The diameter of the modular McKay graph of $SL_n(\mathbb{F}_p)$ with respect to its standard representation

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Plan?

1. Define a McKay graph
2. State the result for the diameter of the McKay graph of interest
3. Motivate the result by looking at some restrictions on the length of paths in the graph
4. Talk about possible generalisations/current work

McKay graphs: notation

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Definition

The McKay graph $\mathcal{M}_F(G, W)$ is the directed graph whose vertices are elements of the set $Irr(G)$. For any two irreducible FG-modules V_1 and V_2 there is an edge from V_1 to V_2 if V_2 occurs as a composition factor of $V_1 \otimes W$.

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If W corresponds to a faithful representation then the graph $\mathcal{M}_F(G, W)$ is connected.

Therefore it makes sense to talk about the diameter (length of the longest geodesic in the graph).

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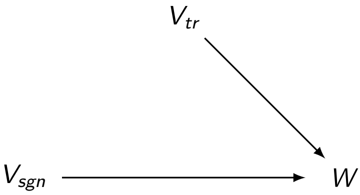

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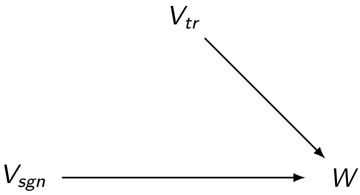

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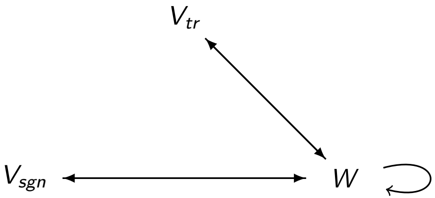

Background

These are exactly the graphs that come up in McKay's correspondence which says that taking G a subgroup of $SU(2)$, $F = \mathbb{C}$ and W to be $\mathbb{C}^2$ then $\mathcal{M}_F(G, W)$ is an affine Dynkin diagram of type A,D or E.

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Theorem (N.)

[3] For $F = \overline{\mathbb{F}}_p, G = SL_n(\mathbb{F}_p)$ and W_n the standard FG-module, the diameter of the graph $\mathcal{M}_F(G, W_n)$ is $\frac{1}{2}(p-1)(n^2-n)$.

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Very different behaviour in this case than conjectured for the case where $F = \mathbb{C}$.

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(1a) and (1b) are proved by showing that certain edges exist.
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Definition

For a weight λ such that $C^{-1}\lambda = \sum a_i \alpha_i$ let $f(\lambda) = na_{n-1}$.

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Trivial FG-module: $V(0,\dots,0)|_G$ where $f(0,\dots,0) = 0$

Standard FG-module: $V(1,0,\dots,0)|_G$ where $f(1,0,\dots,0) = 1$

Steinberg FG-module: $V(p-1,\dots,p-1)|_G$ where $f(p-1,\dots,p-1) = \frac{1}{2}(p-1)(n^2-n)$

PGTC 2020

Restriction on edges

Lemma

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Hence the shortest path from $V(0, \dots, 0)|_G$ to $V(p-1, \dots, p-1)|_G$ is of length at least
$f(p-1, \dots, p-1) - f(0, \dots, 0) = \frac{1}{2}(p-1)(n^2 - n)$.

A note on the proof of (1a) and (1b)

Use a result by Brundan and Kleshchev [Theorem V(iv),[1]] that determines possible composition factors of tensor products of irreducible $FGL_n(\mathbb{F}_p)$-modules with the standard $FGL_n(\mathbb{F}_p)$-module.

When applied to $SL_n(\mathbb{F}_p)$ this gives enough edges to describe a path from each vertex of the graph to another in less than $\frac{1}{2}(p-1)(n^2-n)$ steps.

Possible generalisations (restrictions of the proof)

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The proof of (1) uses the result by Brundan and Kleshchev [Theorem V(iv),[1]] which only says something about composition factors when taking the tensor product with the standard module - there currently no such result of any other $FGL_n(\mathbb{F}_p)$-module.

Thanks!

J. Brundan and A. Kleshchev, *On translation functors for general linear and symmetric groups*, Proc. Lon. Math. Soc. **80** (2000), 75–106.

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Clearly if $\lambda \leq \mu$ then $f(\lambda) \leq f(\mu)$ since $f(\mu - \lambda) \geq 0$.

Claim: If μ is a p-restricted dominant weight, and μ' is a dominant weight such that $V(\mu)|_G$ is a composition factor of $V(\mu')|_G$ then $f(\mu) \leq f(\mu')$.

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This implies the composition factors of $L_{\mathbb{Z}}(\eta) \otimes \mathbb{C}$ are contained in the composition factors of $L_{\mathbb{Z}}(\lambda) \otimes \mathbb{C} \otimes L_{\mathbb{Z}}(1, 0, \ldots, 0) \otimes \mathbb{C}$.

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It follows from the Littlewood-Richardson rule that $f(\eta) \leq f(\lambda) + 1$.

Since whenever $\mu' \leq \eta$, $f(\mu') \leq f(\eta)$ the result follows from the claim.

Thanks!

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