariance of the worldsheet theory, which ensures that the worldsheet metric $\gamma_{\alpha\beta}$ decouples. The action we just wrote down is conformal as a classical action. However this will not generically be the case in the quantum theory. Divergences in the quantum theory require regularization and renormalization and these effects will break conformal invariance by introducing an explicit scale: the renormalization group scale. It turns out that conformal invariance is more or less equivalent to finiteness of the quantum field theory. This restriction leads to equations of motions for the spacetime fields $\phi, g_{\mu\nu}$ and $b_{\mu\nu}$ (which from the worldvolume point of view are just fancy coupling constants). It is beyond the scope of these lectures to show this but the constraints of conformal invariance at the one loop level give equations of motion

$$\begin{aligned} 0 &= R_{\mu\nu} + \frac{1}{4} H_{\mu\lambda\rho} H_{\nu}{}^{\lambda\rho} - 2 D_\mu D_\nu \phi + \mathcal{O}(\alpha') \\ 0 &= D^\lambda H_{\lambda\mu\nu} - 2 D^\lambda \phi H_{\lambda\mu\nu} + \mathcal{O}(\alpha') \\ 0 &= 4 D^2 \phi + 4 (D\phi)^2 - R - \frac{1}{12} H^2 + \mathcal{O}(\alpha') \end{aligned} \tag{3.129}$$

where $H_{\mu\nu\lambda} = 3\partial_{[\mu} b_{\nu\lambda]}$. In general there will be corrections to these equations coming from all orders in perturbation theory, *i.e.* higher powers of α'. However such terms will be higher order spacetime derivatives and can be safely ignored at energy scales below the string scale, *i.e.* for $\alpha' p^2 << 1$.

A string propagating in spacetime has an infinite tower of massive excitations. However all but the lightest (massless) modes will be too heavy to observe in any experiment that we do. Thus in many cases one really just wants to consider the dynamics of the massless modes. This introduces the concept of an effective action. This is a very general concept (ubiquitous in quantum field theory) whereby we introduce an action for the light modes that we are interested in (below some scale M). The action is constructed so that it has all the correct symmetries of the full theory and its equations of motion reproduce the correct scattering amplitudes of the light modes that the full theory predicts. In general effective actions need not be renormalizable and they are not expected to be valid at energy scales above the scale M where the massive modes we've ignored can be excited and can no longer be ignored. Often one says that the massive modes have been integrated out. Meaning that one has performed the path integral over modes with momenta larger than M and is just left with a path integral over the low momentum modes.

In our case we have considered a string propagating in a curved spacetime that can be thought of as a background coming from a non-trivial configuration of its massless modes. In particular in our discussion we implicitly assumed that the massive modes were set to zero. The result was that quantum conformal invariance predicted the equations of motion (3.129). These are the on-shell conditions for a string to propagate in spacetime as derived in the full quantum theory. Note that they pick up an infinite series of α' corrections and also an infinite series of g_s corrections (where we allow the splitting an joining of strings). In other words, at lowest order in α' and g_s these are the equations of motion for the spacetime fields. Furthermore these equations of motion can be derived from the spacetime action

$$S_{effective} = -\frac{1}{2\alpha'^{12}} \int d^{26}x\sqrt{-g}e^{-2\phi}\left(R - 4(\partial\phi)^2 + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right) + \dots \tag{3.130}$$

Problem: Show that the equations of motion of (3.130) are indeed (3.129). You may need to recall that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ and $g^{\mu\nu}\delta R_{\mu\nu} = D_\mu D_\nu \delta g^{\mu\nu} - g_{\mu\nu}D^2\delta g^{\mu\nu}$.

This is therefore the effective action for the massless modes of a closed string. It plays a similar role to Klein Gordon equation played for the point particle (although $S_{effective}$ does not include the infinite tower of string states which isn't there for the point particle). The ellipsis denotes contributions from higher loops which will contain higher numbers of derivatives and which are suppressed by higher powers of α'.

3.10 String Scattering and the Perturbation Expansion

As we have emphasized, string theory is really quantum gravity applied to the string wolrdsheet. Thus when we consider the worldsheet we should also consider different topologies. To do this we first Wick rotate to a Euclidean worldsheet. We will assume that the worldsheet is orientatable (this isn't always the case in string theory but it is good enough for us).

To do this we Wick rotate $\tau \to i\tau$ so that

$$\sigma^+ \to \sigma + i\tau = z \qquad \sigma^- \to \sigma - i\tau = \bar{z} \ . \tag{3.131}$$

The worldsheet metric in conformal frame takes the form

$$ds^2 = e^{2\varphi(\tau,\sigma)}(-d\tau^2 + d\sigma^2) \to e^{2\varphi(\tau,\sigma)}(d\tau^2 + d\sigma^2) = e^{2\varphi(z,\bar{z})}dzd\bar{z} \tag{3.132}$$

Consider a holomorphic change of variables

$$z = z(z') \qquad \bar{z} = \bar{z}(\bar{z}') \tag{3.133}$$

then

$$\begin{aligned} ds^2 &= e^{2\varphi}\frac{dz}{dz'}\frac{d\bar{z}}{d\bar{z}'}dz'd\bar{z}' \\ &= exp\left(2\varphi + \ln\frac{dz}{dz'} + \ln\frac{d\bar{z}}{d\bar{z}'}\right)dz'd\bar{z}' \\ &= e^{2\varphi'}dz'd\bar{z}' \end{aligned} \tag{3.134}$$

Thus holomorphic coordinate transformations are conformal transformations (and vice-versa).

A particular such transformation is

$$z' = e^{-iz} = e^{-i\sigma+\tau} \ . \tag{3.135}$$

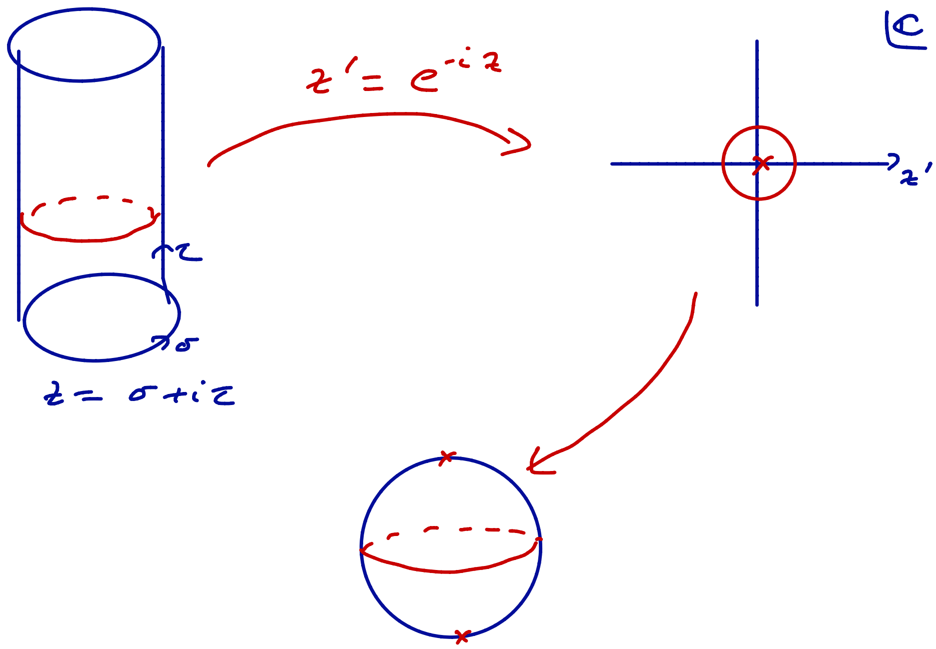

Figure 3.2: Mapping the Cylinder to the Plane and Sphere

This maps the cylinder of the closed string worldsheet to the complex place $\mathbb{C}$ with the origin $\{0\}$ removed. In particular the origin corresponds to $\tau \to -\infty$. Similarly $\tau \to \infty$ is mapped to infinity of $\mathbb{C}$. To include these additional points, corresponding to the string at minus/plus infinty, we can expand

$$\mathbb{C} \to \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1 \cong S^2 \tag{3.136}$$

which is the 2-sphere. Thus we can think of the freely propagating string worldsheet as a two-sphere.

In a path integral formulation of the theory one would consider the partition function or generating function to include a sum over topologies

$$Z = \sum_{topologies} \int D\gamma DX e^{-S_{closed}} \tag{3.137}$$

where $D\gamma$ and DX are the infinite dimensional integrals over the worldvolume metric $\gamma_{\alpha\beta}$ and embedding coordinates X^μ with S_{closed} is given in (3.125).

Let us split the dilaton into its vacuum value ϕ_0 and its fluctuation $\phi = \phi_0 + \phi'$ so

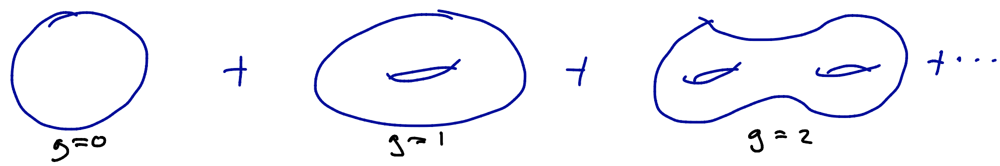

Figure 3.3: The genus expansion

that

$$
\begin{aligned}
Z &= \sum_{topologies} \int_{\Sigma} D\gamma DX e^{-\frac{\phi_0}{4\pi}\int \sqrt{-\gamma}R[\gamma]} e^{-S'_{closed}} \\
&= \sum_{g=0}^{\infty} e^{(2g-2)\phi_0} \int_{\Sigma_g} D\gamma DX e^{-S'_{closed}} \ . \qquad (3.138)
\end{aligned}
$$

Here we have used the fact that compact two-dimensional orientable manifolds are classified by their genus $g = 0, 1, 2, ..$ which is related to the Euler number χ by

$$
2 - 2g = \chi(\Sigma_g) = \frac{1}{4\pi}\int_{\Sigma_g} \sqrt{-\gamma}R[\gamma] \qquad (3.139)
$$

These are known as Riemann surfaces. The genus counts the number of 'holes' in the surface: a sphere has $g = 0$ but a torus has $g = 1$ and a double torus $g = 2$ *etc.*. Next we introduce the string coupling constant $g_s = e^{2\phi_0}$ so that

$$
Z = \sum_{g=0}^{\infty} g_s^{2g-2} \int_{\Sigma_g} D\gamma DX e^{-S'_{closed}} \qquad (3.140)
$$

Thus we find a natural perturbative expansion. For $g_s << 1$ the dominant term in Z comes from $g = 0$ with higher genus surfaces giving a power series in g_s.

Here we find one of the miracles of string theory. The genus expansion represents a splitting and re-joining of strings in an analogous form to how high loop corrections form the perturbative expansion in QFT see figure 3.3

But wait there's more! We can include ingoing and outgoing strings by modifying the Riemann surfaces to include in/out going strings: see figure 3.4

Finally we can use worldsheet conformal transformations to map the asymptotic 'legs' to marked points (τ_i, σ_i) on a compact Riemann surface: see figure 3.5.

Furthermore if we prepare an incoming string into a particular state $|in\rangle$ this corresponds to inserting operators $V(\tau, \sigma)$, commonly called a vertex operators, located at the marked points that correspond to the incoming states. Likewise we can consider out-going strings into a particular state $\langle out|$. This arises as a consequence of what is called the state-operator map in a conformal field theory. This map tells us that there is a one-to-one correspondence between states and operators. As a result the amplitude

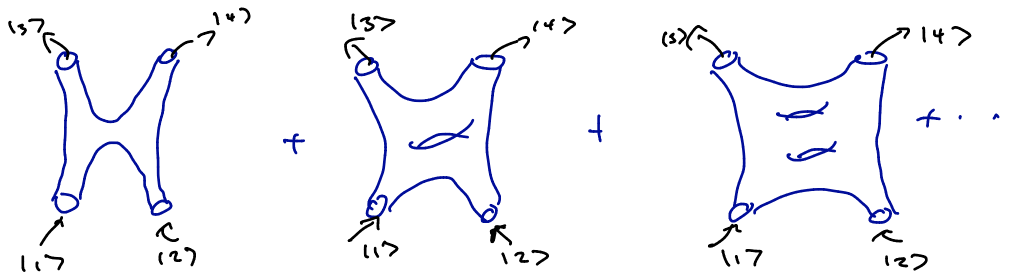

Figure 3.4: Strings Scattering

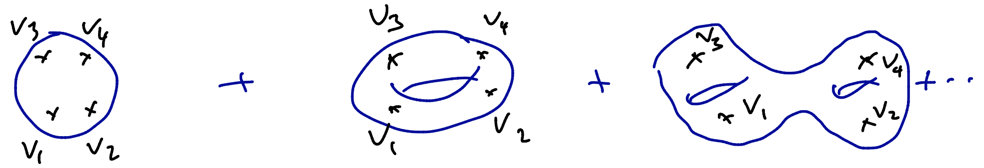

Figure 3.5: Worldsheet correlation functions

for a set of N_{in} incoming strings and $V_1(\tau_1, \sigma_1), ..., V_{N_{in}}(\tau_{N_{in}}, \sigma_{N_{in}})$ to scatter into N_{out} outgoing states $V_1'(\tau_1', \sigma_1'), ..., V_{N_{out}}'(\tau_{N_{out}}', \sigma_{N_{out}}')$ can be mapped to a correlation function in the worldsheet CFT:

$$\langle out|in\rangle = \langle 0|V_1'(\tau_1'\sigma_1')...V_{N_{out}}'(\tau_{N_{out}}'\sigma_{N_{out}}')V_1(\tau_1, \sigma_1)...V_{N_{in}}(\tau_{N_{in}}\sigma_{N_{in}})|0\rangle \tag{3.141}$$

Thus the amplitude for computing string scattering is given by an $(N_{in} + N_{out})$-point correlation function of vertex operators in the worldsheet CFT. One of the most famous results in string theory is that each term in this expansion is UV finite.

In particular the vertex operator for the tachyon with momentum p^μ is simply $V(\tau, \sigma) = e^{ip_\mu X^\mu(\tau,\sigma)}$ and the level one modes (gravitons *etc.*) correspond to $V^{\mu\nu} = \partial_+ X^\mu \partial_- X^\nu e^{ip_\lambda X^\lambda(\tau,\sigma)}$ (all normal ordered of course).

Problem: Write out an expression for the normal ordered form of $e^{ik_\mu X^\mu}$ and show that $\hat{p}^\mu e^{ik_\mu X^\mu}|0,0\rangle = k^\mu e^{ik_\mu X^\mu}|0,0\rangle$.

But note one additional thing: computing these amplitudes does not require new physics beyond the known description of a single string. The reason is it that it is possible to cut these diagrams in such a way that at any given time one only sees individual strings. Thus, unlike point particles, a single string already knows how it interacts with other strings. Our theory is unique: there is no arbitrariness in how we define the interactions.

You might ask why stop at strings? Why not quantize higher dimensional objects? As far as we know our luck runs out. Gravity is dynamical above 3 dimensions and not so easy to deal with (quantizing a (3+1)-dimensional object would require understanding full quantum gravity in four-dimensions which was the problem we originally started with). Furthermore a string has the nice feature that for a fixed length it has a fixed energy. But for higher dimensional dimensional objects it is possible to stretch out in one dimension and shrink in another so as to keep the volume fixed. This leads to infinite valleys in its potential energy where the mass of the object remains constant as it develops 'spikes'. There is also no natural perturbative expansion in terms of Riemann surfaces.

However we will see that higher dimensional objects, branes, do play a crucial role in string theory, just not in the way that strings do. Indeed we do not think that string theory is fundamentally about strings. Strings are just convenient, sometimes, when it makes sense to form such a perturbative expansion.

4 Open Strings and D-Branes

Okay so by now I hope to have convinced you that looking at closed strings leads to a theory of gravity along with a massless Kalb-Ramond field and dilaton, and then an infinite tower of massive fields. All propagating in a spacetime of 26 dimensions. But we need more than this. Where are the non-Abelian gauge interactions and matter fields that make up the Standard Model?

4.1 Quantizing Open Strings

The answers lies in the fact strings come in two varieties: open and closed. Open strings have two end points which traditionally arise at $\sigma = 0$ and $\sigma = \pi$. We must be careful to ensure that the correct boundary conditions are imposed. In particular we must choose boundary conditions so that the boundary value problem is well defined. This requires that

$$\eta_{\mu\nu}\delta X^\mu \partial_\sigma X^\nu = 0 \tag{4.142}$$

at $\sigma = 0, \pi$.

Problem: Show this!

There are essentially two boundary conditions that one can impose. The first is Dirichlet: we hold X^μ fixed at the end points so that $\delta X^\mu = 0$. The second is Neumann: we set $\partial_\sigma X^\mu = 0$ at the end points. The first condition implies that somehow the end points of the string are fixed in spacetime, like a flag to a flag pole. At first glance this seems unphysical and we will ignore it for now, although such boundary conditions turn out to be profoundly important. So we will start by considering second boundary condition, which states that no momentum leaks off the ends of the string. It is known as an NN boundary condition meaning the Neumann condition is selected at both ends.

Let us recall the general solution

$$\hat{X}^\mu = \hat{x}^\mu + \hat{w}^\mu\sigma + 2\alpha'\hat{p}^\mu\tau + \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\left(\frac{a^\mu_n}{n}e^{-in(\tau+\sigma)} + \frac{\tilde{a}^\mu_n}{n}e^{-in(\tau-\sigma)}\right) \tag{4.143}$$

Note the extra factor of 2 in front of $\hat{p}^\mu$. This arises as now $\sigma \in [0,\pi]$ which affects the the spacetime momentum (see below).

The condition that $\partial_\sigma\hat{X}^\mu(\tau,0) = 0$ implies that

$$w^\mu = 0\ , \qquad a^\mu_n = \tilde{a}^\mu_n \tag{4.144}$$

i.e. the left and right oscillators are not independent. If we look at the boundary condition at $\sigma = \pi$ then we determine that

$$\sum_{n\neq 0} a^\mu_n e^{in\tau}\sin(n\pi) = 0 \tag{4.145}$$

Thus n is again an integer (this can change). The mode expansion is therefore

$$X^\mu = x^\mu + 2\alpha' p^\mu\tau + \sqrt{2\alpha'}i\sum_{n\neq 0}\frac{a^\mu_n}{n}e^{in\tau}\cos(n\sigma) \tag{4.146}$$

(Note the slightly redefined value of p^μ as compared to before.)

For the open string the physical states are constrained to satisfy

$$L_n|phys>=0\ , n>0 \qquad \text{and} \qquad (L_0-1)|phys>=0 \tag{4.147}$$

in particular there is only one copy of the constraints required since the $\tilde{L}_n$ constraints will automatically be satisfied. The second condition is the most illuminating as it gives the spacetime mass shell condition. To see this we note that translational invariance $X^\mu \to X^\mu + x^\mu$ gives rise to the conserved current $\hat{P}^\mu = \frac{1}{2\pi\alpha'}\dot{X}^\mu$. This is a worldsheet current and hence the conserved charge (from the worldsheet point of view) is

$$\begin{aligned} p^\mu &= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\dot{X}^\mu \\ &= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma 2p^\mu + \sqrt{2\alpha'}\sum_{n\neq 0} a^\mu_n e^{in\tau}\cos(n\sigma) \\ &= p^\mu \end{aligned} \tag{4.148}$$

where again we have abused notation and confused the operator $\hat{p}^\mu$ that appears in the mode expansion of X^μ with its eigenvalue p^μ which we have now identified with the conserved charge. In any case we do this because we have shown that p^μ is indeed the spacetime momentum of the string. Note that this also explains why we put in the extra factor of 2 in front of $p^\mu\tau$ in the mode expansion.

Again we introduce

$$N = \sum_{n>0} \eta_{\mu\nu} a^{\mu}_{-n} a^{\nu}_{n} \tag{4.149}$$

Which is the analogue of the number operator that appears in the Harmonic oscillator. Of course this is an operator even though we are being lazy and dropping the hat.

Problem: Show that if $N|n\rangle = n|n\rangle$ then $Na^{\mu}_{-m}|n\rangle = (n+m)a^{\mu}_{-m}|n\rangle$

With this definition we can write the physical state condition $(L_0 - 1)|phys >= 0$ as

$$(p_\mu p^\mu + \frac{1}{\alpha'}(N-1))|phys >= 0 \tag{4.150}$$

Thus we can identify the spacetime mass-squared of a physical state to be the eigenvalue of

$$M^2 = \frac{1}{\alpha'}(N-1) \tag{4.151}$$

Of course as before we must not forget the other physical state condition $L_n|phys >= 0$ for $n > 0$. This constraint will take the form of a gauge fixing condition.

4.2 Spectrum and Chan-Paton Indices

Let us consider the lowest lying states.

At level zero we have the vacuum $|0; p >$. We see that the mass-shell condition is

$$p^2 - \alpha'^{-1} = 0 \tag{4.152}$$

The other constraint, $L_n|0; p >= 0$ with $n > 0$, is automatically satisfied. As with the closed string this problem goes away, or is well understood, in superstring theory. So we just ignore it.

Next consider level 1. Here we have

$$|A_\mu >= A_\mu(p) a^{\mu}_{-1}|0; p > \tag{4.153}$$

Since these modes have $N = 1$ it follows from the mass shell condition that they are massless (for $a = 1$!), *i.e.* the L_0 constraint implies that $p^2 A_\mu = 0$. Note that this depends crucially on the fact that $a = 1$. If $a > 1$ then $|A_\mu >$ would be tachyonic whereas if $a < 1$ $|A_\mu >$ would be massive.

But we must also check that $L_n|A >= 0$ for $n > 0$. Thus

$$\begin{aligned} L_n|A_\mu > &= \frac{1}{2} A_\mu \sum_m \eta_{\nu\lambda} a^{\nu}_{n-m} a^{\lambda}_{m} a^{\mu}_{-1}|0; p > \\ &= \frac{1}{2} A_\mu \eta_{\nu\lambda} \sum_{m\leq 1} a^{\nu}_{n-m} a^{\lambda}_{m} a^{\mu}_{-1}|0; p > \\ &= \frac{1}{2} A_\mu \eta_{\nu\lambda} \sum_{n-1\leq m\leq 1} a^{\nu}_{n-m} a^{\lambda}_{m} a^{\mu}_{-1}|0; p > \end{aligned} \tag{4.154}$$

In the second line we've noted that if $m > 1$ we can safely commute a^λ_m past a^μ_{-1} where it annihilates the vacuum. In the third line we've observed that if $n-m > 1$ then we can safely commute a^ν_{n-m} through the other two oscillators to annihilate the vacuum (recall that for $n > 0$ a^ν_{n-m} always commutes through a^λ_m). Thus for $n > 1$ we automatically have $L_n|A_\mu >= 0$. For $n = 1$ we find just two terms

$$\begin{aligned} L_1|A> &= \frac{1}{2}A_\mu\eta_{\nu\lambda}(a^\nu_1 a^\lambda_0 a^\mu_{-1} + a^\nu_0 a^\lambda_1 a^\mu_{-1})|0;p> \\ &= A_\mu a^\mu_0|0;p> \\ &= \sqrt{2\alpha'}p^\mu A_\mu|0;p> \end{aligned} \tag{4.155}$$

Thus we see that $|A_\mu >$ is represent a massless vector mode with $p^\mu A_\mu = 0$. In position space this is just $\partial^\mu A_\mu = 0$ and this looks like the Lorentz gauge condition for an electromagnetic potential.

Indeed recall that before we found the null state, with $p^2 = 0$,

$$\begin{aligned} |\Lambda> &= i\Lambda(p)L_{-1}|0;p> \\ &= i\eta_{\mu\nu}\Lambda a^\mu_0 a^\nu_{-1}|0;p> \\ &= i\sqrt{2\alpha'}p_\mu\Lambda a^\mu_{-1}|0;p> \end{aligned} \tag{4.156}$$

provided that $p^2 = 0$. Thus we must identify $A_\mu \equiv A_\mu + i\sqrt{2\alpha'}p_\mu\Lambda$ which in position space is the electromagnetic gauge symmetry $A_\mu \equiv A_\mu + \sqrt{2\alpha'}\partial_\mu\Lambda$. Again this occurs precisely when $a = 1$, otherwise $L_{-1}|0;p>$ is not a null state and their would not be a gauge symmetry.

There is one more thing that can be done. Since and open string has two preferred points, its end points, we can attach discrete labels to the end points so that the ground state, of the open string carries two indices

$$|0;p,ab\rangle \tag{4.157}$$

where $a = 1,..,N$ refers the $\sigma = 0$ end and $b = 1,...,N$ refers to the $\sigma = \pi$ end. It then follows that all the Fock space elements built out of $|0;p,ab>$ will carry these indices. These are called Chan-Paton indices. The level one states now have the form

$$|A^{ab}_\mu\rangle = A^{ab}_\mu a^\mu_{-1}|0;p,ab\rangle \tag{4.158}$$

The null states take the form

$$|\Lambda^{ab}\rangle = i\Lambda^{ab}L_{-1}|0;p,ab\rangle \tag{4.159}$$

and the gauge symmetry is

$$A^{ab}_\mu \equiv A^{ab}_\mu + \sqrt{2\alpha'}\partial_\mu\Lambda^{ab} \tag{4.160}$$

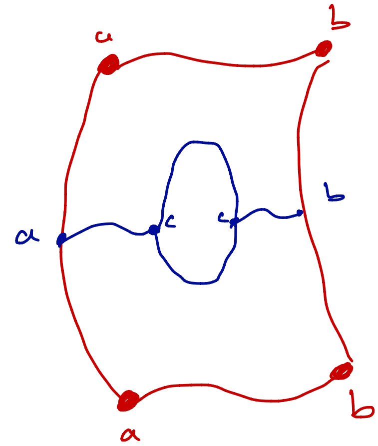

Figure 4.1: Splitting and Rejoining of Open Strings

These are the gauge symmetries of a non-Abelian Yang-Mills field with gauge group $U(N)$ (at lowest order in the fields).

To see why a matrix-multiplication might arise consider a process where a single open string in a state $|a,b\rangle$ splits into an intermediate state of two strings. These will have to have Chan-Paton labels $|a,c\rangle$ and $|c,b\rangle$. Furthermore these two intermediate strings rejoin to form the original string later on (see Figure). In a quantum theory one must sum over all the possible intermediate states. This means that the process can be written schematically as

$$|a,b\rangle \to \sum_c |a,c\rangle \otimes |c,b\rangle \to |a,b\rangle \ . \tag{4.161}$$

In other words the splitting and joining interact behaves like matrix multiplication:

$$A^{ab}_\mu \to \sum_c A^{ac}_\mu A^{cb}_\mu \qquad A^{ac}_\mu, A^{cb}_\mu \to \sum_c A^{ac}_\mu A^{cb}_\mu \ . \tag{4.162}$$

In fact it has been known since the earliest string theory days that the scattering of open strings is indeed described by Yang-Mills gauge theories with gauge group $U(N)$. Perhaps you haven't yet seen Yang-Mills theories yet. We will go over these issues next when we reinterpret open strings in terms of D-branes.

4.3 Background Fields and the Effective Action

Next week need to understand how we can extend the open string to curved backgrounds and also backgrounds that have a non-vanishing gauge field A_μ. This is an extension of what we did above for closed strings. Let us start by not considering the Chan-Paton indices.

As we mentioned the end points of open strings are special and in a sense behave as particles. Thus we can naturally take

$$S_{open} = S_{closed} + \int_{\sigma=\pi} d\tau A_\mu \dot{X}^\mu - \int_{\sigma=0} d\tau A_\mu \dot{X}^\mu \tag{4.163}$$

where S_{closed} is the closed string σ-model that we discussed above including the generalization to a curved background. This is an example of a so-called boundary conformal field theory. Meaning that we still require conformal invariance, even in the presence of the boundary. This breaks the separate left and right moving conformal algebras generated by L_n and $\tilde{L}_n$ to a single copy.

Recall the b-field gauge symmetry of closed strings:

$$b_{\mu\nu} \to b_{\mu\nu} + \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu \tag{4.164}$$

However if the worldsheet has a boundary then this is no-longer a symmetry:

$$\begin{aligned} \delta S_{open} &= -\frac{1}{2\pi\alpha'} \int d^2\sigma \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\mu \lambda_\nu \\ &= -\frac{1}{2\pi\alpha'} \int d^2\sigma \partial_\alpha (\epsilon^{\alpha\beta} \partial_\beta X^\nu \lambda_\nu) \\ &= \frac{1}{2\pi\alpha'} \int_{\sigma=\pi} d\tau \dot{X}^\nu \lambda_\nu - \frac{1}{2\pi\alpha'} \int_{\sigma=0} d\tau \dot{X}^\nu \lambda_\nu \end{aligned} \tag{4.165}$$

which is non-zero. However we sees that it can be canceled by a shift

$$A_\mu \to A_\mu - \frac{1}{2\pi\alpha'} \lambda_\mu \tag{4.166}$$

so that

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{2\pi\alpha'} b_{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.167}$$

is gauge invariant under both

$$\begin{aligned} b_{\mu\nu} &\to b_{\mu\nu} + \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu \\ A_\mu &\to A_\mu + \partial_\mu \Lambda - \frac{1}{2\pi\alpha'} \lambda_\mu \end{aligned} \tag{4.168}$$

Problem: Convince yourself that classically S_{open} still has all the other symmetries that we want.

Introducing boundaries to the string world sheet also affects our counting of the powers of $g_s = e^{\phi_0}$ in the perturbation expansion. In particular the formula for the Euler number for a Riemann surface with b boundary components is

$$\chi = 2 - 2g - b \tag{4.169}$$

Thus with open strings the perturbation expansion contains all powers of g_s (greater than -2), not just even powers. In addition cutting the loop diagrams of open strings will reveal closed strings. Put another way open stings can merge to form closed strings. Thus we can't just consider a theory of open strings, but we can add open strings into a theory of closed strings.

Once again we expect a non-trivial conditions on conformal invariance to arise at higher order in the α' expansion. In general the answer is unknown. However in the special case that $F_{\mu\nu}$ is constant it is known that, to all orders in α', the conditions on $\mathcal{F}$ arise from the effective action

$$S_{eff}^{open} = -\frac{1}{\alpha'^{13}} \int d^{26}x e^{-\phi} \sqrt{-\det(g_{\mu\nu} + 2\pi\alpha\mathcal{F}_{\mu\nu})} + \dots \tag{4.170}$$

Here the ellipsis denote higher order terms that arise when $\partial_\lambda F_{\nu\lambda} \neq 0$. There will also be corrections to the closed string conformal invariance conditions arising away from the boundaries.

In the case of a flat background (with $g_{\mu\nu} = \eta_{\mu\nu}$, $b_{\mu\nu} = 0$ and $\phi = \phi_0$ a constant) we find

$$S_{eff}^{open} = -\frac{1}{\alpha'^{13}} \int d^{26}x \sqrt{-\det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})} + \dots \tag{4.171}$$

which is well-known as the Born-Infeld action for electrodynamics. In particular expanding out the determinant and square-root to lowest non-trivial order gives. Now because for us $\mathrm{tr}({F_\mu}^\nu) = {F_\mu}^\mu = 0$ we must expand to second order

Problem: Assuming $\det e^A = e^{\mathrm{tr}A}$ show that to second order

$$\det(1 + A) = 1 + \mathrm{tr}(A) + \frac{1}{2}(\mathrm{tr(A)})^2 - \frac{1}{2}\mathrm{tr}(\mathrm{A}^2) + \dots \tag{4.172}$$

Thus

$$\det(1 + 2\pi\alpha' {F_\mu}^\nu) = 1 - \frac{1}{2}(2\pi\alpha')^2 {F_\mu}^\nu {F_\nu}^\mu + \dots \tag{4.173}$$

and hence

$$\begin{aligned} S_{eff}^{open} &= -\frac{1}{\alpha'^{13}} e^{-\phi_0} \int d^{26}x \sqrt{1 - \frac{1}{2}(2\pi\alpha')^2 F_{\mu\nu} F^{\nu\mu} + \dots} \\ &= -\frac{1}{\alpha'^{13}} e^{-\phi_0} \int d^{26}x \left(1 + \frac{1}{4}(2\pi\alpha')^2 F_{\mu\nu} F^{\mu\nu} + \dots\right) \end{aligned} \tag{4.174}$$

which looks just right for electromagnetism.

Next we need to consider the effect of Chan-Paton indices. These arise in the conformal field theory by adding labels to the boundaries to each boundary component. The massless level one field is now a matrix-valued vector (one-form would be a better term): A_μ^{ab}, $a, b = 1, 2, .., N$. The conformal invariance conditions are now very complicated in general, leading to an α' expansion in terms higher order corrections which are in general unknown.

However to lowest order in α' and in flat space it is known that the scattering of open strings is captured by a $U(N)$ Yang-Mills gauge theory. So lets look at the lowest order term in the effective action:

$$\begin{aligned} S_{eff}^{open} &= \frac{1}{\alpha'^{13}} \int d^{26}x \left(-Ne^{-\phi_0} - \frac{1}{4}(2\pi\alpha')^2 e^{-\phi_0} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \ldots \right) \\ &= S_{YM} \end{aligned} \tag{4.175}$$

here the constant term arises from taking $N = \mathrm{tr}1$ and now tr is over the Chan-Paton indices: $\mathrm{tr}(X^{ab}) = \sum_a X^{aa}$ for any matrix valued field X^{ab}. Furthermore we have

$$\begin{aligned} F_{\mu\nu}^{ab} &= \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} - i \sum_c \left(A_\mu^{ac} A_\nu^{cb} - A_\nu^{ac} A_\mu^{cb} \right) \\ &= \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} - i[A_\mu, A_\nu]^{ab} \end{aligned} \tag{4.176}$$

This action is now interacting since the gauge fields A_μ^{ab} do not commute. As expected the level one gauge symmetry arising from null states survives in a non-abelian form:

Problem: Show that the Yang-Mills action S_{YM} is invariant under the gauge transformation

$$\delta A_\mu^{ab} = D_\mu A^{ab} = \partial_\mu \Lambda^{ab} - i[A_\mu, \Lambda]^{ab} \tag{4.177}$$

4.4 D-branes

So what is the role of open strings. At first they were rather obscure. But we now realise that open strings are always attached to something: a so-called D-brane. And the dynamics of the D-brane are governed by the open strings. Thus we find a whole new class of dynamical objects in string theory.

By definition a Dp-brane is a $(p+1)$-dimensional worldvolume in spacetime upon which open strings can end. In practice this means that within closed string theory we include objects where open strings can end, *i.e.* we allow for Dirichlet boundary conditions

$$\delta X^I = 0 \qquad I = p+1, ..., 25 \tag{4.178}$$

for the values of μ which are transverse the the D-brane. Along the D-brane directions we impose

$$\partial_m X^\mu = 0 \qquad m = 0, 1, 2, ..., p \tag{4.179}$$

Such open strings therefore have NN boundary conditions for X^m and DD (*i.e.* Dirichlet at both ends) for X^i. In particular the open strings we looked at above have only NN boundary conditions and therefore describe D25-branes, meaning spacetime filling D-branes.

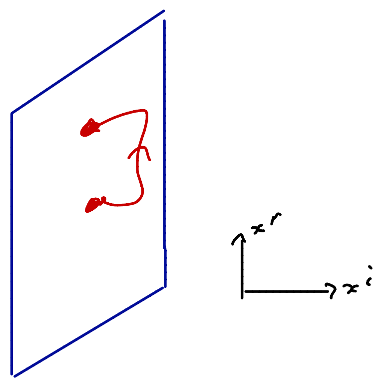

Figure 4.2: A D-brane and its Open String

To be precise consider a Dp-brane parallel to the $x^0, x^1, ..., x^p$ dimensions. This means that it sits at a specific location in the $x^{p+1}, ..., x^{25}$ dimensions, say $(x^{p+1}, ..., x^{25}) = (a^{p+1}, ..., a^{25})$. Thus one imposes Dirichlet boundary conditions on the fields $X^{p+1}, ..., X^{25}$. However the $X^0, ..., X^p$ coordinates are freely allowed to move and hence and are subjected to Neumann boundary conditions, *i.e.* $\partial_\sigma X^m = 0$ at $\sigma = 0, \pi$ and $m = 0, ..., p$. Thus the mode expansion for these fields is as before:

$$X^m = x^m + 2\alpha' p^m \tau + \sqrt{2\alpha'} i \sum_{n\neq 0} \frac{a_n^m}{n} e^{in\tau} \cos(n\sigma) \tag{4.180}$$

On the other hand for the transverse coordinates to the Dp-brane we have the boundary condition that $X^i = a^i$ at $\sigma = 0, \pi$. These are called DD boundary conditions. Starting from the expansion

$$\hat{X}^I = x^I + w^I \sigma + \alpha' p^i \tau + \sqrt{\frac{\alpha'}{2}} i \sum_{n\neq 0} \left(\frac{a_n^i}{n} e^{-in(\tau+\sigma)} + \frac{\tilde{a}_n^i}{n} e^{-in(\tau-\sigma)} \right) \tag{4.181}$$

and setting $\sigma = 0$ we see that $x^I = a^I$ and $p^I = 0$. We also find $a_n^I = -\tilde{a}_n^I$. Next we consider the $\sigma = \pi$ end. Here we find $w^I = 0$ and

$$a_n^i e^{-in\pi} + \tilde{a}_n^i e^{in\pi} = 0 \tag{4.182}$$

Using the fact that $a_n^I = -\tilde{a}_n^I$ we see that we need $\sin n\pi = 0$. Thus once again we see that the n are integers. In summary we find

$$X^I = a^I + \sqrt{2\alpha'} \sum_{n\neq 0} \frac{a_n^i}{n} e^{in\tau} \sin(n\sigma) \tag{4.183}$$

The main difference with the X^m coordinates are the lack of a momentum zero mode p^i. This means that the states of these open strings cannot move away from $x^I = a^I$, but they can move parallel to the Dp-brane. We denote the ground state of this open string by $|D;p>$ to distinguish it from the ground state of other open strings or closed strings.

What are the low lying states? Well they are very similar to before. Only now the $SO(1,25)$ symmetry is broken to $SO(1,p)\times SO(25-p)$. There is still a tachyon $|D;p>$ at level 0 with mass-squared $-1/\alpha'$ (which we continue to ignore). At level 1 there are two types of massless states:

$$|A_m>= A_m a^m_{-1}|D;p> \qquad |Y^I>= \frac{1}{2\pi\alpha'}Y^I a^I_{-1}|D;p> \tag{4.184}$$

where we now have $m=0,1,...,p$ and $I=p+1,...,25$. And of course an infinite tower of massive states. The extra factor of $1/2\pi\alpha'$ is to give Y^I units of length.

Note that the gauge symmetry that we saw above is also suitably reduced. In particular the null state that we used above is now

$$\begin{aligned} i\Lambda L_{-1}|0;p> &= \frac{i}{2}\Lambda\sum_m \eta_{\mu\nu}a^\mu_{-1+m}a^\nu_{-m}|0;p> \\ &= i\sqrt{2\alpha'}p_m\Lambda a^m_{-1}|0;p> \end{aligned} \tag{4.185}$$

The point to note here is that p^μ is only nonvanishing for $\mu=0,1,2,...,p$. Hence the modes $|Y^I>$ are not subject to a gauge symmetry, however $|A_m>$ still plays the role of a gauge Boson. The states $|Y^I>$ have the interpretation as $25-p$ massless scalar fields. They parameterize fluctuations of the Dp-brane in the transverse coordinates.

The importance of D-branes was not appreciated until 1994. Dp-branes should be thought of as solitonic-like states that appear in the closed string theory. As such they are like p-dimensional hypersurfaces in space, which are constant in time. In general we can consider configurations made up of several types of Dp-branes lying in different planes and intersecting with each other. The rules of string theory tell us that for each pair of brane (or for a brane and itself) we must consider the open string that stretches between the two. Each such string leads to additional particle like degrees of freedom.

We can also consider situations with N Dp-branes all parallel to each other. In this case we must label the end points of the open strings by an index $a=1,...,N$ to indicate which Dp-brane they end on. Indeed one sees that this is a geometric origin for the Chan-Paton factors that we discussed about and leads to a $U(N)$ gauge symmetry. The D-brane ground state can therefore be denote by $|D;p,ab>$, with one Chan-Paton index for each end point. It follows that all the states in the Fock space created using the string oscillators will carry ab indices and hence can be thought of as matrix valued.

The modern view on string theory is that one thinks of the bulk, 26-dimensional, dynamics are governed by closed strings, whose massless modes are a graviton, Kalb-Ramond field and dilaton. However in addition there are these soliton like D-brane state. On the worldvolume of these D-branes one finds $U(N)$ gauge vector fields, as well as scalars. It may happen that in some cases the D-branes are spacetime filling, meaning that they

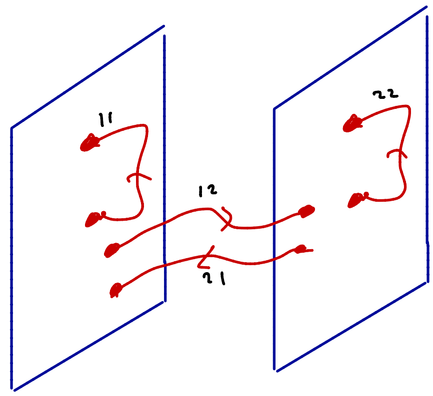

Figure 4.3: Two D-branes and their Open Strings

For example a D0-brane is essentially a point particle. The open strings are confined to end at a particular point in space (but not time). One also has D1-branes which are much like strings themselves. A D25-brane is simply the original notion of open strings, but now these are viewed as a state within the closed string theory.

Problem: Draw a picture of three parallel D-branes located at $x^I = a^I$, $x^I = b^I$ and $x^I = c^I$ and depict all their open strings. Give the corresponding mode expansions for the scalars X^μ.

4.5 The Abelian D-brane Effective action

How can we include the open string massless modes in an effective action? Well if we consider a 25-brane, that is a space-filling D-brane then this is essentially just the original definition of open strings and there is a massless vector A_μ. For D-branes one finds a vector field A_m, living on a $(p+1)$ dimensional subspace, plus the scalars Y^I. The corresponding worldsheet action is

$$S_{Dp}^{open} = S_{closed} + \int_{\sigma=\pi} d\tau (A_\mu \dot{X}^\mu + \frac{1}{2\pi\alpha'} Y^I X'^I) - \int_{\sigma=0} d\tau (A_\mu \dot{X}^\mu + \frac{1}{2\pi\alpha'} Y^I X'^I) \quad (4.186)$$

To obtain the effective action we simply notice that we just set $p^I = 0$, that is $\partial_T = 0$ in the effective action we found above. All this means is that $F_{\mu\nu}$ splits up:

$$\begin{aligned} F_{mn} &= \partial_m A_n - \partial_n A_m \\ F_{mI} &= \frac{1}{2\pi\alpha'} \partial_m Y^I \\ F_{IJ} &= 0 \end{aligned} \quad (4.187)$$

Thus to two derivative order, and in flat spacetime with constant dilaton and vanishing $b_{\mu\nu}$, we find

$$\begin{aligned} S_{Dp} &= -T_p \int d^{p+1}x e^{-\phi} \sqrt{-\det(\eta_{mn} + 2\pi\alpha' F_{mn} + \partial_m Y^I \partial_n Y^J \delta_{IJ})} \\ &= S_{BDI} \end{aligned} \tag{4.188}$$

where $T_p \propto \alpha'^{-(p+1)/2}$ is the tension of the Dp-brane (its exact coefficient can be computed but by other methods).

Problem: Show that substituting (4.187) into (4.171) leads to S_{DBI} (up to an overall constant). Hint: there is a matrix identity

$$\det \begin{pmatrix} M & A \\ -A^T & N \end{pmatrix} = \det N \det(M + A^T N^{-1} A) \tag{4.189}$$

This action had been studied well before string theory and is known as the Dirac-Born-Infeld action. It has two interesting special cases. If we set the transverse scalar fields to zero we recover the Born-Infeld action on the brane,as we saw above but now reduced to $p+1$ dimensions. If the gauge fields are set to zero then we obtain

$$S_{Dp} = -T_p \int d^{p+1}x \sqrt{-\det(\eta_{mn} + \partial_m Y^I \partial_n Y^J \delta_{IJ})} \tag{4.190}$$

Problem: Show that this is a gauge fixed form of the Nambu-Goto action

$$S_{NG} = -T_p \int d^{p+1}x \sqrt{-\det\left(\partial_m X^\mu \partial_n X^\nu \eta_{\mu\nu}\right)} \tag{4.191}$$

where we take $X^m = x^m$ if $m = 0, ..., p$ and $X^I = Y^I$ if $I = p+1, ..., 25$.

This describes an extended object in spacetime whose action is simply its volume and indeed the Y^I give the position of the Dp-brane in the transverse space. Thus indeed we should think of a Dp-brane as a p-dimensional extended object in space, that propagates through time and whose fluctuations are governed by the worldvolume extremisation principle of the Nambu-Goto action.

Thus in general the total spacetime effective action is

$$S = \frac{1}{2\alpha'^{12}} \int d^{26}x \sqrt{-g} e^{-2\phi} \left(R - 4(\partial\phi)^2 + \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right) + \sum_{Dp} S_{Dp} \tag{4.192}$$

where the second term is a sum over all the Dp-branes in the background. We emphasize that this is just the lowest order term in an effective action with otherwise contains contributions from higher powers of α' and $g_s = e^{\phi_0}$.

4.6 Multiple D-Branes and Non-Abelian Gauge Theory

So now let us consider how to incorporate Chan-Paton indices and non-abelian dynamics. This corresponds to having N Dp-branes all parallel to each other. For simplicity we take the background spacetime to be flat Minkowski space.

If we now reduce the 26-dimensional Yang-Mills gauge theory, which simply means setting $\partial_\mu = 0$ for $\mu = I$ we find

$$\begin{aligned} F_{mn} &= \partial_m A_n - \partial_n A_m - i[A_m, A_n] \\ F_{mI} &= \frac{1}{2\pi\alpha'}(\partial_m Y^I - i[A_m, Y^I]) = \frac{1}{2\pi\alpha'} D_m Y^I \\ F_{IJ} &= -\frac{i}{(2\pi\alpha')^2}[Y^I, Y^J] \end{aligned} \tag{4.193}$$

We have dropped the matrix indices in the interest of not cluttering up the equations. All fields should be understood to be matrix valued and carry ab indices which label which Dp-brane each end lives on. In particular they are physical fields and so taken to be hermitian:

$$A_m^\dagger = A_m \quad (Y^I)^\dagger = Y^I \quad \Longleftrightarrow \quad A_m^{ba*} = A_m^{ab} \quad Y^{Iba*} = Y^{Iab} \tag{4.194}$$

To find the dynamics we simply dimensionally reduce the Yang-Mills action from 26 to $p+1$ dimensions:

$$\begin{aligned} S_{Dp} = -T_p \int d^{p+1}x e^{-\phi}\Big(N \quad &+ \quad (\pi\alpha')^2 \mathrm{tr}(F_{mn}F^{mn}) + \frac{1}{2}\mathrm{tr}(D_m Y^I D^m Y^I) \\ &- \frac{1}{16\pi^2\alpha'^2}\sum_{I,J}\mathrm{tr}([Y^I, Y^J][Y^I, Y^J])\Big) \end{aligned} \tag{4.195}$$

where $T_p \sim \alpha'^{-\frac{p+1}{2}}$ is the tension of a Dp-brane *i.e.* the energy per unit p-volume.

Problem: Show that S_{Dp} is invariant under the gauge transformation

$$A_m \to ig\partial_m g^{-1} + gA_m g^{-1} \qquad Y^I \to gY^I g^{-1} \tag{4.196}$$

where $g(x^m)$ is worldvolume-dependent element of $U(N)$.

Now we find a Yang-Mills gauge theory in $p+1$ dimensions coupled to $25-p$ scalar fields which take values in the adjoint of $U(N)$. Thus our stringy physics is getting more interesting.

Note that the combination

$$(A_m^{ab}, Y^{Iab}) = \delta^{ab}(A_m^0, Y^{I0}) \tag{4.197}$$

i.e. matrices proportional to the identity matrix, will commute with everything and therefore represent free degrees of freedom. The Y^I components correspond to the centre of mass of the Dp-branes which is a free field due to the symmetries of Minkowski space.

Finally let us consider the potential on the coordinates Y^I

$$V = -\frac{T_p}{16\pi^2\alpha'^2}\sum_{I,J}\text{Tr}\left([Y^I, Y^J]\right)^2 \tag{4.198}$$

Note that since $(Y^I)^\dagger = Y^I$ we have $[Y^I, Y^J]^\dagger = -[Y^I, Y^J]$. Thus the potential is minus the sum of the square of an anti-Hermitian matrix. Therefore $V \geq 0$. It follows that the vacuum states of this action correspond to $[Y^I, Y^J] = 0$ for all I, J. Therefore, up to a gauge transformation, we can write

$$Y^I = \text{diag}(a_1^I, \dots, a_N^I) \tag{4.199}$$

We interpret the a_a^I as the location of the ath Dp-brane in the x^I direction. In fact there are further gauge transformations of the form

$$g = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad etc. \tag{4.200}$$

i.e. the identity matrix with two rows interchanged. These generate the symmetric group S^N which act to swap $a_a^I \leftrightarrow a_b^I$ for some pair a, b. This tells us that the individual D-branes are indistinguishable, like particles. In particular the vacuum moduli space, that is the space of solutions to $V = 0$ modulo gauge transformations, is is the N-th symmetric product of $\mathbb{R}^{25-9}$:

$$\mathcal{M}_N = Sym^N(\mathbb{R}^{25-9}) = (\mathbb{R}^{25-9})^N/S^N \ . \tag{4.201}$$

Thus even though the open strings that stretch between N parallel D-branes carry N^2 degrees of freedom, the vacuum moduli space just consists of N independent vectors $(a_a^{p+1}, ..., a_a^{25})$ which parameterize the position of the ath D-brane in the transverse space. The other modes will generically be massive as a consequence of the Higgs' effect. In particular their masses will be of the form

$$m_{ab}^2 \sim \sum(a_a^I - a_b^I)(a_a^I - a_b^I) \tag{4.202}$$

Personally I find this stringy interpretation of the Higgs' mechanism very pleasing. The Higgs' mechanism is ubiquitous quantum field theory but this geometric interpretation requires strings as only a string-like object will lead to a mass formula that grows with separation, which is what the Higgs' mechanism requires.

Problem: Find the masses of the scalar fields and gauge fields for the case of two D-branes located at $x^I = a^I$ and $x^I = b^I$. Show that indeed the fields proportional to the identity matrix decouple as a free system.

Geometrically this occurs because the open string that stretches between two separated D-branes must have a non-zero length and hence is massive. However whenever two or more D-brane coincide there is an enhanced symmetry and additional massless fields arise.

4.7 D-Branes and Black Holes

Note that this discussion has only been at the level of the classical action. The existence of a moduli space of vacua implies that we can place branes at any separation hence that there is no force between them. However branes are massive objects and will gravitate towards each other once gravitational effects are taken into account. Thus we expect that in the quantum theory, and when the interactions of closed strings are taken into account, the moduli space will be lifted.

In particular recall the action (4.192)

$$S = \frac{1}{2\alpha'^{12}} \int d^{26}x\sqrt{-g}e^{-2\phi}\left(R - 4(\partial\phi)^2 + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right) + \sum_{Dp} S_{Dp} \tag{4.203}$$

since the S_{Dp} terms involve the metric $g_{\mu\nu}$ they will contribute to the spacetime energy momentum tensor obtained from

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{4.204}$$

and therefore appear as a source in the Einstein equation in the form of a δ-function localized to the brane worldvolume. The Dp-brane also acts as a delta-function source for the dilaton equation of motion. For a Dp-brane localized at $x^I = a^I$ the corresponding solution to the closed string gravitation equations of motion will look like Schwarzschild-like black hole but where the singularity is spread out along a $p+1$ dimensional hypersurface. The dilaton will also have a non-trivial dependence on the radial direction transverse to the Dp-brane. Such solutions are known as dilatonic black branes, or black p-branes. As it stands we do not expect to find solutions where they sit at arbitrary points in space since their gravitational attraction will pull them together.

In superstrings, where the Dp-brane worldvolume theory becomes a maximally supersymmetric Yang-Mills theory, this classical description of the moduli space is preserved in the quantum theory. Furthermore in superstring theories Dp-branes (as well as other types of branes which are not described by open strings) carry a charge with respect to $p+1$-form fields $C_{\mu_0\ldots\mu_p}$, in an analogous way to how Strings couple to $b_{\mu\nu}$:

$$\begin{aligned} S_{Dp} &\to S_{Dp} + S_{WZ} \\ S_{WZ} &= \frac{T_p}{(p+1)!}\int d^{p+1}x\varepsilon^{m_0\ldots m_p}C_{\mu_0\ldots\mu_p}\partial_{m_0}X^{\mu_0}\ldots\partial_{m_p}X^{\mu_{m_p}}\ , \end{aligned} \tag{4.205}$$

where S_{WZ} is known as the Wess-Zumino term. We have seen examples of these terms in both the point particle:

$$S_{WZ} = q \int d\tau \partial_\tau X^\mu A_\mu \ , \tag{4.206}$$

and fundamental string:

$$S_{WZ} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu b_{\mu\nu} \ . \tag{4.207}$$

Note that in these cases T_p is both the mass (well tension which is mass per unit volume) and charge (or more precisely the charge per unit volume). It also introduces the notion of an anti-Dp-brane with opposite charge. This corresponds to the opposite choice of sign for S_{WZ} which in turn can be thought of as a flip in the orientation of the worldvolume.

As with electromagnetism, the $C_{\mu_0 \ldots \mu_p}$ field leads to a repulsive force between Dp-branes. In addition the gravitational attraction and this repulsive force are mediated by massless fields and hence both have the same fall-off with distance:

$$F_{gravity} \sim -\frac{T_p^2}{r^{23-p}} \qquad F_C \sim \frac{T_p^2}{r^{23-p}} \tag{4.208}$$

and these exactly cancel, leading to stable D-branes. And indeed it turns out that one can find stable black hole like solutions in supergravity, the low energy description of superstrings, which are interpreted representing the presence of branes.

We have also neglected the tachyon. Again in superstring theory this can be made to go away. Although there are also very interesting situations where the open string tachyon survives. For example it arises in the spectrum of open strings that stretch between a Dp-brane and an anti Dp-brane. In this case the $C_{\mu_0 \ldots \mu_p}$-field induces an attractive force that adds to the gravitational one, rather than cancelling it. However the resulting instability is relatively well understood. In this case the tachyon represents the instability for the them to annihilate leaving nothing but closed string radiation.

5 The Worldsheet Revisited

So far we have quantized a string in flat D-dimensional spacetime. We have seen that this leads to an infinite tower of states and among the massless modes we find a graviton and diffeomorphism invariance (at least if $a = 1$). Including open strings leads to D-branes as higher-dimensional extended objects whose dynamics are controlled at lowest order by non-Abelian Yang-Mills gauge theories.

Apart from D we have the parameters a and α'. In fact α' is not a parameter, it is a dimensional quantity - it has the dimensions of length-squared - and simply sets the scale. What is important are unitless quantities such as $p^2\alpha'$. For example small momentum means $p^2\alpha' << 1$. We are left with D and a but actually these are fixed: quantum consistency demands that $D = 26$ and $a = 1$. Indeed we have seen that

$a = 1$ is special. We have also described how these theories can be generalized to curved spacetimes. But we haven't been as precise as we might have liked.

Let us now go back and explore some aspects in more detail. Perhaps these easiest way to do this requires us to go to a particular frame and thereby break manifest Lorentz symmetry. But it has the benefit that only the physical states arise.

5.1 Light-cone gauge

The easiest way to see this is to introduce light-cone gauge. Recall that the action we started with had diffeomorphism symmetry. We used this symmetry to fix $\gamma_{\alpha\beta} = e^{2\rho}\eta_{\alpha\beta}$. However there is still a residual symmetry. In particular in terms of the coordinates $\sigma^\pm$ then under a transformation

$$\sigma'^+ = \sigma'^+(\sigma^+) \qquad \sigma'^- = \sigma'^-(\sigma^-) \tag{5.209}$$

we see that $\gamma'_{\alpha\beta} = e^{2\rho'}\eta_{\alpha\beta}$ with

$$\rho' = \rho + \frac{1}{2}\ln\left(\frac{\partial\sigma^+}{\partial\sigma'^+}\frac{\partial\sigma^-}{\partial\sigma'^-}\right) \tag{5.210}$$

i.e. this preserves the conformal gauge. In terms of the worldsheet coordinates σ, τ we see that

$$\tau' = \frac{1}{2}(\sigma'^+ + \sigma'^-) \tag{5.211}$$

and since $\sigma'^\pm$ are arbitrary functions of $\sigma^\pm$ we see that any τ that solves the two-dimensional wave equation can be obtained by such a diffeomorphism. Therefore, without loss of generality, we can choose the worldsheet 'time' coordinate τ to be any of the spacetime coordinates (since these solve the two-dimensional wave-equation). Of course there are many choices but the usual one is to define

$$X^+ = \frac{1}{2}(X^0 + X^{D-1}) \qquad X^- = \frac{1}{2}(X^0 - X^{D-1}) \tag{5.212}$$

and then take

$$X^+ = x^+ + \alpha' p^+ \tau \tag{5.213}$$

This is called light cone gauge.

We observe that in these coordinates the spacetime $\eta_{\mu\nu}$ is

$$\eta_{-+} = \eta_{+-} = -2 \qquad \eta_{ij} = \delta_{ij} \tag{5.214}$$

Next we evaluate the worldsheet energy momentum tensor which was given by

$$\begin{aligned} T_{00} = T_{11} &= \frac{1}{2}\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} + \frac{1}{2}X'^\mu X'^\nu \eta_{\mu\nu} \\ T_{01} = T_{10} &= \dot{X}^\mu X'^\nu \eta_{\mu\nu} \end{aligned} \tag{5.215}$$

Thus we find that

$$\begin{aligned} T_{00} = T_{11} &= -2\alpha' p^+ \dot{X}^- + \frac{1}{2}\dot{X}^i \dot{X}^j \delta_{ij} + \frac{1}{2} X'^i X'^j \delta_{ij} = 0 \\ T_{01} = \hat{T}_{10} &= -2\alpha' p^+ X'^- + \dot{X}^i X'^j \delta_{ij} = 0 \end{aligned} \tag{5.216}$$

where $i, j = 1, 2, 3, ..., D-2$. This allows one to explicitly solve for X^- in term of the mode expansions for X^i.

Problem: Show that with our conventions

$$X^- = \hat{x}^- + \alpha' \hat{p}^- \tau + i \left(\sum_{n \neq 0} \frac{a_n^-}{n} e^{-in\sigma^+} + \frac{\tilde{a}_n^-}{n} e^{-in\sigma^-} \right) \tag{5.217}$$

where

$$a_n^- = \frac{1}{2p^+} \sum_m a^i_{n-m} a^j_m \delta_{ij} \tag{5.218}$$

and the mass-shell constraint is

$$-4\alpha' p^+ p^- + \alpha' p^i p^j \delta_{ij} + 2(N + \tilde{N}) = 0 \tag{5.219}$$

with

$$N + \tilde{N} = \frac{1}{2} \delta_{ij} \sum_{n \neq 0} a^i_n a^j_{-n} + \tilde{a}^i_n \tilde{a}^j_{-n} \tag{5.220}$$

and

$$N = \tilde{N} \tag{5.221}$$

. This last condition is just level matching and arises from demanding that $\hat{X}^-$ is periodic in σ.

To continue we note that in the quantum theory there is a normal ordering ambiguity in the definition of $N + \tilde{N}$ and we must include our constant a again into the definition. Hence we must take (temporarily putting in the :: symbols for normal ordering)

$$: N + \tilde{N} := \delta_{ij} \sum_{n=1}^{\infty} a^i_{-n} a^j_n + \tilde{a}^i_{-n} \tilde{a}^j_n \tag{5.222}$$

However since we have dropped an infinite constant, the intercept a will now show up in the mass shell constraint as

$$-4\alpha' p^+ p^- + \alpha' p^i p^j \delta_{ij} + 2(N + \tilde{N} - 2a) = 0 \tag{5.223}$$

Note that $-4p^+ p^- + p^i p^j \delta_{ij} = \eta_{\mu\nu} p^\mu p^\nu$ so this really just tells us that the mass of a state is

$$M^2 = \frac{2}{\alpha'}(N + \tilde{N} - 2a) \tag{5.224}$$

Where we have dropped the :: to indicate normal ordering.

Note that this breaks the $SO(1,D-1)$ symmetry of our flat target space since we choose X^0 and X^{D-1} whereas any pair will do (so long as one is timelike). Thus we will not see a manifest $SO(1,D-1)$ symmetry but just an $SO(D-2)$ symmetry from rotations of the $\tilde{X}^i$. However it is important to realize that the $SO(1,D-1)$ symmetry is not really broken, we have merely performed a kind of gauge fixing (recall there was this underlying gauge symmetry of the string spectrum). It is just no longer manifest.

5.2 $D=26$, $a=1$

On the other hand the benefit of this procedure is that the physical Hilbert space is manifestly postive definite because we remove the oscillators $a^0_n, \tilde{a}^0_n, a^{D-1}_n, \tilde{a}^{D-1}_n$. This is often a helpful way to determine the physical spectrum of the theory.

For example we can reconsider the low lying states that we constructed above. The ground states are unchanged as they do not involve any oscillators. For the open string we find the $D-2$ states at level one

$$|A_i\rangle = a^i_{-1}|0;p\rangle \tag{5.225}$$

These are the transverse components of a massless gauge field. For the closed string we find, at level one,

$$|G_{ij}\rangle = G_{ij}a^i_{-1}\tilde{a}^j_{-1}|0;p\rangle \tag{5.226}$$

Again G_{ij} splits into

$$G_{ij} = g_{ij} + b_{ij} + \frac{1}{D-2}\delta_{ij}\phi \tag{5.227}$$

with $\delta^{ij}g_{ij}=0$. These correspond to the physical components, in a certain gauge, of the metric, Kalb-Ramond field and dilaton. Note however that there is no remnant at all of gauge symmetry which is a crucial feature that allowed us to indentify the dynamics.

Now formally a is given by

$$\begin{aligned} a &= -\frac{1}{2}\sum_{m=1}^{\infty}[a^i_m, a^j_{-m}]\delta_{ij} \\ &= -\frac{D-2}{2}\sum_{m=1}^{\infty} m\ . \end{aligned} \tag{5.228}$$

This is divergent however it can be regularized in the following manner. We note that

$$a = -\frac{D-2}{2}\zeta(-1)\ , \tag{5.229}$$

where $\zeta(s)$ is the Riemann ζ-function

$$\zeta(s) = \sum_{m=1}^{\infty}\frac{1}{m^s}\ . \tag{5.230}$$

This is analytic for complex s with $\text{Re}(s) > 1$. Thus it can be extended to a holomorphic function of the complex plane, with poles at a discrete number of points. Analytically continuing to $s = -1$ one finds $\zeta(-1) = -1/12$ and hence

$$a = \frac{D-2}{24} \ . \tag{5.231}$$

We have seen that in order to have a sensible theory we must take $a = 1$ (otherwise there are no massless states or nice gauge invariances). Hence we must take $D = 26$.

Perhaps this is not a very satisfactory derivation of the dimension of spacetime. A more convincing argument is the following. Light cone gauge is just a gauge. Therefore although the manifest spacetime Lorentz symmetry is no longer present there is still an $SO(1, D-1)$ Lorentz symmetry, even though only an $SO(D-2)$ subgroup is manifest. In light cone gauge the spacetime Lorentz generators $M^{\mu}{}_{\nu}$ (3.33) split into

$$\begin{aligned}
M^{ij} &= \frac{1}{4\pi\alpha'}\int d\sigma \dot{X}^i X^j - X^i \dot{X}^j \\
M^{+j} &= \frac{1}{4\pi\alpha'}\int d\sigma \dot{X}^+ X^j - X^+ \dot{X}^j \\
M^{-j} &= \frac{1}{4\pi\alpha'}\int d\sigma \dot{X}^- X^j - X^- \dot{X}^j \\
M^{-+} &= \frac{1}{4\pi\alpha'}\int d\sigma \dot{X}^- X^+ - X^- \dot{X}^+ \ .
\end{aligned} \tag{5.232}$$

The quantization procedure preserves $SO(D-2)$ so the commutators $[M^i{}_j, M^k{}_l]$ are as they should be. However problems can arise with $[M^i{}_j, M^+{}_k]$ *etc.*. It is too lengthy a calculation to do here, but one can show that the full $SO(1, D-1)$ Lorentz symmetry, generated by these is preserved in the quantum theory, *i.e.* once normal ordering is taken into account, if and only if $a = 1$ and $D = 26$. You are urged to read the section 2.3 of Green Schwarz and Witten or section 12.5 of Zwiebach where this is shown more detail.

This provides a nice interpretation for a. In lightcone gauge, where only the physical modes are present, each transverse dimension contributes $1/24$ to the value of a. That is to say that each transverse direction to the string acts as a free periodic bosonic degree of freedom on the worldsheet and contributes $1/24$ to the ground state energy:

$$E_0 = \langle 0|L_0|0\rangle \ = \langle 0|\tilde{L}_0|0\rangle \ = \frac{D-2}{24}. \tag{5.233}$$

It can be thought of as the regularised sum of the ground state energies of all the oscillators.

This is also useful if we encounter modes with other boundary conditions, *i.e.* modes where n is not integer. For example consider a scalar with half-integer moding then the

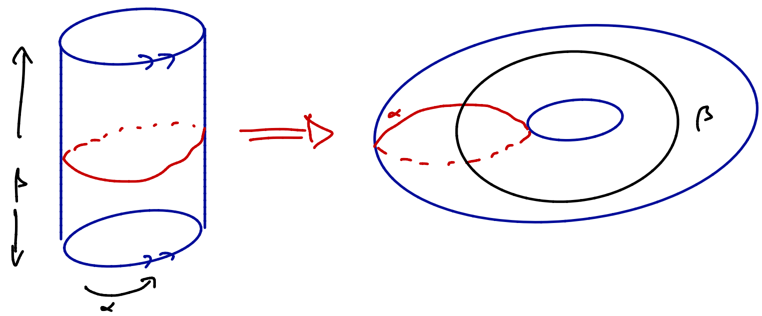

Figure 5.1: A Closed String Loop

ground state energy is

$$\begin{aligned}
\frac{1}{2}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right) &= -\frac{1}{4}\sum_{n=odd}^{\infty} n \\
&= -\frac{1}{4}\left(\sum_{n=1}^{\infty} n - \sum_{n=even}^{\infty} n\right) \\
&= -\frac{1}{4}\left(\sum_{n=1}^{\infty} n - \sum_{m=1}^{\infty} 2m\right) \\
&= \frac{1}{4}\sum_{n=1}^{\infty} n \\
&= -\frac{1}{48}
\end{aligned} \tag{5.234}$$

Problem: Determine the mode expansion for an open string that stretches between a D1-brane located at $x^2 = ... = x^{25} = 0$ and a D25-brane, which fills all of spacetime. By considering light cone gauge (along the direction X^0, X^1) describe the lightest physical states, what is their mass?

5.3 Modular Invariance ($D = 26$, $a = 1$ again)

Let us look at a particular one loop diagram corresponding to a closed string that travels around in a loop: thus the world sheet is a torus with genus $g = 1$. In particular we compactify worldsheet time $\tau \cong \tau + 2\pi\beta$ in addition to $\sigma \cong \sigma + 2\pi\alpha$ for two parameters α, β which describe the geometry of the worldsheet.

Problem: Show that classically $T_{00} = T_{++} + T_{--}$ and $T_{01} = T_{++} - T_{--}$

It follows from this that the energy and momentum along the worldsheet are

$$E = \frac{1}{4\pi\alpha'}\int d\sigma T_{00} = \frac{1}{2}(L_0 + \bar{L}_0)$$
$$P = \frac{1}{4\pi\alpha'}\int d\sigma T_{01} = \frac{1}{2}(L_0 - \bar{L}_0)\ . \tag{5.235}$$

Switching to the quantum theory the Hamiltonian generates translations in time τ:

$$i\frac{d}{d\tau}|Phys\rangle = H|Phys\rangle = \frac{1}{2}(L_0 + \tilde{L}_0 - 2a)|Phys\rangle \tag{5.236}$$

where the $-2a$ arises from normal ordering. While on the other hand $L_0 - \tilde{L}_0$ generates translations in σ

$$i\frac{d}{d\sigma}|Phys\rangle = H|Phys\rangle = \frac{1}{2}(L_0 - \tilde{L}_0)|Phys\rangle \tag{5.237}$$

Thus level matching can be thought of as the statement that the string is invariant along σ.

Let us examine a open-loop closed string diagram corresponding to a closed string that simply propagates straight up. However we identify the beginning and end closed strings. This corresponds to a trace over all string states which have been propagated around the torus:[4]

$$Z_{closed} = \sum_{states} \langle State|e^{-2\pi\beta(L_0+\tilde{L}_0-2a)+2\pi\alpha i(L_0-\tilde{L}_0)}|State\rangle \tag{5.238}$$

where we have Wick rotated to imaginary worldsheet time. The full amplitude is

$$\mathcal{A}_{closed} = \int \frac{d\alpha d\beta}{\beta^2} Z_{closed} \tag{5.239}$$

and one can think of the integral over α as imposing level matching. The reason for the β^{-2} in the measure is due to the Weyl-Petterson metric on the moduli space of torus. In plain english that means that it is invariant under conformal transformations $(\alpha,\beta) \to \lambda(\alpha,\beta)$ as well as the identification $\alpha \sim \alpha + 1$.

This corresponds to torus worldsheet with no insertions (*i.e.* no scattering states). Thus we are computing a one-loop vacuum diagram. It is natural to introduce

$$q = e^{2\pi i z} \qquad z = \alpha + i\beta \tag{5.240}$$

so that

$$Z_{closed} = \int \frac{d^{D-2}p}{(2\pi)^{D-2}} \sum_{states} q^{L_0-a}\bar{q}^{\tilde{L}_0-a} \tag{5.241}$$

[4] Note that unlike section 3.10 this the partition function of the worldsheet theory, not of the whole string theory.

where the sum is over all the physical states created from the oscillators (which are all states in lightcone gauge) and the integral is over the continuous momentum zero-mode.

Consider first a single scalar with no zero-mode but just oscillators $a_{-1}, a_{-2}, ...$ and intercept E_0

$$\begin{aligned} Z_1 &= \sum_{states} \langle State | q^{\sum_l a_{-l} a_l - E_0} | State \rangle \\ &= \sum_{states} \langle State | q^{-E_0} \prod_{l=1}^{\infty} q^{a_{-l} a_l} | State \rangle \end{aligned} \tag{5.242}$$

Each oscillator a_{-l} can be used k times and each $a_{-l} a_l$ contributes l to the exponent. We also need to sum over all k and using $\sum_{k=0}^{\infty} q^{kl} = (1-q^l)^{-1}$ we find

$$Z_1 = q^{-E_0} \prod_{l=1}^{\infty} \frac{1}{1-q^l} \tag{5.243}$$

This is the partition function for a single scalar field (with no zero modes). It is essentially just the product of partition functions for simple harmonic oscillators with frequencies $l = 1, 2, 3, \ldots$

$$\begin{aligned} Z_1 &= \prod_{l=1}^{\infty} Z_{sho}(l) \\ Z_{sho}(l) &= 1 + q_l + q_l^2 + q_l^3 + \ldots = \frac{1}{1-q_l} \ , \end{aligned} \tag{5.244}$$

where $q_l = e^{-2\pi i l \beta} = (e^{-2\pi i \beta})^l$. Each coefficient of $(q_l)^k$ counts the number of states with energy kl. Since there is just one oscillator a_{-l} one finds just one state for each kl:

$$|0\rangle \ , a_{-l}|0\rangle \ , (a_{-l})^2|0\rangle \ , (a_{-l})^3|0\rangle \ \ldots \ . \tag{5.245}$$

The prefactor of q^{-E_0} just represents an overall shift due to the ground state energy.

Having $D-2$ scalar fields just means taking the $(D-2)$-th power

$$Z_{D-2} = (Z_1)^{D-2} = q^{-(D-2)E_0} \prod_{l=1}^{\infty} \left(\frac{1}{1-q^l} \right)^{D-2} \ . \tag{5.246}$$

Returning to the closed string we find, putting in the zero-modes,

$$\begin{aligned} Z_{closed} &= \int \frac{d^{D-2}p}{(2\pi)^{D-2}} e^{-\pi \alpha' \beta p^2} (q\bar{q})^{-a} \prod_{l=1}^{\infty} \left(\frac{1}{1-q^l} \right)^{D-2} \prod_{m=1}^{\infty} \left(\frac{1}{1-\bar{q}^m} \right)^{D-2} \\ &= \left(\frac{1}{2\pi\beta} \right)^{(D-2)/2} (q\bar{q})^{-a} \prod_{l=1}^{\infty} \left(\frac{1}{1-q^l} \right)^{D-2} \prod_{m=1}^{\infty} \left(\frac{1}{1-\bar{q}^m} \right)^{D-2} \ . \end{aligned} \tag{5.247}$$

Here we have used the fact that

$$\int \frac{dp}{2\pi} e^{-\pi\alpha'\beta p^2} = \sqrt{\frac{1}{4\pi^2\alpha'\beta}} \ . \tag{5.248}$$

Let us introduce the Dedekind η-function

$$\eta(z) = q^{1/24} \prod_{l=1}^{\infty} \left(1 - q^l\right) \ , \tag{5.249}$$

so that

$$Z_{closed}(z) = \left(\frac{1}{4\pi^2\alpha' \mathrm{Im}(z)}\right)^{\frac{D-2}{2}} (q\bar{q})^{\frac{D-2}{24}-a} \frac{1}{(\eta(q)\eta(\bar{q}))^{D-2}} \tag{5.250}$$

The η-function has the important property that

$$\eta(z+1) = \eta(z) \qquad \eta(-1/z) = \sqrt{-iz}\,\eta(z) \tag{5.251}$$

The first identity is clear from the defintion of $q = e^{2\pi i z}$. The second is highly non-trivial and we won't prove it here. Thus we find

$$\begin{aligned} Z_{closed}(z+1) &= Z_{closed}(z) \\ Z_{closed}(-1/z) &= \left(\frac{z\bar{z}}{4\pi^2\alpha' \mathrm{Im}(z)}\right)^{\frac{D-2}{2}} (q(-1/z)\bar{q}(-1/z))^{\frac{D-2}{24}-a} \frac{1}{(z\bar{z})^{\frac{D-2}{2}} (\eta(q)\eta(\bar{q}))^{D-2}} \\ &= (q(-1/z)\bar{q}(-1/z))^{\frac{D-2}{24}-a} Z_{closed}(z) \end{aligned} \tag{5.252}$$

Thus only if $D-2 = 24a$ do we find

$$Z_{closed}(-1/z) = Z_{closed}(z) \ . \tag{5.253}$$

Thus since we can think of $a = (D-2)E_0$ we see that $E_0 = 1/24$.

Problem: Show that our complete amplitude is

$$\mathcal{A}_{closed} = \int \frac{dz d\bar{z}}{(\mathrm{Im} z)^2} \left(\frac{1}{4\pi^2\alpha' \mathrm{Im}(z)}\right)^{12} \left(\frac{1}{\eta(q)\eta(\bar{q})}\right)^{12} \ , \tag{5.254}$$

and verify that the integrand is invariant under the action $z \to z+1$ and $z \to -1/z$.

Why is this important? The complex parameter $z = \alpha + i\beta$ parameterizes the shape of the torus $\mathbb{T}^2 = S^1 \times S^1$.[5] Taking $z \to 2z$ doubles the size of both S^1 factors but this action is a Weyl transformation and hence is a symmetry. So one must integrate over z

[5] z is often called τ in the literature and should not be confused with the worldsheet time coordinate. Similarly the z here should not be confused with the complexified worldsheet coordinate $z = \sigma + i\tau$ that we briefly used earlier.

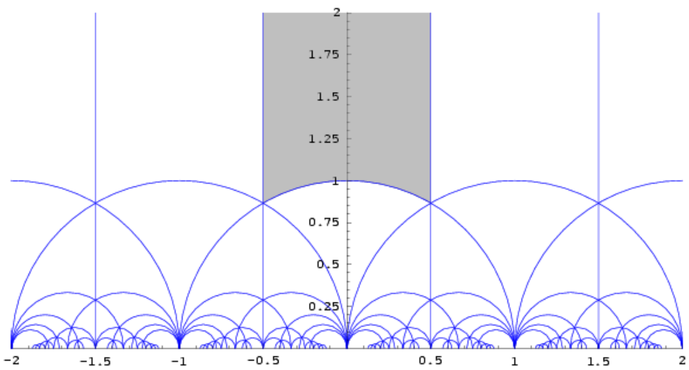

Figure 5.2: A Fundamental Domain of $SL(2,\mathbb{Z})$

modulo Weyl transformations. As shown in the problem the measure factor $dzd\bar{z}/(\text{Im}z)^2$ is invariant and so is the rest of the integrand.

The actions $z \to z+1$ and $z \to -1/z$ generate the so-called modular transformations. For example modular transformation can map the two cycles of the torus into each other. Furthermore the transformations $z \to z+1$ and $z \to -1/z$ generate the group $SL(2,\mathbb{Z})$ which acts as a fractional linear transformation

$$z \to \frac{az+b}{cz+d} \qquad ab - cd = 1 \ , a,b,c,d \in \mathbb{Z} \tag{5.255}$$

Therefore when performing our integral over z to compute $\mathcal{A}_{closed}$ we would find a divergence due to the infinite number of copies related by modular transformations. So rather than integrating $\alpha \in [0, 2\pi)$ and $\beta \in [0, \infty)$ we must restrict to the fundamental domain of $SL(2,\mathbb{Z})$.

This is known as modular invariance and it is a crucial property of the worldsheet theory. Modular transformations are infact residual diffeomorphisms. They are called large diffeomorphisms as they are not continuously connected to the identity *i.e.* they cannot be reached by looking at infinitessimal reparameterizations of the the worldsheet. As we have just seen modular invariance fixes the spacetime dimension (or equivalently the intercept if you didn't like the ζ-function approach above.) If $D - 2 \neq 24a$ then the string theory has a gravitational anomaly: it is not invariant under large diffeomorphisms. Many other things go wrong as a result. This is also a crucial part of the finiteness of string scatting amplitudes: the potentially divergent parts of the loop integrands are removed by modular invariance which restricts the integrals to a smaller range.

5.4 Open/Closed String Duality

Above we considered a torus partition function corresponding to a closed string that propagates in a loop. We could do a similar amplitude for an open string. Here we just

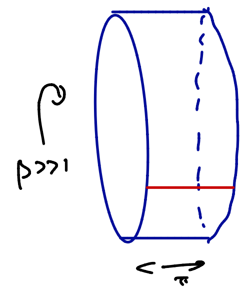

Figure 5.3: An Open String Loop

want to consider an open string that moves around in a closed loop making a cylinder:

$$Z_{open} = \sum_{states} \langle State|e^{-2\pi i\beta(L_0-a)}|State\rangle \ . \tag{5.256}$$

Using conformal invariance we keep the length of the string fixed at π. The appropriate vacuum amplitude is therefore

$$\mathcal{A}_{open} = \int_0^\infty \frac{d\beta}{\beta} Z_{open} \ , \tag{5.257}$$

where again the measure is such that it is invariant under Weyl transformations $\beta \to \lambda\beta$.

Following the steps above it is quite easy to construct Z_{open} (and we take $D = 26, a = 1$) in lightcone gauge:

$$\begin{aligned} Z_{open} &= \int \frac{d^{p-1}p}{(2\pi)^{p-1}} e^{-\pi\alpha'\beta p^2} q^{-1} \prod_{l=1}^{\infty} \left(\frac{1}{1-q^l}\right)^{24} \\ &= \left(\frac{1}{2\pi\alpha'\beta}\right)^{(p-1)/2} \eta^{-24}(q) \ , \end{aligned} \tag{5.258}$$

where $q = e^{-\pi\beta}$. Here we have considered the case of an open string ending on a Dp-brane so that the momenta are $p-1$-dimensional in light cone gauge (please excuse the use of p as the brane dimension and also the momentum).

It is educational to expand this out for small q (large β):

$$Z_{open} = \left(\frac{1}{2\pi\alpha'\beta}\right)^{(p-1)/2} \left(\frac{1}{q} + 24 + 324q + \ldots\right) \ . \tag{5.259}$$

By construction the sum in the brackets arises from the tachyon (q^{-1}) plus the 24 polarization modes of the massless vector (q^0) and then the massive modes which arise at level 1 (q^1) *etc.*

Problem: Obtain (5.259). Identify the states and their degeneracies in term of oscillators. Do the same expansion for the closed string (you need only look at the level-matched terms up to $q\bar{q}$).

There is no notion of modular symmetry for a cylinder but something else interesting happens. For large β (small q) we have a short string moving around in a large circle. The expansion above shows that the dominant contributions come from the light modes. This makes sense as the heavier higher string modes have propagators that fall off very quickly with distance.

What happens for small β (large q)? To perform the expansion we can make use of the fact that (taking $z = i\beta$ in (5.249))

$$\eta(\beta) = \sqrt{1/\beta}\,\eta(1/\beta)\ . \tag{5.260}$$

In other words for small β we introduce $\beta' = 1/\beta$ then

$$\begin{aligned}\eta^{-24}(\beta) &= \beta'^{-12}\eta^{-24}(\beta') \\ &= \beta'^{-12}\left(\frac{1}{q'} + 24 + 324q' + \ldots\right)\ ,\end{aligned} \tag{5.261}$$

where $q' = e^{-\pi\beta'} << 1$ and hence

$$Z_{open} = (2\pi\alpha')^{13-p}\left(\frac{1}{2\pi\alpha'\beta'}\right)^{(25-p)/2}\left(\frac{1}{q'} + 24 + 324q' + \ldots\right)\ . \tag{5.262}$$

What are these states? It is tempting to also view these as massless open string modes but that can't be as we expect the massive string modes to be important now. Furthermore the power of the prefactor indicates that these states are propagating in $25-p$ dimensions, not on the brane worldvolume.

The answer comes by looking at the cylinder diagram the other way around. Rather than thinking of it as a very long open string in a small loop we could consider a small closed string moving along a long interval. So now the diagram is dominated by the lightest closed string states. The fact that we only see half the states arises from the boundary conditions at the ends of the cylinder which require that the left-moving modes are equal to the right moving ones. Thus we are seeing the closed string states (tachyon along with the diagonal modes of the metric) propagating in the transverse space to the Dp-brane!

This is a central feature of string theory. Loops of open strings can been seen as tree level closed strings (but with boundary conditions that restrict the in and outgoing states). In particular it suggests that the gauge theory dynamics of open string scattering has a dual description in terms of gravitationally interacting closed strings in the transverse space. This lies at the heart of much of our modern understanding of topics such as gauge/gravity duality.

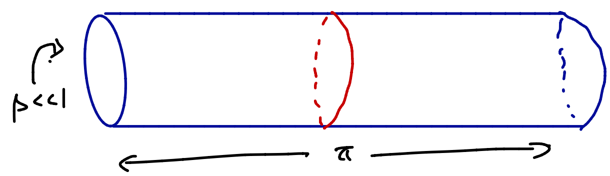

Figure 5.4: A Closed String Propagating

5.5 T-Duality

Let us consider what happens when some spacetime direction, labelled by X^T, is a circle: $X^T \sim X^T + 2\pi R_T$. Returning to our mode expansion we see that X^T need not be single valued but rather

$$X^T(\sigma + 2\pi) = X^T(\sigma) + 2\pi m R_T \ , \tag{5.263}$$

for some integers m. Such a string is wound around the X^T dimension. Thus we see that in our original mode expansion we can have $w^T = nR_T$ for an integer n. Furthermore the momentum around a circle must be quantised (so that the wavefunction is single valued) and hence $p^T = m/R_T$. It then follows that

$$a_0^T = \sqrt{\frac{\alpha'}{2}} m R_T^{-1} + \sqrt{\frac{1}{2\alpha'}} n R_T \qquad \tilde{a}_0^T = \sqrt{\frac{\alpha'}{2}} m R_T^{-1} - \sqrt{\frac{1}{2\alpha'}} n R_T \ , \tag{5.264}$$

and

$$a_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu \ , \tag{5.265}$$

for the non-compact directions. Now that $w^T \neq 0$ we see that

$$\begin{aligned} L_0 - 1 &= \frac{1}{2} a_0^\mu a_0^\nu \eta_{\mu\nu} + N - 1 \\ &= \frac{\alpha'}{4} p^2 + \frac{1}{2} \sum \left(\frac{\alpha'}{2} \left(\frac{m}{R_T} \right)^2 + \frac{1}{2\alpha'} (nR_T)^2 + nm \right) + N - 1 \ . \end{aligned} \tag{5.266}$$

On the other hand we have

$$\begin{aligned} \tilde{L}_0 - 1 &= \frac{1}{2} a_0^\mu a_0^\nu \eta_{\mu\nu} + N - 1 \\ &= \frac{\alpha'}{4} p^2 + \frac{1}{2} \sum \left(\frac{\alpha'}{2} \left(\frac{m}{R_T} \right)^2 + \frac{1}{2\alpha'} (nR_T)^2 - nm \right) + \tilde{N} - 1 \ . \end{aligned} \tag{5.267}$$

Level matching is now slightly shifted to

$$(N - \tilde{N} + nm)|phys\rangle = 0 \ . \tag{5.268}$$

And the spacetime mass shell is $p^2 + M^2 = 0$ with

$$M^2 = \sum \left(\left(\frac{m}{R_T} \right)^2 + \frac{1}{\alpha'^2} (nR_T)^2 \right) + \frac{2}{\alpha'}(N + \tilde{N} - 2) \ . \tag{5.269}$$

You should notice an interesting symmetry. The spectrum is invariant under

$$n \leftrightarrow m \qquad R_T \leftrightarrow \alpha'/R_T \ , \tag{5.270}$$

i.e. under the interchange of momentum and winding quantum numbers along with an inversion of the radii. In fact this symmetry extends to the full interacting theory and is know as T-duality.

It implies that there is a sort of minimum length scale built into string theory as a string on a circle of radius R is equivalent to a string on a circle of radius α'/R. Physically, for distances smaller than $\sqrt{\alpha'}$, the string behaves more and more like an extended object and cannot resolve smaller distances.

We see that for each circular dimension there is a double tower of increasingly massive states (in addition to the exponentially growing tower of states). In particular each mode in the stringy tower of states now carries two extra integral charges. These are the momentum and winding numbers about each compact dimension. It follows that all but the zero-modes are massive, with a mass-squared of order α'^{-1}. Therefore at low energy only the zero-modes will be physically relevant.

Note that the momentum modes get heavier as we shrink the radii whereas the winding mode will get lighter. However so long as the string length $\sqrt{\alpha'}$ is small we can ensure that all these extra massive momentum and winding modes are too massive to observe.

5.6 D-branes and T-duality

How does T-duality affect open strings? For simplicity consider a D1-brane wrapped on a circle direction that is along its worldvolume, call that direction x^T. So the mode expansion for X^T is NN but with a discrete momentum ($n \in \mathbb{Z}$)

$$X^T \ = \ x^T + \alpha' \frac{n}{R_T}\tau + \sqrt{2\alpha'} i \sum_n \frac{a^T_n}{n} e^{-in\tau} \cos n\sigma \ . \tag{5.271}$$

T-duality therefore needs to map this to some kind of string with zero mode $\alpha' n/R_T = n\tilde{R}_T$. This is right for a winding zero-mode but this can only arise for D-branes if the circle is transverse to the brane as NN boundary conditions exclude a winding term. The only option is a D0-brane in the T-dual theory with associated mode DD expansion ($n \in \mathbb{Z}$)

$$\begin{aligned} \tilde{X}^T \ &= \ a^T + n\tilde{R}_T\sigma + \sqrt{2\alpha'} \sum_n \frac{a^T_n}{n} e^{-in\tau} \sin n\sigma \\ \tilde{R}_T \ &= \ \alpha'/R_T \ . \end{aligned} \tag{5.272}$$

Here the integer n counts how many times the open string wraps around the compact direction before ending on the D0-brane. Thus the momentum Fourier modes of the D1-brane wrapped on an S^1 are mapped to winding modes of open strings around the dual S^1 that is transverse to a D0-brane.

More generally we see that under T-duality a Dp-brane wrapped on a circle S^1 is mapped to a D$(p-1)$-brane transverse to the dual circle $\tilde{S}^1$ with radius $\tilde{R} = \alpha/R$. Conversely a Dp-brane transverse to a circle S^1 is mapped to a D$(p+1)$-brane wrapped on the dual circle $\tilde{S}^1$ with radius $\tilde{R}_T = \alpha/R_T$.

In fact putting all these together, looking at the mode expansion in terms of left and right movers, one has

$$\begin{aligned} X^T_L &= \frac{1}{2}x^T_L + \frac{1}{2}(\alpha' m R_T^{-1} + nR_T)(\tau+\sigma) + \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\frac{a^T_n}{n}e^{-in(\tau+\sigma)} \\ X^T_R &= \frac{1}{2}x^T_R + \frac{1}{2}(\alpha' m R_T^{-1} - nR_T)(\tau-\sigma) + \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\frac{\tilde{a}^T_n}{n}e^{-in(\tau-\sigma)} \ , \end{aligned} \tag{5.273}$$

Note that for open strings if X^T is a worldvolume (NN) direction then $n = 0$ whereas if X^T is a transverse (DD) direction $m = 0$. Thus under a T-duality transformation (5.270) one sees that

$$X^T_L \leftrightarrow X^T_L \qquad X^T_R \leftrightarrow -X^T_R \ , \tag{5.274}$$

so in particular the we can summarise things by noting that the T-dual scalar has is $\tilde{X}^T = \tilde{X}^T_L + \tilde{X}^T_R$ with

$$\begin{aligned} \tilde{X}^T_L &= \frac{1}{2}x^T_L + \frac{1}{2}(\alpha' m R_T^{-1} + nR_T)(\tau+\sigma) + \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\frac{a^T_n}{n}e^{-in(\tau+\sigma)} \\ \tilde{X}^T_R &= -\frac{1}{2}x^T_R - \frac{1}{2}(\alpha' m R_T^{-1} - nR_T)(\tau-\sigma) - \sqrt{\frac{\alpha'}{2}}i\sum_{n\neq 0}\frac{\tilde{a}^T_n}{n}e^{-in(\tau-\sigma)} \ . \end{aligned} \tag{5.275}$$

Note that changing the sign of $\hat{x}^T_R$ and $\tilde{a}^T_n$ has no effect for closed strings. However for Dp-branes it swaps NN and DD boundary conditions and replaces the momentum with winding. Thus T-duality is just a symmetry where you swap the sign of the right moving modes.

Let us see how this can be demonstrated on the worldsheet. We consider a simple system where the metric and other fields are flat. Since the action is free we need only look at the part concerning the compact coordinate X^T. Starting with NN boundary conditions we have

$$S_T = -\frac{1}{4\pi\alpha'}\int d^2\sigma R_T^2 \eta^{\alpha\beta}\partial_\alpha X^T \partial_\beta X^T + \int_{\sigma=\pi} d\tau \dot{X}^T A_T - \int_{\sigma=0} d\tau \dot{X}^T A_T \ , \tag{5.276}$$

where we have used a spacetime metric with $g_{TT} = R_T^2$ so that $X^T \sim X^T + 2\pi$. Furthermore A_T is the X^T component of A_μ which, since X^T is periodic, we assume is independent of X^T. Note that X^T does not appear in the action undifferentiated. Therefore let us write

$$\partial_\alpha X^T = F_\alpha \ , \tag{5.277}$$

but we need to impose the identity $\varepsilon^{\alpha\beta}\partial_\alpha F_\beta = 0$ that arises from $[\partial_\alpha, \partial_\beta] = 0$. Therefore we introduce into S_T a Lagrange multiplier term:

$$\begin{aligned} S_T &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(R_T^2\eta^{\alpha\beta}F_\alpha F_\beta + 2\alpha'\varepsilon^{\alpha\beta}\tilde{X}^T\partial_\alpha F_\beta)\right) + \int_{\sigma=\pi} d\tau F_0 A - \int_{\sigma=0} d\tau F_0 A \\ &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(R_T^2\eta^{\alpha\beta}F_\alpha F_\beta - 2\alpha'\varepsilon^{\alpha\beta}\partial_\alpha\tilde{X}^T F_\beta)\right) \\ &\qquad + \int_{\sigma=\pi} d\tau F_0(A_T - 2\pi\tilde{X}^T) - \int_{\sigma=0} d\tau F_0(A_T - 2\pi\tilde{X}^T) \ . \end{aligned} \tag{5.278}$$

If we use the $\tilde{X}^T$ equation of motion then from the first line we find

$$\varepsilon^{\alpha\beta}\partial_\alpha F_\beta = 0 \ , \tag{5.279}$$

which can be solved by writing $F_\alpha = \partial_\alpha X^T$. Substituting back into the action leads to our original action. On the other hand we can use the second line to evaluate the F_α equation of motion:

$$\begin{aligned} 2R_T^2\eta^{\alpha\beta}F_\alpha &= 2\alpha'\varepsilon^{\alpha\beta}\partial_\alpha\tilde{X}^T \\ \Longrightarrow F_\alpha &= \tfrac{\alpha'}{R_T^2}\varepsilon_{\alpha\beta}\partial^\beta\tilde{X}^T \ . \end{aligned} \tag{5.280}$$

Note that the NN boundary condition $\partial_1 X = 0$ corresponds to $F_1 = 0$ and hence $\partial_0\tilde{X}^T = 0$ at $\sigma = 0, \pi$, which is the required DD boundary condition. Substituting this back into the action gives

$$\begin{aligned} S_T &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\frac{\alpha'^2}{R_T^2}\eta^{\alpha\beta}\partial_\alpha\tilde{X}^T\partial_\beta\tilde{X}^T\right) \\ &\qquad -\frac{\alpha'}{R_T^2}\int_{\sigma=\pi} d\tau\partial_1\tilde{X}^T(A - 2\pi\tilde{X}^T) + \frac{\alpha'}{R_T^2}\int_{\sigma=0} d\tau\partial_1\tilde{X}^T(A - 2\pi\tilde{X}^T) \\ &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\frac{\alpha'^2}{R^2}\eta^{\alpha\beta}\partial_\alpha\tilde{X}^T\partial_\beta\tilde{X}^T\right) + \int_{\sigma=\pi} d\tau\partial_1\tilde{X}^T Y - \int_{\sigma=0} d\tau\partial_1\tilde{X}^T Y \ , \end{aligned} \tag{5.281}$$

where $Y = 2\pi\alpha' R_T^{-2}\tilde{X}^T - \alpha' A_T/R_T^2$. This is the correct open string worldsheet action for a coordinate $\tilde{X}^T$ with periodicity $2\pi\tilde{R}_T = 2\pi\sqrt{\alpha'/R_T}$ but now with DD boundary conditions. Furthermore we see that T-duality corresponds to

$$\begin{aligned} R_T &\leftrightarrow \tilde{R}_T = \alpha'/R_T \\ \partial_\alpha X^T &\leftrightarrow \varepsilon_{\alpha\beta}\partial^\beta\tilde{X}^T \ , \end{aligned} \tag{5.282}$$

which in terms of $\sigma^{\pm}$ coordinates indeed means

$$\begin{aligned} R_T &\leftrightarrow \tilde{R}_T = \alpha'/R_T \\ \partial_- X^T &\leftrightarrow \partial_- \tilde{X}^T \\ \partial_+ X^T &\leftrightarrow -\partial_+ \tilde{X}^T \ , \end{aligned} \tag{5.283}$$

i.e. the radius is inverted and the sign of the right movers is changed but the left movers are unchanged.

Problem: Show that under the T-duality transformation $A_T \leftrightarrow Y^T$ the Dp-brane Yang-Mills action turns into the D$(p-1)$-brane action.

6 Superstrings

In the final section let us try to extend the previous sections to the superstring. Conceptually not much changes but there are several additional bells and whistles that need to be considered.

6.1 Type II strings

The starting point for the superstring is include Fermions ψ^μ on the worldsheet so as to construct a supersymmetric action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\eta^{\alpha\beta} + i\bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi^\nu \eta_{\mu\nu} \tag{6.284}$$

where $\bar{\psi} = \psi^T \gamma_0$ and γ^α are real 2×2 matrices that satisfy $\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}$. A convenient choice is $\gamma^0 = i\sigma^2$ and $\gamma^1 = \sigma^1$. This action is also conformally invariant and in addition has the supersymmetry

$$\delta X^\mu = i\bar{\epsilon}\psi^\mu \ , \qquad \delta\psi^\mu = \gamma^\alpha \partial_\alpha X^\mu \epsilon \tag{6.285}$$

for any constant ϵ.

Problem: Show this.

In analogy with the bosonic string this can be seen to arise from gauge fixing a full two-dimensional supergravity theory on the worldsheet coupled to X^μ and ψ^μ. But we don't have time to consider that here.

The mode expansion for the X^μ remains as before with the a^μ_n and $\tilde{a}^\mu_n$ oscillators. When we expand the Fermionic fields we can allow for two types of boundary conditions (let us just consider boundary conditions consistent with a closed string where $\sigma \sim \sigma + 2\pi$ and $X^\mu(\tau,\sigma) = X^\mu(\tau, \sigma + 2\pi)$):

$$\begin{aligned} \text{R} : \qquad \psi^\mu(\tau, \sigma + 2\pi) &= \psi^\mu(\tau,\sigma) \\ \text{NS} : \qquad \psi^\mu(\tau, \sigma + 2\pi) &= -\psi^\mu(\tau,\sigma) \end{aligned} \tag{6.286}$$

these are known as the Ramond and Neveu-Schwarz sectors respectively. Thus we find

$$\begin{aligned} \mathrm{R}: \qquad \psi^{\mu}(\tau, \sigma + 2\pi) &= \sum_{n \in \mathbf{Z}} d_n e^{-in\sigma^+} + \tilde{d}_n e^{-in\sigma^-} \\ \mathrm{NS}: \qquad \psi^{\mu}(\tau, \sigma + 2\pi) &= \sum_{r \in \mathbf{Z}+\frac{1}{2}} b_r e^{-ir\sigma^+} + \tilde{b}_r e^{-ir\sigma^-} \end{aligned} \tag{6.287}$$

One finds that these satisfy the anti-commutation relations

$$\begin{aligned} \{d^{\mu}_m, d^{\nu}_n\} &= \eta^{\mu\nu}\delta_{m,-n} \qquad \{b^{\mu}_r, b^{\nu}_s\} = \eta^{\mu\nu}\delta_{r,-s} \\ \{\tilde{d}^{\mu}_m, \tilde{d}^{\nu}_n\} &= \eta^{\mu\nu}\delta_{m,-n} \qquad \{\tilde{b}^{\mu}_r, \tilde{b}^{\nu}_s\} = \eta^{\mu\nu}\delta_{r,-s} \end{aligned} \tag{6.288}$$

with all other anti-commutators vanishing.

One important consequence of supersymmetry is that the algebra of constraints generated by L_n is enhanced to a super-Virasoro algebra with odd generators G_r and F_n (depending on whether or not one is in the NS or R sector respectively). The super-Virasoro algebra turns our to be (see the references)

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8}m(m^2-1)\delta_{m,-n} \\ [L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s} \end{aligned} \tag{6.289}$$

in the NS sector and

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m,-n} \\ [L_m, F_n] &= \left(\frac{m}{2} - n\right) F_{m+n} \\ \{F_n, F_m\} &= 2L_{m+n} + \frac{D}{2}m^2\delta_{m,-n} \end{aligned} \tag{6.290}$$

in the R sector. Here all operators are normal ordered. Just as before this only affects L_0 and F_0 however there is no associated intercept a for F_0 since it is Fermionic (and in addition this is not allowed by the $\{F_0, F_0\}$ anti-commutator). Note that the Fermionic generators are in effect the 'square-root' of L_n, as we expect in a supersymmetric theory. We won't go into more details here but we must impose the physical constraints for the positive modded generators. Just as L_0 gives a spacetime Klein-Gordon equation, F_0 gives a spacetime Dirac equation.

Let us compute the intercept a. As before we go to light-cone gauge where we fix two of the coordinates X^{μ} and their superpartners ψ^{μ}. We then compute the vacuum energy of the remaining $D-2$ Bosonic and Fermionic oscillators. The result depends on the boundary conditions we use. Noting that the sign of the Fermionic contribution is opposite to that of a Boson one finds

$$\begin{aligned} a_R &= -\frac{D-2}{2}\sum_{n=1}^{\infty} n + \frac{D-2}{2}\sum_{n=1}^{\infty} n \\ &= -\frac{D-2}{2}\left(-\frac{1}{12}+\frac{1}{12}\right) \\ &= 0 \end{aligned} \tag{6.291}$$

The vanishing of a_R is a direct consequence of the fact that there is a Bose-Fermi degeneracy in the R-sector. In particular each periodic Fermion contributes $-\frac{1}{24}$ to a. In the NS sector we find (recall section 5.2)

$$\begin{aligned} a_{NS} &= -\frac{D-2}{2}\sum_{n=1}^{\infty} n + \frac{D-2}{2}\sum_{r=0}^{\infty}\left(r+\frac{1}{2}\right) \\ &= (D-2)\left(\frac{1}{24}+\frac{1}{48}\right) \\ &= \frac{D-2}{16} \end{aligned} \tag{6.292}$$

Note that this shows that each anti-periodic Fermion contributes $\frac{1}{48}$ to a. Having determined the incepts we can now go out of Light cone gauge and consider the covariant theory.

Let us now look at the lightest states. There is a different ground state for each sector which we denote by $|R;p\rangle$ and $|NS;p\rangle$ where p^μ labels the spacetime momentum. As before we assume that these states are annihilated by any oscillator with positive frequency.

We see that $|R;p\rangle$ is massless and hence all the higher level states created from it by the action of a creation operator will be massive with a mass of order the string scale. However the Ramond ground state $|R;p\rangle$ is degenerate. In particular we see that there are Fermion zero-modes d_0^μ which satisfy $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$, $\mu,\nu = 0,...,D-1$ in light cone gauge. This is a Clifford algebra and it is known that there is a unique representation and it is $2^{[\frac{D}{2}]}$-dimensional. Thus the Ramond ground state is in fact a spinor with $2^{[\frac{D}{2}]}$ independent components.

Let us look at the Neveu-Schwarz ground state $|NS,p\rangle$. It is clear that since $a_{NS}>0$ this state is a tachyon. We can then consider the higher level states (for simplicity we just consider open strings)

$$\begin{aligned} &a^\mu_{-1}|NS,p\rangle \qquad M^2 = 1-\frac{D-2}{16} \\ &b^\mu_{-\frac{1}{2}}|NS,p\rangle \qquad M^2 = \frac{1}{2}-\frac{D-2}{16} \end{aligned}$$

Thus the next lightest state is $b^\mu_{-\frac{1}{2}}|NS,p\rangle$ and its mass-squared is $M^2 = -\frac{D-10}{16}$. Thus if $D < 10$ then these states are also tachyonic. However as before the magic (that is gauge symmetries from null states) happens when these states are massless, *i.e.* $D = 10$. In this case the states $a^\mu_{-1}|NS,p\rangle$ are massive. Thus we take $D = 10$ and $a_{NS} = 1/2$. Indeed as before this is forced upon us if we want the $SO(1, D-1)$ Lorentz symmetry of spacetime to be preserved in the quantum theory. And also for modular invariance.

Nevertheless we are still left with some bad features. For one the Neveu-Schwarz ground state is still a tachyon. There is also another puzzling feature: $|NS,p\rangle$ is a spacetime scalar and hence it must be a Boson. We can then construct the spacetime vector $b^\mu_{-\frac{1}{2}}|NS,p\rangle$. From the spacetime point of view this state should be a Boson since it transforms under Lorentz transformations as a vector. However it is created from $|NS,p\rangle$ by a Fermionic operator and thus will obey Fermi-statistics. This is contradictory.

The solution to both these problems is to project out the odd states and in particular $|NS,p\rangle$. This is known as the GSO projection. More specifically we declare that $|NS,p\rangle$ is a Fermionic state. Mathematically we introduce the operator $(-1)^F$ which acts as $(-1)^F|NS,p\rangle = -|NS,p\rangle$ and $\{\psi^\mu, (-1)^F\} = 0$, $[X^\mu, (-1)^F] = 0$. We then project out all Fermionic states, *i.e.* states in the eigenspace $(-1)^F = -1$. Thus $|NS,p\rangle$ and $a^\mu_{-1}|NS,p\rangle$ are removed from the spectrum but the massless states $b^\mu_{-\frac{1}{2}}|NS,p\rangle$ remain.

Let us now consider the Ramond sector states. We already saw that the ground state here is massless but degenerate. Indeed it is a spinor of $SO(1,9)$, that is to say it can be represented by a vector in the 32-dimensional vector space that furnishes a representation of the Clifford algebra relation $\{d^\mu_0, d^\nu_0\} = \eta^{\mu\nu}$, $\mu,\nu = 0,...,9$. We need to discuss how $(-1)^F$ acts here. There is a natural candidate where we take $(-1)^F = \pm\Gamma_{11} = \pm\Gamma_0\Gamma_1...\Gamma_9$, the chirality operator in the 10-dimensional Clifford algebra. Thus after the GSO projection $|R,p\rangle$ is a chiral spinor with 16 independent components. More generally in the Ramond sector we project out states with $(-1)^F = -1$. The GSO projection is also required to ensure modular invariance.

In the Ramond sector of the open superstring either choice of sign is equivalent to the other, it is just a convention. Thus for the open superstring the lightest states are massless and consist of a spacetime vector (and hence a Boson) $b^\mu_{-\frac{1}{2}}|NS,p\rangle$ along with a spacetime Fermion $|R,p\rangle$ which can be identified with a chiral spinor. Note that there is a Bose-Fermi degeneracy: onshell, and gauged fixed we find 8 Bosonic and 8 Fermionic states (Why? - you can see this in lightcone gauge).

Let us consider closed strings. Here the states are essentially obtained by taking a tensor product of left and right moving modes and hence there are four possibilities:

$$\begin{aligned} |NS\rangle_L &\otimes |NS\rangle_R \\ |R\rangle_L &\otimes |R\rangle_R \\ |NS\rangle_L &\otimes |R\rangle_R \\ |R\rangle_L &\otimes |NS\rangle_L \end{aligned} \tag{6.293}$$

In this case the relative sign taken in the GSO projection is important. There are two choices: either we chose the same chirality projector for the left and right moving modes or the opposite. This leads to two distinction theories known as the type IIB and type IIA superstring respectively. The states one find are of the form

$$\begin{array}{lcl} |NS\rangle_L & \otimes & |NS\rangle_R \\ |R+\rangle_L & \otimes & |R-\rangle_R \\ |NS\rangle_L & \otimes & |R-\rangle_R \\ |R+\rangle_L & \otimes & |NS\rangle_L \end{array} \tag{6.294}$$

for type IIA and

$$\begin{array}{lcl} |NS\rangle_L & \otimes & |NS\rangle_R \\ |R+\rangle_L & \otimes & |R+\rangle_R \\ |NS\rangle_L & \otimes & |R+\rangle_R \\ |R+\rangle_L & \otimes & |NS\rangle_L \end{array} \tag{6.295}$$

for type IIB. Here the $\pm$ sign corresponds to the different choice of GSO projector for the left and right moving modes.

The spacetime Bosons come from either the NS-NS or R-R sectors whereas the spacetime Fermions from the NS-R or R-NS sectors. One sees that in the type IIA theory there are Fermionic states with both spacetime chiralities but in the type IIB theory only one chirality appears.

Let us look more closely at the massless Bosonic states. The NS-NS sector is essentially the same as the spectrum of the Bosonic string only now they are created from the vacuum by $b^\mu_{-\frac{1}{2}}$ and $\tilde{b}^\mu_{-\frac{1}{2}}$ rather than a^μ_{-1} and $\tilde{a}^\mu_{-1}$. In particular we still find a graviton, Kalb-Ramond field and a dilaton. This sector is universal to all closed string theories.

However we also have R-R fields. These arise as a tensor product of a left and right spinor ground state. As such they form a 'bi-spinor':

$$F_{\alpha\beta} = |R\pm\rangle_{L\alpha} \quad \otimes \quad |R\pm\rangle_{R\beta} \tag{6.296}$$

Any bi-spinor can be expanded in terms of the associated Γ-matrices:

$$F_{\alpha\beta} = \sum_{p=0}^{10} F_{\mu_1\ldots\mu_p}(\Gamma^{\mu_1\ldots\mu_p}\Gamma^0)_{\alpha\beta} \tag{6.297}$$

Here we have used the fact that $\{1, \Gamma^\mu, \Gamma^{\mu_1\mu_2}, ..., \Gamma^{\mu_1\ldots\mu_{10}}\}$ form a basis of 32×32 matrices and used $C^{-1} = \Gamma^0$ to lower the spinor index. Next we note that

$$\Gamma^{11}\Gamma^{\mu_1\ldots\mu_p} = \frac{1}{(10-p)!}\varepsilon^{\mu_1\ldots\mu_p\nu_1\ldots\nu_{10-p}}\Gamma_{\nu_1\ldots\nu_{10-p}} \tag{6.298}$$

Using the GSO projection on the left movers implies that $(\Gamma^{11})_\gamma{}^\alpha F_{\alpha\beta} = F_{\gamma\beta}$ and hence we see that

$$F^{\mu_1\ldots\mu_p} = \frac{1}{(10-p)!}\varepsilon^{\mu_1\ldots\mu_p\nu_1\ldots\nu_{10-p}}F_{\nu_1\ldots\nu_{10-p}} \tag{6.299}$$

This implies that only the fields with $p \leq 5$ are independent of each other. In addition $F_{\mu_1\ldots\mu_5}$ is self-dual. Finally the GSO projection on the right movers tells us that $F_{\alpha\gamma}(\Gamma^{11})^\gamma{}_\beta = \pm F_{\alpha\beta}$ where the sign is $-$ for type IIA and $+$ for type IIB. This implies that $p = even$ for type IIA and $p = odd$ for type IIB. The physical state conditions, in particular the vanishing of F_0 and $\tilde{F}_0$, imply that $\partial_{[\mu_{p+1}}F_{\mu_1\ldots\mu_p]} = 0$ and $\partial^{\mu_1}F_{\mu_1\ldots\mu_p} = 0$.

We motivated superstrings by considering a worldsheet action that was supersymmetric. However it turns out that, after the GSO projection, these theories also have spacetime supersymmetry with 32 supersymmetry generators, the maximum possible. In particular the massless Fermionic states arising from the NS-R and R-NS sectors give two gravitini and a dilatino.

6.2 Type I and Heterotic String

There are three other possibilities. For example one can introduce open strings. Since open strings can combine into a closed string this theory must also contain closed strings. From the modern perspective introducing open strings means introducing space filling D9-branes in type IIB string theory. However it turns out that this is not consistent (as the D9-branes carry a charge) but this can be cured by also introducing a so-called 'orientifold' of the type IIB string where one also mods out by a spacetime reflection (which also changes the sign on the left-moving fermions on the worldsheet). The result is type I string theory which contains unoriented open strings along with closed strings and, as it turns out, a spacetime $SO(32)$ gauge field coming from 16 D9-branes.

A more bizarre construction is to exploit the fact the left and right moving modes sectors of the string worldsheet do not talk to each other (in a closed string). Thus one could take the left moving modes of a superstring living in 10 dimensions and tensor them with the right moving modes of a Bosonic string, which live in 26 dimensions. Remarkably this can be made to work and leads to two types of string theories known as the Heterotic strings. These theories contain $E_8 \times E_8$ or $SO(32)$ spacetime gauge fields.

Thus the right moving sector contains 16 extra Bosons. A fact about two-dimensions is that a right moving Boson is the same as a pair of right moving Fermions (since the Lorentz group in two dimension splits into two commuting, Abelian, parts that act on left and right movers respectively). This is known as Bosonization (or sometimes Fermionization, depending on your point of view). Since a right moving Fermion is more natural than a right moving Boson we will work with 10 scalars X^μ and left-moving Fermions ψ^μ_-, $\mu = 0, 1, ..., 9$ along with 32 right moving Fermions λ^A_+, $A = 1, ..., 32$. In this case left and right moving means:

$$\gamma_{01}\psi^\mu_- = -\psi^\mu_- \qquad \gamma_{01}\lambda^A_+ = \lambda^A_+ \tag{6.300}$$

The worldsheet action of a Heterotic string is now given by

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X^\mu \partial^\beta X^\nu \eta_{\mu\nu} + i\bar{\psi}^\mu_- \gamma^\alpha \partial_\alpha \psi^\nu_- \eta_{\mu\nu} + i\bar{\lambda}^A_+ \gamma^\alpha \partial_\alpha \lambda^B_+ \delta_{AB} \tag{6.301}$$

This has $(1,0)$ supersymmetry:

Problem: Show that this action is invariant under

$$\begin{aligned} \delta X^\mu &= i\bar{\epsilon}_+ \psi^\mu_- \\ \delta \psi^\mu_+ &= \gamma^\alpha \partial_\alpha X^\mu \epsilon_+ \\ \delta \lambda^A_- &= 0 \end{aligned} \tag{6.302}$$

provided that $\gamma_{01}\epsilon_+ = \epsilon_+$.

Problem: Show that the action can be written as

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X^\mu \partial^\beta X^\nu \eta_{\mu\nu} + i(\psi^\mu_-)^T(\partial_\tau - \partial_\sigma)\psi^\nu_- \eta_{\mu\nu} + i(\lambda^A_+)^T(\partial_\tau + \partial_\sigma)\lambda^B_+ \delta_{AB} \tag{6.303}$$

So that ψ^μ_- and λ^A_+ are indeed left and right-moving respectively.

Quantization proceeds much as before, but with all the bells and whistles turned on. The scalars are expanded in terms left and right moving oscillators a^μ_n and $\tilde{a}^\mu_n$. The ψ^μ_- have NS and R sectors with left moving oscillators b^μ_r and d^μ_n. And λ^A_+ has an expansion in terms of right moving oscillators $\tilde{b}^A_r$ and $\tilde{d}^A_n$ for NS and R sectors respectively. In the left moving sector we have $a_{NS} = 1/2$ and $a_R = 0$, just as for the type II superstrings. In the right moving sector we have (going to light cone gauge removes two X^μ fields but none of the λ^A_+ fields)

$$\begin{aligned} \tilde{a}_{NS} &= 8\cdot\frac{1}{24} + 32\cdot\frac{1}{48} = 1 \\ \tilde{a}_R &= 8\cdot\frac{1}{24} - 32\cdot\frac{1}{24} = -1 \end{aligned} \tag{6.304}$$

In particular we see that the right moving Ramond vacuum is massive.

Again the GSO projection is need to give modular invariance and to get rid of the tachyons. Let us look at the massless modes. For the left moving sector again we must take states of the form $b^\mu_{-\frac{1}{2}}|NS\rangle_L$ and $|R\rangle_L$, where again $|R\rangle_L$ is a degenerate spinor ground state with 8 physical states. However in the right moving sector we need only consider the NS states of the form $\tilde{a}^\mu_{-1}|NS\rangle_R$ and $b^A_{-\frac{1}{2}}b^B_{-\frac{1}{2}}|NS\rangle_R$.

Looking at the massless spacetime Bosons we find the metric, dilaton and Kalb-Ramond field from $b^\mu_{-\frac{1}{2}}|NS\rangle_L \otimes \tilde{a}^\mu_{-1}|NS\rangle_R$. However we also obtain a vector state $b^\mu_{-\frac{1}{2}}|NS\rangle_L \otimes b^A_{-\frac{1}{2}}b^B_{-\frac{1}{2}}|NS\rangle_R$. This vector state has index structure $A_\mu{}^{AB}$ and can indeed be identified with a 10-dimensional gauge field. The Fermionic states then give gravitini, dilitino and gauginos. The resulting theory has 16 spacetime supersymmetries: half of the maximum of 32 that the type II theories enjoy.

Finally modular invariance and anomaly cancelation (the spacetime spectrum is chiral and for a general gauge group has anomalies) fixes the possible gauge groups to be either $E_8 \times E_8$ or $SO(32)$.

6.3 Supergravity as the Spacetime Effective Action

The superstrings have a spacetime supersymmetry and include gravity. Therefore their low energy effective actions are those of a supergravity. Such theories are so tightly constrained by their symmetries that, at least to lowest order in derivatives, their action is unique and known. In particular the Bosonic section of these theories is given by

$$\begin{aligned} S_{IIA} &= \frac{1}{\alpha'^4}\int d^{10}x\sqrt{-g}\left(e^{-2\phi}(R+4(\partial\phi)^2-\frac{1}{12}H_3^2)-\frac{1}{4}F_2^2-\frac{1}{48}F_4^2\right)+\ldots \\ S_{IIB} &= \frac{1}{\alpha'^4}\int d^{10}x\sqrt{-g}\left(e^{-2\phi}(R+4(\partial\phi)^2-\frac{1}{12}H_3^2)-\frac{1}{2}F_1^2-\frac{1}{12}F_3^2-\frac{1}{240}F_5^2\right)+\ldots \end{aligned}$$

where the ellipsis denotes additional terms (known as Chern-Simons terms) and the subscript $n = 1, 2, 3, 4, 5$ indicates the number of anti-symmetric indices of the field strength $F_n = F_{\mu_1\ldots\mu_n}$. Note that in the S_{IIB} case there is field strength $F_\mu = \partial_\mu a$ which can be thought of as arising from an additional scalar. In addition the equation of motion that arises from S_{IIB} must be supplemented by the constraint that the five-index field strength $F_{\mu_1\mu_2\mu_3\mu_4\mu_5}$ is self-dual:

$$F_{\mu_1\mu_2\mu_3\mu_4\mu_5} = \frac{1}{5!}\sqrt{-g}\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\nu_1\nu_2\nu_3\nu_4\nu_5}F^{\nu_1\nu_2\nu_3\nu_4\nu_5} \tag{6.305}$$

We can also construct (in limited detail) the effective action for the Heterotic and type I superstrings. These are fixed by supersymmetry and gauge symmetry to be of the form

$$S_I = \frac{1}{\alpha'^4}\int d^{10}x\sqrt{-g}e^{-2\phi}\left(R+4(\partial\phi)^2-\frac{1}{12}H_3^2-\frac{1}{4}\mathrm{tr}(F)^2+\ldots\right) \tag{6.306}$$

where again the ellipsis denotes Fermionic and Green-Schwarz terms that are crucial for anomaly cancelation.

We saw that bosonic string, when compactified on a circle, admits a duality known as T-duality. In the superstring case one finds that type IIA string theory on a circle of radius R is equivalent to type IIB string theory on a circle of radius α'/R. However one finds more remarkable dualities. It turns out that the type IIB supergravity has a symmetry $\phi \leftrightarrow -\phi$.[6] From the point of view of the string theory this is suggests a

[6] This is simplifying things if the R-R-scalar a is not zero but a more general statement is true in that case.

duality between strongly coupled strings with g_s large and weakly coupled stings with g_s small. This self-duality of the type IIB string is known as S-duality.

What happens in the strong coupling limit, $g_s \to \infty$ of the type IIA superstring? Well is it conjectured that $\sqrt{\alpha'}e^{2\phi/3}$ can be interpreted as the radius of an extra, eleventh, dimension. There is a unique supergravity theory in eleven dimensions and indeed the type IIA string effective action comes from dimensional reduction of this theory on a circle. However there is now a great deal of evidence that the whole of type IIA string theory arises as an expansion of an eleven-dimensional theory about zero-radius (in on of its dimensions). This theory is known as M-theory and is rather poorly understood. However it's existence does seem be justified. The lowest order term is in a derivative expansion is fixed by supersymmetry to be

$$S_M = \frac{1}{\kappa^9}\int d^{11}x\sqrt{-g}(R - \frac{1}{48}G_4^2) + \ldots \tag{6.307}$$

where again the ellipsis denotes Chern-Simons and Fermionic terms. One also finds the Heterotic $E_8 \times E_8$ string by compactification of M-theory on a line interval.

Furthermore it promises to be very powerful as it controls not only the strong coupling limit of the type IIA string but, as a consequence of duality, the strong coupling limit of all the five known string theories. Thus one no longer thinks of there being five separate string theories but instead one unique theory, M-theory, which contains five different perturbative descriptions depending on what one considers to be a small parameter.

6.4 Branes and M-theory

Appendix: Conventions

We work in D-dimensional spacetime with "mostly plus" signature

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & & & & \\ & +1 & & & \\ & & +1 & & \\ & & & \ddots & \\ & & & & +1 \end{pmatrix} \tag{6.308}$$

We use Greek indices from the middle of the alphabet for D-dimensional spacetime x^μ, $\mu = 0, 1, , 2, ..., D-1$ and Roman ones for space alone x^i, $i = 1, 2, ..., D-1$. We use Greek letters from the beginning of the alphabet for worldsheet coordinates σ^α, $\alpha = 0, 1$ say. Repeated indices are summed over. For a metric $\gamma_{\alpha\beta}$ we use $\gamma = \det(\gamma_{\alpha\beta})$. In two dimensions there is the anti-symmetric ϵ-symbol $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ which is defined to have $\epsilon^{01} = 1$. We use $a, b = 1, ..., N$ to label parallel D-branes, *i.e.* as Chan-Paton indices. We use $m, n = 0, 1, ..., p$ for the worldvolume coordinates of a p-brane and $I, J = p+1, ..., D-1$ for the transverse coordinates. We use units where $\hbar = c = 1$.

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