Introduction to Supersymmetry

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Contents

1 Introduction

Particle Physics is the study of matter at the smallest scales that can be accessed by experiment. Currently energy scales are as high as $100GeV$ which corresponds to distances of $10^{-16}cm$ (recall that the atomic scale is about $10^{-9}cm$ and the nucleus is about $10^{-13}cm$). Our understanding of Nature up to this scale is excellent[1]. Indeed it must be one of the most successful and accurate scientific theories and goes by the least impressive name "The Standard Model of Elementary Particle Physics". The mathematical framework for such a theory is a relativistic quantum field theory and in particular a quantum gauge theory.

There are two essential ingredients into relativistic quantum field theories: Special relativity and quantum mechanics. The success of special relativity and quantum mechanics are particularly astounding. In some sense what our understanding of particle physics has taught us is that reality is ultimately ruled by quantum mechanics and the Lorentz group is the most fundamental structure we know about spacetime.

Quantum mechanics remains largely untouched in modern theoretical physics. However mathematically there is something deeper than the Lorentz Lie-algebra. This is the super-Lorentz algebra or simply supersymmetry. It is possible to construct interacting relativistic quantum field theories whose spacetime symmetry group is larger than the Lorentz group. These theories are called supersymmetric and exhibit a novel kind of symmetry where Fermions and Bosons are related to each other.

Supersymmetric theories turn out to be very interesting. Since they have more and deeper symmetries they are generally more tractable to solve quantum mechanically. Indeed almost all theoretical progress in understanding gauge theories such as those that arise in the Standard Model have come through studying their supersymmetric cousins. Supersymmetry has also grown-up hand in hand with String Theory but it is logically independent. However the successes of String Theory have also been brought using supersymmetry and hence supersymmetry has, like String Theory, become a central theme in modern theoretical particle physics.

Beyond the abstract mathematical and theoretical beauty of supersymmetry there are phenomenological reasons studying supersymmetric extensions of the Standard Model. There is currently a great deal of interest focused on the LHC (Large Hadron Collider) in CERN. The great hope is that new physics, beyond that predicted by the Standard Model, will be observed. One of the main ideas, in fact probably the most popular, is that supersymmetry will be observed. There are at least three main reasons for this:

- The Hierarchy problem: The natural scale of the Standard Model is the electroweak scale which is at about $1TeV$ (hence the excitement about the LHC). In a quantum field theory physical parameters, such as the mass of the Higg's Boson, get renormalized by quantum effects. Why then is the Higg's mass not renormalized up to the Planck scale? To prevent this requires and enormous amount of

[1] This ignores important issues that arise in large and complex systems such as those that are studied in condensed matter physics.

fine-tuning. However in a supersymmetric model these renormalizations are less severe and fine-tuning is not required (or at least is not as bad).

- Unification: Another key idea about beyond the Standard Model is that all the gauge fields are unified into a single, simple gauge group at some high scale, roughly $10^{15}GeV$. Mathematically this is possible with an $SU(5)$ or $SO(10)$ gauge group. Although the electromagnetic, strong and weak coupling constants differ at low energy, they 'run' with energy and meet at about $10^{15}GeV$. That any two of them should meet is trivial but that all three meet at the same scale is striking and gives further physical evidence for unification. Well in fact they don't quite meet in the Standard Model but they do in a supersymmetric version.

- Dark Matter: It would appear that most, roughly 70%, of the matter floating around in the universe is not the stuff that makes up the Standard Model. Supersymmetry predicts many other particles other than those observed in the Standard Model, the so so-called superpartners, and the lightest superpartner (LSP) is considered a serious candidate for dark matter.

If supersymmetry is observed in Nature it will be a great triumph of theoretical physics. Indeed the origin of supersymmetry is in string theory and the two fields have been closely linked since their inception. If not one can always claim that supersymmetry is broken at a higher energy (although in so doing the arguments in favour of supersymmetry listed above will cease to be valid). Nevertheless supersymmetry has been a very fruitful subject of research and has taught us a great deal about mathematics and quantum field theory. For example supersymmetric quantum field theories, especially those with extended supersymmetry, can be exactly solved (in some sense) at the perturbative and non-perturbative levels. Hopefully this course will convince the student that supersymmetry is a beautiful and interesting subject.

2 The Lorentz Algebra, Clifford Algebras and Spinors

Matter is made of Fermions. Since the details are crucial before proceeding it is necessary to review in detail the formalism that is needed to describe spinors and Fermions. We shall now do this. It is helpful to generalize to spacetime with D dimensions. The details of spinors vary slightly from dimension to dimension (although conceptually things are more or less the same). To help highlight the differences between vectors and spinors it is useful to consider a general dimension.

Fermions first appeared with Dirac who thought that the equation of motion for an electron should be first order in derivatives. Hence, for a free electron, where the equation should be linear, it must take the form

$$(\gamma^\mu\partial_\mu - M)\psi = 0 \tag{2.1}$$

Acting on the left with $(\gamma^\mu\partial_\mu + M)$ one finds

$$(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - M^2)\psi = 0 \tag{2.2}$$

This should be equivalent to the Klein Gordon equation (which is simply the mass-shell condition $E^2 - p^2 - m^2 = 0$)

$$(\partial^2 - m^2)\psi = 0 \tag{2.3}$$

Thus we see that we can take $m = M$ to be the mass and, since $\partial_\mu\partial_\nu\psi = \partial_\nu\partial_\mu\psi$, we also require that

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu} \tag{2.4}$$

This seemingly innocent condition is in fact quite deep. It first appeared in Mathematics in the geometrical work of Clifford (who was a student at King's). The next step is to find representations of this relation which reveals an 'internal' spin structure to Fermions.

2.1 Clifford Algebras

Introducing Fermions requires that we introduce a set of γ-matrices. These furnish a representation of the Clifford algebra, which is generically taken to be over the complex numbers, whose generators satisfy the relation

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{2.5}$$

Note that we have suppressed the spinor indices α, β. In particular the right hand side is proportional to the identity matrix in spinor space. We denote spinor indices by α, β.... Although we will only be interested in the four-dimensional Clifford algebra in this course it is instructive to consider Clifford algebras in a variety of dimensions. Each dimension has its own features and these often play an important role in the supersymmetric theories that can arise.

One consequence of this relation is that the γ-matrices are traceless (at least for $D > 1$). To see this we evaluate

$$\begin{aligned}
2\eta_{\mu\nu}\mathrm{Tr}(\gamma_\lambda) &= \mathrm{Tr}(\{\gamma_\mu, \gamma_\nu\}\gamma_\lambda) \\
&= \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\lambda + \gamma_\nu\gamma_\mu\gamma_\lambda) \\
&= \mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\lambda + \gamma_\mu\gamma_\lambda\gamma_\nu) \\
&= \mathrm{Tr}(\gamma_\mu\{\gamma_\nu, \gamma_\lambda\}) \\
&= 2\eta_{\nu\lambda}\mathrm{Tr}(\gamma_\mu)
\end{aligned} \tag{2.6}$$

Choosing $\mu = \nu \neq \lambda$ immediately implies that $\mathrm{Tr}(\gamma_\lambda) = 0$

Theorem: In even dimensions there is only one non-trivial irreducible representation of the Clifford algebra, up to conjugacy, *i.e.* up to a transformation of the form $\gamma_\mu \to U\gamma_\mu U^{-1}$. In particular the (complex) dimension of this representation is $2^{D/2}$, *i.e.* the γ-matrices will be $2^{D/2} \times 2^{D/2}$ complex valued matrices.

Without loss of generality one can choose a representation such that

$$\gamma_0^\dagger = -\gamma_0 \ , \qquad \gamma_i^\dagger = \gamma_i \tag{2.7}$$

which can be written as

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 \tag{2.8}$$

An even-dimensional Clifford algebra naturally lifts to a Clifford algebra in one dimension higher. In particular one can show that

$$\gamma_{D+1} = c\gamma_0\gamma_1 ... \gamma_{D-1} \tag{2.9}$$

anti-commutes with all the γ_μ's. Here c is a constant which we can fix, up to sign, by taking $\gamma_{D+1}^2 = 1$. In particular a little calculation shows that

$$\gamma_{D+1}^2 = -(-1)^{D(D-1)/2} c^2 \tag{2.10}$$

Here the first minus sign comes from γ_0^2 whereas the others come from anti-commuting the different γ_μ's through each other. In this way we find that

$$c = \pm i(-1)^{D(D-1)/4} \tag{2.11}$$

Thus we construct a Clifford Algebra in $(D+1)$-dimensions. It follows that the dimension (meaning the range of the spinor indices $\alpha, \beta ...$) of a Clifford algebra in $(D+1)$-dimensions is the same as the dimension of a Clifford algebra in D-dimensions when D is even.

In odd dimensions there are two inequivalent representations. To see this one first truncates down one dimension. This leads to a Clifford algebra in a even dimension which is therefore unique. We can then construct the final γ-matrix using the above procedure. This leads to two choices depending on the choice of sign above. Next we observe that in odd-dimensions γ_{D+1}, defined as the product of all the γ-matrices, commutes with all the γ_μ's. Hence by Shur's lemma it must be proportional to the identity. Under conjugacy one therefore has $\gamma_{D+1} \to U\gamma_{D+1}U^{-1} = \gamma_{D+1}$. The constant of proportionality is determined by the choice of sign we made to construct the final γ-matrix. Since this is unaffected by conjugation we find two representation we constructed are inequivalent.

We can also construct a Clifford algebra in $D+2$ dimensions using the Clifford algebra in D dimensions. Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.12}$$

be the ubiquitous Pauli matrices. If we have the Clifford algebra in D-dimensions given by γ_μ, $\mu = 0, 1, 2, ..., D-1$ then let

$$\begin{aligned} \Gamma_\mu &= 1 \otimes \gamma_\mu \\ \Gamma_D &= \sigma_1 \otimes \gamma_{D+1} \\ \Gamma_{D+1} &= \sigma_3 \otimes \gamma_{D+1} \end{aligned} \tag{2.13}$$

where we have used Γ_μ for $(D+2)$-dimensional γ-matrices. One readily sees that this gives a Clifford algebra in $(D+2)$-dimensions. Note that this gives two algebras corresponding to the two choices of sign for γ_{D+1}. However these two algebras are equivalent under conjugation by $U = \sigma_2 \otimes 1$. This is to be expected from the uniqueness of an even-dimensional Clifford algebra representation.

Having constructed essentially unique γ-matrices for a given dimension there are two special things that can happen. We have already seen that in even dimensions one finds an "extra" Hermitian γ-matrix, γ_{D+1} (so in four dimensions this is the familiar γ_5). Since this is Hermitian it has a basis of eigenvectors with eigenvalues ± 1 which are called the chirality. Indeed since the γ-matrices are traceless half of the eigenvalues are $+1$ and the other half -1. We can therefore write any spinor ψ uniquely as

$$\psi = \psi_+ + \psi_- \tag{2.14}$$

where $\psi_\pm$ has γ_{D+1} eigenvalue ± 1. A spinor with a definite γ_{D+1} eigenvalue is called a Weyl spinor.

The second special case occurs when the γ-matrices can be chosen to be purely real. In which case it is possible to chose the spinors to also be real. A real spinor is called a Majorana spinor.

Either of these two restrictions will cut the number of independent spinor components in half. In some dimensions it is possible to have both Weyl and Majorana spinors simultaneously. These are called Majorana-Weyl spinors. This reduces the number of independent spinor components to a quarter of the original size. Spinors without any such restrictions are called Dirac spinors. Which restrictions are possible in which dimensions comes in a pattern which repeats itself for dimensions D modulo 8.

Let us illustrate this by starting in low dimensions and work our way up. We will give concrete example of γ-matrices but it is important to bare in mind that these are just choices - there are other choices.

2.1.1 D=1

If there is only one dimension, time, then the Clifford algebra is the simple relation $(\gamma_0)^2 = -1$. In other words $\gamma_0 = i$ or one could also have $\gamma_0 = -i$. It is clear that there is no Majorana representation.

2.1.2 D=2

Here the γ-matrices can be taken to be

$$\begin{aligned}\gamma_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \gamma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\end{aligned} \tag{2.15}$$

One can easily check that $\gamma_0^2 = -\gamma_1^2 = -1$ and $\gamma_0\gamma_1 = -\gamma_1\gamma_0$.

Here we have a real representation so that we can choose the spinors to also be real. We can also construct $\gamma_3 = c\gamma_0\gamma_1$ and it is also real:

$$\gamma_3 = -\gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.16}$$

Thus we can have Weyl spinors, Majorana spinors and Majorana-Weyl spinors. These will have 2, 2 and 1 real independent components respectively whereas a Dirac spinor will have 2 complex, *i.e.* 4 real, components.

2.1.3 D=3

Here the γ-matrices can be constructed from $D = 2$ and hence a natural choice is

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \gamma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \gamma_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{2.17}$$

(we could also have taken the opposite sign for γ_2). These are just the Pauli matrices (up to a factor of i for γ_0). Since we are in an odd dimension there are no Weyl spinors but we can choose the spinors to be Majorana with only 2 real independent components.

2.1.4 D=4

Following our discussion above a natural choice is

$$\begin{aligned} \gamma_0 &= 1 \otimes i\sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ \gamma_1 &= 1 \otimes \sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ \gamma_2 &= \sigma_1 \otimes \sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ \gamma_3 &= \sigma_3 \otimes \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{2.18}$$

By construction this is a real basis of γ-matrices. Therefore we can chose to have Majorana, *i.e.* real, spinors.

Since we are in an even dimension we can construct the chirality operator $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Note the factor of i which is required to ensure that $\gamma_5^2 = 1$. Thus in our basis γ_5 is purely imaginary and, since it is Hermitian, it must be anti-symmetric. This means that it cannot be diagonalized over the reals. Of course since it is Hermitian it can be diagonalized over the complex numbers, *i.e.* there is another choice of γ-matrices for which γ_5 is real and diagonal but in this basis the γ_μ cannot all be real.

Thus in four dimensions we can have Majorana spinors or Weyl spinors but not both simultaneously. In many books, especially those that focus on four-dimensions, a Weyl basis of spinors is used. Complex conjugation then acts to flip the chirality. However we prefer to use a Majorana basis whenever possible (in part because it applies to more dimensions).

2.2 Lorentz and Poincare Algebras

We wish to construct relativistic theories which are covariant with respect to the Lorentz and Poincare symmetries. These consists of translations along with the Lorentz transformations (which in turn contain rotations and boosts). In particular the theory is invariant under the infinitesimal transformations

$$x^\mu \to x^\mu + a^\mu + \omega^\mu{}_\nu x^\nu \qquad i.e. \qquad \delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu \tag{2.19}$$

Here a^μ generates translations and $\omega^\mu{}_\nu$ generates Lorentz transformations. The principle of Special relativity requires that the spacetime proper distance $\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$ between two points is invariant under these transformations. Expanding to first order in $\omega^\mu{}_\nu$ tells us that

$$\begin{aligned}
\Delta s^2 &\to \eta_{\mu\nu}(\Delta x^\mu + \omega^\mu{}_\lambda \Delta x^\lambda)(\Delta x^\nu + \omega^\nu{}_\rho \Delta x^\rho) \\
&= \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu + \eta_{\mu\nu}\omega^\mu{}_\lambda \Delta x^\lambda \Delta x^\nu + \eta_{\mu\nu}\omega^\nu{}_\rho \Delta x^\mu \Delta x^\rho \\
&= \Delta s^2 + (\omega_{\mu\nu} + \omega_{\nu\mu})\Delta x^\mu \Delta x^\nu
\end{aligned} \tag{2.20}$$

where we have lowered the index on $\omega^\mu{}_\nu$. Thus we see that the Lorentz symmetry requires $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

Next we consider the algebra associated to such generators. To this end we want to know what happens if we make two Poincare transformations and compare the difference, *i.e.* we consider $\delta_1\delta_2 x^\mu - \delta_2\delta_1 x^\mu$. First we calculate

$$\delta_1\delta_2 x^\mu = \omega_1^\mu{}_\nu a_2^\nu + \omega_1^\mu{}_\lambda \omega_2^\lambda{}_\nu x^\nu \tag{2.21}$$

from which we see that

$$(\delta_1\delta_2 - \delta_2\delta_1)x^\mu = (\omega_1^\mu{}_\lambda a_2^\lambda - \omega_2^\mu{}_\lambda a_1^\lambda) + (\omega_1^\mu{}_\lambda \omega_2^\lambda{}_\nu - \omega_2^\mu{}_\lambda \omega_1^\lambda{}_\nu)x^\nu \tag{2.22}$$

This corresponds to a new Poincare transformation with

$$a^{\mu} = \omega_{1\ \lambda}^{\mu} a_2^{\lambda} - \omega_{2\ \lambda}^{\mu} a_1^{\lambda} \qquad \omega^{\mu}{}_{\nu} = \omega_{1\ \lambda}^{\mu} \omega_{2\ \nu}^{\lambda} - \omega_{2\ \lambda}^{\mu} \omega_{1\ \nu}^{\lambda} \tag{2.23}$$

note that $\omega_{(\mu\nu)} = \frac{1}{2}(\omega_{\mu\nu} + \omega_{\nu\mu}) = 0$ so this is indeed a Poincare transformation.

More abstractly we think of these transformations as being generated by linear operators P_μ and $M_{\mu\nu}$ so that

$$\delta x^{\mu} = i a^{\nu} P_{\nu}(x^{\mu}) + \frac{i}{2}\omega^{\nu\lambda} M_{\nu\lambda}(x^{\mu}) \tag{2.24}$$

The factor of $\frac{1}{2}$ arises because of the anti-symmetry (one doesn't want to count the same generator twice). The factors of i are chosen for later convenience to ensure that the generators are Hermitian. These generators can then also be though of as applying on different objects, *e.g.* spacetime fields rather than spacetime points. In other words we have an abstract algebra and its action on x^μ is merely one representation.

This abstract object is the Poincare algebra and it defined by the commutators

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [P_\mu, M_{\nu\lambda}] &= -i\eta_{\mu\nu} P_\lambda + i\eta_{\mu\lambda} P_\nu \\ [M_{\mu\nu}, M_{\lambda\rho}] &= -i\eta_{\nu\lambda} M_{\mu\rho} + i\eta_{\mu\lambda} M_{\nu\rho} - i\eta_{\mu\rho} M_{\nu\lambda} + i\eta_{\nu\rho} M_{\mu\lambda} \end{aligned} \tag{2.25}$$

which generalizes (2.22).

Problem: Using (2.24) and(2.25) show that (2.22) is indeed reproduced.

The Poincare group has two clear pieces: translations and Lorentz transformations. It is not quite a direct product because of the non-trivial commutator $[P_\mu, M_{\nu\lambda}]$. It is a so-called a semi-direct product. Translations by themselves form an Abelian and non-compact subgroup. On physical grounds one always takes $P_\mu = -i\partial_\mu$. This seems obvious from the physical interpretation of spacetime. Mathematically the reason for this is simply Taylor's theorem for a function $f(x^\mu)$:

$$\begin{aligned} f(x+a) &= f(x) + \partial_\mu f(x) a^\mu + \ldots \\ &= f(x) + i a^\mu P_\mu f(x) + \ldots \end{aligned} \tag{2.26}$$

Thus acting by P_μ will generate a infinitessimal translation. Furthermore Taylor's theorem is the statement that finite translations are obtained from exponentiating P_μ:

$$\begin{aligned} f(x+a) &= e^{ia^\mu P_\mu} f(x) \\ &= f(x) + a^\mu \partial_\mu f(x) + \frac{1}{2!} a^\mu a^\nu \partial_\mu \partial_\nu f(x) + \ldots \end{aligned} \tag{2.27}$$

However the other part, the Lorentz group, is non-Abelian and admits interesting finite-dimensional representations. For example the Standard Model contains a scalar field $H(x)$ (the Higg's Boson) which carries a trivial representation and also vector fields $A_\mu(x)$ (*e.g.* photons) and spinor fields $\psi_\alpha(x)$ (*e.g.* electrons). A non-trivial representation of the Lorentz group implies that the field carries some kind of index. In the two cases above these are μ and α respectively. The Lorentz generators then act as matrices with two such indices (one lowered and one raised). Different representations mean that there are different choices for these matrices which still satisfies (2.25). For example in the vector representation one can take

$$(M_{\mu\nu})^\lambda{}_\rho = i\eta_{\mu\rho}\delta^\lambda_\nu - i\delta^\lambda_\mu\eta_{\nu\rho} \tag{2.28}$$

Notice the dual role of μ, ν indices as labeling both the particular Lorentz generator as well as it's matrix components. Whereas in the spinor representation we have

$$(M_{\mu\nu})_\alpha{}^\beta = -\frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta = -\frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\mu\gamma_\nu)_\alpha{}^\beta \tag{2.29}$$

Here $(\gamma_\mu)_\alpha{}^\beta$ are the Dirac γ-matrices. However in either case it is important to realize that the defining algebraic relations (2.25) are reproduced.

Problem: Verify that these two representation of $M_{\mu\nu}$ do indeed satisfy the Lorentz subalgebra of (2.25).

2.3 Spinors

Having defined Clifford algebras we next need to discuss the properties of spinors in greater detail. We will see later that $M_{\mu\nu} = \frac{i}{2}\gamma_{\mu\nu}$ gives a representation of the Lorentz algebra, known as the spinor representation. A spinor is simply an object that transforms in the spinor representation of the Lorentz group (it is a section of the spinor bundle over spacetime). Hence it carries a spinor index α. From our definitions this means that under a Lorentz transformation generated by $\omega^{\mu\nu}$, a spinor ψ_α transforms as

$$\delta\psi_\alpha = \frac{1}{4}\omega^{\mu\nu}(\gamma_{\mu\nu})_\alpha{}^\beta\psi_\beta \tag{2.30}$$

Note that we gives spinors a lower spinor index. As such the γ-matrices naturally come with one upper and one lower index, so that matrix multiplication requires contraction on one upper and one lower index.

Let us pause for a moment to consider a finite Lorentz transformation. To begin with consider an infinitesimal rotation by an angle θ in the (x^1, x^2)-plane,

$$\delta\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \theta\begin{pmatrix} x^0 \\ -x^2 \\ x^1 \\ x^3 \end{pmatrix} \tag{2.31}$$

i.e.

$$\omega^{12} = -\omega_{21} = \theta \qquad M_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.32}$$

A finite rotation is obtained by exponentiating M_{12}:

$$x^\mu \to (e^{\omega^{\lambda\rho}M_{\lambda\rho}})^\mu{}_\nu x^\nu \tag{2.33}$$

Since $M_{12}^2 = -1$ one finds that, using the same proof as the famous relation $e^{i\theta} = \cos\theta + i\sin\theta$,

$$e^{\theta M_{12}} = \cos\theta + M_{12}\sin\theta \tag{2.34}$$

In particular we see that if $\theta = 2\pi$ then $e^{2\pi M_{12}} = 1$ as expected.

How does a spinor transform under such a rotation? The infinitesimal transformation generated by ω^{12} is, by definition,

$$\delta\psi = \frac{1}{4}\omega^{\mu\nu}\gamma_{\mu\nu}\psi = \frac{1}{2}\theta\gamma_{12}\psi \tag{2.35}$$

If we exponentiate this we find

$$\psi \to e^{\frac{1}{2}\theta\gamma_{12}}\psi = \cos(\theta/2) + \gamma_{12}\sin(\theta/2) \tag{2.36}$$

We see that now, if $\theta = 2\pi$, then $\psi \to -\psi$. Thus we recover the well known result that under a rotation by 2π a spinor (such as an electron) picks up a minus sign.

Let us now try to contract spinor indices to obtain Lorentz scalars. It follows that the Hermitian conjugate transforms as

$$\delta\psi^\dagger = \frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_\nu^\dagger\gamma_\mu^\dagger = \frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_0\gamma_{\nu\mu}\gamma_0 = \frac{1}{4}\psi^\dagger\omega^{\mu\nu}\gamma_0\gamma_{\mu\nu}\gamma_0 \tag{2.37}$$

Here we have ignored the spinor index. Note that the index structure is $(\gamma_0\gamma_{\mu\nu}\gamma_0)_\alpha{}^\beta$ and therefore it is most natural to write $(\psi^\dagger)^\alpha = \psi^{*\alpha}$ with an upstairs index.

However we would like to contract two spinors to obtain a scalar. One can see that the naive choice

$$\lambda^\dagger\psi = \lambda^{*\alpha}\psi_\alpha \tag{2.38}$$

will not be a Lorentz scalar due to the extra factors of γ_0 that appear in $\delta\lambda^\dagger$ as compared to $\delta\psi$. To remedy this one defines the Dirac conjugate

$$\bar\lambda = \lambda^\dagger\gamma_0 \tag{2.39}$$

In which case on finds that, under a Lorentz transformation,

$$\delta\bar\lambda = -\frac{1}{4}\bar\lambda\omega^{\mu\nu}\gamma_{\mu\nu} \tag{2.40}$$

and hence

$$\begin{aligned} \delta(\bar{\lambda}\psi) &= \delta\bar{\lambda}\psi + \bar{\lambda}\delta\psi \\ &= -\frac{1}{4}\bar{\lambda}\omega^{\mu\nu}\gamma_{\mu\nu}\psi + \frac{1}{4}\bar{\lambda}\omega^{\mu\nu}\gamma_{\mu\nu}\psi \\ &= 0 \end{aligned} \tag{2.41}$$

Thus we have found a Lorentz invariant way to contract spinor indices.

Note that from two spinors we can construct other Lorentz covariant objects such as vectors and anti-symmetric tensors:

$$\bar{\lambda}\gamma_\mu\psi\ , \quad \bar{\lambda}\gamma_{\mu\nu}\psi\ , \ldots \tag{2.42}$$

Problem: Show that $V_\mu = \bar{\lambda}\gamma_\mu\psi$ is a Lorentz vector, *i.e.* show that $\delta V_\mu = \omega_\mu{}^\nu V_\nu$ under the transformation (2.30).

So far our discussion applied to general Dirac spinors. In much of this course we will be interested in Majorana spinors where the γ_μ are real. The above discussion is then valid if we replace the Hermitian conjugate $\dagger$ with the transpose T so that $\gamma_\mu^T = -\gamma_0\gamma_\mu\gamma_0^{-1}$. More generally such a relationship always exists because if $\{\gamma_\mu\}$ is a representation of the Clifford algebra then so is $\{-\gamma_\mu^T\}$. Therefore, since there is a unique representation up to conjugacy, there must exist a matrix C such that $-\gamma_\mu^T = C\gamma_\mu C^{-1}$. C is called the charge conjugation matrix. The point here is that in the Majorana case it is possible to find a representation in which C coincides with Dirac conjugation matrix γ_0.

Problem: Show that, for a general Dirac spinor in any dimension, $\lambda^T C\psi$ is Lorentz invariant, where C is the charge conjugation matrix.

One way to think about charge conjugation is to view the matrix $C^{\alpha\beta}$ as a metric on the spinor indices with inverse $C^{-1}_{\alpha\beta}$. In which case $\psi^\alpha = \psi_\beta C^{\beta\alpha}$.

Finally we note that spinor quantum fields are Fermions in quantum field theory (this is the content of the spin-statistics theorem). This means that spinor components are anti-commuting Grassmann variables

$$\psi_\alpha\psi_\beta = -\psi_\beta\psi_\alpha \tag{2.43}$$

We also need to define how complex conjugation acts. Since ultimately in the quantum field theory the fields are elevated to operators we take the following convention for complex conjugation

$$(\psi_\alpha\psi_\beta)^* = \psi_\beta^*\psi_\alpha^* \tag{2.44}$$

which is analogous to the Hermitian conjugate. This leads to the curious result that, even for Majorana spinors, one has that

$$(\bar{\psi}\chi)^* = (\psi_\alpha^* C^{\alpha\beta}\chi_\beta)^* = \chi_\beta C^{\alpha\beta}\psi_\alpha = -\psi_\alpha C^{\alpha\beta}\chi_\beta = -\bar{\psi}\chi \tag{2.45}$$

is pure imaginary!

3 Supersymmetry and its Consequences

3.1 Symmetries, A No-go Theorem and How to Avoid It

Quantum field theories are essentially what you get from the marriage of quantum mechanics with special relativity (assuming locality). A central concept of these ideas is the notion of symmetry. And indeed quantum field theories are thought of and classified according to their symmetries.

The most important symmetry is of course the Poincare group of special relativity which we have already discussed. To say that the Poincare algebra is fundamental in particle physics means that everything is assumed fall into some representation of this algebra. The principle of relativity then asserts that the laws of physics are covariant with respect to this algebra.

The Standard Model and other quantum field theories also have other important symmetries. Most notably gauge symmetries that we have discussed above. These symmetries imply that there is an additional Lie-algebra with a commutation relation of the form

$$[T_r, T_s] = i f_{rs}{}^{t} T_t \tag{3.46}$$

where the T_r are Hermitian generators and $f_{rs}{}^{t}$ are the structure constants. This means that every field in the Standard model Lagrangian also carries a representation of this algebra. If this is a non-trivial representation then there is another 'internal' index on the field. For example the quarks are in the fundamental (*i.e.* three-dimensional) representation of $SU(3)$ and hence, since they are spacetime spinors, the field carries the indices $\psi^{a}_{\alpha}(x)$.

Finally we recall Noether's theorem which asserts that for every continuous symmetry of a Lagrangian one can construct a conserved charge. Suppose that a Lagrangian $\mathcal{L}(\Phi_A, \partial_\alpha \Phi_A)$, where we denoted the fields by Φ_A, has a symmetry: $\mathcal{L}(\Phi_A) = \mathcal{L}(\Phi_A + \delta\Phi_A)$. This implies that

$$\frac{\partial \mathcal{L}}{\partial \Phi_A}\delta\Phi_A + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\delta\partial_\alpha\Phi_A = 0 \tag{3.47}$$

This allows us to construct a current:

$$J^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\delta\Phi_A \tag{3.48}$$

which is, by the equations of motion,

$$\begin{aligned} \partial_\alpha J^\alpha &= \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\right)\delta\Phi_A + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\partial_\alpha\delta\Phi_A \\ &= \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi_A)}\right)\delta\Phi_A - \frac{\partial \mathcal{L}}{\partial \Phi_A}\delta\Phi_A \\ &= 0 \end{aligned} \tag{3.49}$$

conserved. This means that the integral over space of J^0 is a constant defines a charge

$$Q = \int_{space} J^0 \tag{3.50}$$

which is conserved

$$\begin{aligned}\frac{dQ}{dt} &= \int_{space} \partial_0 J^0 \\ &= -\int_{space} \partial_i J^i \\ &= 0\end{aligned}$$

Thus one can think of symmetries and conservations laws as being more or less the same thing.

So the Standard Model of Particle Physics has several symmetries built into it (*e.g.* $SU(3) \times SU(2) \times U(1)$) and this means that the various fields carry representations of various algebras. These algebras split up into those associated to spacetime (Poincare) and those which one might call internal (such as the gauge symmetry algebra). In fact the split is a direct product in that

$$[P_\mu, T_a] = [M_{\mu\nu}, T_a] = 0 \tag{3.51}$$

where T_a refers to any internal generator. Physically this means the conserved charges of these symmetries are Lorentz scalars.

Since the Poincare algebra is so central to our understanding of spacetime it is natural to ask if this direct product is necessarily the case or if there is, in principle, some deeper symmetry that has a non-trivial commutation relation with the Poincare algebra. This question was answered by Coleman and Mandula:

Theorem: In any spacetime dimension greater than two the only interacting quantum field theories have Lie algebra symmetries which are a direct product of the Poincare algebra with an internal symmetry.

In other words the Poincare algebra is apparently as deep as it gets. There are no interacting theories which have conserved charges that are not Lorentz scalars. Intuitively the reasons is that tensor-like charge must be conserved in any interaction and this is simply too restrictive as the charges end up being proportional to (products of) the momenta. Thus one finds that the individual momenta are conserved, rather than the total momentum.

But one shouldn't stop here. A no-go theorem is only as good as its assumptions. This theorem has several assumptions, for example that there are a finite number of massive particles and no massless ones. However the key assumption of the Coleman-Mandula theorem is that the symmetry algebra should be a Lie-algebra. We recall that a Lie-algebra can be thought of as the tangent space at the identity of a continuous group, so that, an infinitessimal group transformation has the form

$$g = 1 + i\epsilon A \tag{3.52}$$

where A is an element of the Lie-algebra and ϵ is an infinitessimal parameter. The Lie-algebra is closed under a bilinear operation, the Lie-bracket,

$$[A, B] = -[B, A] \tag{3.53}$$

subject to the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{3.54}$$

If we relax this assumption then there is something deeper - Supersymmetry. So how do we relax it since Lie-algebras are inevitable whenever you have continuous symmetries and because of Noether's theorem we need a continuous symmetry to give a conserved charge?

The way to proceed is to note that quantum field theories such as the Standard Model contain two types of fields: Fermions and Bosons. These are distinguished by the representation of the field under the Lorentz group. In particular a fundamental theorem in quantum field theory - the spin-statistics theorem - asserts that Bosons must carry representations of the Lorentz group with integer spins and their field operators must commute outside of the light-cone whereas Fermions carry half-odd-integer spins and their field operators are anti-commuting. This means that the fields associated to Fermions are not ordinary (so-called c-number) valued field but rather Grassmann variables that satisfy

$$\psi_1(x)\psi_2(x) = -\psi_2(x)\psi_1(x) \tag{3.55}$$

So a way out of this no-go theorem is to find a symmetry that relates Bosons to Fermions. Such a symmetry will require that the 'infinitessimal' generating parameter is a Grassmann variable and hence will not lead to a Lie-algebra. More precisely the idea is to consider a Grassmann generator (with also carries a spinor index) and which requires a Grassmann valued spinorial parameter. One then is lead to something called a superalgebra, or a $\mathbf{Z}_2$-graded Lie-algebra. This means that the generators can be labeled as either even and odd. The even generators behave just as the generators of a Lie-algebra and obey commutation relations. An even and an odd generator will also obey a commutator relation. However two odd generators will obey an anti-commutation relation. The even-ness or odd-ness of this generalized Lie-bracket is additive modulo two: the commutator of two even generators is even, the anti-commutator of two odd generators is also even, whereas the commutator of an even and an odd generator is odd. Schematically, the structure of a superalgebra takes the form

$$\begin{aligned} [even, even] &\sim even \\ [even, odd] &\sim odd \\ \{odd, odd\} &\sim even \end{aligned} \tag{3.56}$$

In particular one does not consider things that are the sum of an even and an odd generator (at least physicists don't but some Mathematicians might), nor does the commutator of two odd generators, or anti-commutator of two even generators, play any role. Just as in Lie-algebras there is a Jacobi identity. It is a little messy as whether or not one takes a commutator or anti-commutator depends on the even/odd character of the generator. It can be written as

$$(-1)^{ac}[A,[B,C]_\pm]_\pm + (-1)^{ba}[B,[C,A]_\pm]_\pm + (-1)^{cb}[C,[A,B]_\pm]_\pm = 0 \tag{3.57}$$

where $a, b, c \in \mathbf{Z}_2$ are the gradings of the generators A, B, C respectively and $[\ ,\ ]_\pm$ is a commutator or anti-commutator according to the rule (3.56).

There is a large mathematical literature on superalgebras as abstract objects. However we will simply focus on the case most relevant for particle physics. In particular the even elements will be the Poincare generators $P_\mu, M_{\nu\lambda}$ and the odd elements supersymmetries Q_α. The important point here is that the last line in (3.56) takes the form

$$\{Q, Q\} \sim P + M \tag{3.58}$$

(in fact one typically finds only P or M on the right hand side, and in this course just P). Thus supersymmetries are the square-root of Poincare transformations. Thus there is a sensible algebraic structure that is "deeper" that the Poincare group. Surely this is worth of study.

One final comment is in order. Although we have found a symmetry that underlies the Poincare algebra one generally still finds that supersymmetries commute with the other internal symmetries. Thus a refined version of the Coleman-Mandula theorem still seems to apply and states that the symmetry algebra of a non-trivial theory is at most the direct product of the superalgebra and an internal Lie-algebra.[2]

3.2 Elementary Consequences of Supersymmetry

The exact details of the supersymmetry algebra vary from dimension to dimension, depending on the details of Clifford algebras, however the results below for four-dimensions are qualitatively unchanged. If there are Majorana spinors then the algebra is, in addition to the Poincare algebra relations (2.25),[3]

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= -2(\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \\ [Q_\alpha, P_\mu] &= 0 \\ [Q_\alpha, M_{\mu\nu}] &= \frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta \end{aligned} \tag{3.59}$$

[2]Note that one should be careful here, while this statement is true in spirit it is imprecise and in some sense counter examples can be found (*e.g.* in gauged supergavity).

[3]More precisely this is the minimal $N = 1$ super-Poincare algebra. One can have N-extended supersymmetry algebras and centrally extended supersymmetry algebras. There are also superalgebras based on other Bosonic algebras than the Poincare algebra, *e.g.*, the anti-de-Sitter algebra.

The primary relation is the first line. The second line simply states that the Q_α's are invariant under translations and the third line simply states that they are spacetime spinors.

At first sight one might wonder why there is a C^{-1} on the right hand side. The point is that this is used to lower the second spinor index. Furthermore it is clear that the left hand side is symmetric in α and β and therefore the right hand side must also be symmetric. To see that this is the case we observe that, since we have assumed a Majorana basis where $C = -C^T = \gamma_0$,

$$(\gamma_\mu C^{-1})^T = (C^{-1})^T \gamma_\mu^T = -(C^{-1})^T C \gamma_\mu C^{-1} = \gamma_\mu C^{-1} \tag{3.60}$$

is indeed symmetric.

Let us take the trace of the primary supersymmetry relation

$$\sum_\alpha \{Q_\alpha, Q_\alpha\} = 8P_0 \tag{3.61}$$

Here we have used the fact that $C^{-1} = \gamma^0$, $\mathrm{Tr}(\gamma_{\mu\nu}) = 0$ and $\mathrm{Tr}(1) = 2^2$. We can identify $P_0 = E$ with the energy and hence we see that

$$E = \frac{1}{4}\sum_\alpha Q_\alpha^2 \tag{3.62}$$

Since Q_α is Hermitian it follows that the energy is positive definite. Furthermore the only states with $E = 0$ must have $Q_\alpha|0>= 0$, *i.e.* they must preserve the supersymmetry.

Supersymmetry, like other symmetries in quantum field theory, can be spontaneously broken. This means that the vacuum state $|vacuum>$, *i.e.* the state of lowest energy, does not satisfy $Q_\alpha|vacuum>= 0$. We see that in a supersymmetric theory this will be the case if and only if the vacuum energy is positive.

Next let us consider the representations of supersymmetry. First we observe that since $[P_\mu, Q_\alpha] = 0$ we have $[P^2, Q_\alpha] = 0$. Thus P^2 is a Casmir, that is to say irreducible representations of supersymmetry (*i.e.* of the Q's) all carry the same value of $P^2 = -m^2$. Thus all the particles in a supermultiplet (*i.e.* in a irreducible representation) have the same mass.

Let us first consider a massive supermultiplet. We can therefore go to the rest frame where $P_\mu = (m, 0, 0, 0)$. In this case the algebra becomes

$$\{Q_\alpha, Q_\beta\} = 2m\delta_{\alpha\beta} \tag{3.63}$$

We can of course assume that $m > 0$ and rescale $\tilde{Q}_\alpha = m^{-1/2}Q_\alpha$ which gives

$$\{\tilde{Q}_\alpha, \tilde{Q}_\beta\} = 2\delta_{\alpha\beta} \tag{3.64}$$

This is just a Clifford algebra in 4 Euclidean dimensions! As such we know that it has $2^{4/2} = 4$ states. We can construct the analogue of γ_5:

$$(-1)^F = Q_1 Q_2 Q_3 ... Q_4 \tag{3.65}$$

Since we are in 4 Euclidean dimensions we have that $((-1)^F)^2 = 1$. Again $(-1)^F$ is traceless and Hermitian. Therefore it has 2 eigenvalues equal to $+1$ and 2 equal to -1. What is the significance of these eigenvalues? Well if $|\pm>$ is a state with $(-1)^F$ eigenvalue ± 1 then $Q_\alpha|\pm>$ will satisfy

$$(-1)^F Q_\alpha|\pm>= -Q_\alpha(-1)^F|\pm>= \mp Q_\alpha|\pm> \tag{3.66}$$

Thus acting by Q_α will change the sign of the eigenvalue. However since Q_α is a Fermionic operator it will map Fermions to Bosons and vise-versa. Thus $(-1)^F$ measures whether or not a state is Fermionic or Bosonic. Since it is traceless we see that a supermutliplet contains an equal number of Bosonic and Fermionic states. This is only true on-shell since we have assumed that we are in the rest frame.

Next let us consider massless particles. Here we can go to a frame where $P_\mu = (E, E, 0, 0)$ so that the supersymmetry algebra becomes

$$\{Q_\alpha, Q_\beta\} = 2E(\delta_{\alpha\beta} + (\gamma_{01})_{\alpha\beta}) \tag{3.67}$$

We observe that $\gamma_{01}^2 = 1$ and also that $\mathrm{Tr}(\gamma_{01}) = 0$. Therefore the matrix $1 - \gamma_{01}$ has half its eigenvalues equal to 0 and the others equal to 2. It follows the algebra splits into two pieces:

$$\{Q_{\alpha'}, Q_{\beta'}\} = 4E\delta_{\alpha'\beta'} \qquad \{Q_{\alpha''}, Q_{\beta''}\} = 0 \tag{3.68}$$

where the primed and doubled primed indices only take on 2 values each. Again by rescaling, this time $\tilde{Q}_{\alpha'} = (2E)^{-1/2}Q_{\alpha'}$ we recover a Clifford algebra but in 2 dimensions. Thus there are just 2 states. Again we find that half are Fermions and the other half Bosons.

Finally we note that the condition $[Q_\alpha, M_{\mu\nu}] = \frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta$ implies that states in a supermultiplet will have spins that differ in steps of 1/2. In an irreducible mutliplet there is a unique state $|j_{max}>$ with maximal spin (actually helicity). The remaining states therefore have spins $j_{max} - 1/2, j_{max} - 1, ...$

It should be noted that often these multiplets will not be CPT complete. For example if they are constructed by acting with lowering operators on a highest helicity state then the tend to have more positive helicity states than negative ones. Therefore in order to obtain a CPT invariant theory, as is required by Lorentz invariance, one has to add in a CPT mirror multiplet (for example based on using raising operators on a lowest helicity state).

In higher dimensions the number of states in a supermultiplet grows exponentially quickly. This is essentially because the number of degrees of freedom of a spinor grow exponentially quickly. However the number of degrees of freedom of Bosonic fields (such as scalars and vectors) do not grow so quickly, if at all, when the spacetime dimension is increase. Although one can always keep adding in extra scalar modes to keep the Bose-Fermi degeneracy this becomes increasingly unnatural. In fact one finds that if we only wish to consider theories with spins less than two (*i.e.* do not include gravity) then the highest spacetime dimension for which there exists supersymmetric theories is $D = 10$. If we do allow for gravity then this pushes the limit up to $D = 11$.

3.3 Weyl Notation

In these notes we have used a Majorana representation, since it applies in more dimensions. However often in four-dimensions a so-called Weyl basis is used. Here the spinors are expanded as eigenstates of γ_5 and hence are complex. Given a Majorana spinor λ_M we can construct a Weyl spinor by taking

$$\lambda_W = \frac{1}{2}(1+\gamma_5)\lambda_M \ ,$$

which is complex. Clearly this satisfies $\gamma_5\lambda_W = \lambda_W$. Furthermore, since γ_5 is purely imaginary in four dimensions we have

$$\gamma_5\lambda_W^* = -\lambda_W^*$$

Thus complex conjugation flips the chirality of a Weyl spinor.

Problem: Show that in four-dimensions, where Q_α is a Majorna spinor, the first line of the supersymmetry algebra (3.59) can be written as

$$\begin{aligned}
\{Q_{W\alpha}, Q_{W\beta}\} &= 0 \\
\{Q^*_{W\alpha}, Q^*_{W\beta}\} &= 0 \\
\{Q_{W\alpha}, Q^*_{W\beta}\} &= -((1+\gamma_5)\gamma_\mu C^{-1})_{\alpha\beta}P_\mu \\
\{Q^*_{W\alpha}, Q_{W\beta}\} &= -((1-\gamma_5)\gamma_\mu C^{-1})_{\alpha\beta}P_\mu
\end{aligned} \tag{3.69}$$

where $Q_{W\alpha}$ is a Weyl spinor and $Q^*_{W\alpha}$ is its complex conjugate.

In Weyl notation one chooses a different basis for the four-dimensional Clifford Algebra. In particular one writes, in terms of block 2×2 matrices,

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \qquad \gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{3.70}$$

where σ_i are the Pauli matrices. Note that the charge conjugation matrix, defined by $\gamma_\mu^T = -C\gamma_\mu C^{-1}$, is no longer $C=\gamma_0$. Rather we find

$$C = \gamma_0\gamma_2\gamma_5 \ .$$

Since Weyl spinors only have two independent components one usually introduces a new notation: $a, \dot{a} = 1,2$ so that a general 4-component Dirac spinor is decomposed in terms of two complex Weyl spinors as

$$\psi_D = \begin{pmatrix} \lambda_a \\ \chi_{\dot{a}} \end{pmatrix} \tag{3.71}$$

i.e. the first two indices are denoted by a and the second two by $\dot{a}$.

Let us define $\sigma^{\mu}_{a\dot{b}} = \frac{1}{2}((1-\gamma_5)\gamma^{\mu}C^{-1})_{a\dot{b}}$, $\bar{\sigma}^{\mu}_{\dot{a}b} = \frac{1}{2}((1+\gamma_5)\gamma^{\mu}C^{-1})_{\dot{a}b}$. In this case the algebra is

$$\begin{aligned}
\{Q_a, Q_b\} &= 0 \\
\{Q^*_{\dot{a}}, Q^*_{\dot{b}}\} &= 0 \\
\{Q_a, Q^*_{\dot{b}}\} &= -2\sigma^{\mu}_{a\dot{b}}P_{\mu} \\
\{Q^*_{\dot{a}}, Q_b\} &= -2\bar{\sigma}^{\mu}_{\dot{a}b}P_{\mu}
\end{aligned} \tag{3.72}$$

Here we have dropped the subscript W since the use of a and $\dot{a}$ indices implies that we are talking about Weyl spinors. This form for the algebra appears in many text books and is also known as the two-component formalism.

Problem: Show that

$$\begin{aligned}
(\sigma_{\mu})_{a\dot{b}} &= \left(\delta_{a\dot{b}}, \sigma^i_{a\dot{b}}\right) \\
(\bar{\sigma}_{\mu})_{a\dot{b}} &= \left(\delta_{a\dot{b}}, -\sigma^i_{a\dot{b}}\right)
\end{aligned} \tag{3.73}$$

And therefore

$$\gamma_{\mu} = \begin{pmatrix} 0 & \bar{\sigma}_{\mu} \\ -\bar{\sigma}_{\mu} & 0 \end{pmatrix} . \tag{3.74}$$

Recall that we defined two Lorentz invariant contractions of spinors; Dirac: $\psi^{\dagger}\gamma_0\psi$ and Majorana: $\psi^T C\psi$? In the Majorana notation with real spinors these are manifestly the same (but not if ψ isn't real). In two component notation these are

$$\begin{aligned}
\psi^{\dagger}\gamma_0\psi &= \lambda^{\dagger}\chi - \chi^{\dagger}\lambda \\
\psi^T C\psi &= \lambda^T\sigma^2\chi + \chi^T\sigma^2\lambda .
\end{aligned}$$

Finally, what is a Majorana spinor in this notation? Well its one for which the Dirac conjugate and Majorana conjugate coincide:

$$\psi^{\dagger}\gamma_0 = \psi^T C .$$

Taking the transpose leads to $\psi^* = \gamma_2\gamma_5\psi$. In terms of two-component spinors this gives:

$$\lambda^* = -\sigma^2\chi , \qquad \chi^* = \sigma^2\lambda .$$

4 Super-Yang Mills

We can now start to construct a version of Yang-Mills theory that has supersymmetry. Since we must have a gauge field in the adjoint representation we see that supersymmetry will force us to have a Fermion that is also in adjoint representation. We can then add Fermions in other representations provided that we also include scalar superpartners for them.

4.1 Super-Maxwell

We start our first construction of a supersymmetric theory by looking at a very simple theory: Electromagnetism coupled to a Majorana Fermion in the adjoint. Since the adjoint of $U(1)$ is trivial the Fermion is chargeless and we have a free theory!

Why these fields? We just saw that the simplest supermultiplet is massless with 2 real Fermions and 2 real Bosons on-shell. Furthermore since supersymmetries commute with any internal symmetries we see that the Fermions need to be in the same representation of the gauge group as the Bosons. In Maxwell theory the gauge field is in the adjoint so the Fermion must also be in the adjoint.

Let us now check that the number of degrees of freedom is correct. We fix the gauge to Lorentz gauge $\partial^\mu A_\mu = 0$. Maxwell's equation is then just $\partial^2 A_\mu = 0$. However this only partially fixes the gauge since we can also take $A_\mu \to A_\mu + \partial_\mu \theta$ so long as $\partial^2\theta = 0$. This allow us to remove one component of A_μ, say A_3. However imposing $\partial^\mu A_\mu = 0$ provides a further constraint leaving 2 degrees of freedom. In particular in momentum space choosing $p^\mu = (E, 0, 0, E)$ we see that $p^\mu A_\mu = E(A_0 + A_3) = 0$ and hence $A_0 = A_3 = 0$ leaving just A_1 and A_2.

For the Fermion λ we have the Dirac equation $\gamma^\mu \partial_\mu \lambda = 0$. In momentum space this is $p^\mu \gamma_\mu \lambda = 0$. Choosing $p^\mu = (E, 0, 0, E)$ we find

$$E(\gamma_0 + \gamma_3)\lambda = 0 \tag{4.75}$$

For $E \neq 0$ this implies $\gamma_{03}\lambda = \lambda$. Since γ_{03} is traceless and squares to one we see that this projects out 2 of the 4 components of λ.

The action is

$$S_{SuperMaxwell} = -\int d^4x \; \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\lambda}\gamma^\mu \partial_\mu \lambda \tag{4.76}$$

where $\bar{\lambda} = \lambda^T C$. Not very exciting except that it has the following symmetry

$$\begin{aligned} \delta A_\mu &= i\bar{\epsilon}\gamma_\mu \lambda \\ \delta\lambda &= -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon \end{aligned} \tag{4.77}$$

To see this we first note that, since $C\gamma^\mu$ is symmetric,

$$\begin{aligned} \delta\bar{\lambda}\gamma^\mu \partial_\mu \lambda &= \partial_\mu(\delta\bar{\lambda}\gamma^\mu\lambda) - \partial_\mu \delta\bar{\lambda}\gamma^\mu \lambda \\ &= \partial_\mu(\delta\bar{\lambda}\gamma^\mu\lambda) + \bar{\lambda}\gamma^\mu \partial_\mu \delta\lambda \end{aligned} \tag{4.78}$$

We can drop the total derivative term in the action and find

$$\begin{aligned} \delta S &= -\int d^4x F^{\mu\nu}\partial_\mu \delta A_\nu + i\bar{\lambda}\gamma^\rho \partial_\rho \delta\lambda \\ &= -\int d^4x F^{\mu\nu} i\bar{\epsilon}\gamma_\nu \partial_\mu \lambda - \frac{i}{2}\bar{\lambda}\gamma^\rho \partial_\rho F_{\mu\nu}\gamma^{\mu\nu}\epsilon \end{aligned} \tag{4.79}$$

To continue we note that $C\gamma^\mu$ is symmetric and $\gamma^\rho\gamma^{\mu\nu} = \gamma^{\rho\mu\nu} + \eta^{\rho\mu}\gamma^\nu - \eta^{\rho\nu}\gamma^\mu$. Thus we have

$$\delta S = -\int d^4x - F^{\mu\nu} i\partial_\mu\bar{\lambda}\gamma_\nu\epsilon - \frac{i}{2}\bar{\lambda}\partial_\rho F_{\mu\nu}(\gamma^{\rho\mu\nu} + 2\eta^{\rho\mu}\gamma^\nu)\epsilon \tag{4.80}$$

Now $\gamma^{\rho\mu\nu}\partial_\rho F_{\mu\nu} = \gamma^{\rho\mu\nu}\partial_{[\rho}F_{\mu\nu]} = 0$ so we are left with

$$\begin{aligned} \delta S &= -\int d^4x - F^{\mu\nu} i\bar{\partial}_\mu\lambda\gamma_\nu\epsilon - i\bar{\epsilon}\partial^\mu F_{\mu\nu}\gamma^\nu\lambda \\ &= \int d^4x \partial_\mu(iF^{\mu\nu}\bar{\lambda}\gamma^\nu\epsilon) \\ &= 0 \end{aligned} \tag{4.81}$$

Our next task is to show that these symmetries do indeed close into the supersymmetry algebra. First we compute the closure on the gauge field

$$\begin{aligned} [\delta_1, \delta_2]A_\mu &= i\bar{\epsilon}_2\gamma_\mu(\frac{1}{2}F^{\lambda\rho}\gamma_{\lambda\rho}\epsilon_2) - (1 \leftrightarrow 2) \\ &= i\bar{\epsilon}_2(\frac{1}{2}\gamma_{\mu\lambda\rho} + \eta_{\mu\lambda}\gamma_\rho)F^{\lambda\rho}\epsilon_1 - (1 \leftrightarrow 2) \end{aligned} \tag{4.82}$$

Now consider the spinor contractions in the first term. We note that

$$(C\gamma_{\mu\lambda\rho})^T = -C\gamma_{\rho\lambda\mu}C^{-1}C^T = C\gamma_{\rho\lambda\mu} = -C\gamma_{\mu\lambda\rho} \tag{4.83}$$

Thus $\epsilon_2\gamma_{\mu\lambda\rho}\epsilon_1$ is symmetric under $1 \leftrightarrow 2$ and hence doesn't contribute to the commutator. Hence

$$\begin{aligned} [\delta_1, \delta_2]A_\mu &= -2i\bar{\epsilon}_2\gamma^\nu\epsilon_1 F_{\mu\nu} \\ &= (2i\bar{\epsilon}_2\gamma^\nu\epsilon_1)\partial_\nu A_\mu - \partial_\mu(2i\bar{\epsilon}_2\gamma^\nu\epsilon_1 A_\nu) \end{aligned} \tag{4.84}$$

We recognize the first term as a translation and the second a gauge transformation. Thus the supersymmetry algebra closes correctly on A_μ.

Next we must look at the Fermions. Here we find

$$\begin{aligned} [\delta_1, \delta_2]\lambda &= -2\partial_\mu(\frac{i}{2}\bar{\epsilon}_1\gamma_\nu\lambda)\gamma^{\mu\nu}\epsilon_2 - (1 \leftrightarrow 2) \\ &= -i\gamma^{\mu\nu}(\bar{\epsilon}_1\gamma_\nu\partial_\mu\lambda)\epsilon_2 - (1 \leftrightarrow 2) \end{aligned} \tag{4.85}$$

The problem here is that the spinor index on λ is contracted with $\bar{\epsilon}_1$ on the right hand side and the free spinor index comes from ϵ_2 whereas the left hand side has a free spinor coming from λ. There is a way to rewrite the right hand side using the so-called Fierz identity, valid for any three, anti-commuting, spinors ρ, ψ and χ in four spacetime dimensions,

$$\begin{aligned} (\bar{\rho}\psi)\chi_\alpha &= -\frac{1}{4}(\bar{\rho}\chi)\psi_\alpha - \frac{1}{4}(\bar{\lambda}\gamma_5\chi)\gamma_5\psi_\alpha - \frac{1}{4}(\bar{\rho}\gamma_\mu\chi)(\gamma^\mu\psi)_\alpha \\ &\quad + \frac{1}{4}(\bar{\rho}\gamma_\mu\gamma_5\chi)(\gamma^\mu\gamma_5\psi)_\alpha + \frac{1}{8}(\bar{\rho}\gamma_{\mu\nu}\chi)(\gamma^{\mu\nu}\psi)_\alpha \end{aligned} \tag{4.86}$$

The proof of this identity is given in appendix B and you are strongly encouraged to read it. The point of this identity is that the free spinor index is moved from being on χ on the left hand side to being on ψ on the right hand side.

Returning to the case at hand we can take $\rho = \epsilon_1$, $\chi = \epsilon_2$ and $\psi = \gamma_\nu\partial_\mu\lambda$. This leads to

$$
\begin{aligned}
[\delta_1, \delta_2]\lambda &= \frac{i}{4}\gamma^{\mu\nu}(\bar\epsilon_1\epsilon_2)\gamma_\nu\partial_\mu\lambda + \frac{i}{4}\gamma^{\mu\nu}(\bar\epsilon_1\gamma_5\epsilon_2)\gamma_5\gamma_\nu\partial_\mu\lambda + \frac{i}{4}\gamma^{\mu\nu}\left(\bar\epsilon_1\gamma_\rho\epsilon_2\right)\gamma^\rho\gamma_\nu\partial_\mu\lambda \\
&\quad -\frac{i}{4}\gamma^{\mu\nu}\left(\bar\epsilon_1\gamma_\rho\gamma_5\epsilon_2\right)\gamma^\rho\gamma_5\gamma_\nu\partial_\mu\lambda - \frac{i}{8}\gamma^{\mu\nu}\left(\bar\epsilon_1\gamma_{\rho\sigma}\epsilon_2\right)\gamma^{\rho\sigma}\gamma_\nu\partial_\mu\lambda - (1 \leftrightarrow 2)
\end{aligned}
\tag{4.87}
$$

Problem: Show that

$$
\begin{aligned}
\bar\epsilon_1\epsilon_2 - \bar\epsilon_2\epsilon_1 &= 0 \\
\bar\epsilon_1\gamma_5\epsilon_2 - \bar\epsilon_2\gamma_5\epsilon_1 &= 0 \\
\bar\epsilon_1\gamma_\rho\gamma_5\epsilon_2 - \bar\epsilon_2\gamma_\rho\gamma_5\epsilon_1 &= 0 \\
\bar\epsilon_1\gamma_{\rho\sigma}\epsilon_2 + \bar\epsilon_2\gamma_{\rho\sigma}\epsilon_1 &= 0
\end{aligned}
\tag{4.88}
$$

Given this we have

$$
[\delta_1, \delta_2]\lambda = \frac{i}{2}\left(\bar\epsilon_1\gamma_\rho\epsilon_2\right)\gamma^{\mu\nu}\gamma^\rho\gamma_\nu\partial_\mu\lambda - \frac{i}{4}\left(\bar\epsilon_1\gamma_{\rho\sigma}\epsilon_2\right)\gamma^{\mu\nu}\gamma^{\rho\sigma}\gamma_\nu\partial_\mu\lambda \tag{4.89}
$$

Now look at the first term

$$
\begin{aligned}
\gamma^{\mu\nu}\gamma^\rho\gamma_\nu &= -\gamma^{\mu\nu}\gamma_\nu\gamma^\rho + 2\gamma^{\mu\rho} \\
&= -3\gamma^\mu\gamma^\rho + 2\gamma^{\mu\rho} \\
&= -3\eta^{\mu\rho} - \gamma^{\mu\rho} \\
&= -4\eta^{\mu\rho} + \gamma^\rho\gamma^\mu
\end{aligned}
\tag{4.90}
$$

And the second

$$
\begin{aligned}
\gamma^{\mu\nu}\gamma^{\rho\sigma}\gamma_\nu &= [\gamma^{\mu\nu}, \gamma^{\rho\sigma}]\gamma_\nu + \gamma^{\rho\sigma}\gamma^{\mu\nu}\gamma_\nu \\
&= 2(\eta^{\nu\rho}\gamma^{\mu\sigma} - \eta^{\mu\rho}\gamma^{\nu\sigma} + \eta^{\mu\sigma}\gamma^{\nu\rho} - \eta^{\nu\sigma}\gamma^{\mu\rho})\gamma_\nu + 3\gamma^{\rho\sigma}\gamma^\mu \\
&= 2\gamma^{\mu\sigma}\gamma^\rho + 6\eta^{\mu\rho}\gamma^\sigma - 6\eta^{\mu\sigma}\gamma^\rho - 2\gamma^{\mu\rho}\gamma^\sigma + 3\gamma^{\rho\sigma}\gamma^\mu \qquad (4.91) \\
&= 2\gamma^{\mu\sigma\rho} + 2\gamma^\mu\eta^{\rho\sigma} + 4\eta^{\mu\rho}\gamma^\sigma - 2\gamma^{\mu\rho\sigma} - 2\gamma^\mu\eta^{\rho\sigma} - 4\eta^{\mu\sigma}\gamma^\rho + 3\gamma^{\rho\sigma}\gamma^\mu \\
&= 4\gamma^{\sigma\rho\mu} + 4\eta^{\mu\rho}\gamma^\sigma - 4\eta^{\mu\sigma}\gamma^\rho + 3\gamma^{\rho\sigma}\gamma^\mu \\
&= 4\gamma^{\sigma\rho}\gamma^\mu + 3\gamma^{\rho\sigma}\gamma^\mu \\
&= -\gamma^{\rho\sigma}\gamma^\mu
\end{aligned}
\tag{4.92}
$$

In the second line we used a result from the problems that showed $-\frac{i}{2}\gamma_{\mu\nu}$ satisfy the Lorentz algebra.

Putting this altogether we find

$$[\delta_1, \delta_2]\lambda = -2i\,(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\,\partial_\mu\lambda + \frac{i}{2}\,(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\,\gamma_\nu\gamma^\mu\partial_\mu\lambda \\ +\frac{i}{4}\,(\bar{\epsilon}_1\gamma_{\rho\sigma}\epsilon_2)\,\gamma^{\rho\sigma}\gamma^\mu\partial_\mu\lambda \tag{4.93}$$

We recognize the first term as a translation (the same one since $\bar{\epsilon}_1\gamma^\mu\epsilon_2 = -\bar{\epsilon}_2\gamma^\mu\epsilon_1$). Since λ has a trivial gauge transformation we do not expect anything else. But there clearly is stuff. However this extra stuff vanishes if the Fermion is on-shell: $\gamma^\mu\partial_\mu\lambda = 0$. Thus we say that the supersymmetry algebra closes on-shell.

This is good enough for us since this course is classical (indeed it is often good enough in the quantum theory too). In fact we can see that it couldn't have closed off-shell since the degrees of freedom don't match. In particular A_μ has four degrees of freedom but one is removed by a gauge transformation whereas λ_α also has four degrees of freedom but none can be removed by a gauge transformation. On-shell however A_μ has two degrees of freedom and λ also has two.

4.2 Super-Yang-Mills

Our next task is to find an interacting supersymmetric theory. To this end we try to generalize the previous action to an arbitrary Lie group G. In particular we take have a gauge field A_μ and Fermion λ, both of which are in the adjoint representation

$$S_{susyYM} = -\frac{1}{g^2_{YM}}\int d^4x\;\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}, F^{\mu\nu}) + \frac{i}{2}\mathrm{Tr}(\bar{\lambda}, \gamma^\mu D_\mu\lambda) \tag{4.94}$$

with $D_\mu\lambda = \partial_\mu\lambda - i[A_\mu, \lambda]$. The natural guess for the supersymmetry transformation is

$$\delta A_\mu = i\bar{\epsilon}\gamma_\mu\lambda \\ \delta\lambda = -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon \tag{4.95}$$

Note that although this looks the same as in the Abelian case above it is in fact rather complicated and interacting. Nevertheless the steps to prove invariance are very similar but more involved.

The first thing to note is that there is a term in δS coming from $\bar{\lambda}\gamma^\mu[\delta A_\mu, \lambda]$ that is cubic in λ. This is the only term that is cubic in λ and hence must vanish:

$$\mathrm{Tr}(\bar{\lambda}, \gamma^\mu[(\bar{\epsilon}\gamma_\mu\lambda), \lambda]) = 0 \tag{4.96}$$

Problem: Using the Fierz identity show that, in four dimensions,

$$\mathrm{Tr}(\bar{\lambda}, \gamma^\mu[(\bar{\epsilon}\gamma_\mu\lambda), \lambda]) = f^{abc}\bar{\lambda}_c\gamma^\mu(\bar{\epsilon}\gamma_\mu\lambda_a)\lambda_b = 0. \tag{4.97}$$

This is a crucial condition, without it we would be sunk. In fact it is only true in a few dimensions ($D = 3, 4, 6, 10$) and hence what is called pure super-Yang Mills (*i.e.* super Yang-Mills with the minimum number of fields) only exists in these dimensions. Super-Yang-Mills theories exist in lower dimensions but they always contain additional fields such as scalars (you can construct them by compactification one of the pure theories). Ultimately the reason for this is that these are the only dimensions where one can match up the number of Bose and Fermi degrees of freedom on-shell.

Okay now we can proceed. We first note that

$$\mathrm{Tr}(\bar{\delta}\lambda, \gamma^\mu D_\mu \lambda) = \partial_\mu \mathrm{Tr}(\bar{\delta}\lambda, \gamma_\mu \lambda) + \mathrm{Tr}(\bar{\lambda}, \gamma^\mu D_\mu \delta\lambda) \tag{4.98}$$

We have see that this is true when $D_\mu = \partial_\mu$ so we now need to check the A_μ term. The left hand side gives

$$\begin{aligned} -i\mathrm{Tr}(\bar{\delta}\lambda, \gamma^\mu [A_\mu, \lambda]) &= i\mathrm{Tr}([A_\mu, \bar{\lambda}], \gamma^\mu \delta\lambda) \\ &= -i\mathrm{Tr}([\bar{\lambda}, A_\mu], \gamma^\mu \delta\lambda) \\ &= -i\mathrm{Tr}(\bar{\lambda}, \gamma^\mu [A_\mu, \delta\lambda]) \end{aligned} \tag{4.99}$$

and this is indeed the right hand side. Note that in the first line we used the fact that $C\gamma^\mu$ is symmetric to interchange the two spinors with a minus sign and in the last line we used the fact that $\mathrm{Tr}([A, B], C) = \mathrm{Tr}(A, [B, C])$

Thus we find that, up to boundary terms,

$$\delta S = -\int d^4x \frac{1}{2}\mathrm{Tr}(F^{\mu\nu}, \delta F_{\mu\nu}) + i\mathrm{Tr}(\bar{\lambda}, \gamma^\rho D_\rho \delta\lambda)$$

Next we need to compute

$$\begin{aligned} \delta F_{\mu\nu} &= \partial_\mu \delta A_\nu - i[A_\mu, \delta A_\nu] - (\mu \leftrightarrow \nu) \\ &= i\bar{\epsilon}\gamma_\nu D_\mu \lambda - (\mu \leftrightarrow \nu) \end{aligned} \tag{4.100}$$

Thus we have

$$\delta S = -\int d^4x i\mathrm{Tr}(F^{\mu\nu}, \bar{\epsilon}\gamma_\nu D_\mu \lambda) - \frac{i}{2}\mathrm{Tr}(\bar{\lambda}, \gamma^\rho D_\rho \gamma_{\mu\nu} F^{\mu\nu} \epsilon) \tag{4.101}$$

Again we can use the identity

$$\gamma_\rho \gamma_{\mu\nu} = \gamma_{\rho\mu\nu} + \eta_{\rho\mu}\gamma_\nu - \eta_{\rho\nu}\gamma_\mu \tag{4.102}$$

so that we find

$$\begin{aligned} \delta S = & -\int d^4x - i\mathrm{Tr}(F^{\mu\nu}, D_\mu \bar{\lambda}\gamma_\nu \epsilon) - i\mathrm{Tr}(\bar{\lambda}, D_\mu F^{\mu\nu}\gamma_\nu \epsilon) \\ & -\frac{i}{2}\mathrm{Tr}(\bar{\lambda}, \gamma^{\rho\mu\nu} D_\rho F_{\mu\nu} \epsilon) \end{aligned} \tag{4.103}$$

The first line adds up to a total derivative and hence can be dropped. This leaves us the the final line. This indeed vanishes because of the so-called Bianchi identity

$$D_{[\mu}F_{\nu\lambda]} = 0 \tag{4.104}$$

Problem: Prove the Bianchi identity $D_{[\mu}F_{\nu\lambda]} = 0$, where $D_\mu F_{\nu\lambda} = \partial_\mu F_{\nu\lambda} - i[A_\mu, F_{\nu\lambda}]$.

Thus we have established that the action is supersymmetric. It is also important to show that the supersymmetry variations close (on-shell). Let us consider the gauge field first. Little changes from the above Abelian calculation and we find

$$\begin{aligned}
[\delta_1, \delta_2]A_\mu &= -(2i\bar{\epsilon}_2\gamma^\nu\epsilon_1)F_{\mu\nu} \\
&= 2i(\bar{\epsilon}_2\gamma^\nu\epsilon_1)\partial_\nu A_\mu - D_\mu(2i\bar{\epsilon}_2\gamma^\nu\epsilon_1 A_\nu)
\end{aligned} \tag{4.105}$$

Here we have used the fact that $F_{\mu\nu} = D_\mu A_\nu - \partial_\nu A_\mu$.

Problem: Show that the transformations (7.50) close on-shell on the Fermions to Poincare transformations and gauge transformations.

4.3 $N = 4$ Super-Yang-Mills

The theory we have constructed has so-called $N = 1$ Supersymmetry because there is a single supersymmetry generator Q. On the other hand this sometimes called $\mathcal{N} = 4$ supersymmetry since Q is a spinor which has four components Q_α, $\alpha = 1, 2, 3, 4$ in four dimensions. As we have seen the dimension of the spinor representions various from dimension to dimension. In particular above four dimensions there will be more than $\mathcal{N} = 4$ supersymmetries. This suggests that even in four dimensions one can have more than $\mathcal{N} = 4$. In fact the maximum number of supersymmetries is limit to $\mathcal{N} = 16$ which means that there must be four separate supersymmetries Q^I_α, $I = 1, 2, 3, 4$.

There is essentially a unique theory with such maximal supersymmetry and which doesn't include gravity (basically because with more supersymmetries one requires higher spins and hence requires gravity). This theory was predicted to exist by string theory and constructed in the 1970's. In the 1980's it was shown that this theory is a conformal field theory, *i.e.*that it is a finite quantum field theory - the first known example. It has become of central importance in the last 10 or so years as it is the prime example of the AdS/CFT correspondence. Thus it is a very interesting theory to study.

What is the easiest way to construct such a theory? Well we can start with super-Yang-Mills is a higher dimension where the spinors have more components and 16 is the number of components of a Majorana-Weyl spinor in 10 dimensions (recall that a general Dirac spinor in 10 dimensions has 32 complex components, Majorana reduces this to 32 real components and Weyl then reduces this to 16).

In 10 dimensions the super-Yang-Mills action is

$$S_{susyYM} = -\frac{1}{g_{YM}^2}\int d^{10}x \; \frac{1}{4}\mathrm{tr}(F_{mn}, F^{mn}) + \frac{i}{2}\mathrm{Tr}(\bar{\Lambda}, \Gamma^m D_m \Lambda) \tag{4.106}$$

This is just as before except that the indices $m, n, = 0, ..., 9$ and the spinors Λ and Γ_m are those of 10 dimensions, *i.e.* 32-component. As mentioned we can chose a Majorana basis and also, simultaneously, restrict to Weyl spinors $\Gamma_{11}\Lambda = \Lambda$, where $\Gamma_{11} = \Gamma_0\Gamma_1...\Gamma_9$ is the 10-dimensional analogue of γ_5.

To prove that this is supersymmetric we can follow the same argument that we did for the four-dimensional case. The variations are taken to be

$$\begin{aligned} \delta A_m &= i\bar{\varepsilon}\Gamma_m\Lambda \\ \delta\Lambda &= -\frac{1}{2}F_{mn}\Gamma^{mn}\varepsilon \; . \end{aligned} \tag{4.107}$$

Note that the preserve $\Gamma_{11}\Lambda = \Lambda$ we must also impose

$$\Gamma_{11}\varepsilon = \varepsilon$$

The only time that the dimension of spacetime showed up was in the cubic variation. As mentioned above, in 10-dimensions, we also have that

$$\mathrm{Tr}(\bar{\Lambda}, \Gamma^m[(\bar{\varepsilon}\Gamma_m\Lambda), \Lambda]) = f^{abc}\bar{\Lambda}_c\Gamma^m(\bar{\varepsilon}\Gamma_m\Lambda_a)\Lambda_b = 0. \tag{4.108}$$

provided that $\Gamma_{11}\Lambda = \Lambda$ and $\Gamma_{11}\varepsilon = \varepsilon$.

Problem: Show this. You may assume the Fierz transformation in 10 dimensions is (why?)

$$\begin{aligned} (\bar{\chi}\psi)\lambda \;=\; & -\frac{1}{32}\Big[(\bar{\chi}\lambda)\psi + (\bar{\chi}\Gamma_{11}\lambda)\Gamma^{11}\psi + (\bar{\chi}\Gamma_m\lambda)\Gamma^m\psi - \frac{1}{2!}(\bar{\chi}\Gamma_{mn}\lambda)\Gamma^{mn}\psi \\ & -(\bar{\chi}\Gamma_m\Gamma_{11}\lambda)\Gamma^m\Gamma_{11}\psi - \frac{1}{3!}(\bar{\chi}\Gamma_{mnp}\lambda)\Gamma^{mnp}\psi - \frac{1}{2!}(\bar{\chi}\Gamma_{mn}\Gamma_{11}\lambda)\Gamma^{mn}\Gamma_{11}\psi \\ & +\frac{1}{4!}(\bar{\chi}\Gamma_{mnpq}\lambda)\Gamma^{mnpq}\psi + \frac{1}{4!}(\bar{\chi}\Gamma_{mnp}\Gamma_{11}\lambda)\Gamma^{mnp}\Gamma_{11}\psi + \frac{1}{5!}(\bar{\chi}\Gamma_{mnpqr}\lambda)\Gamma^{mnpqr}\psi \\ & +\frac{1}{4!}(\bar{\chi}\Gamma_{mnpq}\Gamma_{11}\lambda)\Gamma^{mnpq}\Gamma_{11}\psi\Big] \end{aligned} \tag{4.109}$$

Thus the 10-dimensional action is supersymmetric. In fact we should also check that the supersymmetry closes on-shell. The calculation for the gauge fields is just as it was in 4-dimensions. For the Fermions we again need to use the Fierz transformation. This introduces several more terms but nevertheless it all works out (this is to be expected as the Lagrangian is invariant, hence what ever the supersymmetry algebra closes into must be a symmetry of the Lagrangian too).

Problem: Show that in ten-dimensions, with $\Gamma_{11}\Lambda = \Lambda$, the transformations close on-shell on Λ.

Our next task is to dimensionally reduce this theory to 4 dimensions. All this means is that we simply imagine that there is no motion along the $x^4, x^5, ..., x^9$ directions. This is related to the idea of compactification except that we don't imagine there is an infinite tower of Kaluza-Klein states. We are just using this as a trick to obtain a theory in 4 dimensions with $N = 4$ supersymmetry.

Let us consider the Bosons. We have the 10-dimensional adjoint-valued gauge vector field A_m. From the 4-dimensional point of view this can be viewed as a vector gauge field A_μ, $\mu = 0, 1, 2, 3$ along with 6 scalar adjoint-valued fields $\phi_A = A_A$, $A = 4, 5, .., 9$. We note that if we assume that there are no derivatives then under a gauge transformation (that only depends on x^μ) we have

$$A'_\mu = -i\partial_\mu g g^{-1} + gA_\mu g \qquad \phi'_A = g\phi g^{-1} \tag{4.110}$$

Thus indeed the components $\phi_A = A_A$ behave as scalar fields from the 4-dimensional point of view. In addition the field strength reduces to

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \\ F_{\mu A} &= \partial_\mu \phi_A - i[A_\mu, \phi_A] = D_\mu \phi_A \\ F_{AB} &= -i[\phi_A, \phi_B] \end{aligned} \tag{4.111}$$

The 10-dimensional kinetic term can be written as

$$\frac{1}{4}\mathrm{Tr}(F_{mn}, F^{mn}) = \frac{1}{4}\mathrm{Tr}(F_{\mu\nu}, F^{\mu\nu}) + \frac{1}{2}\sum_A \mathrm{Tr}(D_\mu\phi_A, D^\mu\phi_A) - \frac{1}{4}\sum_{A,B}\mathrm{Tr}([\phi_A, \phi_B], [\phi_A, \phi_B]) \tag{4.112}$$

Thus the Bosonic part of the action reduces that of a gauge field and six adjoint-valued scalars in 4 dimensions, along with the potential

$$V = -\frac{1}{4}\sum_{A,B}\mathrm{Tr}([\phi_A, \phi_B], [\phi_A, \phi_B]) \ . \tag{4.113}$$

Next we need to look at the Fermions. We can write the Fermionic term as

$$\frac{i}{2}\mathrm{Tr}(\bar\Lambda, \Gamma^m D_m\Lambda) = \frac{i}{2}\mathrm{Tr}(\bar\Lambda, \Gamma^\mu D_\mu\Lambda) + \frac{1}{2}\mathrm{Tr}(\bar\Lambda, \Gamma^A[\phi_A, \Lambda]) \tag{4.114}$$

The second term is a Yukawa-type term in 4 dimensions. The 4-dimensional action of $N = 4$ super-Yang-Mills is

$$\begin{aligned} S_{N=4SYM} = & -\frac{1}{g^2}\int d^4x \frac{1}{4}\mathrm{Tr}(F_{\mu\nu}, F^{\mu\nu}) + \frac{1}{2}\sum_A \mathrm{Tr}(D_\mu\phi_A, D^\mu\phi_A) + \frac{i}{2}\mathrm{Tr}(\bar\Lambda, \Gamma^\mu D_\mu\Lambda) \\ & + \frac{1}{2}\mathrm{Tr}(\bar\Lambda, \Gamma^A[\phi_A, \Lambda]) - \frac{1}{4}\sum_{A,B}\mathrm{Tr}([\phi_A, \phi_B], [\phi_A, \phi_B]) \end{aligned} \tag{4.115}$$

In principle we are done. However it is a good idea to rewrite the Fermion Λ in terms of 4-dimensional spinors.

To reduce the Fermions we decompose the 10-dimensional Clifford algebra in terms of the 4-dimensional γ_μ's as

$$\begin{aligned}\Gamma_\mu &= \gamma_\mu \otimes 1 \\ \Gamma_A &= \gamma_5 \otimes \rho_A\end{aligned} \tag{4.116}$$

Here ρ_A are a Euclidean Clifford algebra in 6-dimensions which we take to be pure imaginary so that Γ_m are Majorana (recall that γ_5 is pure imaginary in a four-dimensional Majorana basis). This is indeed possible and is called a pseudo-Majorana representation. For example we could take

$$\begin{aligned}\rho_1 &= 1 \otimes 1 \otimes \sigma_2 \qquad & \rho_2 = 1 \otimes \sigma_2 \otimes \sigma_3 \\ \rho_3 &= \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \qquad & \rho_4 = \sigma_2 \otimes \sigma_1 \otimes \sigma_3 \\ \rho_5 &= \sigma_3 \otimes \sigma_2 \otimes \sigma_1 \qquad & \rho_6 = \sigma_2 \otimes 1 \otimes \sigma_1 .\end{aligned} \tag{4.117}$$

Since σ_2 is pure imaginary and σ_1 and σ_3 are real we have a pseudo-Majorana representation of 8×8 matrices.

Similarly we decompose spinors as

$$\begin{aligned}\Lambda &= \lambda_I \otimes \eta^I \\ \varepsilon &= \epsilon_I \otimes \eta^I\end{aligned} \tag{4.118}$$

where η^I are a basis of spinors in six-dimensions (which are 8-dimensional). However we note that we require Λ and ε to be chiral with respect to $\Gamma_{11} = -i\gamma_5 \otimes \rho^1 ... \rho^6$. Thus the chirality of η^I with respect to $i\rho^1...\rho^6$ needs to be correlated with the chirality of λ_I with respect to γ_5. This projects out half of the six-dimensional spinors and so there are only four independent values of I. To see this we note that since the η_I are a basis we can write

$$-i\rho_1 ... \rho_6 \eta_I = R_I{}^J \eta_J$$

for some pure imaginary 8×8 matrix $R_I{}^J$. Thus the chirality constraint becomes

$$\lambda^I = \gamma_5 R_J{}^I \lambda^J$$

A similar constraint applies to ϵ^I too. This means that there are no longer 8 independent λ^I and ϵ^I but rather just 4. Finally we assume that these are normalized to $(\eta^I)^T \eta^J = \delta^{IJ}$.

We can now compute

$$\begin{aligned}\bar{\Lambda}\Gamma^\mu D_\mu \Lambda &= \lambda_I^T \otimes (\eta^I)^T (C\gamma^\mu \otimes 1) D_\mu \lambda_J \otimes \eta_J \\ &= \delta^{IJ} \bar{\lambda}_I \gamma^\mu D_\mu \lambda_J\end{aligned} \tag{4.119}$$

and

$$\begin{aligned}
\bar{\Lambda}\Gamma_A[\phi^A,\Lambda] &= \lambda_I^T \otimes (\eta^I)^T (C\gamma^5 \otimes \rho^A)[\phi^A, \lambda_J \otimes \eta^J] \\
&= (\eta^I)^T \rho_A \eta^J \bar{\lambda}_I \gamma_5 [\phi^A, \lambda_J] \\
&= \rho_A^{IJ} \bar{\lambda}_I \gamma_5 [\phi^A, \lambda_J]
\end{aligned} \tag{4.120}$$

where $\rho_A^{IJ} = (\eta^I)^T \rho_A \eta^J$ are the chiral-chiral matrix components of ρ_A. Thus the action can be written as

$$\begin{aligned}
S_{N=4SYM} &= -\frac{1}{g^2}\int d^4x \frac{1}{4}\mathrm{Tr}(F_{\mu\nu}, F^{\mu\nu}) + \frac{1}{2}\sum_A \mathrm{Tr}(D_\mu\phi_A, D^\mu\phi_A) + \frac{i}{2}\delta^{IJ}\mathrm{Tr}(\bar{\lambda}_I, \gamma^\mu D_\mu \lambda_J) \\
&\quad + \frac{1}{2}\rho_A^{IJ}\mathrm{Tr}(\bar{\lambda}_I\gamma_5[\phi^A, \lambda_J]) - \frac{1}{4}\sum_{A,B}\mathrm{Tr}([\phi_A, \phi_B], [\phi_A, \phi_B])
\end{aligned} \tag{4.121}$$

Note that we have raised and lowed IJ indices freely with δ^{IJ} and δ_{IJ}.

Our last task is to write the supersymmetry transformations in terms of 4-dimensional spinors.

Problem: Show that the ten-dimensional supersymmetry

$$\begin{aligned}
\delta A_m &= i\bar{\varepsilon}\Gamma_m\Lambda \\
\delta\Lambda &= -\frac{1}{2}F_{mn}\Gamma^{mn}\varepsilon \ .
\end{aligned} \tag{4.122}$$

becomes

$$\begin{aligned}
\delta A_\mu &= i\bar{\epsilon}_I\Gamma_\mu\lambda^I \\
\delta\phi_A &= -\bar{\epsilon}_I\gamma_5\lambda_J\rho_A^{IJ} \\
\delta\lambda^I &= -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon^I - \gamma^\mu\gamma_5 D_\mu\phi^A\rho_A^{IJ}\epsilon_J + \frac{i}{2}[\phi_A, \phi_B]\rho_{AB}^{JI}\epsilon_I \ .
\end{aligned} \tag{4.123}$$

where $\rho_{AB}^{JI} = (\eta^J)^T\rho_{AB}\eta^I$. Here we see that there are indeed 4 supersymmetry parameters ϵ_I.

Thus we find a theory in 4 dimensions with one vector field (spins $= \pm 1$) 4 Fermions (spin $= \pm 1/2$) and 6 scalars (spin 0). In fact this is what we expect if we generalize our previous discussion to $N = 4$ supersymmetry. In the fixed (massless) momentum frame there are $4 \times 4 = 16$ supersymmetry generators:

$$\{Q_\alpha^I, Q_\beta^J\} = 2E(1-\gamma_{01})_{\alpha\beta}\delta^{IJ}$$

However since γ_{01} can be diagonalized to the form $\mathrm{diag}(1,1,-1,-1)$ we see that we can find 8 linear combinations of the Q_α^I, which we denote by $Q_{\ddot{\alpha}}$ that satisfy $\{Q_{\ddot{\alpha}}, Q_{\ddot{\beta}}\} = 0$. These must act trivially $Q_{\ddot{\alpha}}|state\rangle = 0$. Thus there are 8 nontrivial $Q's$ that we denote by $Q_{\dot{\alpha}}$, $\dot{\alpha} = 1, ..., 8$ and satisfy

$$\{Q_{\dot{\alpha}}, Q_{\dot{\beta}}\} = 4E\delta_{\dot{\alpha}\dot{\beta}}$$

If we rewrite these as

$$\mathcal{Q}_1 = Q_1 + iQ_2 \qquad \mathcal{Q}_2 = Q_3 + iQ_4 \qquad \mathcal{Q}_3 = Q_5 + iQ_6 \qquad \mathcal{Q}_4 = Q_7 + iQ_8$$

then the algebra becomes

$$\{\mathcal{Q}_I, \mathcal{Q}_J\} = \{\mathcal{Q}_I^\dagger, \mathcal{Q}_J^\dagger\} = 0 \qquad \{\mathcal{Q}_I, \mathcal{Q}_J^\dagger\} = 4E\delta_{IJ}$$

This is four copies of the algebra of Fermionic harmonic oscillator with creation and annihilation operators. To construct the represention we start with a highest spin state $|s>$. The $\mathcal{Q}_I$ lower the spin by $1/2$ whereas the $\mathcal{Q}_I^\dagger$ raise it. Thus $\mathcal{Q}_I^\dagger|s>= 0$. Thus the states and their helicites are obtained by acting on $|s>$ with $\mathcal{Q}_I$:

$$s \qquad |s> \tag{4.124}$$

$$s-1/2 \qquad \mathcal{Q}_I|s> \tag{4.125}$$

$$s-1 \qquad \mathcal{Q}_I\mathcal{Q}_J|s> \tag{4.126}$$

$$s-3/2 \qquad \mathcal{Q}_I\mathcal{Q}_J\mathcal{Q}_K|s> \tag{4.127}$$

$$s-2 \qquad \mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3\mathcal{Q}_4|s> \tag{4.128}$$

Note that since the $\mathcal{Q}_I$ anticommute these states must be antisymmetric in I, J, K. Thus there are $1, 4, 6, 4, 1$ states in each row respectively leading to $2^4 = 16$ states. Note that we require $|s| \leq 1$ to remain in field theory without gravity. Therefore we see that we must have precisely $s = 1$ in order that the lowest state has $s \geq -1$. We then find a vector (with states $|1>$ and $|-1>$), 6 scalars (states $|0>$) and 4 Fermions (states $|1/2>$).

5 Appendix A: Conventions

In these notes we are generally in 4 spacetime dimensions labeled by x^μ, $\mu = 0, 1, 2, ..., 3$. When we only want to talk about the spatial components we use x^i with $i = 1, ..., 3$. We use the the "mostly plus" convention for the metric:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{5.1}$$

Spinor indices will in general be denoted by $\alpha, \beta = 1, ..., 4$. When we talk about Weyl spinors we will use the spinor indices $a, \dot{a} = 1, 2$. We will briefly talk about more general D dimensions. In this case $\mu = 0, 1, 2, ..., D-1$ and $\alpha, \beta = 1, ..., 2^{[D/2]}$.

We also assume, according to the spin-statistics theorem, that spinorial quantities and fields are Grassmann variables, *i.e.* anti-commuting. We will typically use Greek symbols for Fermionic Grassmann fields, $\psi, \lambda, ...$ and Roman symbols for Bosonic c-number fields.

6 Appendix B: The Fierz Transformation

The γ-matrices have several nice properties. Out of them one can construct the additional matrices

$$1, \gamma_\mu, \gamma_\mu\gamma_{D+1}, \gamma_{\mu\nu}, \gamma_{\mu\nu}\gamma_{D+1}, ... \tag{6.1}$$

where $\gamma_{\mu\nu\lambda...}$ is the anti-symmetric product over the given indices with weight one, *e.g.*

$$\gamma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \tag{6.2}$$

Because of the relation $\gamma_0\gamma_1...\gamma_{D-1} \propto \gamma_{D+1}$ not all of these matrices are independent. The list stops when the number of indices is bigger than $D/2$. It is easy to convince yourself that the remaining ones are linearly independent.

Problem: Using the fact that $< M_1, M_2 >= \mathrm{Tr}(M_1^\dagger M_2)$ defines a complex inner product, convince yourself that the set (6.1), where the number of spacetime indices is no bigger than $D/2$, is a basis for the space of $2^{[D]/2} \times 2^{[D]/2}$ matrices ($[D/2]$ is the integer part of $D/2$).

Thus any matrix can be written in terms of γ-matrices. In particular one can express

$$\delta_\alpha^\beta \delta_\gamma^\delta = \sum_{\Gamma\Gamma'} c_{\Gamma\Gamma'} (\gamma_\Gamma)_\gamma^{\ \beta} (\gamma_{\Gamma'})_\alpha^{\ \delta} \tag{6.3}$$

for some constants $c_{\Gamma\Gamma'}$. Here Γ and Γ' are used as a indices that range over all independent γ-matrix products in (6.1).

To proceed one must determine the coefficients $c_{\Gamma\Gamma'}$. To do this we simply multiply (6.3) by $(\gamma_{\Gamma''})_\beta^{\ \gamma}$ which gives

$$(\gamma_{\Gamma''})_\alpha^{\ \delta} = \sum_{\Gamma\Gamma'} c_{\Gamma\Gamma'} \mathrm{Tr}(\gamma_\Gamma\gamma_{\Gamma''})(\gamma_{\Gamma'})_\alpha^{\ \delta} \tag{6.4}$$

Now we have observed that $\mathrm{Tr}(\gamma_\Gamma\gamma_{\Gamma''}) = 0$ unless $\Gamma = \Gamma''$ so we find

$$(\gamma_{\Gamma''})_\alpha^{\ \delta} = \sum_{\Gamma''\Gamma'} c_{\Gamma''\Gamma'} \mathrm{Tr}(\gamma_{\Gamma''}^2)(\gamma_{\Gamma'})_\alpha^{\ \delta} \tag{6.5}$$

From here we see that $c_{\Gamma''\Gamma'} = 0$ unless $\Gamma' = \Gamma''$ and hence

$$c_{\Gamma\Gamma} = \frac{1}{\mathrm{Tr}(\gamma_\Gamma^2)} = \pm_\Gamma \frac{1}{2^{[D/2]}} \tag{6.6}$$

Here the $\pm_\Gamma$ arises because $\gamma_\Gamma^2 = \pm 1$ and $2^{[D/2]} = \mathrm{Tr}(1)$ is the dimension of the representation of the Clifford algebra.

The point of doing all this is that the index contractions have been swapped and hence one can write

$$
\begin{aligned}
(\bar{\lambda}\psi)\chi_\alpha &= \bar{\lambda}^\gamma \psi_\delta \chi_\beta \delta_\alpha^\beta \delta_\gamma^\delta \\
&= -\sum_\Gamma c_{\Gamma\Gamma} \bar{\lambda}^\gamma (\gamma_\Gamma)_\gamma^{\ \beta} \chi_\beta (\gamma_\Gamma)_\alpha^{\ \delta} \psi_\delta \\
&= -\frac{1}{2^{[D/2]}} \sum_\Gamma \pm_\Gamma \left(\bar{\lambda}\gamma_\Gamma \chi\right) (\gamma_\Gamma \psi)_\alpha
\end{aligned}
\tag{6.7}
$$

here the minus sign out in front arises because we must interchange the order of ψ and χ which are anti-commuting. This is called a Fierz rearrangement and it has allowed us to move the free spinor index from χ to ψ. Its draw back is that it becomes increasingly complicated as the spacetime dimension D increases, but generally speaking there isn't an alternative so you just have to slog it out.

In particular consider four dimensions. The independent matrices are

$$
1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \gamma_{\mu\nu} \tag{6.8}
$$

One can see that this is the case by noting that $\gamma_{\mu\nu5} = \frac{i}{2}\varepsilon_{\mu\nu\lambda\rho}\gamma^{\lambda\rho}$. You can check that the Fierz identity is

$$
\begin{aligned}
(\bar{\lambda}\psi)\chi_\alpha = & -\frac{1}{4}(\bar{\lambda}\chi)\psi_\alpha - \frac{1}{4}(\bar{\lambda}\gamma_5\chi)\gamma_5\psi_\alpha - \frac{1}{4}\left(\bar{\lambda}\gamma_\mu\chi\right)(\gamma^\mu\psi)_\alpha \\
& +\frac{1}{4}\left(\bar{\lambda}\gamma_\mu\gamma_5\chi\right)(\gamma^\mu\gamma_5\psi)_\alpha + \frac{1}{8}\left(\bar{\lambda}\gamma_{\mu\nu}\chi\right)(\gamma^{\mu\nu}\psi)_\alpha
\end{aligned}
\tag{6.9}
$$

note the extra factor of $\frac{1}{2}$ in the last term that is there is ensure that $\gamma_{\mu\nu}$ and $\gamma_{\nu\mu}$ don't contribute twice. We will use this at various points in the course.

Problem: Show that in three dimensions the Fierz rearrangement is

$$
(\bar{\lambda}\psi)\chi_\alpha = -\frac{1}{2}(\bar{\lambda}\chi)\psi_\alpha - \frac{1}{2}\left(\bar{\lambda}\gamma_\mu\chi\right)(\gamma^\mu\psi)_\alpha \tag{6.10}
$$

Using this, show that in the special case that $\lambda = \chi$ one simply has

$$
(\bar{\lambda}\psi)\lambda_\alpha = -\frac{1}{2}(\bar{\lambda}\lambda)\psi_\alpha \tag{6.11}
$$

for Majorana spinors. Convince yourself that this is true by considering the explicit 3D γ-matrices above and letting

$$
\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{6.12}
$$

What is the Fierz rearrangement in two dimensions (Hint: this last part should take you very little time)?

7 Appendix C: Solutions to Problems

Problem: Using

$$\delta x^\mu = ia^\nu P_\nu(x^\mu) + \frac{i}{2}\omega^{\nu\lambda}M_{\nu\lambda}(x^\mu) \tag{7.1}$$

and

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [P_\mu, M_{\nu\lambda}] &= -i\eta_{\mu\nu}P_\lambda + i\eta_{\mu\lambda}P_\nu \\ [M_{\mu\nu}, M_{\lambda\rho}] &= -i\eta_{\nu\lambda}M_{\mu\rho} + i\eta_{\mu\lambda}M_{\nu\rho} - i\eta_{\mu\rho}M_{\nu\lambda} + i\eta_{\nu\rho}M_{\mu\lambda} \end{aligned} \tag{7.2}$$

show that

$$(\delta_1\delta_2 - \delta_2\delta_1)x^\mu = (\omega_1{}^\mu{}_\lambda a_2^\lambda - \omega_2{}^\mu{}_\lambda a_1^\lambda) + (\omega_1{}^\mu{}_\lambda \omega_2{}^\lambda{}_\nu - \omega_2{}^\mu{}_\lambda \omega_1{}^\lambda{}_\nu)x^\nu \tag{7.3}$$

is indeed reproduced.

Solution: Just calculate

$$\begin{aligned} [\delta_1, \delta_2] &= -a_1^\mu a_2^\nu [P_\mu, P_\nu] - \frac{1}{2}\omega_1^{\mu\nu} a_2^\lambda [M_{\mu\nu}, P_\lambda] - \frac{1}{2} a_1^\lambda \omega_2^{\mu\nu}[P_\lambda, M_{\mu\nu}] - \frac{1}{4}\omega_1^{\mu\nu}\omega_2^{\lambda\rho}[M_{\mu\nu}, M_{\lambda\rho}] \\ &= -\frac{i}{2}\omega_1^{\mu\nu}a_{2\mu}P_\nu + \frac{i}{2}\omega_1^{\mu\nu}a_{2\nu}P_\mu + \frac{i}{2}\omega_2^{\mu\nu}a_{1\mu}P_\nu - \frac{i}{2}\omega_2^{\mu\nu}a_{1\nu}P_\mu \\ &\quad + \frac{i}{4}(\omega_1^{\mu\lambda}\omega_{2\lambda}{}^\nu - \omega_1^{\lambda\mu}\omega_{2\lambda}{}^\nu + \omega_1^{\lambda\mu}\omega_2{}^\nu{}_\lambda - \omega_1^{\mu\lambda}\omega_2{}^\nu{}_\lambda)M_{\mu\nu} \\ &= i(\omega_1^{\mu\nu}a_{2\nu} - \omega_2^{\mu\nu}a_{1\nu})P_\mu + \frac{i}{2}(\omega_1^{\mu\lambda}\omega_{2\lambda}{}^\nu - \omega_2^{\mu\lambda}\omega_{1\lambda}{}^\nu)M_{\mu\nu} \end{aligned} \tag{7.4}$$

Comparing this with

$$\delta x^\mu = ia^\nu P_\nu(x^\mu) + \frac{i}{2}\omega^{\nu\lambda}M_{\nu\lambda}(x^\mu) \tag{7.5}$$

tells us that

$$a^\mu = (\omega_2{}^\mu{}_\lambda a_1^\lambda - \omega_1{}^\mu{}_\lambda a_2^\lambda)\ , \qquad \omega^{\mu\nu} = (\omega_1{}^\mu{}_\lambda \omega_2{}^\lambda{}_\nu - \omega_2{}^\mu{}_\lambda \omega_1{}^\lambda{}_\nu) \tag{7.6}$$

as required.

Problem: Verify that the two representations

$$\begin{aligned} (M_{\mu\nu})^\lambda{}_\rho &= i\eta_{\mu\rho}\delta_\nu^\lambda - i\delta_\mu^\lambda \eta_{\nu\rho} \\ (M_{\mu\nu})_\alpha{}^\beta &= \frac{i}{2}(\gamma_{\mu\nu})_\alpha{}^\beta = -\frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\mu\gamma_\nu)_\alpha{}^\beta \end{aligned} \tag{7.7}$$

do indeed satisfy the Lorentz algebra

$$[M_{\mu\nu}, M_{\lambda\rho}] = -i\eta_{\nu\lambda}M_{\mu\rho} + i\eta_{\mu\lambda}M_{\nu\rho} - i\eta_{\mu\rho}M_{\nu\lambda} + i\eta_{\nu\rho}M_{\mu\lambda} \tag{7.8}$$

Solution: In the first case we find

$$\begin{aligned}(M_{\mu\nu})^{\sigma}{}_{\tau}(M_{\lambda\rho})^{\tau}{}_{\theta} &= (i\eta_{\mu\tau}\delta^{\sigma}_{\nu} - i\delta^{\sigma}_{\mu}\eta_{\nu\tau})(i\eta_{\lambda\theta}\delta^{\tau}_{\rho} - i\delta^{\tau}_{\lambda}\eta_{\rho\theta}) \\ &= -\eta_{\mu\rho}\eta_{\lambda\theta}\delta^{\sigma}_{\nu} + \eta_{\mu\lambda}\delta^{\sigma}_{\nu}\eta_{\rho\theta} + \delta^{\sigma}_{\mu}\eta_{\nu\rho}\eta_{\lambda\theta} - \delta^{\sigma}_{\mu}\eta_{\nu\lambda}\eta_{\rho\theta}\end{aligned} \tag{7.9}$$

and

$$\begin{aligned}(M_{\lambda\rho})^{\sigma}{}_{\tau}(M_{\mu\nu})^{\tau}{}_{\theta} &= (i\eta_{\lambda\tau}\delta^{\sigma}_{\rho} - i\delta^{\sigma}_{\lambda}\eta_{\rho\tau})(i\eta_{\mu\theta}\delta^{\tau}_{\nu} - i\delta^{\tau}_{\mu}\eta_{\nu\theta}) \\ &= -\eta_{\lambda\nu}\eta_{\mu\theta}\delta^{\sigma}_{\rho} + \eta_{\lambda\mu}\delta^{\sigma}_{\rho}\eta_{\nu\theta} + \delta^{\sigma}_{\lambda}\eta_{\rho\nu}\eta_{\mu\theta} - \delta^{\sigma}_{\lambda}\eta_{\rho\mu}\eta_{\nu\theta}\end{aligned} \tag{7.10}$$

Combining these gives

$$\begin{aligned}[M_{\mu\nu}, M_{\lambda\rho}]^{\sigma}{}_{\theta} &= -\eta_{\mu\rho}\eta_{\lambda\theta}\delta^{\sigma}_{\nu} + \eta_{\mu\lambda}\delta^{\sigma}_{\nu}\eta_{\rho\theta} + \delta^{\sigma}_{\mu}\eta_{\nu\rho}\eta_{\lambda\theta} - \delta^{\sigma}_{\mu}\eta_{\nu\lambda}\eta_{\rho\theta} \\ &\quad +\eta_{\lambda\nu}\eta_{\mu\theta}\delta^{\sigma}_{\rho} - \eta_{\lambda\mu}\delta^{\sigma}_{\rho}\eta_{\nu\theta} - \delta^{\sigma}_{\lambda}\eta_{\rho\nu}\eta_{\mu\theta} + \delta^{\sigma}_{\lambda}\eta_{\rho\mu}\eta_{\nu\theta} \\ &= -i\eta_{\nu\lambda}(i\eta_{\mu\theta}\delta^{\sigma}_{\rho} - i\delta^{\sigma}_{\mu}\eta_{\rho\theta}) + i\eta_{\nu\rho}(i\delta^{\sigma}_{\lambda}\eta_{\mu\theta} - i\delta^{\sigma}_{\mu}\eta_{\lambda\theta}) \\ &\quad -i\eta_{\mu\rho}(i\delta^{\sigma}_{\lambda}\eta_{\nu\theta} - i\eta_{\lambda\theta}\delta^{\sigma}_{\nu}) + i\eta_{\mu\lambda}(i\delta^{\sigma}_{\rho}\eta_{\nu\theta} - i\delta^{\sigma}_{\nu}\eta_{\rho\theta})\end{aligned} \tag{7.11}$$

which is the correct relation.

In the second case we first note that

$$\begin{aligned}\gamma_{\mu\nu}\gamma_{\lambda\rho} &= \gamma_{\mu\nu\lambda\rho} + \eta_{\nu\lambda}\gamma_{\mu\rho} - \eta_{\mu\lambda}\gamma_{\nu\rho} + \eta_{\mu\rho}\gamma_{\nu\lambda} - \eta_{\nu\rho}\gamma_{\mu\lambda} + \eta_{\nu\lambda}\eta_{\mu\rho} - \eta_{\nu\rho}\eta_{\mu\lambda} \\ \gamma_{\lambda\rho}\gamma_{\mu\nu} &= \gamma_{\lambda\rho\mu\nu} + \eta_{\rho\mu}\gamma_{\lambda\nu} - \eta_{\lambda\mu}\gamma_{\rho\nu} + \eta_{\lambda\nu}\gamma_{\rho\mu} - \eta_{\rho\nu}\gamma_{\lambda\mu} + \eta_{\rho\mu}\eta_{\lambda\nu} - \eta_{\nu\rho}\eta_{\mu\lambda}\end{aligned} \tag{7.12}$$

To prove this you can either work it out using the Clifford relation $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ or simply test the various cases $\mu = \lambda, \nu = \rho$, $\mu = \lambda, \nu \neq \rho$..... From this wee see that

$$[\gamma_{\mu\nu}, \gamma_{\lambda\rho}] = 2\eta_{\nu\lambda}\gamma_{\mu\rho} - 2\eta_{\mu\lambda}\gamma_{\nu\rho} + 2\eta_{\mu\rho}\gamma_{\nu\lambda} - 2\eta_{\nu\rho}\gamma_{\mu\lambda} \tag{7.13}$$

Multiplying through by $(\frac{i}{2})^2$ we find

$$[-\frac{i}{2}\gamma_{\mu\nu}, -\frac{i}{2}\gamma_{\lambda\rho}] = i\eta_{\nu\lambda}\frac{i}{2}\gamma_{\mu\rho} - i\eta_{\mu\lambda}\frac{i}{2}\gamma_{\nu\rho} + i\eta_{\mu\rho}\frac{i}{2}\gamma_{\nu\lambda} - i\eta_{\nu\rho}\frac{i}{2}\gamma_{\mu\lambda} \tag{7.14}$$

which is indeed the correct relation.

Problem: Show that $V_\mu = \bar{\lambda}\gamma_\mu\psi$ is a Lorentz vector, *i.e.* show that $\delta V_\mu = \omega_\mu{}^\nu V_\nu$ under the transformation $\delta\psi = \frac{1}{4}\omega^{\lambda\rho}\gamma_{\lambda\rho}\psi$.

Solution: We have that

$$\begin{aligned}\delta V_\mu &= \delta\bar{\lambda}\gamma_\mu\psi + \bar{\lambda}\gamma_\mu\delta\psi \\ &= -\frac{1}{4}\omega^{\lambda\rho}\bar{\lambda}(\gamma_{\lambda\rho}\gamma_\mu - \gamma_\mu\gamma_{\lambda\rho})\psi\end{aligned} \tag{7.15}$$

So we need to evaluate $[\gamma_{\lambda\rho}, \gamma_\mu]$ to this end we observe that

$$\begin{aligned}\gamma_{\lambda\rho}\gamma_\mu &= \gamma_{\lambda\rho\mu} + \eta_{\rho\mu}\gamma_\lambda - \eta_{\lambda\mu}\gamma_\rho \\ \gamma_\mu\gamma_{\lambda\rho} &= \gamma_{\mu\lambda\rho} + \eta_{\lambda\mu}\gamma_\rho - \eta_{\rho\mu}\gamma_\lambda\end{aligned} \tag{7.16}$$

why (think about the possible cases)? This implies that

$$[\gamma_{\lambda\rho}, \gamma_\mu] = 2\eta_{\rho\mu}\gamma_\lambda - 2\eta_{\lambda\mu}\gamma_\rho \tag{7.17}$$

Putting this back in leads to

$$\begin{aligned}\delta V_\mu &= -\frac{1}{4}\omega^{\lambda\rho}\bar{\lambda}(2\eta_{\rho\mu}\gamma_\lambda - 2\eta_{\lambda\mu}\gamma_\rho)\psi \\ &= \omega_\mu{}^\rho\bar{\lambda}\gamma_\rho\psi\end{aligned} \tag{7.18}$$

Problem: Show that, for a general Dirac spinor in any dimension, $\lambda^T C\psi$ is Lorentz invariant, where C is the charge conjugation matrix.

Solution: Under a Lorentz transformation

$$\delta\psi = \frac{1}{4}\omega^{\mu\nu}\gamma_{\mu\nu}\psi\ , \qquad \delta\lambda = \frac{1}{4}\omega^{\mu\nu}\gamma_{\mu\nu}\lambda\ , \tag{7.19}$$

Therefore

$$\delta\lambda^T = \frac{1}{4}\omega^{\mu\nu}\lambda^T\gamma_\nu^T\gamma_\mu^T = \frac{1}{4}\lambda^T\omega^{\mu\nu}C\gamma_{\nu\mu}C^{-1} = -\frac{1}{4}\lambda^T\omega^{\mu\nu}C\gamma_{\mu\nu}C^{-1} \tag{7.20}$$

and hence

$$\delta\lambda^T C = -\frac{1}{4}\lambda^T\omega^{\mu\nu}C\gamma_{\mu\nu} \tag{7.21}$$

Finally we see that

$$\begin{aligned}\delta(\lambda^T C\psi) &= \delta\lambda^T C\psi + \lambda^T C\delta\psi \\ &= -\frac{1}{4}\lambda^T\omega^{\mu\nu}C\gamma_{\mu\nu}\psi + \frac{1}{4}\lambda^T C\omega^{\mu\nu}\gamma_{\mu\nu}\psi \\ &= 0\end{aligned} \tag{7.22}$$

Problem: Why are their factors of i in the Fermionic terms of the action.

Solution: This is to ensure that the action is real. Consider the mass term

$$(im\bar{\psi}\psi)^* = -im(\psi_\alpha^* C^{\alpha\beta}\psi_\beta)^* = -im\psi_\beta^*(C^{\alpha\beta})^*\psi_\alpha = im\psi_\beta^* C^{\beta\alpha}\psi_\alpha = im\bar{\psi}\psi \tag{7.23}$$

where we have used that C is anti-Hermitian. Next consider the kinetic terms

$$\begin{aligned}
\mathrm{Tr}(i\bar{\psi},\gamma^\mu D_\mu\psi)^* &= -ih_{ab}(\partial_\mu\psi^b_\beta - i\psi^c_\beta A^r_\mu (T_r)^b{}_c)^*(C\gamma^\mu)^{\alpha\beta}\psi^a_\alpha \\
&= -ih_{ab}\partial_\mu\bar{\psi}^b\gamma^\mu\psi^a + h_{ab}\psi^{c*}_\beta A^r_\mu (T_r)_c{}^b (C\gamma^\mu)^{\beta\alpha}\psi^a_\alpha \\
&= -ih_{ab}\partial_\mu(\bar{\psi}^a\gamma^\mu\psi^b) + ih_{ab}\bar{\psi}^a\gamma^\mu\partial_\mu\psi^b + h_{bc}\bar{\psi}^b\gamma^\mu A^r_\mu (T_r)^c{}_a\psi^a \\
&= -ih_{ab}\partial_\mu(\bar{\psi}^a\gamma^\mu\psi^b) + ih_{cd}\bar{\psi}^c\gamma^\mu D_\mu\psi^d \\
&\cong \mathrm{Tr}(i\bar{\psi},\gamma^\mu D_\mu\psi)
\end{aligned}$$

Here we've used the fact the fact that $(T_r)^a{}_b$ is Hermitian and also that $C\gamma^\mu$ real (we are in a Majorana basis) and symmetric:

$$(C\gamma^\mu)^T = (\gamma^\mu)^T C^\dagger = -C\gamma^\mu C^{-1}C^T = C\gamma^\mu \tag{7.24}$$

Problem: Show that in four-dimensions, where Q_α is a Majorna spinor, the supersymmetry algebra

$$\{Q_\alpha, Q_\beta\} = -2(\gamma^\mu C^{-1})_{\alpha\beta}P_\mu \tag{7.25}$$

can be written as

$$\begin{aligned}
\{Q_{W\alpha}, Q_{W\beta}\} &= 0 \\
\{Q^*_{W\alpha}, Q^*_{W\beta}\} &= 0 \\
\{Q_{W\alpha}, Q^*_{W\beta}\} &= -((1+\gamma_5)\gamma_\mu C^{-1})_{\alpha\beta}P_\mu \\
\{Q^*_{W\alpha}, Q_{W\beta}\} &= -((1-\gamma_5)\gamma_\mu C^{-1})_{\alpha\beta}P_\mu
\end{aligned} \tag{7.26}$$

where $Q_{W\alpha}$ is a Weyl spinor and $Q^*_{W\alpha}$ is its complex conjugate. (Hint: Weyl spinors are chiral and are obtained from Majorana spinors Q_M through $Q_W = \frac{1}{2}(1+\gamma_5)Q_M$, $Q^*_W = \frac{1}{2}(1-\gamma_5)Q_M$.)

Solution: By definition we have $Q_\alpha = Q_{W\alpha} + Q^*_{W\alpha}$ and hence

$$\{Q_{W\alpha}, Q_{W\beta}\} + \{Q^*_{W\alpha}, Q_{W\beta}\} + \{Q_{W\alpha}, Q^*_{W\beta}\} + \{Q^*_{W\alpha}, Q^*_{W\beta}\} = -2(\gamma^\mu C^{-1})_{\alpha\beta}P_\mu \tag{7.27}$$

Next we note that

$$(1\mp\gamma_5)(1\pm\gamma_5) = 1-\gamma_5^2 = 0 \tag{7.28}$$

and

$$\frac{1}{2}(1\pm\gamma_5)\frac{1}{2}(1\pm\gamma_5) = \frac{1}{4}(1\pm 2\gamma_5+\gamma_5^2) = \frac{1}{2}(1+\gamma_5) \tag{7.29}$$

Thus if we multiply the left and right hand side by $\frac{1}{2}(1+\gamma_5)_\gamma{}^\alpha$ and $\frac{1}{2}(1+\gamma_5)_\delta{}^\beta$ we find

$$\begin{aligned}
\{Q_{W\gamma}, Q_{W\delta}\} &= -\frac{1}{2}((1+\gamma_5)\gamma^\mu C^{-1})_{\gamma\beta}(1+\gamma_5)_\delta{}^\beta P_\mu \\
&= -\frac{1}{2}((1+\gamma_5)\gamma^\mu C^{-1})_{\gamma\beta}(1-\gamma_5)^\beta{}_\delta P_\mu
\end{aligned} \tag{7.30}$$

where we have used the fact that γ_5 is antisymmetric (recall it is pure imaginary and Hermitian). The right hand side vanishes because it is equal to $(1+\gamma_5)\gamma^\mu C^{-1}(1-\gamma_5) = (1+\gamma_5)(1-\gamma_5)\gamma^\mu C^{-1} = 0$. Thus

$$\{Q_{W\gamma}, Q_{W\delta}\} = 0 \tag{7.31}$$

Similarly multiplying by $\frac{1}{2}(1-\gamma_5)_\gamma{}^\alpha$ and $\frac{1}{2}(1-\gamma_5)_\delta{}^\beta$ shows that $\{Q^*_{W\gamma}, Q^*_{W\delta}\} = 0$

Next we multiply through by $\frac{1}{2}(1+\gamma_5)_\gamma{}^\alpha$ and $\frac{1}{2}(1-\gamma_5)_\delta{}^\beta$ to find

$$\begin{aligned}
\{Q_{W\gamma}, Q^*_{W\delta}\} &= -\frac{1}{2}((1+\gamma_5)\gamma^\mu C^{-1})_{\gamma\beta}(1-\gamma_5)_\delta{}^\beta P_\mu \\
&= -\frac{1}{2}((1+\gamma_5)\gamma^\mu C^{-1}(1+\gamma_5))_{\gamma\delta} P_\mu \\
&= -((1+\gamma_5)\gamma^\mu C^{-1})_{\gamma\delta} P_\mu
\end{aligned} \tag{7.32}$$

Similarly we find

$$\begin{aligned}
\{Q^*_{W\gamma}, Q_{W\delta}\} &= -\frac{1}{2}((1-\gamma_5)\gamma^\mu C^{-1})_{\gamma\beta}(1+\gamma_5)_\delta{}^\beta P_\mu \\
&= -\frac{1}{2}((1-\gamma_5)\gamma^\mu C^{-1}(1-\gamma_5))_{\gamma\delta} P_\mu \\
&= -((1-\gamma_5)\gamma^\mu C^{-1})_{\gamma\delta} P_\mu
\end{aligned} \tag{7.33}$$

Problem: Show that

$$\begin{aligned}
(\sigma_\mu)_{a\dot{b}} &= \left(\delta_{a\dot{b}}, \sigma^i_{a\dot{b}}\right) \\
(\bar{\sigma}_\mu)_{a\dot{b}} &= \left(\delta_{a\dot{b}}, -\sigma^i_{a\dot{b}}\right)
\end{aligned} \tag{7.34}$$

Solution: Since $C^{-1} = -\gamma_0$ we have that

$$\gamma_0 C^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \gamma_i C^{-1} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \tag{7.35}$$

Thus

$$\begin{aligned}
\frac{1}{2}(1+\gamma_5)\gamma_0 C^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
\frac{1}{2}(1+\gamma_5)\gamma_i C^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = \begin{pmatrix} \sigma^i & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned} \tag{7.36}$$

By definition the top left entry of these matrices are $(\sigma_\mu)_{a\dot{b}}$ and so

$$(\sigma_0)_{a\dot{b}} = \delta_{a\dot{b}} \qquad (\sigma_i)_{a\dot{b}} = \sigma^i_{a\dot{b}} \tag{7.37}$$

The last equation is an identity by construction. Similarly one has

$$\begin{aligned} \frac{1}{2}(1-\gamma_5)\gamma_0 C^{-1} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \frac{1}{2}(1-\gamma_5)\gamma_i C^{-1} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma^i \end{pmatrix} \end{aligned} \tag{7.38}$$

and hence

$$(\bar{\sigma}_0)_{\dot{a}b} = \delta_{\dot{a}b} \qquad (\bar{\sigma}_i)_{\dot{a}b} = -\sigma^i_{\dot{a}b} \tag{7.39}$$

Problem: Show that

$$\begin{aligned} \bar{\epsilon}_1\epsilon_2 - \bar{\epsilon}_2\epsilon_1 &= 0 \\ \bar{\epsilon}_1\gamma_5\epsilon_2 - \bar{\epsilon}_2\gamma_5\epsilon_1 &= 0 \\ \bar{\epsilon}_1\gamma_\rho\gamma_5\epsilon_2 - \bar{\epsilon}_2\gamma_\rho\gamma_5\epsilon_1 &= 0 \\ \bar{\epsilon}_1\gamma_{\rho\sigma}\epsilon_2 + \bar{\epsilon}_2\gamma_{\rho\sigma}\epsilon_1 &= 0 \end{aligned} \tag{7.40}$$

Solution: In all these cases we are looking at something of the form

$$\bar{\epsilon}_1\Gamma\epsilon_2 = \epsilon_{1\alpha}(C\Gamma)^{\alpha\beta}\epsilon_1 \tag{7.41}$$

Since $\epsilon_{1\alpha}\epsilon_{2\beta} = -\epsilon_{2\alpha}\epsilon_{1\beta}$ we see that these identities correspond to whether or not $C\Gamma$ is anti-symmetric (first three) or symmetric (last one).

In the first case we have $\Gamma = 1$ and hence $C\Gamma = C$ is anti-symmetric. Similarly in the second case we have $(C\gamma_5)^T = \gamma_5^T C^T = \gamma_5 C = -C\gamma_5$ is antisymmetric. In the third case $\Gamma = \gamma_\rho\gamma_5$. Thus

$$(C\gamma_\rho\gamma_5)^T = \gamma_5(\gamma_\rho)^T C = -\gamma_5(C\gamma_\rho C^{-1})C = -C\gamma_\rho\gamma_5 \tag{7.42}$$

as required. In the fourth case we have $\Gamma = \gamma_{\rho\sigma}$ and so

$$\begin{aligned} (C\gamma_{\rho\sigma})^T &= -\frac{1}{2}(\gamma_\sigma)^T(\gamma_\rho)^T C + \frac{1}{2}(\gamma_\rho)^T(\gamma_\rho)^T C \\ &= -\frac{1}{2}(C\gamma_\sigma C^{-1})(C\gamma_\rho C^{-1})C + \frac{1}{2}(C\gamma_\rho C^{-1})(C\gamma_\sigma C^{-1})C \\ &= -C\gamma_{\sigma\rho} \\ &= C\gamma_{\rho\sigma} \end{aligned} \tag{7.43}$$

Problem: Using the Fierz identity show that, in four dimensions,

$$\mathrm{Tr}(\bar{\lambda}, \gamma^\mu[(\bar{\epsilon}\gamma_\mu\lambda), \lambda]) = f^{abc}\bar{\lambda}_c\gamma^\mu(\bar{\epsilon}\gamma_\mu\lambda_a)\lambda_b = 0. \tag{7.44}$$

Solution: We rewrite this as

$$\begin{aligned} f^{abc}\bar\lambda_c\gamma^\mu(\bar\epsilon\gamma_\mu\lambda_a)\lambda_b &= -f^{abc}\bar\lambda_c\gamma^\mu(\bar\lambda_a\gamma_\mu\epsilon)\lambda_b \\ &= \frac{1}{4}f^{abc}\sum_\Gamma\bar\lambda_c\gamma^\mu(\bar\lambda_a\Gamma\lambda_b)\Gamma\gamma_\mu\epsilon \end{aligned} \tag{7.45}$$

Here have used the Fierz identity and the sum is over all the relevant matrices: $\Gamma \propto 1, \gamma_5, \gamma_\nu, \gamma_{\nu5}, \gamma_{\nu\lambda}$, with the correct signs and factors assumed. However since f^{abc} is anti-symmetric in a, b the only terms that contribute are where $(C\Gamma)^T = C\Gamma$. This reduces the sum to just $\Gamma = \gamma_\mu$ and $\Gamma = \gamma_{\mu\lambda}$ restoring the right factors we have

$$\begin{aligned} f^{abc}\bar\lambda_c\gamma^\mu(\bar\epsilon\gamma_\mu\lambda_a)\lambda_b &= \frac{1}{4}f^{abc}\bar\lambda_c\gamma^\mu(\bar\lambda_a\gamma_\nu\lambda_b)\gamma^\nu\gamma_\mu\epsilon \\ &\quad -\frac{1}{8}f^{abc}\bar\lambda_c\gamma^\mu(\bar\lambda_a\gamma_{\nu\lambda}\lambda_b)\gamma^{\nu\lambda}\gamma_\mu\epsilon \end{aligned} \tag{7.46}$$

Next we need the following two identities

$$\begin{aligned} \gamma^\mu\gamma^\nu\gamma_\mu &= \gamma^\mu(2\delta^\nu_\mu - \gamma_\mu\gamma^\nu) \\ &= 2\gamma^\nu - 4\gamma^\nu \\ &= -2\gamma^\nu \end{aligned} \tag{7.47}$$

and

$$\begin{aligned} \gamma^\mu\gamma^{\nu\lambda}\gamma_\mu &= \gamma^\mu(\gamma_\mu\gamma^{\nu\lambda} + [\gamma^{\nu\lambda}, \gamma_\mu]) \\ &= 4\gamma^{\nu\lambda} + 2\gamma^\mu(\delta^\lambda_\mu\gamma^\nu - \delta^\nu_\mu\gamma^\lambda) \\ &= 4\gamma^{\nu\lambda} + 4\gamma^{\lambda\nu} \\ &= 0 \end{aligned} \tag{7.48}$$

Thus we find

$$\begin{aligned} f^{abc}\bar\lambda_c\gamma^\mu(\bar\epsilon\gamma_\mu\lambda_a)\lambda_b &= -\frac{1}{2}f^{abc}\bar\lambda_c\gamma^\nu(\bar\lambda_a\gamma_\nu\lambda_b)\epsilon \\ &= \frac{1}{2}f^{abc}\bar\epsilon\gamma^\nu\lambda_c(\bar\lambda_a\gamma_\nu\lambda_b) \end{aligned}$$

A quick inspection shows that the right hand side is $-1/2$ times the left hand side. Thus they must both vanish, as required.

Problem: Prove the Bianchi identity $D_{[\mu}F_{\nu\lambda]} = 0$, where $D_\mu F_{\nu\lambda} = \partial_\mu F_{\nu\lambda} - i[A_\mu, F_{\nu\lambda}]$.

Solution: It is straight forward to expand this

$$\begin{aligned} 3!D_{[\mu}F_{\nu\lambda]} &= \partial_\mu(\partial_\nu A_\lambda - \partial_\lambda A_\nu - i[A_\nu, A_\lambda]) - i[A_\mu, \partial_\nu A_\lambda - \partial_\lambda A_\nu - i[A_\nu, A_\lambda]]) \pm \text{cyclic} \\ &= -i[\partial_\mu A_\nu, A_\lambda] - i[A_\nu, \partial_\mu A_\lambda] - i[A_\mu, \partial_\nu A_\lambda] + i[A_\mu, \partial_\lambda A_\nu] - [A_\mu, [A_\nu, A_\lambda]] \pm \text{cyclic} \\ &= i[A_\lambda, \partial_\mu A_\nu] - i[A_\nu, \partial_\mu A_\lambda] - i[A_\mu, \partial_\nu A_\lambda] + i[A_\mu, \partial_\lambda A_\nu] \pm \text{cyclic} \\ &= 2i[A_\lambda, \partial_\mu A_\nu] - 2i[A_\mu, \partial_\nu A_\lambda] \pm \text{cyclic} \\ &= 0 \end{aligned} \tag{7.49}$$

In the second line we used the fact that second derivatives commute and the third line the $[A_\mu, [A_\nu, A_\lambda]]$ terms vanish by the Jacobi identity. In the last two lines we used the cyclic sum to swap the indices.

Problem: Show that the transformations

$$\begin{aligned} \delta A_\mu &= i\bar{\epsilon}\gamma_\mu\lambda \\ \delta\lambda &= -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon \end{aligned} \tag{7.50}$$

close on-shell on the Fermions to Poincare transformations and gauge transformations.

Solution: First compute

$$[\delta_1, \delta_2]\lambda = -\frac{1}{2}\delta_1 F_{\mu\nu}\gamma^{\mu\nu}\epsilon_2 - (1 \leftrightarrow 2)$$

Now

$$\begin{aligned} \delta_1 F_{\mu\nu} &= \partial_\mu\delta_1 A_\nu - \partial_\nu\delta_1 A_\mu - i[\delta_1 A_\mu, A_\nu] - i[A_\mu, \delta_1 A_\nu] \\ &= D_\mu\delta_1 A_\nu - D_\nu\delta_1 A_\mu \end{aligned} \tag{7.51}$$

Thus we find

$$\begin{aligned} [\delta_1, \delta_2]\lambda &= -D_\mu\delta_1 A_\nu\gamma^{\mu\nu}\epsilon_2 - (1 \leftrightarrow 2) \\ &= -i(\bar{\epsilon}_1\gamma_\nu D_\mu\lambda)\gamma^{\mu\nu}\epsilon_2 - (1 \leftrightarrow 2) \end{aligned} \tag{7.52}$$

The rest just follows the previous discussion in the Abelian case with $\partial_\mu \to D_\mu$. As a result of Feirz we find

$$\begin{aligned} [\delta_1, \delta_2]\lambda &= -2i\,(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\,D_\mu\lambda - \frac{i}{2}\,(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\,\gamma_\nu\gamma^\mu D_\mu\lambda \\ &\quad -\frac{i}{4}\,(\bar{\epsilon}_1\gamma_{\rho\sigma}\epsilon_2)\,\gamma^{\rho\sigma}\gamma^\mu D_\mu\lambda \end{aligned} \tag{7.53}$$

The on-shell equation is $\gamma^\mu D_\mu\lambda = 0$ so that, on-shell, we find

$$[\delta_1, \delta_2]\lambda = -2i\,(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\,\partial_\mu\lambda + 2\,(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\,[A_\mu, \lambda] \tag{7.54}$$

Here the first term is a translation and the second term a gauge transformation.

Problem: Show that, in ten-dimensions, with $\Gamma_{11}\Lambda = \Lambda$,

$$\mathrm{Tr}(\bar{\Lambda}, \Gamma^m[(\bar{\epsilon}\Gamma_m\Lambda), \Lambda]) = f^{abc}\bar{\Lambda}_c\Gamma^m(\bar{\epsilon}\Gamma_m\Lambda_a)\Lambda_b = 0. \tag{7.55}$$

You may assume the Fierz transformation in 10 dimensions is (why?)

$$\begin{aligned}(\bar\chi\psi)\lambda &= -\frac{1}{32}\Big[(\bar\chi\lambda)\psi+(\bar\chi\Gamma_{11}\lambda)\Gamma^{11}\psi+(\bar\chi\Gamma_m\lambda)\Gamma^m\psi-\frac{1}{2!}(\bar\chi\Gamma_{mn}\lambda)\Gamma^{mn}\psi \\ &\quad -(\bar\chi\Gamma_m\Gamma_{11}\lambda)\Gamma^m\Gamma_{11}\psi-\frac{1}{3!}(\bar\chi\Gamma_{mnp}\lambda)\Gamma^{mnp}\psi-\frac{1}{2!}(\bar\chi\Gamma_{mn}\Gamma_{11}\lambda)\Gamma^{mn}\Gamma_{11}\psi \\ &\quad +\frac{1}{4!}(\bar\chi\Gamma_{mnpq}\lambda)\Gamma^{mnpq}\psi+\frac{1}{4!}(\bar\chi\Gamma_{mnp}\Gamma_{11}\lambda)\Gamma^{mnp}\Gamma_{11}\psi+\frac{1}{5!}(\bar\chi\Gamma_{mnpqr}\lambda)\Gamma^{mnpqr}\psi \\ &\quad +\frac{1}{4!}(\bar\chi\Gamma_{mnpq}\Gamma_{11}\lambda)\Gamma^{mnpq}\Gamma_{11}\psi\Big] \qquad (7.56)\end{aligned}$$

Solution: We perform a Fierz transformation

$$\begin{aligned}f^{abc}\bar\Lambda_c\Gamma^m(\bar\epsilon\Gamma_m\Lambda_a)\Lambda_b &= -f^{abc}\bar\Lambda_c\Gamma^m(\bar\Lambda_a\Gamma_m\epsilon)\Lambda_b \\ &= \frac{1}{32}f^{abc}\bar\Lambda_c\sum_\Gamma\Big((\bar\Lambda_a\Gamma\Lambda_b)\Gamma^m\Gamma\Gamma_m\epsilon\Big) \qquad (7.57)\end{aligned}$$

Where the sum is over all the Γ's that appear in the Fierz transformation. We note that only terms which are anti-symmetric in a, b contribute and this corresponds to terms where $C\Gamma$ is symmetric. A little thinking shows that the options are

$$\Gamma = \Gamma_m, \Gamma_{11}, \Gamma_{mn}, \Gamma_m\Gamma_{11}, \Gamma_{mnpqr}, \Gamma_{mnpq}\Gamma_{11} \qquad (7.58)$$

In addition since $\Gamma_{11}\Lambda=\Lambda$ the only terms which survive are those for which $[C\Gamma,\Gamma_{11}]=0$ since if $\{C\Gamma,\Gamma_{11}\}=0$ then

$$\bar\Lambda_a\Gamma\Lambda_b=\Lambda_a^T C\Gamma\Gamma_{11}\Lambda_b=-\Lambda_a^T\Gamma_{11}C\Gamma\Lambda_b=-\bar\Lambda_a\Gamma\Lambda_b \qquad (7.59)$$

and hence vanishes. This leaves us with

$$\Gamma=\Gamma_m,\Gamma_m\Gamma_{11},\Gamma_{mnpqr} \qquad (7.60)$$

and so

$$\begin{aligned}f^{abc}\bar\Lambda_c\Gamma^m(\bar\epsilon\Gamma_m\Lambda_a)\Lambda_b &= \frac{1}{32}f^{abc}\bar\Lambda_c\Big((\bar\Lambda_a\Gamma_n\Lambda_b)\Gamma^m\Gamma^n\Gamma_m\epsilon \qquad (7.61) \\ &\quad +(\bar\Lambda_a\Gamma_n\Gamma_{11}\Lambda_b)\Gamma^m\Gamma^n\Gamma_m\Gamma_{11}\epsilon+\frac{1}{5!}(\bar\Lambda_a\Gamma_{npqrs}\Lambda_b)\Gamma^m\Gamma^{npqrs}\Gamma_m\epsilon\Big) \\ &= \frac{1}{32}f^{abc}\bar\Lambda_c\sum_\Gamma\Big(2(\bar\Lambda_a\Gamma_n\Lambda_b)\Gamma^m\Gamma^n\Gamma_m\epsilon+\frac{1}{5!}(\bar\Lambda_a\Gamma_{npqrs}\Lambda_b)\Gamma^m\Gamma^{npqrs}\Gamma_m\epsilon\Big)\end{aligned}$$

No the second term vanishes because

$$\Gamma^m\Gamma^{npqrs}\Gamma_m=0 \qquad (7.62)$$

To see this note that, for any fixed values of n,p,q,r,s there are five values of m for which $\{\Gamma^{npqrs},\Gamma^m\}=0$ (namely $m=n,p,q,r,s$) and five values of m for which $[\Gamma^{npqrs},\Gamma^m]=0$ (namely $m\neq n,p,q,r,s$). Hence we have

$$\begin{aligned} f^{abc}\bar\Lambda_c\Gamma^m(\bar\epsilon\Gamma_m\Lambda_a)\Lambda_b &= \frac{1}{16}f^{abc}\bar\Lambda_c(\bar\Lambda_a\Gamma_n\Lambda_b)\Gamma^m\Gamma^n\Gamma_m\epsilon \\ &= -\frac{8}{16}f^{abc}\bar\Lambda_c(\bar\Lambda_a\Gamma_n\Lambda_b)\Gamma^n\epsilon \\ &= \frac{1}{2}f^{abc}(\bar\epsilon\Gamma_n\Lambda_c)(\bar\Lambda_a\Gamma^n\Lambda_b) \end{aligned} \tag{7.63}$$

However the left-hand-side is

$$f^{abc}(\bar\epsilon\Gamma_n\Lambda_a)(\bar\Lambda_c\Gamma^n\Lambda_b)=f^{cba}(\bar\epsilon\Gamma_n\Lambda_c)(\bar\Lambda_a\Gamma^n\Lambda_b)=-f^{abc}(\bar\epsilon\Gamma_n\Lambda_c)(\bar\Lambda_a\Gamma^n\Lambda_b) \tag{7.64}$$

since $f^{abc}=-f^{cba}$. Thus we see that $f^{abc}(\bar\epsilon\Gamma_n\Lambda_c)(\bar\Lambda_a\Gamma^n\Lambda_b)=0$.

Problem: Show that in ten-dimensions, with $\Gamma_{11}\Lambda=\Lambda$, the transformations

$$\begin{aligned} \delta A_m &= i\bar\epsilon\Gamma_m\Lambda \\ \delta\Lambda &= -\frac{1}{2}F_{mn}\Gamma^{mn}\epsilon\ . \end{aligned} \tag{7.65}$$

close on-shell on Λ.

Solution: We start by following the four-dimensional calculation:

$$[\delta_1,\delta_2]\Lambda = -\frac{1}{2}\delta_1F_{mn}\Gamma^{mn}\varepsilon_2-(1\leftrightarrow 2)$$

Now

$$\begin{aligned} \delta_1F_{mn} &= \partial_m\delta_1A_n-\partial_n\delta_1A_m-i[\delta_1A_m,A_n]-i[A_m,\delta_1A_n] \\ &= D_n\delta_1A_n-D_n\delta_1A_m \end{aligned} \tag{7.66}$$

Thus we find

$$\begin{aligned} [\delta_1,\delta_2]\Lambda &= -D_m\delta_1A_n\Gamma^{mn}\varepsilon_2-(1\leftrightarrow 2) \\ &= -i(\bar\varepsilon_1\Gamma_nD_m\Lambda)\Gamma^{mn}\varepsilon_2-(1\leftrightarrow 2) \end{aligned} \tag{7.67}$$

Next we need to use the Fierz identity. As in the previous question we write

$$[\delta_1,\delta_2]\Lambda = +\frac{i}{32}\sum_\Gamma(\bar\varepsilon_1\Gamma\varepsilon_2)\Gamma^{mn}\Gamma\Gamma_nD_m\Lambda-(1\leftrightarrow 2)$$

Again we only need the contributions that are anti-symmetric under $1\leftrightarrow 2$ and this restricts to $\Gamma=\Gamma_p,\Gamma_{11},\Gamma_{pq},\Gamma_p\Gamma_{11},\Gamma_{pqrst},\Gamma_{pqrs}\Gamma_{11}$. Furthermore the only non-zero terms are when $[C\Gamma,\Gamma_{11}]=0$ and hence $\Gamma=\Gamma_p,\Gamma_p\Gamma_{11},\Gamma_{pqrst}$. Thus

$$\begin{aligned} [\delta_1,\delta_2]\Lambda = &+\frac{i}{16}\Big((\bar\varepsilon_1\Gamma_p\varepsilon_2)\Gamma^{mn}\Gamma^p\Gamma_nD_m\Lambda-(\bar\varepsilon_1\Gamma_p\Gamma_{11}\varepsilon_2)\Gamma^{mn}\Gamma^p\Gamma_{11}\Gamma_nD_m\Lambda \\ &+\frac{1}{5!}(\bar\varepsilon_1\Gamma_{pqrst}\varepsilon_2)\Gamma^{mn}\Gamma^{pqrst}\Gamma_nD_m\Lambda\Big) \end{aligned} \tag{7.68}$$

Note the factor of 16 from anti-symmetry under $1 \leftrightarrow 2$. Since $\Gamma_{11}\varepsilon_2 = \varepsilon_2$ and $\Gamma_{11}\Lambda = \Lambda$ we see the the first two terms are equal. Thus we have to compute two expressions.

$$
\begin{aligned}
\Gamma^{mn}\Gamma^p\Gamma_n &= (\Gamma^m\Gamma^n + \eta^{mn})(-\Gamma_n\Gamma^p + 2\delta^p_n) \\
&= -10\Gamma^m\Gamma^p - \Gamma^m\Gamma^p + 2\Gamma^m\Gamma^p + 2\eta^{mn} \\
&= -9\Gamma^m\Gamma^p + 2\eta^{mn} \\
&= 9\Gamma^p\Gamma^m - 16\eta^{mn}
\end{aligned}
\tag{7.69}
$$

and

$$
\begin{aligned}
\Gamma^{mn}\Gamma^{pqrst}\Gamma_n &= (\Gamma^m\Gamma^n + \eta^{mn})\Gamma^{pqrst}\Gamma_n \\
&= \Gamma^m(\Gamma^n\Gamma^{pqrst}\Gamma_n) + \Gamma^{pqrst}\Gamma^m \\
&= \Gamma^{pqrst}\Gamma^m
\end{aligned}
\tag{7.70}
$$

Here we have used the previous result that $\Gamma^n\Gamma^{pqrst}\Gamma_n = 0$ since for half the values of n Γ_n commutes with Γ^{pqrst} whereas for the other half it anti-commutes. Thus we have

$$
\begin{aligned}
[\delta_1, \delta_2]\Lambda &= -2i(\bar\varepsilon_1\Gamma^m\varepsilon_2)D_m\Lambda + \frac{9i}{8}(\bar\varepsilon_1\Gamma^n\varepsilon_2)\Gamma_n\Gamma^m D_m\Lambda \\
&\quad + \frac{i}{5!\cdot 16}(\bar\varepsilon_1\Gamma_{pqrst}\varepsilon_2)\Gamma^{pqrst}\Gamma^m D_m\Lambda
\end{aligned}
\tag{7.71}
$$

Thus we indeed find

$$
[\delta_1, \delta_2]\Lambda = +2i(\bar\varepsilon_2\Gamma^m\varepsilon_1)D_m\Lambda \tag{7.72}
$$

when the Fermions are on-shell; $\Gamma^m D_m\Lambda = 0$.

Problem: Show that the ten-dimensional supersymmetry

$$
\begin{aligned}
\delta A_m &= i\bar\varepsilon\Gamma_m\Lambda \\
\delta\Lambda &= -\frac{1}{2}F_{mn}\Gamma^{mn}\varepsilon \ .
\end{aligned}
\tag{7.73}
$$

becomes

$$
\begin{aligned}
\delta A_\mu &= i\bar\epsilon_I\Gamma_\mu\lambda^I \\
\delta\phi_A &= -\bar\epsilon_I\gamma_5\lambda_J\rho_A^{IJ} \\
\delta\lambda^I &= -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon^I - \gamma^\mu\gamma_5 D_\mu\phi^A\rho_A^{IJ}\epsilon_J + \frac{i}{2}[\phi_A, \phi_B]\rho_{AB}^{JI}\epsilon_I \ .
\end{aligned}
\tag{7.74}
$$

where $\rho_{AB}^{JI} = (\eta^J)^T\rho_{AB}\eta^I$.

Solution: Setting $m = \mu$ in δA_m gives

$$
\begin{aligned}
\delta A_\mu &= i(\bar\epsilon_I \otimes (\eta^I)^T)(\gamma_\mu \otimes 1)(\lambda_J \otimes \eta^J) \\
&= i\bar\epsilon_I\gamma_\mu\lambda_J((\eta^I)^T \otimes \eta^J) \\
&= i\bar\epsilon_I\gamma_\mu\lambda^I
\end{aligned}
\tag{7.75}
$$

Next we set $m = A$ in δA_m to find

$$\begin{aligned}
\delta\phi_A &= i(\bar\epsilon_I \otimes (\eta^I)^T)(\gamma_5 \otimes 1)(\lambda_J \otimes \eta^J) \\
&= i\bar\epsilon_I\gamma_5\lambda_J((\eta^I)^T \otimes \eta^J) \\
&= i\bar\epsilon_I\gamma_5\lambda^I
\end{aligned} \tag{7.76}$$

Finally we have $\delta\Lambda = \delta\lambda_I \otimes \eta^I$. Since $(\eta^I)^T\eta^J = \delta^{IJ}$ we can read off $\delta\lambda_J$ from

$$\delta\lambda_J \otimes 1 = (1 \otimes (\eta^J)^T)\delta\Lambda \tag{7.77}$$

Hence, recalling that $F_{\mu A} = D_\mu\phi_A$, we see that

$$\begin{aligned}
\delta\lambda_J &= (1 \otimes (\eta^J)^T)\Big(-\frac{1}{2}F_{\mu\nu}(\gamma^{\mu\nu} \otimes 1) - D_\mu\phi_A(\gamma^\mu\gamma_5 \otimes \rho^A) + \frac{i}{2}[\phi_A, \phi_B](1 \otimes \rho^{AB})\Big)\epsilon_I \otimes \eta^I \\
&= -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon_J - D_\mu\phi_A\gamma^\mu\gamma_5\epsilon_I(\eta^J)^T\rho_A\eta_I + \frac{i}{2}[\phi_A, \phi_B]\epsilon_I(\eta^J)^T\rho^{AB}\eta_I \\
&= -\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon_J - D_\mu\phi_A\gamma^\mu\gamma_5\rho_A^{JI}\epsilon_I + \frac{i}{2}[\phi_A, \phi_B]\rho_{AB}^{JI}\epsilon_I
\end{aligned} \tag{7.78}$$

where $\rho_{AB}^{JI} = (\eta^J)^T\rho_{AB}\eta^I$.

Problem: Using the fact that $< M_1, M_2 >= \mathrm{Tr}(M_1^\dagger M_2)$ defines a complex inner product, convince yourself that the set

$$1, \gamma_{D+1}, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu5} \tag{7.79}$$

where the number of spacetime indices is no bigger than $D/2$, is a basis for the space of $2^{D/2} \times 2^{D/2}$ matrices.

Solution: We denote a general element of this set by γ_Γ. Since $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$ and $\gamma_\Gamma^2 = \pm 1$ we see that $\gamma_\Gamma^\dagger\gamma_\Gamma = 1$. Hence up to a constant (which is $2^{D/2}$, the dimension of the representation) all these elements have unit length.

The key point is that, since $\gamma_{D+1} \propto \epsilon^{\mu_1\mu_2\mu_3\ldots}\gamma_{\mu_1\mu_2\mu_3\ldots}$, one can always express any $\gamma_{\mu_1\mu_2\mu_3\ldots\mu_n}$ which has $n > D/2$ spacetime indices as proportional to $\gamma_{\mu_1\mu_2\mu_3\ldots,\mu_{D/2-n}}$, possibly including one factor of γ_{D+1}, which has no more than $D/2$ indices. Hence we have that

$$\gamma_\Gamma^\dagger\gamma_{\Gamma'} = \sum_{\Gamma''} C_{\Gamma''}\gamma_{\Gamma''} \tag{7.80}$$

where number of spacetime indices on any given γ_Γ matrix is no bigger than $D/2$. If we take the trace, and use the fact that only the identity has a non-vanishing trace, then we see that all the γ_Γ are orthogonal.

Finally we need to count the number of $\gamma_\Gamma' s$. If D is even the list is

$$1, \gamma_{D+1}, \gamma_\mu, \gamma_\mu\gamma_{D+1}, \gamma_{\mu\nu}, \gamma_{\mu\nu}\gamma_{D+1}\ldots, \gamma_{\mu_1\ldots\mu_{D/2}} \tag{7.81}$$

(note that $\gamma_{\mu_1 \dots \mu_{D/2}}$ appears but $\gamma_{\mu_1 \dots \mu_{D/2}} \gamma_{D+1}$ will not) which contains

$$1 + 1 + D + D + \binom{D}{2} + \binom{D}{2} + \dots + \binom{D}{D/2} + 0 = \sum_{k=0}^{D} \binom{D}{k} = 2^D \tag{7.82}$$

elements. If D is odd then $\gamma_{D+1} = 1$ and hence the list is

$$1, \gamma_\mu, \gamma_{\mu\nu}, \dots \tag{7.83}$$

which has

$$1 + D + \binom{D}{2} + \dots + \binom{D}{(D-1)/2} = \frac{1}{2} \sum_{k=0}^{D} \binom{D}{k} = \frac{1}{2} 2^D = 2^{D-1} \tag{7.84}$$

elements. In either case this is the number of $2^{[D]/2} \times 2^{[D]/2}$ matrices.

Problem: Show that in three dimensions the Fierz rearrangement is

$$(\bar{\lambda}\psi)\chi_\alpha = -\frac{1}{2}(\bar{\lambda}\chi)\psi_\alpha - \frac{1}{2}\left(\bar{\lambda}\gamma_\mu\chi\right)(\gamma^\mu\psi)_\alpha \tag{7.85}$$

Using this, show that in the special case that $\lambda = \chi$ one simply has

$$(\bar{\lambda}\psi)\lambda_\alpha = -\frac{1}{2}(\bar{\lambda}\lambda)\psi_\alpha \tag{7.86}$$

for Majorana spinors. Convince yourself that this is true by considering the explicit 3D γ-matrices above and letting

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} , \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{7.87}$$

What is the Fierz rearrangement in two dimensions (Hint: this last part should take you very little time)?

Solution: In three dimensions the γ-matrix products are

$$1 \ , \gamma_\mu \ , \gamma_{\mu\nu} \ , \gamma_{\mu\nu\lambda} \tag{7.88}$$

However one has that $\gamma_{\mu\nu\lambda} \propto \epsilon_{\mu\nu\lambda} 1$ and also that $\gamma_{\mu\nu} \propto \epsilon_{\mu\nu\lambda}\gamma^\lambda$. Thus only 1 and γ_μ are independent matrices. Indeed since there are four of these and they are 2×2 matrices this is the correct counting. Furthermore all of these matrices square to one, except for γ_0 which squares to minus one. Thus our discussion in the lectures tells us that

$$\delta_\alpha^\beta \delta_\gamma^\delta = \frac{1}{2}(1)_\gamma^{\ \beta}(1)_\alpha^{\ \delta} + \frac{1}{2}(\gamma_\mu)_\gamma^{\ \beta}(\gamma^\mu)_\alpha^{\ \delta} \tag{7.89}$$

Note that the minus sign in γ_0^2 is automatically taken care of by the minus sign involved in raising the $\mu = 0$ index. Thus we have

$$\begin{aligned} (\bar{\lambda}\psi)\chi_\alpha &= \bar{\lambda}^\alpha \psi_\beta \chi_\delta \delta^\beta_\alpha \delta^\delta_\gamma \\ &= -\frac{1}{2}(\bar{\lambda}\chi)\psi_\alpha - \frac{1}{2}\left(\bar{\lambda}\gamma_\mu\chi\right)(\gamma^\mu\psi)_\alpha \end{aligned} \tag{7.90}$$

where the minus sign is due to interchanging the order of ψ and χ.

For the case that $\lambda = \chi$ we note that, for a Majorana basis in three-dimensions,

$$(C\gamma_\mu)^T = (\gamma_0\gamma_\mu)^T = \gamma_\mu^T \gamma_0^T = -\gamma_0\gamma_\mu\gamma_0\gamma_0 = C\gamma_\mu \tag{7.91}$$

Thus, since λ_α is anti-commuting and $C\gamma_\mu$ is symmetric, we have $\bar{\lambda}\gamma_\mu\lambda = 0$ and the result follows.

Let us compute this in the explicit matrix representation:

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \gamma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \gamma_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{7.92}$$

We start with the right hand side:

$$\begin{aligned} \bar{\lambda}\lambda &= (\lambda_1, \lambda_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \\ &= \lambda_1\lambda_2 - \lambda_2\lambda_1 \\ &= 2\lambda_1\lambda_2 \end{aligned} \tag{7.93}$$

so

$$-\frac{1}{2}(\bar{\lambda}\lambda)\psi_1 = -\lambda_1\lambda_2\psi_1 \qquad \text{and} \qquad -\frac{1}{2}(\bar{\lambda}\lambda)\psi_2 = -\lambda_1\lambda_2\psi_2 \tag{7.94}$$

Next look at the left hand side

$$\begin{aligned} \bar{\lambda}\psi &= (\lambda_1, \lambda_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\ &= \lambda_1\psi_2 - \lambda_2\psi_1 \end{aligned} \tag{7.95}$$

Since $\lambda_\alpha^2 = 0$ we find

$$\begin{aligned} (\bar{\lambda}\psi)\lambda_1 &= (\lambda_1\psi_2 - \lambda_2\psi_1)\lambda_1 \\ &= -\lambda_2\psi_1\lambda_1 \\ &= -\lambda_1\lambda_2\psi_1 \end{aligned} \tag{7.96}$$

similarly

$$
\begin{aligned}
(\bar{\lambda}\psi)\lambda_2 &= (\lambda_1\psi_2 - \lambda_2\psi_1)\lambda_2 \\
&= \lambda_1\psi_2\lambda_2 \\
&= -\lambda_1\lambda_2\psi_2
\end{aligned}
\tag{7.97}
$$

and thats what we had to show.

To find the Fierz rearrangement in two dimensions we note that the two-dimensional Clifford algebra is essentially the same as the three-dimensional Clifford algebra, only now γ_2 is called γ_3 and treated as a chirality matrix (and not one of that γ_μ). All this just amounts to separating out γ_2 for the others and calling it γ_3. This leads to

$$
(\bar{\lambda}\psi)\chi_\alpha = -\frac{1}{2}(\bar{\lambda}\chi)\psi_\alpha - \frac{1}{2}\left(\bar{\lambda}\gamma_\mu\chi\right)(\gamma^\mu\psi)_\alpha - \frac{1}{2}\left(\bar{\lambda}\gamma_3\chi\right)(\gamma_3\psi)_\alpha \tag{7.98}
$$

where now $\mu = 0, 1$.

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