ON ADDITIVE TIME-CHANGES OF FELLER PROCESSES

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ABSTRACT. In this note we generalise the Phillips theorem [1] on the subordination of Feller processes by Lévy subordinators to the class of additive subordinators (i.e. subordinators with independent but possibly nonstationary increments). In the case where the original Feller process is Lévy we also express the time-dependent characteristics of the subordinated process in terms of the characteristics of the Lévy process and the additive subordinator.

1. INTRODUCTION

One of the established devices for building statistically relevant market models is that of the stochastic change of time-scale (e.g. Carr et al. [2]). Such a time change may be modelled as an independent additive subordinator $Z = \{Z_t\}_{t \geq 0}$, i.e. an increasing stochastic process with independent possibly nonstationary increments. If we subordinate a time-homogeneous Markov process $X = \{X_t\}_{t \geq 0}$ by Z, the resulting process $Y = \{X_{Z_t}\}_{t \geq 0}$ is a Markov process that will in general be time-inhomogeneous. The main result of this note shows that if X is a Feller process and Z satisfies some regularity assumptions, then Y is a time-inhomogeneous Feller process. The generator of Y is expressed in terms of the generator of X and the characteristics of Z. In the special case where X is a Lévy process it is shown that Y is an additive process with characteristics that are given explicitly in terms of the characteristics of X and of the additive subordinator Z. The explicit knowledge of the generator of Y is desirable from the viewpoint of pricing theory because contingent claims in the time-inhomogeneous market model Y can be evaluated using algorithms that are based on the explicit form of the generator of the underlying process (see for example [3]).

2. TIME-CHANGED FELLER PROCESSES

Throughout the paper we assume that $X = \{X_t\}_{t \geq 0}$ is a càdlàg Feller process with the state-space $\mathbb{R}^n$ for some $n \in \mathbb{N}$ and the infinitesimal generator $\mathcal{L}$ defined on a dense subspace $\mathcal{D}(\mathcal{L})$ in the Banach space of all continuous functions $C_0(\mathbb{R}^n)$ that vanish at infinity with norm $\|f\|_\infty := \sup_{x \in \mathbb{R}^n} |f(x)|$. The corresponding semigroup $(P_t)_{t \geq 0}$ is given by $P_t f(x) = \mathbf{E}^x[f(X_t)]$ for any $f \in C_0(\mathbb{R}^n)$, where the expectation is taken with respect to the law of X started at $X_0 = x$ (see Ethier and Kurtz [5] for the definition and properties of Feller semigroups).

Let $Z = \{Z_t\}_{t \geq 0}$ be an additive process, independent of X, with the Laplace exponent $\overline{\psi}_t(u) = \log \mathbf{E}[\mathrm{e}^{-uZ_t}]$ given by $\overline{\psi}_t(u) := \int_0^t \psi_s(u)\mathrm{d}s$, where $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and for all $s \in \mathbb{R}_+$, $u \in \mathbb{C}$ we have $\int_{(0,\infty)} (1 \wedge r)g(s,r)\mathrm{d}r < \infty$ and

$$(1) \qquad \psi_s(u) \;=\; -u\beta(s) + \int_{(0,\infty)} (\mathrm{e}^{-ur} - 1)g(s,r)\mathrm{d}r \quad \text{if} \quad \Re(u) \geq 0.$$

In other words Z is a càdlàg process with nondecreasing paths such that the random variable $Z_t - Z_s$ is independent of Z_u for all $0 \leq u \leq s < t$ (see Jacod and Shiryaev [4] Ch. II, Sec. 4c, for a systematic treatment of additive processes).

In this paper we are interested in the process $(D, Y) = \{(D_t, Y_t)\}_{t \geq 0}$ defined by $D_t := D_0 + t$ and $Y_t := X_{Z_{D_t}}$ for some $D_0 \in \mathbb{R}_+$.

Theorem 1. *The process (D, Y) is Feller with the state-space $\mathbb{R}_+ \times \mathbb{R}^n$ and infinitesimal generator $\mathcal{L}'$, defined on a dense subspace of the Banach space $C_0(\mathbb{R}_+ \times \mathbb{R}^n)$ of continuous functions that vanish at infinity, given by*

$$\mathcal{L}'f(s,x) = \frac{\partial f}{\partial s}(s,x) + \beta(s)\mathcal{L}f_s(x) + \int_{(0,\infty)} [P_r f_s(x) - f(s,x)]\, g(s,r)\mathrm{d}r,$$

where $f \in C_0(\mathbb{R}_+ \times \mathbb{R}^n)$ such that $f_s(\cdot) := f(s,\cdot) \in \mathcal{D}(\mathcal{L})$ $\forall s \in \mathbb{R}_+$ and the functions $(s,x) \mapsto \mathcal{L}f_s(x)$ and $(s,x) \mapsto \frac{\partial f}{\partial s}(s,x)$ are continuous and vanish at infinity.

If Z is a Lévy subordinator, Theorem 1 reduces to the well-known Philips [1] theorem. If X is a Lévy process, then the time-changed process is an additive process with characteristics determined by those of Z and X.

Proposition 1. *Let X be a Lévy process with $X_0 = 0$ and characteristic triplet (c, Q, ν), where $c \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n \times n}$ a nonnegative symmetric matrix and ν a measure on $\mathbb{R}^n \backslash \{0\}$ such that $\int_{\mathbb{R}^n \backslash \{0\}} (|x|^2 \wedge 1)\nu(\mathrm{d}x)$. The process Y defined above (with $D_0 = 0$) is additive with càdlàg paths, jump measure*

$$\widetilde{\nu}_s(\mathrm{d}x) = \beta(s)\nu(\mathrm{d}x) + \int_{(0,\infty)} \mathbf{P}(X_r \in \mathrm{d}x)g(s,r)\mathrm{d}r,$$

nonnegative symmetric matrix $\widetilde{Q}_s = \beta(s)Q$, drift

$$\widetilde{c}_s = \beta(s)c + \int_{(0,\infty)} \mathbf{E}[X_r \mathbf{I}_{\{|X_r| \leq 1\}}]g(s,r)\mathrm{d}r$$

and characteristic exponent $\overline{\Psi}_t(u) = \int_0^t \Psi_s(u)\mathrm{d}s$ (recall that $\mathbf{E}[\mathrm{e}^{\mathrm{i}u \cdot Y_t}] = \mathrm{e}^{\overline{\Psi}_t(u)}$ for all $u \in \mathbb{R}^n$) where

$$\Psi_s(u) = \mathrm{i}u \cdot \widetilde{c}_s - \frac{1}{2}u \cdot \widetilde{Q}_s u + \int_{\mathbb{R}^n \backslash \{0\}} \left[\mathrm{e}^{\mathrm{i}u \cdot x} - 1 - \mathrm{i}(u \cdot x)\mathbf{I}_{\{|x| \leq 1\}}\right]\widetilde{\nu}_s(\mathrm{d}x).$$

3. Example: a symmetric self-decomposable process

Suppose that Y is an additive process, considered in [2] as a model for the risky security, with no drift or Gaussian component and jump density

$$g_Y(t, y) = h_\nu(|y|/t^\gamma) \frac{\gamma}{\nu t^{\gamma+1}}, \quad \text{where} \quad h_\nu(y) = \frac{1}{\nu} \exp(-y/\nu) \mathbf{I}_{\{y>0\}}.$$

Then in law the process Y is equal to a Brownian motion time-changed by an independent additive subordinator Z with $\beta \equiv 0$ and jump density

$$g(t, r) = a_t e^{-r/b_t}, \quad \text{where} \quad a_t = \frac{\gamma}{\nu^3 t^{2\gamma+1}}, \quad b_t = 2\nu^2 t^{2\gamma}.$$

It is clear from Proposition 1 that $\widetilde{c}_t = \widetilde{Q}_t = 0$ for all $t \in \mathbb{R}_+$ and that the moment-generating functions of measures $\widetilde{\nu}_t(\mathrm{d}x)$ and $g_Y(t, x)\mathrm{d}x$ coincide

$$\int_{\mathbb{R}\setminus\{0\}} e^{\lambda x} \, \widetilde{\nu}_t(\mathrm{d}x) = \frac{2\gamma}{\nu t(1 - \lambda^2 \nu^2 t^{2\gamma})} = \int_{\mathbb{R}\setminus\{0\}} e^{\lambda x} g_Y(t, x)\mathrm{d}x$$

for $|\lambda| < 1/\nu t^\gamma$. This implies that the two additive processes coincide in law.

4. Proofs

4.1. **Proof of Proposition 1.** Let $\Psi_X(u)$ denote the characteristic exponent of the Lévy process X, i.e. $\mathbf{E}[\exp(iu \cdot X_s)] = \exp(s\Psi_X(u))$ for any $u \in \mathbb{R}^n$. Since X and Z are independent processes with independnent increments, for any sequence of positive real numbers $0 \le t_0 < \ldots < t_m$ and vectors $u_1, \ldots, u_m \in \mathbb{R}^n$ it follows that

$$
\begin{aligned}
\mathbf{E}\left[e^{i\sum_{i=1}^m u_i \cdot (Y_{t_i} - Y_{t_{i-1}})}\right] &= \mathbf{E}\left[\mathbf{E}\Big[\prod_{i=1}^m e^{iu_i \cdot (X_{Z_{t_i}} - X_{Z_{t_{i-1}}})} \,\Big|\, Z_{t_0}, \ldots, Z_{t_m}\Big]\right] \\
&= \mathbf{E}\left[\prod_{i=1}^m e^{(Z_{t_i} - Z_{t_{i-1}})\Psi_X(u_i)}\right] \\
&= \prod_{i=1}^m \mathbf{E}\left[e^{iu_i \cdot (Y_{t_i} - Y_{t_{i-1}})}\right].
\end{aligned}
$$

Hence the process Y also has independent increments. Since Y is clearly càdlàg (as X and Z are), it is an additive process.

Finally, we have to determine the characteristic curve of Y. An argument similar to the one above implies that the characteristic function of Y_t equals

$$\mathbf{E}[e^{iu \cdot Y_t}] = \mathbf{E}[e^{\Psi_X(u)Z_t}] = e^{\int_0^t \psi_s(-\Psi_X(u))\mathrm{d}s} \quad \text{for any} \quad u \in \mathbb{R}^n.$$

The last equality holds since $\Re(\Psi_X(u)) \le 0$ for all u and the integral in (1) is well-defined. It is not difficult to prove that for any Lévy process X started at 0 there exists a constant $C > 0$ such that the inequality holds

$$\max\left\{\mathbf{P}(|X_r| > 1), |\mathbf{E}[X_r \mathbf{I}_{\{|X_r| \le 1\}}]|, \mathbf{E}[|X_r|^2 \mathbf{I}_{\{|X_r| \le 1\}}]\right\} \le C(r \wedge 1) \quad \forall r \in \mathbb{R}_+$$

(see e.g. Lemma 30.3 in Sato [6]). Therefore, since $\int_0^\infty g(s,r)(r \wedge 1)\mathrm{d}r < \infty$ by assumption, we have

$$\int_0^\infty g(s,r) \max \left\{ \mathbf{P}(|X_r| > 1), |\mathbf{E}[X_r \mathbf{I}_{\{|X_r|\leq 1\}}]|, \mathbf{E}[|X_r|^2 \mathbf{I}_{\{|X_r|\leq 1\}}] \right\} \mathrm{d}r < \infty.$$

We can thus define the measure $\widetilde{\nu}_s(\mathrm{d}x)$, the vector $\widetilde{c}_s$, the matrix $\widetilde{Q}_s$ and the function $\Psi_s(u)$ by the formulae in Proposition 1. The Lévy-Khintchine representation

$$\Psi_X(u) = \mathrm{i}u \cdot c - \frac{1}{2}u \cdot Qu + \int_{\mathbb{R}^n \setminus \{0\}} \left[\mathrm{e}^{\mathrm{i}u \cdot x} - 1 - \mathrm{i}(u \cdot x)\mathbf{I}_{\{|x|\leq 1\}} \right] \nu(\mathrm{d}x)$$

and Fubini's theorem, which applies by the inequality above, yield the following calculation, whcih concludes the proof of the proposition:

$$
\begin{aligned}
\psi_s(-\Psi_X(u)) &= \beta(s)\Psi_X(u) + \int_0^\infty (\mathbf{E}[\mathrm{e}^{\mathrm{i}u \cdot X_r}] - 1)g(s,r)\mathrm{d}r \\
&= \beta(s)\Psi_X(u) + \mathrm{i}u \cdot \int_0^\infty \mathbf{E}[X_r \mathbf{I}_{\{|X_r|\leq 1\}}]g(s,r)\mathrm{d}r \\
&+ \int_0^\infty (\mathbf{E}[\mathrm{e}^{\mathrm{i}u \cdot X_r}] - 1 - \mathrm{i}u \cdot \mathbf{E}[X_r \mathbf{I}_{\{|X_r|\leq 1\}}])g(s,r)\mathrm{d}r = \Psi_s(u).
\end{aligned}
$$

4.2. **Proof of Theorem 1.** Note first that the paths of the process (D,Y) are càdlàg. In what follows we prove that (D,Y) is a Markov process that satisfies the Feller property and find the generator of its semigroup.

1. Markov property. For any $g \in C_0(\mathbb{R}_+ \times \mathbb{R}^n)$ define

$$Q_t g(s,x) := \mathbf{E}[g(D_t, Y_t)|D_0 = s, Y_0 = x] = \mathbf{E}[g(s+t, X_{Z_{s+t}})|X_{Z_s} = x].$$

Let $\lambda_{s,s+t}(\mathrm{d}r) := \mathbb{P}(Z_{s+t} - Z_s \in \mathrm{d}r)$ denote the law of the increment of Z which may have an atom at 0. Then, since X and Z are independent processes and the increments of Z are independent of the past, it follows from the definition that

$$Q_t g(s,x) = \int_{[0,\infty)} \mathbf{E}[g(s+t, X_{Z_s+r})|X_{Z_s} = x]\lambda_{s,s+t}(\mathrm{d}r).$$

Define for any $v \in \mathbb{R}_+$ a σ-algebra $\mathcal{G}_v = \sigma(X_l : l \in [0,v])$. Then for a Borel set $A \in \mathcal{B}(\mathbb{R}^n)$ and any $X_0 = x_0 \in \mathbb{R}^n$ the Markov property of X yields

$$
\begin{aligned}
\mathbf{E}^{x_0}[g(t+s, X_{Z_s+r})\mathbf{I}_{\{X_{Z_s} \in A\}}] &= \int_{[0,\infty)} \mathbf{E}^{x_0}[g(t+s, X_{v+r})\mathbf{I}_{\{X_v \in A\}}]\lambda_{0,s}(\mathrm{d}v) \\
&= \int_{[0,\infty)} \mathbf{E}^{x_0}\left[\mathbf{E}[g(t+s, X_{v+r})|\mathcal{G}_v]\mathbf{I}_{\{X_v \in A\}}\right]\lambda_{0,s}(\mathrm{d}v) \\
&= \mathbf{E}^{x_0}\left[\mathbf{E}^{X_{Z_s}}[g(t+s, X_r)]\mathbf{I}_{\{X_{Z_s} \in A\}}\right].
\end{aligned}
$$

Hence we get $\mathbf{E}[g(t+s, X_{Z_s+r})|X_{Z_s}] = \mathbf{E}^{X_{Z_s}}[g(t+s, X_r)]$ a.s. for any $r \in \mathbb{R}_+$ and the following identity holds

$$(2) \qquad Q_t g(s,x) = \int_{[0,\infty)} \mathbf{E}^x[g(s+t, X_r)]\lambda_{s,s+t}(\mathrm{d}r).$$

A similar argument and the monotone class theorem imply that, if $\Phi_s = \sigma(X_{Z_l} : l \in [0, s])$, then

$$\mathbf{E}[g(t + s, X_{r+Z_s})|\Phi_s] = \mathbf{E}^{X_{Z_s}}[g(t + s, X_r)] \quad \text{a.s.}$$

The process (D, Y), started at $(0, x_0)$, satisfies

$$\mathbf{E}[g(D_{s+t}, Y_{s+t})|\mathcal{F}_s] = \mathbf{E}[g(t + s, X_{Z_{s+t}})|\mathcal{F}_s] = Q_t g(s, X_{Z_s}) = Q_t g(D_s, Y_s)$$

and is therefore Markov with the semigroup $(Q_t)_{t\geq 0}$.

2. Feller property. Since (D, Y) and Z are right-continuous, identity (2) implies that $\lim_{t\searrow 0} Q_t f(s, x) = f(s, x)$ for each $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^n$. It is well-known that in this case pointwise convergence implies convergence in the Banach space $(C_0(\mathbb{R}_+ \times \mathbb{R}^n), \|\cdot\|_\infty)$. It also follows from representation (2), the dominated convergence theorem and the Feller property of X that a continuous function $(s, x) \mapsto Q_t g(s, x)$ tends to zero at infinity for any $g \in C_0(\mathbb{R}_+ \times \mathbb{R}^n)$. Hence (D, Y) is a Feller process.

3. Infinitesimal generator of the semigroup $(Q_t)_{t\geq 0}$. As before, let $\lambda_{s,s+t}$ be the law of the increment $Z_{s+t} - Z_s$ and let ψ_s be as in (1). Let $(t_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $(0, \infty)$ that converges to zero. Denote by $\widehat{\mu}_n$ the Laplace transform of a compound Poisson process with Lévy measure $t_n^{-1}\lambda_{s,s+t_n}$. Hence we find for any $u \in \mathbb{C}$ that satisfies $\Re(u) \geq 0$

$$\begin{aligned} \widehat{\mu}_n(u) &= \exp\left(\frac{1}{t_n}\int_0^\infty (e^{-ur} - 1)\lambda_{s,s+t_n}(dr)\right) \\ &= \exp\left(t_n^{-1}(e^{\int_s^{s+t_n}\psi_v(u)dv} - 1)\right). \end{aligned}$$

Since the function $t \mapsto \int_0^t \psi_{s+v}(u)dv$ is right-differentiable at zero with derivative $\psi_s(u)$, we get

$$\lim_{n\to\infty}\widehat{\mu}_n(u) = \exp(\psi_s(u)).$$

It is clear from (1) that $\exp(\psi_s(u))$ is a Laplace transform of an infinitely divisible distribution with Lévy measure $g(s, r)dr$. Therefore by Theorem 8.7 in [6] for every continuous bounded function $k : \mathbb{R} \to \mathbb{R}$ that vanishes on a neighbourhood of zero we get

$$(3) \qquad \lim_{n\to\infty} t_n^{-1}\int_0^\infty k(r)\lambda_{s,s+t_n}(dr) = \int_0^\infty k(r)g(s, r)dr.$$

Furthermore the same theorem implies that for any continuous function h such that $h(r) = 1 + o(|r|)$ for $|r| \to 0$ and $h(r) = O(1/|r|)$ for $|r| \to \infty$ we have

$$(4) \qquad \lim_{n\to\infty} t_n^{-1}\int_0^\infty rh(r)\lambda_{s,s+t_n}(dr) = \beta(s) + \int_0^\infty rh(r)g(s, r)dr.$$

A key observation is that (3) and (4) together imply that (3) holds for every continuous bounded function k that satisfies $k(r) = o(|r|)$ as $r \searrow 0$.

Claim. Let the function $f \in C_0(\mathbb{R}_+ \times \mathbb{R}^n)$ satisfy the assumptions of Theorem 1. Then for any $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ the limit holds

$$
\begin{aligned}
\lim_{t \searrow 0} t^{-1}(Q_t f - f)(s, x) &= \frac{\partial f}{\partial s}(s, x) + \beta(s)\mathcal{L}f_s(x) \\
&+ \int_0^\infty [P_r f_s(x) - f_s(x)]g(s, r)\mathrm{d}r.
\end{aligned}
$$

To prove this claim recall first that $(P_t)_{t \geq 0}$ is the semigroup of X and note that the identity holds

$$
(Q_t f - f)(s, x) = \mathbf{E}^{s,x}[f(D_t, Y_t) - f(D_0, Y_0)] + \int_0^\infty [P_r f_s(x) - f_s(x)]\lambda_{s,s+t}(\mathrm{d}r).
$$

If we divide this expression by t and take the limit as $t \searrow 0$, the first term converges to the partial derivative $\frac{\partial f}{\partial s}(s, x)$ by the dominated convergence theorem (recall that the paths of Y are right-continuous).

Choose a function h as above, define $D(r) := P_r f_s(x) - f_s(x)$ and express the second term as

$$
\begin{aligned}
t^{-1}\int_0^\infty D(r)\lambda_{s,s+t}(\mathrm{d}r) &= t^{-1}\int_0^\infty D(r)(1 - h(r))\lambda_{s,s+t}(\mathrm{d}r) \\
&+ t^{-1}\int_0^\infty (D(r) - r\mathcal{L}f_s(x))h(r)\lambda_{s,s+t}(\mathrm{d}r) \\
&+ \mathcal{L}f_s(x)t^{-1}\int_0^\infty rh(r)\lambda_{s,s+t}(\mathrm{d}r).
\end{aligned}
$$

The first and second integrals on the right-hand side converge by (3) to

$$
\int_0^\infty D(r)(1 - h(r))g(s, r)\mathrm{d}r \quad \text{and} \quad \int_0^\infty (D(r) - r\mathcal{L}f_s(x))h(r)g(s, r)(\mathrm{d}r)
$$

respectively and the third integral converges by (4) to

$$
\mathcal{L}f_s(x)\int_0^\infty rh(r)g(s, r)\mathrm{d}r.
$$

This proves the claim.

Since $(Q_t)_{t \geq 0}$ is a strongly continuous contraction semigroup on the function space $C_0(\mathbb{R}_+ \times \mathbb{R}^n)$ with some generator $\mathcal{L}'$, if the pointwise limit in the claim exists and is in $C_0(\mathbb{R}_+ \times \mathbb{R}^n)$ for some continuous function f that vanishes at infinity, then f is in the domain of $\mathcal{L}'$ and $\mathcal{L}'f$ equals this limit (see e.g. Lemma 31.7 in [6]). This concludes the proof of the theorem.

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