MARTINGALE PROPERTY OF GENERALIZED STOCHASTIC EXPONENTIALS

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Abstract. For a real Borel measurable function b, which satisfies certain integrability conditions, it is possible to define a stochastic integral of the process $b(Y)$ with respect to a Brownian motion W, where Y is a diffusion driven by W. It is well know that the stochastic exponential of this stochastic integral is a local martingale. In this paper we consider the case of an arbitrary Borel measurable function b where it may not be possible to define the stochastic integral of $b(Y)$ directly. However the notion of the stochastic exponential can be generalized. We define a non-negative process Z, called *generalized stochastic exponential*, which is not necessarily a local martingale. Our main result gives deterministic necessary and sufficient conditions for Z to be a local, true or uniformly integrable martingale.

1. Introduction

A *stochastic exponential* of X is a process $\mathcal{E}(X)$ defined by

$$\mathcal{E}(X)_t = \exp\left\{X_t - X_0 - \frac{1}{2}\langle X\rangle_t\right\}$$

for some continuous local martingale X, where $\langle X\rangle$ denotes a quadratic variation of X. It is well known that the process $\mathcal{E}(X)$ is also a continuous local martingale. Sufficient conditions for the martingale property of $\mathcal{E}(X)$ have been studied extensively in the literature because this question appears naturally in many situations. Novikov's and Kazamaki's sufficient conditions (see [9] and [6]) for $\mathcal{E}(X)$ to be a martingale are particularly well-known. Novikov's condition consists in $\mathbb{E}\exp\{(1/2)\langle X\rangle_t\} < \infty$, $t \in [0,\infty)$. Kazamaki's condition consists in that $\exp\{(1/2)X\}$ should be a submartingale. Novikov's criterion is of narrower scope than Kazamaki's one but often easier to apply. In this respect let us note that none of conditions $\mathbb{E}\exp\{(1/2-\varepsilon)\langle X\rangle_t\} < \infty$ (with $\varepsilon > 0$) is sufficient for $\mathcal{E}(X)$ to be a martingale (see Liptser and Shiryaev [7, Ch. 6]). For a further literature review see the bibliographical notes in the monographs Karatzas and Shreve [5, Ch. 3], Liptser and Shiryaev [7, Ch. 6], Protter [10, Ch. III], and Revuz and Yor [11, Ch. VIII].

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In the case of one dimensional processes, necessary and sufficient conditions for the process $\mathcal{E}(X)$ to be a martingale were recently studied by Engelbert and Senf in [3], Blei and Engelbert in [1] and Mijatović and Urusov in [8]. In [3] X is a general continuous local martingale and the characterisation is given in terms of the Dambis-Dubins-Schwartz time-change that turns X into a Brownian motion. In [1] X is a strong Markov continuous local martingale and the condition is deterministic, expressed in terms of the speed measure of X.

In [8] the local martingale X is of the form $X_t = \int_0^t b\,(Y_u)\,\mathrm{d}W_u$ for some measurable function b and a one-dimensional diffusion Y with drift μ and volatility σ driven by a Brownian motion W. In order to define the stochastic integral X, an assumption that the function $\frac{b^2}{\sigma^2}$ is locally integrable on the entire state space of the process Y is required. Under this restriction the characterization of the martingale property of $\mathcal{E}(X)$ is studied in [8], where the necessary and sufficient conditions are deterministic and are expressed in terms of functions μ, σ and b only.

In the present paper we consider an arbitrary Borel measurable function b. In this case the stochastic integral X can only be defined on some subset of the probability space. However, it is possible to define a non-negative possibly discontinuous process Z, known as a generalized stochastic exponential, on the entire probability space. It is a consequence of the definition that, if the function b satisfies the required local integrability condition, the process Z coincides with $\mathcal{E}(X)$. We show that the process Z is not necessarily a local martingale. In fact Z is a local martingale if and only if it is continuous. We find a deterministic necessary and sufficient condition for Z to be a local martingale, which is expressed in terms of local integrability of the quotient $\frac{b^2}{\sigma^2}$ multiplied by a linear function. We also characterize the processes Z that are true martingales and/or uniformly integrable martingales. All the necessary and sufficient conditions are deterministic and are given in terms of functions μ, σ and b.

The paper is structured as follows. In Section 2 we define the notion of generalized stochastic exponential and study its basic properties. The main results are stated in Section 3, where we give a necessary and sufficient condition for the process Z defined by (8) and (12) to be a local martingale, a true martingale or a uniformly integrable martingale. Finally, in Section 4 we prove Theorem 3.4 that is central in obtaining the deterministic characterisation of the martingale property of the process Z. Appendix A contains an auxiliary fact that is used in Section 2.

2. Definition of Generalized Stochastic Exponential

Let $J = (l, r)$ be our state space, where $-\infty \le l < r \le \infty$. Let us define a J-valued diffusion Y on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ driven by a stochastic differential equation

$$\mathrm{d}Y_t = \mu(Y_t)\,\mathrm{d}t + \sigma(Y_t)\,\mathrm{d}W_t, \quad Y_0 = x_0 \in J,$$

where W is a $(\mathcal{F}_t)$-Brownian motion, and μ and σ real, Borel measurable functions defined on J that satisfy the Engelbert–Schmidt conditions

$$\sigma(x) \neq 0 \quad \forall x \in J, \tag{1}$$

$$\frac{1}{\sigma^2}, \frac{\mu}{\sigma^2} \in L^1_{\mathrm{loc}}(J). \tag{2}$$

With $L^1_{\mathrm{loc}}(J)$ we denote the class of locally integrable functions, i.e. real Borel measurable functions defined on J that are integrable on every compact subset of J. Engelbert–Schmidt conditions guarantee existence of a weak solution that might exit the interval J and is unique in law (see [5, Chapter 5]). Denote by ζ the exit time of Y. In addition, we assume that the boundary points are absorbing, i.e. the solution Y stays at the boundary point at which it exits on the set $\{\zeta < \infty\}$. Let us note that we assume that $(\mathcal{F}_t)$ is generated neither by Y nor by W.

We would like to define a process X as a stochastic integral of a process $b(Y)$ with respect to Brownian motion W, where $b : J \to \mathbb{R}$ is an arbitrary Borel measurable function. Before further discussion, we should establish if the stochastic integral can be defined.

Define a set

$$A = \{x \in J;\ \tfrac{b^2}{\sigma^2} \notin L^1_{\mathrm{loc}}(x)\},$$

where $L^1_{\mathrm{loc}}(x)$ denotes a space of real, Borel measurable functions f with $\int_{x-\varepsilon}^{x+\varepsilon} |f(y)|\mathrm{d}y < \infty$ for some $\varepsilon > 0$. Then A is closed and its complement is a union of open intervals. Furthermore we can define maps α and β on $J \setminus A$ so that

$$\alpha(x), \beta(x) \in A \cup \{l, r\} \quad \text{and} \quad x \in (\alpha(x), \beta(x)) \subseteq J \setminus A. \tag{3}$$

In other words $\alpha(x)$ is the point in $A \cup \{l, r\}$ that is closest to x from the left side and $\beta(x)$ is the closest point in $A \cup \{l, r\}$ from the right side. Let us also note that the equality

$$L^1_{\mathrm{loc}}(I) = \bigcap_{x\in I} L^1_{\mathrm{loc}}(x) \tag{4}$$

holds for any interval I by a simple compactness argument.

For any $x, y \in J$ we define the stopping times

$$\tau_x = \inf\{t \geq 0;\ Y_t = x\}, \tag{5}$$

$$\tau_{x,y} = \tau_x \wedge \tau_y, \tag{6}$$

with the convention $\inf \emptyset = \infty$, where $c \wedge d := \min\{c, d\}$. Define the stopping time $\zeta^A = \zeta \wedge \tau_A$, where

$$\tau_A = \inf\{t \geq 0;\ Y_t \in A\}. \tag{7}$$

Then for all $t \geq 0$ we have

$$\int_0^t b^2 (Y_u)\, \mathrm{d}u < \infty \ \ \mathbb{P}\text{-a.s. on } \{t < \zeta^A\}.$$

This follows from Proposition A.1 and the fact that a continuous process Y on $\{t < \zeta^A\}$ reaches only values in an open interval that is a component of the complement of A, where $\frac{b^2}{\sigma^2}$ is locally integrable (note that (4) is applied here).

Let us define $A_n = \{x \in J;\ \rho(x, A \cup \{l, r\}) \leq \frac{1}{n}\}$, where $\rho(x, y) = |\arctan x - \arctan y|$, $x, y \in \bar{J}$, and set $\zeta_n^A = \inf\{t \geq 0;\ Y_t \in A_n\}$. Since $\zeta_n^A < \zeta^A$ on the set $\{\zeta^A < \infty\}$, we have $\int_0^{t\wedge\zeta_n^A} b^2(Y_u)\mathrm{d}u < \infty$ $\mathbb{P}$-a.s. Thus, we can define the stochastic integral $\int_0^{t\wedge\zeta_n^A} b(Y_u)\mathrm{d}W_u$ for every n. Since the integrals $\int_0^{t\wedge\zeta_n^A} b(Y_u)\mathrm{d}W_u$ and $\int_0^{t\wedge\zeta_{n+1}^A} b(Y_u)\mathrm{d}W_u$ coincide on $\{t < \zeta_n^A\}$ and $\zeta_n^A \nearrow \zeta^A$, we can define $\int_0^{t\wedge\zeta^A} b(Y_u)\mathrm{d}W_u$ as a $\mathbb{P}$-a.s limit of integrals

$$\int_0^{t\wedge\zeta^A} b(Y_u)\mathrm{d}W_u = \lim_{n\to\infty} \int_0^{t\wedge\zeta_n^A} b(Y_u)\mathrm{d}W_u \text{ on } \left\{t < \zeta^A\right\} \cup \left\{\int_0^{\zeta^A} b^2(Y_u)\mathrm{d}u < \infty\right\}.$$

In the case where A is not empty or Y exits the interval J, the stochastic exponential cannot be defined. However, we can define a *generalized stochastic exponential* Z in the following way for every $t \in [0, \infty)$

$$Z_t = \begin{cases} \exp\{\int_0^t b(Y_u)\mathrm{d}W_u - \frac{1}{2}\int_0^t b^2(Y_u)\mathrm{d}u\} & , t < \zeta^A \\ \exp\{\int_0^\zeta b(Y_u)\mathrm{d}W_u - \frac{1}{2}\int_0^\zeta b^2(Y_u)\mathrm{d}u\} & , t \geq \zeta^A = \zeta, \int_0^\zeta b^2(Y_u)\,\mathrm{d}u < \infty \\ 0 & , t \geq \zeta^A = \tau_A \text{ or} \\ & \ \ t \geq \zeta^A = \zeta, \int_0^\zeta b^2(Y_u)\,\mathrm{d}u = \infty \end{cases} \tag{8}$$

The different behaviour of Z on $\{t \geq \zeta^A = \zeta\}$ and $\{t \geq \zeta^A = \tau_A\}$ follows from the fact that after the exit time ζ the process Y is stopped, while this does not happen after τ_A. The definition of the set A and Proposition A.1 imply that the integral $\int_0^t b^2(Y_u)\,\mathrm{d}u$ is infinite on the event $\{t > \tau_A\}$ $\mathbb{P}$-a.s. Therefore, we set $Z = 0$ on the set $\{t \geq \zeta^A = \tau_A\}$.

Let us define the processes

$$\bar{Z}_t = \exp\left\{\int_0^{t\wedge\zeta^A} b\left(Y_u\right)\mathrm{d}W_u - \frac{1}{2}\int_0^{t\wedge\zeta^A} b^2\left(Y_u\right)\mathrm{d}u\right\}, \tag{9}$$

where we set $\bar{Z}_t = 0$ for $t \geq \zeta^A$ on $\{\zeta^A < \infty, \int_0^{\zeta^A} b^2(Y_u)\mathrm{d}u = \infty\}$. The process $\bar{Z}$ has continuous trajectories: continuity at time ζ^A on the set $\{\zeta^A < \infty, \int_0^{\zeta^A} b^2(Y_u)\,du = \infty\}$ follows from the Dambis–Dubins–Schwarz theorem on stochastic intervals; see Th. 1.6 and Ex. 1.18 [11, Ch. V]. Note further that $\bar{Z}_t$ is strictly positive on the event $\left\{t < \zeta^A\right\} \cup \left\{\int_0^{\zeta^A} b^2(Y_u)\mathrm{d}u < \infty\right\}$ $\mathbb{P}$-a.s. and is equal to zero on its complement.

Lemma 2.1. *The process $\bar{Z} = (\bar{Z}_t)_{t\geq 0}$ defined in* (9) *is a continuous local martingale.*

Remark. Fatou's lemma and Lemma 2.1 imply that the process $\bar{Z}$ is a continuous non-negative supermartingale.

Proof. If the starting point x_0 of the diffusion Y is in A, then $\tau_A = 0$ $\mathbb{P}$-a.s and $\bar{Z}$ is constant and hence a local martingale. If $x_0 \in J \setminus A$, then pick a decreasing (resp. increasing) sequence $(\alpha_n)_{n\in\mathbb{N}}$ (resp. $(\beta_n)_{n\in\mathbb{N}}$) in the open interval $(\alpha(x_0), \beta(x_0))$ such that $\alpha_n \searrow \alpha(x_0)$ (resp. $\beta_n \nearrow \beta(x_0)$), where $\alpha(x_0), \beta(x_0)$ are defined in (3). Assume also that $\alpha_n < x_0 < \beta_n$ for all $n \in \mathbb{N}$. Note that $\tau_{\alpha_n,\beta_n} \nearrow \zeta^A$ $\mathbb{P}$-a.s.

The process $M^n = (M^n_t)_{t\in[0,\infty)}$, defined by

$$M^n_t = \int_0^{t\wedge\tau_{\alpha_n,\beta_n}} b(Y_u)\mathrm{d}W_u \qquad \text{for all} \quad t \in [0,\infty),$$

is a local martingale. Therefore its stochastic exponential $\mathcal{E}(M^n)$ is also a local martingale and, since $\tau_{\alpha_n,\beta_n} < \zeta^A$ $\mathbb{P}$-a.s., the equality $\bar{Z}_{t\wedge\tau_{\alpha_n,\beta_n}} = \mathcal{E}(M^n)_t$ holds $\mathbb{P}$-a.s. for all $t \geq 0$. For any $m \in \mathbb{N}$ define the stopping time $\eta_m = \inf\{t \geq 0; \bar{Z}_t \geq m\}$. The stopped process $\bar{Z}^{\eta_m\wedge\tau_{\alpha_n,\beta_n}}$ is a bounded local martingale and hence a martingale. Furthermore note that $\eta_m \nearrow \infty$ $\mathbb{P}$-a.s. as m tends to infinity. To prove that $\bar{Z}^{\eta_m}$ is a martingale for each $m \in \mathbb{N}$, note that the process $\bar{Z}$ is stopped at ζ^A, which implies the almost sure limit $\lim_{n\to\infty} \bar{Z}_t^{\eta_m\wedge\tau_{\alpha_n,\beta_n}} = \bar{Z}_t^{\eta_m}$ for every $t \in [0,\infty)$. Since $\bar{Z}_t^{\eta_m\wedge\tau_{\alpha_n,\beta_n}} \leq m$ $\mathbb{P}$-a.s., the conditional dominated convergence theorem implies the martingale property of $\bar{Z}^{\eta_m}$ for every $m \in \mathbb{N}$. This concludes the proof. □

Define the process $S = (S_t)_{t\in[0,\infty)}$ by

$$S_t = \exp\left\{\int_0^{\tau_A} b\left(Y_u\right)\mathrm{d}W_u - \frac{1}{2}\int_0^{\tau_A} b^2\left(Y_u\right)\mathrm{d}u\right\}\mathbb{1}_{\{t\geq\tau_A, \int_0^{\tau_A} b^2(Y_u)\mathrm{d}u<\infty\}}. \tag{10}$$

Note that for any $t \in [0, \infty)$, $\mathbb{P}$-a.s. on the set $\{\tau_A \le t\}$ we have $\tau_A < \zeta$ and hence the integrals in (10) are well-defined. We can express Z as

$$Z = \bar{Z} - S. \tag{11}$$

It is clear from this representation that Z is not necessarily a continuous process. Moreover, since that paths of S are non-decreasing, we have

$$\mathbb{E}[Z_t|\mathcal{F}_s] \le \bar{Z}_s - \mathbb{E}[S_t|\mathcal{F}_s] \le \bar{Z}_s - S_s = Z_s.$$

It follows that Z is a non-negative supermartingale and we can define

$$Z_\infty = \lim_{t\to\infty} Z_t. \tag{12}$$

Note further that if $x_0 \in A$, we have $Z \equiv 0$.

A path of the process Z defined by (8) and (12) is equal to a path of a stochastic exponential if $\zeta^A = \infty$. Otherwise, if $\zeta^A < \infty$, it has one of the following forms:

(*i*) $\tau_A < \zeta$ and $\int_0^{\tau_A} b^2(Y_t)\mathrm{d}t < \infty$ (see Figure 1);
(*ii*) $\zeta^A < \infty$ and $\int_0^{\zeta^A} b^2(Y_t)\mathrm{d}t = \infty$ (see Figure 2);
(*iii*) $\zeta < \tau_A$ and $\int_0^{\zeta} b^2(Y_t)\mathrm{d}t < \infty$ (see Figure 3).

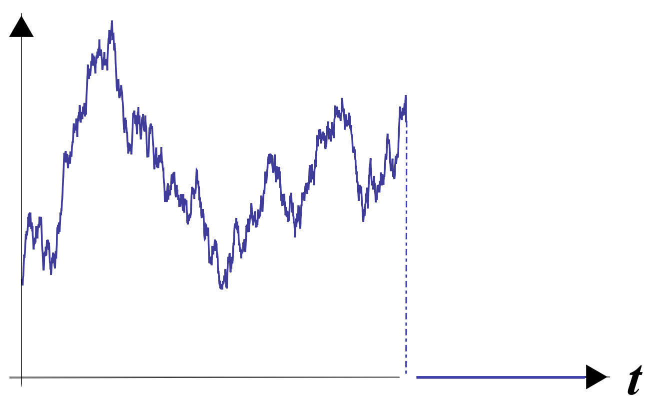

FIGURE 1. If $\tau_A < \zeta$, then the process Z is positive up to time τ_A and is equal to zero afterwards. If the integral $\int_0^{\tau_A} b^2(Y_t)\mathrm{d}t$ is finite, then Z_t approaches a positive value as t approaches τ_A. Therefore, there is a jump at $t = \tau_A$.

3. Main Results

The case $A = \emptyset$ was studied by Mijatović and Urusov in [8]. We generalize their result for the case where $A \neq \emptyset$.

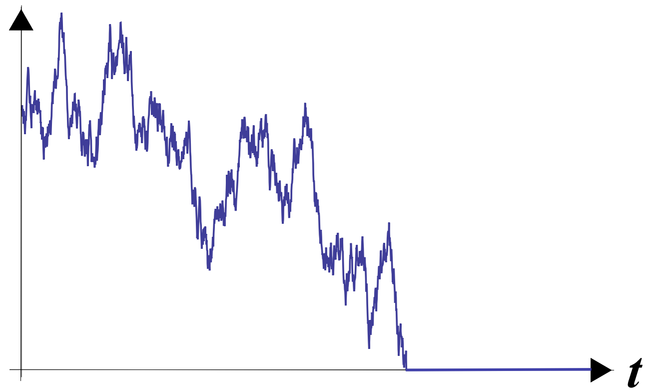

FIGURE 2. If $\zeta^A < \infty$ and $\int_0^{\zeta^A} b^2(Y_t)\mathrm{d}t = \infty$, then the process Z is zero after the time ζ^A. Since the limit of Z_t is zero as t approaches ζ^A, there is no jump.

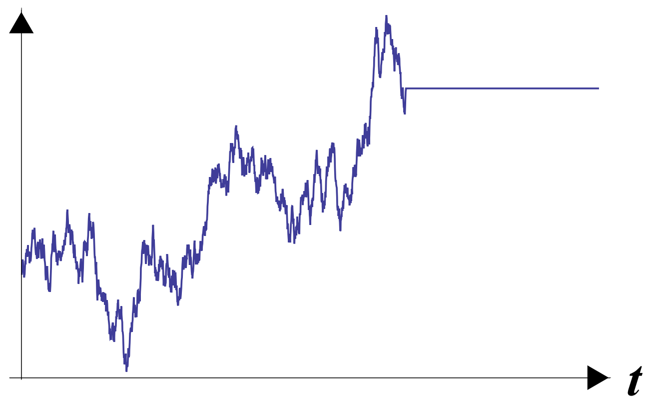

FIGURE 3. If $\zeta < \tau_A$, the process Z is stopped after the exit time. Since $\int_0^{\zeta} b^2(Y_t)\mathrm{d}t$ is finite, Z_t is equal to a positive constant for $t \geq \zeta$.

3.1. **The Case $A = \emptyset$.** In this case we have

$$\frac{b^2}{\sigma^2} \in L^1_{\mathrm{loc}}(J). \tag{13}$$

The generalized stochastic exponential Z defined by (8) and (12) can now be written as

$$Z_t = \exp\left\{\int_0^{t\wedge\zeta} b\left(Y_u\right)\mathrm{d}W_u - \frac{1}{2}\int_0^{t\wedge\zeta} b^2\left(Y_u\right)\mathrm{d}u\right\},$$

where we set $Z_t = 0$ for $t \geq \zeta$ on $\{\zeta < \infty, \int_0^{\zeta} b^2\left(Y_u\right)\mathrm{d}u = \infty\}$. Note that in this case Z is a local martingale by Lemma 2.1, since in this case $\zeta^A = \zeta$ and hence $Z = \bar{Z}$.

Let us now define an auxiliary J-valued diffusion $\widetilde{Y}$ governed by the SDE

$$\mathrm{d}\widetilde{Y}_t = (\mu + b\sigma)\left(\widetilde{Y}_t\right)\mathrm{d}t + \sigma\left(\widetilde{Y}_t\right)\mathrm{d}\widetilde{W}_t, \quad \widetilde{Y}_0 = x_0,$$

on some probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\in[0,\infty)}, \widetilde{\mathbb{P}})$. The coefficients $\mu + b\sigma$ and σ satisfy Engelbert–Schmidt conditions since $\frac{b}{\sigma} \in L^1_{\mathrm{loc}}(J)$ (this follows from (13)). Hence the SDE has a weak solution, unique in law and possibly explosive. As with diffusion Y, we denote by $\widetilde{\zeta}$ the exit time of $\widetilde{Y}$ and assume that the boundary points are absorbing.

For an arbitrary $c \in J$ we define the scale functions $s, \widetilde{s}$ and their derivatives $\rho, \widetilde{\rho}$:

$$\begin{aligned}
\rho(x) &= \exp\left\{-\int_c^x \frac{2\mu(y)}{\sigma^2(y)}\,\mathrm{d}y\right\}, \quad x \in J,\\
\widetilde{\rho}(x) &= \rho(x)\exp\left\{-\int_c^x \frac{2b(y)}{\sigma(y)}\,\mathrm{d}y\right\}, \quad x \in J,\\
s(x) &= \int_c^x \rho\,(y)\,\mathrm{d}y, \quad x \in \bar{J},\\
\widetilde{s}(x) &= \int_c^x \widetilde{\rho}\,(y)\,\mathrm{d}y, \quad x \in \bar{J}.
\end{aligned} \tag{14}$$

For any $a \in (l, r]$, $L^1_{\mathrm{loc}}(a-)$ denotes the set of all Borel measurable functions $f : J \to \mathbb{R}$ such that there exists $b < a$, $b \in J$, with $\int_b^a |f\,(x)|\,\mathrm{d}x < \infty$. Similarly, we define the set $L^1_{\mathrm{loc}}(a+)$ for any $a \in [l, r)$.

We say that the endpoint r is *good* if

$$s(r) < \infty \quad \text{and} \quad \frac{(s(r) - s)b^2}{\rho\sigma^2} \in L^1_{\mathrm{loc}}(r-).$$

It is equivalent to show that

$$\widetilde{s}(r) < \infty \quad \text{and} \quad \frac{(\widetilde{s}(r) - \widetilde{s})b^2}{\widetilde{\rho}\sigma^2} \in L^1_{\mathrm{loc}}(r-).$$

The endpoint l is *good* if

$$s(l) > -\infty \quad \text{and} \quad \frac{(s - s(l))b^2}{\rho\sigma^2} \in L^1_{\mathrm{loc}}(l+),$$

or equivalently

$$\widetilde{s}(l) > -\infty \quad \text{and} \quad \frac{(\widetilde{s} - \widetilde{s}(l))b^2}{\widetilde{\rho}\sigma^2} \in L^1_{\mathrm{loc}}(l+).$$

If an endpoint is not good, we say it is *bad*. The good and bad endpoints were introduced in [8], where one can also find the proof of equivalences above.

We will use the following terminology:

$$\begin{aligned}
&\widetilde{Y}\ \textit{exits at } r \text{ means } \widetilde{\mathbb{P}}(\widetilde{\zeta} < \infty, \lim_{t\nearrow\widetilde{\zeta}} \widetilde{Y}_t = r) > 0;\\
&\widetilde{Y}\ \textit{exits at } l \text{ means } \widetilde{\mathbb{P}}(\widetilde{\zeta} < \infty, \lim_{t\nearrow\widetilde{\zeta}} \widetilde{Y}_t = l) > 0.
\end{aligned}$$

Define

$$\widetilde{v}(x) = \int_c^x \frac{\widetilde{s}(x) - \widetilde{s}(y)}{\widetilde{\rho}(y)\sigma^2(y)}\,\mathrm{d}y, \quad x \in J, \tag{15}$$

and

$$\widetilde{v}(r) = \lim_{x \nearrow r} \widetilde{v}(x), \quad \widetilde{v}(l) = \lim_{x \searrow l} \widetilde{v}(x) \tag{16}$$

(note that $\widetilde{v}$ is decreasing on $(l, c]$ and increasing on $[c, r)$).

Feller's test for explosions (see [5, Chapter 5, Theorem 5.29]) tells us that:

(i) $\widetilde{Y}$ exits at the boundary point r if and only if

$$\widetilde{v}(r) < \infty.$$

It is equivalent to check (see [2, Chapter 4.1])

$$\widetilde{s}(r) < \infty \quad \text{and} \quad \frac{\widetilde{s}(r) - \widetilde{s}}{\widetilde{\rho}\sigma^2} \in L^1_{\text{loc}}(r-);$$

(ii) $\widetilde{Y}$ exits at the boundary point l if and only if

$$\widetilde{v}(l) < \infty,$$

which is equivalent to

$$\widetilde{s}(l) > -\infty \quad \text{and} \quad \frac{\widetilde{s} - \widetilde{s}(l)}{\widetilde{\rho}\sigma^2} \in L^1_{\text{loc}}(l+).$$

Remark. The endpoint r (resp. l) is bad whenever one of the processes Y and $\widetilde{Y}$ exits at r (resp. l) and the other does not.

The following theorems are reformulations of Theorems 2.1 and 2.3 in [8].

Theorem 3.1. *Let the functions μ, σ and b satisfy conditions* (1), (2) *and* (13). *Then the process Z is a martingale if and only if $\widetilde{Y}$ does not exit at the bad endpoints.*

Theorem 3.2. *Let the functions μ, σ and b satisfy conditions* (1), (2) *and* (13). *Then Z is a uniformly integrable martingale if and only if one of the conditions* $(a) - (d)$ *below is satisfied:*

(a) *$b = 0$ a.e. on J with respect to the Lebesgue measure;*

(b) *r is good and $\widetilde{s}(l) = -\infty$;*

(c) *l is good and $\widetilde{s}(r) = \infty$;*

(d) *l and r are good.*

3.2. **The Case $A \neq \emptyset$.** In the rest of the paper we assume that

$$x_0 \notin A,$$

since $x_0 \in A$ implies that $Z \equiv 0$. The following example shows that even when A is not empty we can get a martingale or a uniformly integrable martingale defined by (8) and (12).

Example 3.3. (i) Let us consider the state space $J = \mathbb{R}$, coefficients of the SDE $\mu = 0, \sigma = 1$, starting point of the diffusion $x_0 > 0$ and function $b(x) = \frac{1}{x}$ for $x \in \mathbb{R} \setminus \{0\}$ and $b(0) = 0$. Then $A = \{0\}$ and $Y_t = W_t, W_0 = x_0$. Using Itô's formula and the fact that Brownian motion does not exit at infinity, we get for $t < \tau_0$

$$\begin{aligned} Z_t &= \exp\left\{ \int_0^t \frac{1}{W_u}\,\mathrm{d}W_u - \frac{1}{2}\int_0^t \frac{1}{W_u^2}\,\mathrm{d}u \right\} \\ &= \frac{1}{x_0} W_t \end{aligned}$$

and $Z_t = 0$ for $t \geq \tau_0$. Hence, $Z_t = \frac{1}{x_0} W_{t\wedge\tau_0}$ that is a martingale.

(ii) Using the same functions μ, σ and b as above on a state space $J = (-\infty, x_0 + 1)$ we get

$$Z_t = \frac{1}{x_0} W_{t\wedge\tau_{0,x_0+1}},$$

which is a uniformly integrable martingale.

Let for any $x \in J \setminus A$ the points $\alpha(x), \beta(x) \in A \cup \{l, r\}$ be as in (3). Then $\frac{b^2}{\sigma^2} \in L^1_{\mathrm{loc}}(\alpha(x), \beta(x))$. Therefore, on $(\alpha(x), \beta(x))$ functions μ, σ and b satisfy the same conditions as in the previous subsection.

For any starting point $x_0 \in J \setminus A$ we can define an auxiliary diffusion $\widetilde{Y}$ with state space $(\alpha(x_0), \beta(x_0))$ driven by the SDE

$$\mathrm{d}\widetilde{Y}_t = (\mu + b\sigma)\left(\widetilde{Y}_t\right)\mathrm{d}t + \sigma\left(\widetilde{Y}_t\right)\mathrm{d}\widetilde{W}_t, \quad \widetilde{Y}_0 = x_0,$$

on some probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\in[0,\infty)}, \widetilde{\mathbb{P}})$. There exists a unique weak solution of this equation since coefficients satisfy the Engelbert–Schmidt conditions.

As in the previous subsection we can define good and bad endpoints. Let functions $\rho, \widetilde{\rho}, s, \widetilde{s}$ and $\widetilde{v}$ be defined by (14), (15) and (16) with $c \in (\alpha(x_0), \beta(x_0))$. We say that the endpoint $\beta(x_0)$ is *good* if

$$s(\beta(x_0)) < \infty \quad \text{and} \quad \frac{(s(\beta(x_0)) - s)b^2}{\rho\sigma^2} \in L^1_{\mathrm{loc}}(\beta(x_0)-).$$

It is equivalent to show and sometimes easier to check that

$$\widetilde{s}(\beta(x_0)) < \infty \quad \text{and} \quad \frac{(\widetilde{s}(\beta(x_0)) - \widetilde{s})b^2}{\widetilde{\rho}\sigma^2} \in L^1_{\text{loc}}(\beta(x_0)-).$$

The endpoint $\alpha(x_0)$ is *good* if

$$s(\alpha(x_0)) > -\infty \quad \text{and} \quad \frac{(s - s(\alpha(x_0)))b^2}{\rho\sigma^2} \in L^1_{\text{loc}}(\alpha(x_0)+),$$

or equivalently

$$\widetilde{s}(\alpha(x_0)) > -\infty \quad \text{and} \quad \frac{(\widetilde{s} - \widetilde{s}(\alpha(x_0)))b^2}{\widetilde{\rho}\sigma^2} \in L^1_{\text{loc}}(\alpha(x_0)+).$$

If an endpoint is not good, we say it is *bad*.

The following theorem plays a key role in all that follows.

Theorem 3.4. *Let $\alpha : J \setminus A \to A \cup \{l, r\}$ be the function defined by* (3) *and assume that the starting point $x_0 \in J \setminus A$ of diffusion Y satisfies $\alpha(x_0) > l$. Denote $\alpha_0 = \alpha(x_0) \in A$. Then:*

(*a*) $(x - \alpha_0)\frac{b^2}{\sigma^2}(x) \in L^1_{\text{loc}}(\alpha_0+) \iff \int_0^{\tau_{\alpha_0}} b^2(Y_t)\,\mathrm{d}t < \infty$ $\mathbb{P}$-*a.s. on* $\{\tau_{\alpha_0} = \tau_A < \infty\}$;

(*b*) $(x - \alpha_0)\frac{b^2}{\sigma^2}(x) \notin L^1_{\text{loc}}(\alpha_0+) \iff \int_0^{\tau_{\alpha_0}} b^2(Y_t)\,\mathrm{d}t = \infty$ $\mathbb{P}$-*a.s. on* $\{\tau_{\alpha_0} = \tau_A < \infty\}$.

Remarks. (i) The assumption $\alpha(x_0) > l$ means that Theorem 3.4 deals with the situation where the set A contains points to the left of the starting point x_0.

(ii) Clearly, Theorem 3.4 has its analogue for $\beta_0 = \beta(x_0) < r$, i.e. the case when A contains points to the right of x_0. The deterministic criterion in this case takes the form

$$(\beta_0 - x)\frac{b^2}{\sigma^2}(x) \in L^1_{\text{loc}}(\beta_0-) \iff \int_0^{\tau_{\beta_0}} b^2(Y_t)\,\mathrm{d}t < \infty \ \mathbb{P}\text{-a.s. on } \{\tau_{\beta_0} = \tau_A < \infty\}, \tag{17}$$

$$(\beta_0 - x)\frac{b^2}{\sigma^2}(x) \notin L^1_{\text{loc}}(\beta_0-) \iff \int_0^{\tau_{\beta_0}} b^2(Y_t)\,\mathrm{d}t = \infty \ \mathbb{P}\text{-a.s. on } \{\tau_{\beta_0} = \tau_A < \infty\}. \tag{18}$$

(iii) Note that $\mathbb{P}(\tau_{\alpha_0} = \tau_A < \infty) > 0$. Indeed, if $\beta_0 = r$, then $\{\tau_{\alpha_0} = \tau_A < \infty\} = \{\tau_{\alpha_0} < \infty\}$; if $\beta_0 < r$, then $\{\tau_{\alpha_0} = \tau_A < \infty\} = \{\tau_{\alpha_0} < \tau_{\beta_0}\}$. In both cases $\mathbb{P}(\tau_{\alpha_0} = \tau_A < \infty) > 0$ by [2, Theorem 2.11].

(iv) Since the diffusion Y starts at x_0, $\mathbb{P}$-a.s. we have the following implications

$$\alpha_0, \beta_0 \in A \implies \tau_A = \tau_{\alpha_0,\beta_0}, \tag{19}$$

$$\alpha_0 \in A, \beta_0 = r \implies \tau_A = \tau_{\alpha_0}, \tag{20}$$

$$\alpha_0 = l, \beta_0 \in A \implies \tau_A = \tau_{\beta_0}. \tag{21}$$

The case $\alpha_0 = l, \beta_0 = r$ cannot occur since $A \neq \emptyset$.

(v) Note that right-hand sides of the equivalences in (*a*) and (*b*) in Theorem 3.4 are not negations of each other. If it does not hold that the integral $\int_0^{\tau_{\alpha_0}} b^2(Y_t)\,\mathrm{d}t$ is finite $\mathbb{P}$-a.s. on the set $\{\tau_{\alpha_0} = \tau_A < \infty\}$, then it must be infinite on some subset of $\{\tau_{\alpha_0} = \tau_A < \infty\}$ of positive probability, which may be strictly smaller than $\mathbb{P}(\tau_{\alpha_0} = \tau_A < \infty)$.

For any starting point $x_0 \in J \setminus A$ of diffusion Y, define the set

$$B(x_0) = \left\{ x \in J \cap \{\alpha_0, \beta_0\}; \int_0^{\tau_x} b^2(Y_t)\,\mathrm{d}t = \infty \;\; \mathbb{P}\text{-a.s. on } \{\tau_x = \tau_A < \infty\} \right\}.$$

Note that $B(x_0)$ is contained in A and that it contains at most two points. Theorem 3.4 implies that $\alpha_0 \in B(x_0)$ if and only if the deterministic condition in (b) is satisfied. Similarly $\beta_0 \in B(x_0)$ is equivalent to the deterministic condition in (18). Therefore Theorem 3.4 yields a deterministic description of the set $B(x_0)$.

We can now give a deterministic characterisation for a generalized stochastic exponential Z to be a local martingale and a true martingale.

Theorem 3.5. *(i) The generalized stochastic exponential Z is a local martingale if and only if $\alpha(x_0), \beta(x_0) \in B(x_0) \cup \{l, r\}$.*

(ii) The generalized stochastic exponential Z is a martingale if and only if Z is a local martingale and at least one of the conditions (a)-(b) below is satisfied and at least one of the conditions (c)-(d) below is satisfied:

(a) *$\widetilde{Y}$ does not exit at $\beta(x_0)$, i.e. $\widetilde{v}(\beta(x_0)) = \infty$ or, equivalently, we have*

$$\widetilde{s}(\beta(x_0)) = \infty \qquad \text{or} \qquad \left(\widetilde{s}(\beta(x_0)) < \infty \text{ and } \frac{\widetilde{s}(\beta(x_0)) - \widetilde{s}}{\widetilde{\rho}\sigma^2} \notin L^1_{\mathrm{loc}}(\beta(x_0)-) \right);$$

(b) *$\beta(x_0)$ is good,*

(c) *$\widetilde{Y}$ does not exit at $\alpha(x_0)$, i.e. $\widetilde{v}(\alpha(x_0)) = -\infty$ or, equivalently, we have*

$$\widetilde{s}(\alpha(x_0)) = -\infty \qquad \text{or} \qquad \left(\widetilde{s}(\alpha(x_0)) > -\infty \text{ and } \frac{\widetilde{s} - \widetilde{s}(\alpha(x_0))}{\widetilde{\rho}\sigma^2} \notin L^1_{\mathrm{loc}}(\alpha(x_0)+) \right);$$

(d) *$\alpha(x_0)$ is good.*

Remark. Part *(ii)* of Theorem 3.5 says that Z is a martingale if and only if the $(\alpha(x_0), \beta(x_0))$-valued process $\widetilde{Y}$ can exit only at the good endpoints.

Proof. *(i)* We can write $Z = \bar{Z} - S$ as in (11). The process $\bar{Z}$ is a continuous local martingale by Lemma 2.1. Suppose that Z is a local martingale. Then S can be written as a sum of two local martingales and therefore, it is also a local martingale. It follows that S is a supermartingale (since it is non-negative). Since $\zeta^A > 0$ (we are assuming that $x_0 \in J \setminus A$) and $S_0 = 0$, S should be almost surely equal to 0. By definition (10) of S

this happens if and only if $\mathbb{P}\left(\tau_A < \infty, \int_0^{\tau_A} b^2(Y_u)\mathrm{d}u < \infty\right) = 0$, which is by the definition of the set $B(x_0)$ and (19)–(21) equivalent to $\alpha(x_0), \beta(x_0) \in B(x_0) \cup \{l, r\}$.

(ii) To get at least a local martingale S needs to be zero $\mathbb{P}$-a.s. Then $Z = \bar{Z}$. Since the values of Y on $[0, \zeta^A)$ do not exit the interval $(\alpha(x_0), \beta(x_0))$, the conditions of Theorem 3.1 are satisfied and the result follows. □

Similarly, we can characterize uniformly integrable martingale. We can use characterization in Theorem 3.2 for the process $\bar{Z}$ defined by (9). As above, for $\alpha(x_0), \beta(x_0) \in B(x_0) \cup \{l, r\}$ the process Z defined by (8) and (12) coincides with $\bar{Z}$. Otherwise, Z is not even a local martingale.

Theorem 3.6. *The process Z is a uniformly integrable martingale if and only if Z is a local martingale and at least one of the conditions $(a) - (d)$ below is satisfied:*

- (a) $b = 0$ *a.e. on* $(\alpha(x_0), \beta(x_0))$ *with respect to the Lebesgue measure;*
- (b) $\alpha(x_0)$ *is good and* $\widetilde{s}(\beta(x_0)) = \infty$*;*
- (c) $\beta(x_0)$ *is good and* $\widetilde{s}(\alpha(x_0)) = -\infty$*;*
- (d) $\alpha(x_0)$ *and* $\beta(x_0)$ *are good.*

The following remark simplifies the application of Theorems 3.5 and 3.6 in specific situations.

Remark. If $\alpha(x_0) \in B(x_0)$, then $\alpha(x_0)$ is not a good endpoint. Indeed, if $s(\alpha(x_0)) > -\infty$, then we can write

$$\frac{(s(x) - s(\alpha(x_0)))b^2(x)}{\rho(x)\sigma^2(x)} = \frac{(s(x) - s(\alpha(x_0)))}{(x - \alpha(x_0))\rho(x)}(x - \alpha(x_0))\frac{b^2}{\sigma^2}(x).$$

The first fraction is bounded away from zero, since it is continuous for $x > \alpha(x_0)$ and has a limit equal to 1 as x approaches $\alpha(x_0)$. Since $\alpha(x_0) \in B(x_0)$, (b) of Theorem 3.4 implies $(x - \alpha(x_0))\frac{b^2}{\sigma^2}(x) \notin L^1_{\mathrm{loc}}(\alpha(x_0)+)$. Therefore $\frac{(s - s(\alpha(x_0)))b^2}{\rho\sigma^2} \notin L^1_{\mathrm{loc}}(\alpha(x_0)+)$ and the conclusion follows. Similarly, $\beta(x_0) \in B(x_0)$ implies that $\beta(x_0)$ is not a good endpoint.

4. Proof of Theorem 3.4

For the proof of Theorem 3.4 we first consider the case of a Brownian motion. Let W be a Brownian motion with $W_0 = x_0$. Denote by $L_t^y(W)$ the local time of W at time t and level y. Let us consider a Borel function $b : \mathbb{R} \to \mathbb{R}$ and set

$$A = \{x \in \mathbb{R} \,:\, b^2 \notin L^1_{\mathrm{loc}}(x)\}.$$

We assume that $x_0 \notin A$ and define α_0, β_0 ($\alpha_0 < \beta_0$) so that

$$\alpha_0, \beta_0 \in A \cup \{-\infty, \infty\} \quad \text{and} \quad x_0 \in (\alpha_0, \beta_0) \subseteq \mathbb{R} \setminus A.$$

We additionally assume that $\alpha_0 > -\infty$. Below we use the notations $\tau_x^W, \tau_{x,y}^W$, and τ_A^W for the stopping times defined by (5), (6) and (7) respectively with Y replaced by W.

Lemma 4.1. *If* $(x-\alpha_0)b^2(x) \in L^1_{\text{loc}}(\alpha_0+)$, *then*

$$\int_0^{\tau_{\alpha_0}^W} b^2(W_t)\,\mathrm{d}t < \infty \ \mathbb{P}\text{-}a.s. \ on \ \{\tau_{\alpha_0}^W = \tau_A^W\}.$$

Remark. In the setting of Lemma 4.1 we have $\mathbb{P}(\tau_{\alpha_0}^W < \infty) = 1$ since Brownian motion reaches every level in finite time almost surely. Therefore the events $\{\tau_{\alpha_0}^W = \tau_A^W\}$ and $\{\tau_{\alpha_0}^W = \tau_A^W < \infty\}$ are equal. Cf. with the formulation of Theorem 3.4.

Proof. Let $(\beta_n)_{n\in\mathbb{N}}$ be an increasing sequence such that $x_0 < \beta_n < \beta_0$ and $\beta_n \nearrow \beta_0$. By [11, Chapter VII, Corollary 3.8] we get

$$\mathbb{E}\left[\int_0^{\tau_{\alpha_0}^W \wedge \tau_{\beta_n}^W} b^2(W_t)\,\mathrm{d}t\right] = \tfrac{\beta_n - x_0}{\beta_n - \alpha_0} \int_{\alpha_0}^{x_0} (y-\alpha_0) b^2(y)\,\mathrm{d}y + \tfrac{x_0-\alpha_0}{\beta_n-\alpha_0} \int_{x_0}^{\beta_n} (\beta_n - y) b^2(y)\,\mathrm{d}y$$

for every β_n. Both integrals are finite since $b^2 \in L^1_{\text{loc}}(\alpha_0, \beta_0)$ and $(x-\alpha_0)b^2(x) \in L^1_{\text{loc}}(\alpha_0+)$. Thus, we have $\mathbb{E}[\int_0^{\tau_{\alpha_0}^W \wedge \tau_{\beta_n}^W} b^2(W_t)\,\mathrm{d}t] < \infty$ and therefore $\int_0^{\tau_{\alpha_0}^W \wedge \tau_{\beta_n}^W} b^2(W_t)\,\mathrm{d}t < \infty$ almost surely for every n. It remains to note that $\mathbb{P}$-a.s. on $\{\tau_{\alpha_0}^W = \tau_A^W\}$ we have $\tau_{\alpha_0}^W < \tau_{\beta_n}^W$ for sufficiently large n. This concludes the proof. □

Lemma 4.2. *If* $\int_0^{\tau_{\alpha_0}^W} b^2(W_t)\,\mathrm{d}t < \infty$ *on a set* U *with* $\mathbb{P}(U) > 0$, *then* $(x-\alpha_0)b^2(x) \in L^1_{\text{loc}}(\alpha_0+)$.

Proof. The idea of the proof comes from [4]. Using the occupation times formula we can write

$$\int_0^{\tau_{\alpha_0}^W} b^2(W_t)\,\mathrm{d}t = \int_{\alpha_0}^{\infty} b^2(y) L^y_{\tau_{\alpha_0}^W}(W)\,\mathrm{d}y \geq \int_{\alpha_0}^{x_0} b^2(y) L^y_{\tau_{\alpha_0}^W}(W)\,\mathrm{d}y.$$

Let us define a process $R_y = \frac{1}{y-\alpha_0} L^y_{\tau_{\alpha_0}^W}(W)$. Then R is positive and we have

$$\int_0^{\tau_{\alpha_0}^W} b^2(W_t)\,\mathrm{d}t \geq \int_{\alpha_0}^{x_0} R_y (y-\alpha_0) b^2(y)\,\mathrm{d}y. \tag{22}$$

By [11, Chapter VI, Proposition 4.6], Laplace transform of R_y is

$$\mathbb{E}[\exp\{-\lambda R_y\}] = \frac{1}{1+2\lambda} \quad \text{for every } y.$$

Hence, every random variable R_y has exponential distribution with $\mathbb{E}[R_y] = 2$.

Denote by L an indicator function of a measurable set. We can write

$$\mathbb{E}[LR_y] = \mathbb{E}\left[L\int_0^\infty \mathbb{1}_{\{R_y>u\}}\mathrm{d}u\right] = \int_0^\infty \mathbb{E}[L\mathbb{1}_{\{R_y>u\}}]\mathrm{d}u.$$

By Jensen's inequality we get a lower bound for the integrand

$$\begin{aligned}\mathbb{E}[L\mathbb{1}_{\{R_y>u\}}] &= \mathbb{E}[(L-\mathbb{1}_{\{R_y\le u\}})^+]\\ &\ge (\mathbb{E}[L]-\mathbb{P}[R_y\le u])^+\\ &= (\mathbb{E}[L]+e^{-\frac{u}{2}}-1)^+.\end{aligned}$$

Hence,

$$\mathbb{E}[LR_y] \ge \int_0^\infty (\mathbb{E}[L]+e^{-\frac{u}{2}}-1)^+\mathrm{d}u = C, \tag{23}$$

where C is a strictly positive constant if $\mathbb{E}[L]$ is strictly positive.

Let us suppose that we can choose L, so that $\mathbb{E}[L\int_0^{\tau_{\alpha_0}^W} b^2(W_t)\,\mathrm{d}t]$ is finite. Using Fubini's Theorem and inequalities (22) and (23), we get

$$\mathbb{E}\left[L\int_0^{\tau_{\alpha_0}^W} b^2(W_t)\,\mathrm{d}t\right] \ge \int_{\alpha_0}^{x_0}\mathbb{E}[LR_y](y-\alpha_0)b^2(y)\,\mathrm{d}y \ge C\int_{\alpha_0}^{x_0}(y-\alpha_0)b^2(y)\,\mathrm{d}y.$$

Therefore, $(y-\alpha_0)b^2(y)\in L^1_{\mathrm{loc}}(\alpha_0+)$ if we can find an indicator function L such that $\mathbb{E}[L]$ is strictly positive and $\mathbb{E}[L\int_0^{\tau_{\alpha_0}^W} b^2(W_t)\,\mathrm{d}t]$ is finite.

Since $\int_0^{\tau_{\alpha_0}^W} b^2(W_t)\,\mathrm{d}t<\infty$ on a set with positive measure, such L exists. Indeed, denote by L_n an indicator function of the set $U_n=\{\int_0^{\tau_{\alpha_0}^W} b^2(W_t)\,\mathrm{d}t\le n\}$. Then, for every integer n, we have $\mathbb{E}[L_n\int_0^{\tau_{\alpha_0}^W} b^2(W_t)\,\mathrm{d}t]<\infty$. Since the sequence $(U_n)_{n\in\mathbb{N}}$ is increasing, $U\subseteq\bigcup_{n\in\mathbb{N}}U_n$ and $\mathbb{P}(U)>0$, there exists an integer N such that $\mathbb{P}(U_N)>0$ and therefore $\mathbb{E}[L_N]>0$. □

Now we return to the setting of Section 2.

Proof of Theorem 3.4. Suppose that $\mu\equiv 0$ and σ is arbitrary. Since Y_t is a continuous local martingale, by the Dambis–Dubins–Schwarz theorem we have $Y_t=B_{\langle Y\rangle_t}$ for a Brownian motion B with $B_0=x_0$, defined possibly on an enlargement of the initial probability space. Using the substitution $u=\langle Y\rangle_t$, we get

$$\int_0^{\tau_{\alpha_0}^Y} b^2(Y_t)\,\mathrm{d}t = \int_0^{\tau_{\alpha_0}^Y}\frac{b^2}{\sigma^2}(Y_t)\,\mathrm{d}\langle Y\rangle_t = \int_0^{\langle Y\rangle_{\tau_{\alpha_0}^Y}}\frac{b^2}{\sigma^2}(B_u)\,\mathrm{d}u \quad \mathbb{P}\text{-a.s.} \tag{24}$$

Furthermore, it is easy to see from $Y_t=B_{\langle Y\rangle_t}$ that

$$\langle Y\rangle_{\tau_{\alpha_0}^Y}=\tau_{\alpha_0}^B \quad \mathbb{P}\text{-a.s. on} \quad \{\tau_{\alpha_0}^Y<\infty\} \tag{25}$$

and

$$\{\tau^Y_{\alpha_0} = \tau^Y_A < \infty\} \subseteq \{\tau^B_{\alpha_0} = \tau^B_A\} \quad \mathbb{P}\text{-a.s.} \tag{26}$$

Now (24)–(26) imply the theorem in the case $\mu \equiv 0$ and σ arbitrary.

It only remains to prove the general case when both μ and σ are arbitrary. Let $\widetilde{Y}_t = s(Y_t)$, where s is the scale function of Y. Then $\widetilde{Y}$ satisfies SDE

$$\mathrm{d}\widetilde{Y}_t = \widetilde{\sigma}\left(\widetilde{Y}_t\right)\mathrm{d}W_t,$$

where $\widetilde{\sigma}(x) = s'(q(x))\sigma(q(x))$ and q is the inverse of s.

Define $\widetilde{b} = b \circ q$. Since s is strictly increasing, $s(\alpha_0) > -\infty$ by the assumption $\alpha_0 > l$, and $\widetilde{Y}_{\tau_{\alpha_0}} = s(Y_{\tau_{\alpha_0}}) = s(\alpha_0)$, it follows that the equality $\tau^Y_{\alpha_0} = \tau^{\widetilde{Y}}_{s(\alpha_0)}$ holds $\mathbb{P}$-a.s. Then we have

$$\int_0^{\tau^Y_{\alpha_0}} b^2(Y_t)\,dt = \int_0^{\tau^{\widetilde{Y}}_{s(\alpha_0)}} \widetilde{b}^2\left(\widetilde{Y}_t\right)dt.$$

Besides, for some small positive ε we have

$$\int_{s(\alpha_0)}^{s(\alpha_0+\varepsilon)} \frac{\widetilde{b}^2(x)}{\widetilde{\sigma}^2(x)}\,(x - s(\alpha_0))\,\mathrm{d}x = \int_{\alpha_0}^{\alpha_0+\varepsilon} \frac{b^2(y)}{\sigma^2(y)}\frac{s(y)-s(\alpha_0)}{s'(y)}\,\mathrm{d}y.$$

The fraction $\frac{s(y)-s(\alpha_0)}{s'(y)(y-\alpha_0)}$ is continuous for $y > \alpha_0$ and tends to 1 as $y \searrow \alpha_0$. Hence it is bounded and bounded away from zero on $(\alpha_0, \alpha_0+\varepsilon]$. It follows that $(x-\alpha_0)\frac{b^2}{\sigma^2}(x) \in L^1_{\mathrm{loc}}(\alpha_0+)$ if and only if $(x - s(\alpha_0))\frac{\widetilde{b}^2}{\widetilde{\sigma}^2}(x) \in L^1_{\mathrm{loc}}(s(\alpha_0)+)$. Then the result follows from the first part of the proof. □

Remark. It is interesting to note that, in fact, both sets in (26) are $\mathbb{P}$-a.s. equal. One can prove the reverse inclusion using the Engelbert–Schmidt construction of solutions of SDEs and the fact that $\alpha_0 > l$.

Appendix A.

Let Y be a J-valued diffusion starting from x_0 with a drift μ and volatility σ that satisfy the Engelbert–Schmidt conditions. Let $b : J \to \mathbb{R}$ be a Borel measurable function and let $(c,d) \subseteq J$, $c < x_0 < d$. Recall that, for any $x, y \in J$, the stopping times τ_x and $\tau_{x,y}$ are defined in (5) and (6). We now extend the definition in (6) by setting $\tau_{c.d} := \zeta$ if $c = l, d = r$; $\tau_{c.d} := \tau_c \wedge \zeta$ if $c > l, d = r$; $\tau_{c.d} := \zeta \wedge \tau_d$ if $c = l, d < r$.

Proposition A.1. *(i) The condition*

$$\frac{b^2}{\sigma^2} \in L^1_{\mathrm{loc}}(c,d)$$

implies that for all $t \in [0,\infty)$,

$$\int_0^t b^2(Y_u)\,\mathrm{d}u < \infty \ \mathbb{P}\text{-}a.s. \ \text{on} \ \{t < \tau_{c,d}\}.$$

(ii) For any $\alpha \in J$ *such that* $\frac{b^2}{\sigma^2} \notin L^1_{\mathrm{loc}}(\alpha)$ *we have*

$$\int_0^t b^2(Y_u)\,\mathrm{d}u = \infty \ \mathbb{P}\text{-}a.s. \ \text{on} \ \{\tau_\alpha < t < \zeta\}.$$

Remark. Let us note that $\mathbb{P}(\tau_\alpha < \infty) > 0$ by [2, Theorem 2.11]. Clearly, $\{\tau_\alpha < \infty\} = \{\tau_\alpha < \zeta\}$. Hence there exists $t \in [0,\infty)$ such that $\mathbb{P}(\tau_\alpha < t < \zeta) > 0$.

Proof. (i) Using the occupation times formula, $\mathbb{P}$-a.s. we get

$$\int_0^t b^2(Y_u)\mathrm{d}u = \int_0^t \frac{b^2}{\sigma^2}(Y_u)\mathrm{d}\langle Y\rangle_u = \int_J \frac{b^2}{\sigma^2}(y)L_t^y(Y)\mathrm{d}y, \quad t \in [0,\zeta). \tag{27}$$

$\mathbb{P}$-a.s. on the set $\{t < \tau_{c.d}\}$ the function $y \mapsto L_t^y(Y)$ is cádlág (see [11, Chapter VI, Theorem 1.7] with a compact support in the interval (c,d). Now the first statement follows from (27).

(ii)We have

$$\int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \frac{b^2}{\sigma^2}(y)\mathrm{d}y = \infty \ \text{ for all } \varepsilon > 0.$$

By [2, Theorem 2.7], we have for any $t \geq 0$

$$L_t^\alpha(Y) > 0 \text{ and } L_t^{\alpha-}(Y) > 0 \ \ \mathbb{P}\text{-a.s. on} \quad \{\tau_\alpha < t < \zeta\}.$$

Then, $\mathbb{P}$-a.s. on $\{\tau_\alpha < t < \zeta\}$, there exists $\varepsilon > 0$ such that the function $y \mapsto L_t^y(Y)$ is bounded away from zero on the interval $(\alpha-\varepsilon, \alpha+\varepsilon)$. It follows from (27) that $\int_0^t b^2(Y_u)\,\mathrm{d}u = \infty$ $\mathbb{P}$-a.s on $\{\tau_\alpha < t < \zeta\}$. This concludes the proof. □

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