CONVERGENCE OF INTEGRAL FUNCTIONALS OF ONE-DIMENSIONAL DIFFUSIONS

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Abstract. In this expository paper we describe the pathwise behaviour of the integral functional $\int_0^t f(Y_u)\,du$ for any $t \in [0,\zeta]$, where ζ is (a possibly infinite) exit time of a one-dimensional diffusion process Y from its state space, f is a nonnegative Borel measurable function and the coefficients of the SDE solved by Y are only required to satisfy weak local integrability conditions. Two proofs of the deterministic characterisation of the convergence of such functionals are given: the problem is reduced in two different ways to certain path properties of Brownian motion where either the Williams theorem and the theory of Bessel processes or the first Ray-Knight theorem can be applied to prove the characterisation.

1. Introduction

The *Engelbert–Schmidt zero-one law* states that for a Brownian motion B and any nonnegative Borel function f the following statements are equivalent:

(a) $\mathsf{P}\left(\int_0^t f(B_s)ds < \infty \text{ for all } t \in [0,\infty)\right) > 0$;

(b) $\mathsf{P}\left(\int_0^t f(B_s)ds < \infty \text{ for all } t \in [0,\infty)\right) = 1$;

(c) the function f is locally integrable on $\mathbb{R}$.

This important property has a plethora of applications. For example it constitutes an important step in the Engelbert–Schmidt construction of weak solutions of one-dimensional SDEs. The proof of the zero-one law can be found in monograph [14, Ch. 3] or original article [7]. Note that the equivalences between (a), (b) and (c) do not contain any information about the behaviour of the integral when local integrability of the function f fails on a subset of $\mathbb{R}$. The precise description of the explosion time of this integral functional was given in [8, Lem. 1].

In this paper we investigate a related problem of the convergence of the integral functional

$$\int_0^t f(Y_u)\,du, \quad t \in [0,\zeta],$$

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of a one-dimensional J-valued diffusion Y that solves an SDE up to an exit time ζ. The coefficients of the SDE are required to satisfy only some weak local integrability conditions on the open interval J (see (2.2) and (2.3) for the precise form of the Engelbert–Schmidt conditions satisfied by the coefficients) and the function $f\colon J \to [0,\infty]$ is assumed to be Borel measurable. The main results in this paper (see Theorems 2.7, 2.11 and 2.12) study the integral functional as a process, identify the stopping time after which the integral explodes, and give a deterministic criterion for the convergence of the integral functional at this stopping time. It turns out that this stopping time is the first time the process Y hits the set where the local integrability condition, analogous to (c) above, fails.

The proof of the results consists of two steps. The first step, which uses only basic properties of diffusion processes and their local times, reduces the original problem to a question about the convergence of an integral functional of Brownian motion. In the second step we give two proofs for the characterisation of the convergence of this integral functional of Brownian motion: (i) Williams' theorem (see [16, Ch. VII, Cor. 4.6]) and the result on integral functionals of Bessel processes from Cherny [4] are applied; (ii) a direct approach based on the first Ray-Knight theorem (see [16, Ch. XI, Th. 2.2]) is followed.

In [12] Engelbert and Tittel investigate the convergence of the integral functionals of the form $\int_0^t f(X_s)\,ds$, where f is a nonnegative Borel function and X a strong Markov continuous local martingale. The analytic condition that characterises the convergence of the integral functionals is given in terms of the speed measure of X. The diffusion Y considered in this paper is not necessarily a local martingale. However, the process $s(Y)$, where s is the scale function of Y, is and our characterisation theorems can be deduced from the ones in [12]. The proofs of the main results in [12] are based on Lemma 3.1 in [12], attributed to Jeulin [13], which, together with the Ray-Knight theorem, implies a version of a zero-one law for Brownian local time integrated in the space variable against a measure on $\mathbb{R}$ (see also Assing [1]). This zero-one law for Brownian local time is closely related to key Lemma 4.1. The result in Lemma 4.1 first appeared implicitly in the paper of Engelbert and Schmidt [9, p. 225–226] and was stated explicitly by Assing and Senf [3, Lem. 2]. The proofs in [9] and [3] rest on an application of the Ray-Knight theorem and Shepp's [17] dichotomy result for Gaussian processes. This dichotomy argument was later replaced by the abstract but elementary lemma of Jeulin [13] (see Assing and Schmidt [2, Lem. A1.7]).

The emphasis in the present paper is on understanding the pathwise behaviour of the integral functionals of one-dimensional diffusions directly from the pathwise properties of Brownian motion. Our proofs are short and are based on a simple direct approach which reduces the problem to Brownian motion where either the Williams theorem and Cherny's results from [4] or an idea from Delbaen and Shirakawa [6], which circumvents the lemma of Jeulin [13] mentioned above, and an application of the first Ray-Knight theorem complete the task.

The rest of the paper is organised as follows. Section 2 describes the setting and states the main results. In Section 3 we show how to reduce the main theorems to a problem for Brownian motion. Section 4 gives the characterisation for the convergence of integral functionals of Brownian motion. Sections 5 and 6 give the two proofs of Lemma 4.1.

2. The Setting and Main Results

2.1. First we introduce some common notations used in the sequel. Let us consider an open interval $J = (l, r) \subseteq \mathbb{R}$.

- By $\overline{J}$ we denote $[l, r]$.
- By $L^1_{\mathrm{loc}}(J)$ we denote the set of Borel functions $J \to [-\infty, \infty]$, which are locally integrable on J, i.e. integrable on compact subsets of J.
- For $x \in J$, $L^1_{\mathrm{loc}}(x)$ denotes the set of Borel functions $f\colon J \to [-\infty, \infty]$ such that $\int_{x-\varepsilon}^{x+\varepsilon} |f(y)|\,dy < \infty$ for some $\varepsilon > 0$.
- Let $\alpha \in [l, r)$, $\beta \in (l, r]$. By $L^1_{\mathrm{loc}}(\alpha+)$ we denote the set of Borel functions $f\colon J \to [-\infty, \infty]$ such that $\int_\alpha^z |f(y)|\,dy < \infty$ for some $z \in J$, $z > \alpha$. The notation $L^1_{\mathrm{loc}}(\beta-)$ is introduced similarly.

We will need the following statement. Its proof is straightforward.

Lemma 2.1. $L^1_{\mathrm{loc}}(J) = \bigcap_{x \in J} L^1_{\mathrm{loc}}(x)$.

2.2. Let the state space be $J = (l, r)$, $-\infty \le l < r \le \infty$, and $Y = (Y_t)_{t \in [0,\infty)}$ be a J-valued solution of the one-dimensional SDE

$$dY_t = \mu(Y_t)\,dt + \sigma(Y_t)\,dW_t, \quad Y_0 = x_0, \tag{2.1}$$

on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,\infty)}, \mathsf{P})$, where $x_0 \in J$ and W is an $(\mathcal{F}_t, \mathsf{P})$-Brownian motion. We allow Y to exit its state space J at a finite time in a continuous way. The exit time is denoted by ζ. That is to say, P-a.s. on $\{\zeta = \infty\}$ the trajectories of Y do not exit J, while P-a.s. on $\{\zeta < \infty\}$ we have: either $\lim_{t \uparrow \zeta} Y_t = r$ or $\lim_{t \uparrow \zeta} Y_t = l$. Then we need to specify the behaviour of Y after ζ on $\{\zeta < \infty\}$. In what follows we assume that on $\{\zeta < \infty\}$ the process Y stays at the endpoint of J where it exits after ζ, i.e. l and r are by convention absorbing boundaries.

Throughout the paper it is assumed that the coefficients μ and σ in (2.1) satisfy the Engelbert–Schmidt conditions

$$\sigma(x) \neq 0 \ \ \forall x \in J, \tag{2.2}$$

$$\frac{1}{\sigma^2}, \frac{\mu}{\sigma^2} \in L^1_{\mathrm{loc}}(J). \tag{2.3}$$

Under (2.2) and (2.3) SDE (2.1) has a weak solution, unique in law, which possibly exits J (see [8], [11], or [14, Ch. 5, Th. 5.15]). The Engelbert–Schmidt conditions are reasonable weak assumptions:

any locally bounded Borel function μ and locally bounded away from 0 Borel function σ on J satisfy (2.2) and (2.3).

Finally, the reason for considering an arbitrary interval $J \subseteq \mathbb{R}$ as a state space, and not just $\mathbb{R}$ itself, is that there are natural examples, where the Engelbert–Schmidt conditions hold only on a subset of $\mathbb{R}$. Consider for example geometric Brownian motion

$$dY_t = aY_t\,dt + bY_t\,dW_t, \quad Y_0 = x_0 > 0 \tag{2.4}$$

($a, b \in \mathbb{R}$, $b \neq 0$), with its natural state space $J = (0, \infty)$. If we were to take $J = \mathbb{R}$ here, both (2.2) and (2.3) would be violated. Even though we can replace the diffusion coefficient $\sigma(y) = by$ in (2.4) by $\sigma(y) = by + I_{\{0\}}(y)$, which does not affect solutions of (2.4) but fixes the problem with (2.2), the issue with (2.3) cannot be resolved in this way. On the other hand, any state space $J = (l, r)$ with $0 \le l < x_0 < r \le \infty$ is a possible choice when working with SDE (2.4); however, if $l > 0$ or $r < \infty$, the convention above implies we have a stopped geometric Brownian motion.

2.3. Now we state some well-known results about the behaviour of one-dimensional diffusions with the coefficients satisfying the Engelbert–Schmidt conditions that will be extensively used in the sequel. Let us also note that these results do not hold beyond the Engelbert–Schmidt conditions.

Let s denote the scale function of Y and ρ the derivative of s, i.e.

$$\rho(x) = \exp\left\{-\int_c^x \frac{2\mu}{\sigma^2}(y)\,dy\right\}, \quad x \in J, \tag{2.5}$$

$$s(x) = \int_c^x \rho(y)\,dy, \quad x \in \overline{J}, \tag{2.6}$$

for some $c \in J$. In particular, s is an increasing C^1-function $J \to \mathbb{R}$ with a strictly positive absolutely continuous derivative, while $s(r)$ (resp. $s(l)$) may take value ∞ (resp. $-\infty$).

For $a \in \overline{J}$ let us define the stopping time

$$\tau_a^Y = \inf\{t \in [0, \infty) \colon Y_t = a\} \qquad (\inf \emptyset := \infty). \tag{2.7}$$

Proposition 2.2. *For any $a \in J$ we have $\mathsf{P}(\tau_a^Y < \infty) > 0$.*

Even though it is assumed in Proposition 2.2 that $a \in J$, we stress that τ_a^Y is defined for any $a \in \overline{J}$, which will be needed in Remark 2.8 below.

Further let us consider the sets

$$\begin{aligned} A &= \left\{\zeta = \infty,\ \limsup_{t\to\infty} Y_t = r,\ \liminf_{t\to\infty} Y_t = l\right\}, \\ B_r &= \left\{\zeta = \infty,\ \lim_{t\to\infty} Y_t = r\right\}, \\ C_r &= \left\{\zeta < \infty,\ \lim_{t\uparrow\zeta} Y_t = r\right\}, \\ B_l &= \left\{\zeta = \infty,\ \lim_{t\to\infty} Y_t = l\right\}, \\ C_l &= \left\{\zeta < \infty,\ \lim_{t\uparrow\zeta} Y_t = l\right\}. \end{aligned}$$

Proposition 2.3. *Either* $\mathsf{P}(A) = 1$ *or* $\mathsf{P}(B_r \cup B_l \cup C_r \cup C_l) = 1$.

Proposition 2.4. *(i)* $\mathsf{P}(B_r \cup C_r) = 0$ *holds if and only if* $s(r) = \infty$.
(ii) $\mathsf{P}(B_l \cup C_l) = 0$ *holds if and only if* $s(l) = -\infty$.

In particular, we get that $\mathsf{P}(A) = 1$ holds if and only if $s(r) = \infty$, $s(l) = -\infty$.

Proposition 2.5. *Assume that* $s(r) < \infty$. *Then either* $\mathsf{P}(B_r) > 0$, $\mathsf{P}(C_r) = 0$ *or* $\mathsf{P}(B_r) = 0$, $\mathsf{P}(C_r) > 0$. *Furthermore, we have*

$$\mathsf{P}\left(\lim_{t\uparrow\zeta} Y_t = r,\ \ Y_t > a\ \forall t \in [0, \zeta)\right) > 0$$

for any $a < x_0$.

Propositions 2.2–2.5 are well-known and follow from the Engelbert–Schmidt construction of solutions (see e.g. [11] or [14, Ch. 5.5]) or can be deduced from the results in [10, Sec. 1.5].

Proposition 2.6 (Feller's test for explosions)**.** *We have* $\mathsf{P}(B_r) = 0$, $\mathsf{P}(C_r) > 0$ *if and only if*

$$s(r) < \infty \quad \text{and} \quad \frac{s(r) - s}{\rho\sigma^2} \in L^1_{\mathrm{loc}}(r-).$$

Clearly, Propositions 2.5 and 2.6, which contain statements about the behaviour of one-dimensional diffusions at the endpoint r, have their analogues for the behaviour at l. Feller's test for explosions in this form is taken from [5, Sec. 4.1]. For a different (but equivalent) form see e.g. [14, Ch. 5, Th. 5.29].

2.4. In this paper we study convergence of the integral functional

$$\int_0^t f(Y_u)\,du, \quad t \in [0, \zeta], \tag{2.8}$$

where $f\colon J \to [0, \infty]$ is a nonnegative Borel function. In this subsection we reduce the study of convergence of (2.8) in general to that of convergence of the integral

$$\int_0^\zeta f(Y_u)\,du \tag{2.9}$$

for a nonnegative Borel function $f\colon J \to [0,\infty]$ such that $\frac{f}{\sigma^2} \in L^1_{\text{loc}}(J)$. In the next subsection we formulate the answer to the latter problem.

Let us consider the set

$$D = \left\{ x \in J\colon \frac{f}{\sigma^2} \notin L^1_{\text{loc}}(x) \right\}$$

and note that D is a closed subset in J. Let us further define the stopping time

$$\eta_D = \zeta \wedge \inf\{t \in [0,\infty)\colon Y_t \in D\} \qquad (\inf \emptyset := \infty).$$

Theorem 2.7. P-*a.s. we have:*

$$\int_0^t f(Y_u)\,du < \infty, \quad t \in [0,\eta_D), \tag{2.10}$$

$$\int_0^t f(Y_u)\,du = \infty, \quad t \in (\eta_D,\zeta]. \tag{2.11}$$

Remark 2.8. After Theorem 2.7 it remains only to study the convergence of the integral

$$\int_0^{\eta_D} f(Y_u)\,du.$$

If $x_0 \in D$, then $\eta_D \equiv 0$, and the integral is clearly zero. Let us assume that $x_0 \notin D$ and set

$$\alpha = \sup(l, x_0) \cap D \qquad (\sup \emptyset := l),$$

$$\beta = \inf(x_0, r) \cap D \qquad (\inf \emptyset := r).$$

It is easy to see that $\eta_D = \tau^Y_\alpha \wedge \tau^Y_\beta$. Now if we consider $I := (\alpha,\beta)$ as a new state space for Y, then $\tau^Y_\alpha \wedge \tau^Y_\beta$ will be the new exit time, and we will have $\frac{f}{\sigma^2} \in L^1_{\text{loc}}(I)$ by Lemma 2.1. This concludes the reduction of the study of the convergence of (2.8) to that of the convergence of (2.9).

In order to prove Theorem 2.7 we need some additional notation. Since Y is a continuous semimartingale up to the exit time ζ, one can define its local time $\{L^y_t(Y); y \in J, t \in [0,\zeta)\}$ on the stochastic interval $[0,\zeta)$ for any $y \in J$ in the usual way (e.g. via the obvious generalization of [16, Ch. VI, Th. 1.2]). It follows from Theorem VI.1.7 in [16] that the random field $\{L^y_t(Y); y \in J, t \in [0,\zeta)\}$ admits a modification such that the map $(y,t) \mapsto L^y_t(Y)$ is a.s. continuous in t and cadlag in y.[1] As usual we always work with such a modification. Let us further recall that a.s. on $\{t < \zeta\}$ the function $y \mapsto L^y_t(Y)$ has a compact support in J and hence is bounded as a cadlag function with a compact support.

We will need the following result.

Lemma 2.9 (Theorem 2.7 in [5])**.** *Let* $a \in J$. *Then*

$$L^a_t(Y) > 0 \quad \textit{and} \quad L^{a-}_t(Y) > 0 \quad \mathsf{P}\text{-}\textit{a.s. on } \{\tau^Y_a < t < \zeta\}.$$

[1]Moreover, it can be proved that for a diffusion Y driven by (2.1) under conditions (2.2) and (2.3), any such modification is, in fact, a.s. jointly continuous in (t,y); see [15, Proposition A.1].

Remarks 2.10. (i) By Proposition 2.2 we have $\mathsf{P}(\tau_a^Y < \zeta) > 0$. Hence, there exists $t \in (0, \infty)$ such that $\mathsf{P}(\tau_a^Y < t < \zeta) > 0$.

(ii) Let us note that the result of Lemma 2.9 no longer holds if the coefficients μ and σ of (2.1) fail to satisfy the Engelbert–Schmidt conditions (see Theorem 2.6 in [5]).

Proof of Theorem 2.7. By the occupation times formula, P-a.s. we have

$$\int_0^t f(Y_u)\,du = \int_0^t \frac{f}{\sigma^2}(Y_u)\,d\langle Y, Y\rangle_u = \int_J \frac{f}{\sigma^2}(y) L_t^y(Y)\,dy, \quad t \in [0, \zeta). \tag{2.12}$$

Then (2.10) follows from the fact that P-a.s. on $\{t < \zeta\}$ the function $y \mapsto L_t^y(Y)$ is a cadlag function with a compact support in J.

As for (2.11), it immediately follows from (2.12) and Lemma 2.9 in the case $x_0 \in D$. If $x_0 \notin D$, we first observe that $\eta_D = \tau_\alpha^Y \wedge \tau_\beta^Y$ (see Remark 2.8), and hence $\{\eta_D < \zeta\} = \{\tau_\alpha^Y < \zeta\} \cup \{\tau_\beta^Y < \zeta\}$. If $\mathsf{P}(\tau_\alpha^Y < \zeta) > 0$, then, since D is closed we have $\alpha \in D$ (note that $(l, x_0) \cap D \neq \emptyset$ in this case because otherwise $\alpha = l$ and $\mathsf{P}(\tau_\alpha^Y < \zeta) = 0$). Thus, (2.11) on $\{\tau_\alpha^Y < \zeta\}$ follows from (2.12) and Lemma 2.9 applied with $a = \alpha \in D$. Similarly we get (2.11) on $\{\tau_\beta^Y < \zeta\}$. This concludes the proof. □

2.5. As pointed out in the previous subsection, it remains to study the convergence of the integral

$$\int_0^\zeta f(Y_u)\,du \tag{2.13}$$

for a nonnegative Borel function $f\colon J \to [0, \infty]$ satisfying

$$\frac{f}{\sigma^2} \in L^1_{\text{loc}}(J). \tag{2.14}$$

This study is performed in the following two theorems, where we separately treat the cases $\mathsf{P}(A) = 1$ and $\mathsf{P}(B_r \cup B_l \cup C_r \cup C_l) = 1$ (see Propositions 2.3 and 2.4). Below ν_L denotes the Lebesgue measure on J.

Theorem 2.11. *Assume that the function $f\colon J \to [0, \infty]$ satisfies (2.14). Let $s(r) = \infty$ and $s(l) = -\infty$.*

(i) If $\nu_L(f > 0) = 0$, then

$$\int_0^\zeta f(Y_u)\,du = 0 \quad \mathsf{P}\text{-a.s.}$$

(ii) If $\nu_L(f > 0) > 0$, then

$$\int_0^\zeta f(Y_u)\,du = \infty \quad \mathsf{P}\text{-a.s.}$$

Let us also note that $\zeta = \infty$ P-a.s. in the case $s(r) = \infty$, $s(l) = -\infty$.

In the remaining case $s(l) > -\infty$ or $s(r) < \infty$ we have

$$\Omega = \left\{\lim_{t \uparrow \zeta} Y_t = r\right\} \cup \left\{\lim_{t \uparrow \zeta} Y_t = l\right\} \quad \mathsf{P}\text{-a.s.}$$

In the following theorem we investigate the convergence of (2.13) on $\{\lim_{t\uparrow\zeta} Y_t = r\}$. To this end we need to assume $s(r) < \infty$ because otherwise $\mathsf{P}(\lim_{t\uparrow\zeta} Y_t = r) = 0$ by Proposition 2.4.

Theorem 2.12. *Assume that the function $f\colon J \to [0,\infty]$ satisfies (2.14). Let $s(r) < \infty$.*

(i) If

$$\frac{(s(r)-s)f}{\rho\sigma^2} \in L^1_{\text{loc}}(r-),$$

then

$$\int_0^\zeta f(Y_u)\,du < \infty \quad \mathsf{P}\text{-}a.s.\ on\ \left\{\lim_{t\uparrow\zeta} Y_t = r\right\}.$$

(ii) If

$$\frac{(s(r)-s)f}{\rho\sigma^2} \notin L^1_{\text{loc}}(r-),$$

then

$$\int_0^\zeta f(Y_u)\,du = \infty \quad \mathsf{P}\text{-}a.s.\ on\ \left\{\lim_{t\uparrow\zeta} Y_t = r\right\}.$$

Clearly, Theorem 2.12 has its analogue that describes the convergence of (2.13) on $\{\lim_{t\uparrow\zeta} Y_t = l\}$.

3. Proofs of Theorems 2.11 and 2.12

In this section we prove Theorems 2.11 and 2.12. In the latter proof we apply Lemma 4.1 below, which will be proved in the next sections.

Let us set

$$\widetilde{Y}_t = s(Y_t), \quad t \in [0,\zeta). \tag{3.1}$$

Then

$$d\widetilde{Y}_t = \widetilde{\sigma}(\widetilde{Y}_t)\,dW_t, \quad t \in [0,\zeta), \tag{3.2}$$

where

$$\widetilde{\sigma}(x) = (\rho\sigma)\circ s^{-1}(x), \quad x \in (s(l), s(r)).$$

In particular, $\widetilde{Y}$ is a continuous local martingale on the stochastic interval $[0,\zeta)$. By the Dambis–Dubins–Schwarz theorem, there exists a Brownian motion B starting from $s(x_0)$ (possibly on an enlargement of the initial probability space) such that

$$\widetilde{Y}_t = B_{\langle\widetilde{Y},\widetilde{Y}\rangle_t} \quad \mathsf{P}\text{-a.s.}, \quad t \in [0,\zeta). \tag{3.3}$$

Let us also introduce the function

$$\widetilde{f}(x) = f\circ s^{-1}(x), \quad x \in (s(l), s(r)).$$

Proof of Theorem 2.11. Here $s(r)=\infty$ and $s(l)=-\infty$. Hence $\zeta=\infty$ P-a.s. and, moreover, $\mathsf{P}(A)=1$ (see Propositions 2.3 and 2.4). Then (3.1) implies that

$$\mathsf{P}\left(\limsup_{t\to\infty}\widetilde{Y}_t=\infty,\ \liminf_{t\to\infty}\widetilde{Y}_t=-\infty\right)=1.$$

Now it follows from (3.3) that $\langle\widetilde{Y},\widetilde{Y}\rangle_\infty=\infty$ P-a.s. We have

$$\begin{aligned}\int_0^\infty f(Y_u)\,du&=\int_0^\infty \widetilde{f}(\widetilde{Y}_u)\,du=\int_0^\infty\frac{\widetilde{f}}{\widetilde{\sigma}^2}\left(B_{\langle\widetilde{Y},\widetilde{Y}\rangle_u}\right)d\langle\widetilde{Y},\widetilde{Y}\rangle_u\\&=\int_0^\infty\frac{\widetilde{f}}{\widetilde{\sigma}^2}(B_v)\,dv=\int_{\mathbb{R}}\frac{\widetilde{f}}{\widetilde{\sigma}^2}(x)L^x_\infty(B)\,dx\quad\mathsf{P}\text{-a.s.}\end{aligned}$$

The first equality above is clear (we used $\zeta=\infty$ P-a.s.), the second follows from (3.2) and (3.3), the third is due to the continuity of $\langle\widetilde{Y},\widetilde{Y}\rangle$ and the fact that $\langle\widetilde{Y},\widetilde{Y}\rangle_\infty=\infty$ P-a.s., and the last one follows from the occupation times formula ($L^x_t(B)$ denotes the local time of the Brownian motion B at time t and at level x). It remains to note that $\nu_L(f>0)>0$ is equivalent to $\nu_L(\widetilde{f}>0)>0$ and that for a Brownian local time, P-a.s. it holds $L^x_\infty(B)\equiv\infty$ $\forall x\in\mathbb{R}$ (see e.g. [16, Ch. VI, § 2]). The proof is completed. □

Proof of Theorem 2.12. Here $s(r)<\infty$, i.e. $\mathsf{P}(\lim_{t\uparrow\zeta}Y_t=r)>0$. let us set $R:=\{\lim_{t\uparrow\zeta}Y_t=r\}$ and observe that (3.1) and (3.3) imply

$$R\equiv\left\{\lim_{t\uparrow\zeta}Y_t=r\right\}=\left\{\lim_{t\uparrow\zeta}\widetilde{Y}_t=s(r)\right\}=\left\{\lim_{t\uparrow\zeta}B_{\langle\widetilde{Y},\widetilde{Y}\rangle_t}=s(r)\right\}.$$

In particular,

$$\langle\widetilde{Y},\widetilde{Y}\rangle_\zeta=\tau^B_{s(r)}\quad\mathsf{P}\text{-a.s. on }R,\tag{3.4}$$

where $\tau^B_{s(r)}$ denotes the hitting time of the level $s(r)$ by the Brownian motion B. Let us note that ζ may be finite or infinite on R (see Propositions 2.5 and 2.6 for details), but it follows from (3.4) that $\langle\widetilde{Y},\widetilde{Y}\rangle_\zeta$ is in either case finite on R. Similarly to the previous proof we get

$$\int_0^\zeta f(Y_u)\,du=\int_0^{\tau^B_{s(r)}}\frac{\widetilde{f}}{\widetilde{\sigma}^2}(B_v)\,dv\quad\mathsf{P}\text{-a.s. on }R.\tag{3.5}$$

The question of convergence of the integral in the right-hand side of (3.5) is studied in Lemma 4.1 below. It is easy to obtain from (2.14) that $\frac{\widetilde{f}}{\widetilde{\sigma}^2}\in L^1_{\text{loc}}(s(J))$, which means that Lemma 4.1 can be applied (see (4.1)). Thus, to study the convergence of the integral in the right-hand side of (3.5) we need to check whether

$$(s(r)-x)\frac{\widetilde{f}}{\widetilde{\sigma}^2}(x)\in L^1_{\text{loc}}(s(r)-)$$

(the notation "$f(x) \in \mathfrak{M}$" for a function f and a class of functions $\mathfrak{M}$ is understood to be synonymous to "$f \in \mathfrak{M}$"). We have

$$\int_{s(\cdot)}^{s(r)} \frac{(s(r)-x)f(s^{-1}(x))}{\rho^2(s^{-1}(x))\sigma^2(s^{-1}(x))}\,dx = \int_{\cdot}^{r} \frac{(s(r)-s(y))f(y)}{\rho(y)\sigma^2(y)}\,dy$$

(recall that $s' = \rho$). Now the statement of Theorem 2.12 follows from (3.5) and Lemma 4.1. □

4. The Setting and Notation in the Brownian Case

It remains to prove Lemma 4.1 below. From now on let us consider a Brownian motion B starting from $x_0 \in \mathbb{R}$. We will extensively use the notation τ_a^B ($a \in \mathbb{R}$) for the stopping time defined as in (2.7). Below we use the notation "$f(x) \in \mathfrak{M}$" for a function f and a class of functions $\mathfrak{M}$ as a synonym for "$f \in \mathfrak{M}$".

Lemma 4.1. *Let B be a Brownian motion starting from $x_0 \in \mathbb{R}$ and $x_0 < r < \infty$. Assume that the function $f\colon I \to [0,\infty]$ with $I := (-\infty, r)$ satisfies*

$$f \in L^1_{\text{loc}}(I). \tag{4.1}$$

(i) If $(r-x)f(x) \in L^1_{\text{loc}}(r-)$, then

$$\int_0^{\tau_r^B} f(B_u)\,du < \infty \quad \mathsf{P}\text{-a.s.}$$

(ii) If $(r-x)f(x) \notin L^1_{\text{loc}}(r-)$, then

$$\int_0^{\tau_r^B} f(B_u)\,du = \infty \quad \mathsf{P}\text{-a.s.}$$

In Sections 5 and 6 we give two different proofs of Lemma 4.1.

5. First Proof of Lemma 4.1

This method is based on Williams' theorem (see [16, Ch. VII, Cor. 4.6]) and Cherny's investigation of convergence of integral functionals of Bessel processes (see [4]).

By the occupation times formula and (4.1), P-a.s. we get

$$\int_0^t f(B_u)\,du < \infty, \quad t \in [0, \tau_r^B). \tag{5.1}$$

By $\rho = (\rho_t)_{t\in[0,\infty)}$ we denote a three-dimensional Bessel process starting from 0. Let us set

$$\xi = \sup\{t \in [0,\infty)\colon \rho_t = r - x_0\}$$

(note that ξ is a finite random variable because $\rho_t \to \infty$ a.s.). By Williams' theorem,

$$\text{Law}\left(r - B_{\tau_r^B - t}; t \in [0, \tau_r^B)\right) = \text{Law}\left(\rho_t; t \in [0,\xi)\right), \tag{5.2}$$

where "Law" means distribution. It follows from Theorem 2.2 in [4] that, for a nonnegative function g,

(A) $xg(x) \in L^1_{\mathrm{loc}}(0+)$ implies that a.s. it holds

$$\exists \varepsilon > 0 \quad \int_0^\varepsilon g(\rho_u)\,du < \infty;$$

(B) $xg(x) \notin L^1_{\mathrm{loc}}(0+)$ implies that a.s. it holds

$$\forall \varepsilon > 0 \quad \int_0^\varepsilon g(\rho_u)\,du = \infty.$$

By (5.1), (5.2) and (A), (B), the question reduces to whether $xf(r-x) \in L^1_{\mathrm{loc}}(0+)$, or, equivalently, to whether $(r-x)f(x) \in L^1_{\mathrm{loc}}(r-)$. This concludes the proof.

6. Second Proof of Lemma 4.1

We take the idea for this proof from Theorem 1.4 in Delbaen and Shirakawa [6]. The method is based on the first Ray-Knight theorem (see [16, Ch. XI, Th. 2.2]).

By the occupation times formula,

$$\int_0^{\tau_r^B} f(B_u)\,du = \int_{-\infty}^r f(x) L^x_{\tau_r^B}(B)\,dx \quad \mathsf{P}\text{-a.s.} \tag{6.1}$$

Since P-a.s. the mapping $x \mapsto L^x_{\tau_r^B}(B)$ is a continuous function with a compact support in $\mathbb{R}$, we get from (4.1) that

$$\int_{-\infty}^{x_0} f(x) L^x_{\tau_r^B}(B)\,dx < \infty \quad \mathsf{P}\text{-a.s.}$$

By (6.1), the question of whether $\int_0^{\tau_r^B} f(B_u)\,du$ is finite reduces to the question of whether $\int_{x_0}^r f(x) L^x_{\tau_r^B}(B)\,dx$ is finite, or, equivalently, to

$$\text{whether} \quad \int_0^{r-x_0} f(r-u) L^{r-u}_{\tau_r^B}(B)\,du \quad \text{is finite.} \tag{6.2}$$

Let $\overline{W}$ and $\widetilde{W}$ be independent Brownian motions starting from 0. Let us set

$$\eta_t = \overline{W}_t^2 + \widetilde{W}_t^2, \tag{6.3}$$

i.e. $\eta = (\eta_t)_{t \in [0,\infty)}$ is a squared two-dimensional Bessel process starting from 0. It follows from the first Ray-Knight theorem that

$$\operatorname{Law}\left(L^{r-u}_{\tau_r^B}; u \in [0, r-x_0]\right) = \operatorname{Law}\left(\eta_u; u \in [0, r-x_0]\right). \tag{6.4}$$

In what follows we prove that, for a Brownian motion W starting from 0,

(A) $xf(r-x) \in L^1_{\mathrm{loc}}(0+)$ implies that

$$\int_0^{r-x_0} f(r-u) W_u^2\,du < \infty \quad \text{a.s.};$$

(B) $xf(r-x) \notin L^1_{\text{loc}}(0+)$ implies that

$$\int_0^{r-x_0} f(r-u) W_u^2 \, du = \infty \quad \text{a.s.}$$

Together with (6.2)–(6.4) this will complete the proof of Lemma 4.1.

By Fubini's theorem,

$$\mathsf{E} \int_0^{r-x_0} f(r-u) W_u^2 \, du = \int_0^{r-x_0} f(r-u) u \, du,$$

so (A) is immediate (recall that (4.1) holds).

In order to prove (B) we assume that

$$\mathsf{P}\left(\int_0^{r-x_0} f(r-u) W_u^2 \, du < \infty \right) > 0. \tag{6.5}$$

Then there exists a sufficiently large $M < \infty$ such that

$$\gamma := \mathsf{P}(R) > 0, \quad \text{where} \quad R := \left\{ \int_0^{r-x_0} f(r-u) W_u^2 \, du \le M \right\}.$$

Let us note that, for any positive δ and u, the probability

$$\mathsf{P}(W_u^2 \ge \delta^2 u) = \mathsf{P}(|W_u/\sqrt{u}| \ge \delta) = \mathsf{P}(|N(0,1)| \ge \delta)$$

does not depend on u. We take a sufficiently small $\delta > 0$ such that

$$\mathsf{P}(|N(0,1)| \ge \delta) \ge 1 - \frac{\gamma}{2}.$$

Then, for any u,

$$\mathsf{E}(W_u^2 I_R) \ge \delta^2 u \mathsf{P}(R \cap \{W_u^2 \ge \delta^2 u\}) \ge \frac{\gamma}{2} \delta^2 u.$$

By Fubini's theorem,

$$\mathsf{E}\left[I_R \int_0^{r-x_0} f(r-u) W_u^2 \, du \right] = \int_0^{r-x_0} f(r-u) \mathsf{E}(W_u^2 I_R) \, du \ge \frac{\gamma}{2} \delta^2 \int_0^{r-x_0} f(r-u) u \, du.$$

The left-hand side is finite as on the event R the integral is not greater than M. Thus, (6.5) implies $uf(r-u) \in L^1_{\text{loc}}(0+)$, which proves (B) and completes the proof of Lemma 4.1.

References

[1] S. Assing. *Homogene stochastische Differentialgleichungen mit gewöhnlicher Drift*. Promotionsschrift. Friedrich-Schiller-Universität, Jena, 1994.

[2] S. Assing and W. Schmidt. *Continuous strong Markov processes in dimension one: A stochastic calculus approach*, volume 1688 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998.

[3] S. Assing and T. Senf. On stochastic differential equations without drift. *Stochastics and Stochastics Reports*, 36:21–39, 1997.

[4] A. Cherny. Convergence of some integrals associated with Bessel processes. *Theory Probab. Appl.*, 45(2):195–209, 2001.

[5] A. Cherny and H.-J. Engelbert. *Singular Stochastic Differential Equations*, volume 1858 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2005.

[6] F. Delbaen and H. Shirakawa. No arbitrage condition for positive diffusion price processes. *Asia-Pacific Financial Markets*, 9:159–168, 2002.

[7] H.-J. Engelbert and W. Schmidt. On the behaviour of certain functionals of the Wiener process and applications to stochastic differential equations. In *Stochastic Differential Systems (Visegrád, 1980)*, volume 36 of *Lecture Notes in Control and Information Sci.*, pages 47–55. Springer, Berlin, 1981.

[8] H.-J. Engelbert and W. Schmidt. On one-dimensional stochastic differential equations with generalized drift. In M. Metivier and E. Pardoux, editors, *Stochastic differential systems (Marseille-Luminy, 1984)*, volume 69 of *Lecture Notes in Control and Inform. Sci.*, pages 143–155. Springer, Berlin, 1985.

[9] H.-J. Engelbert and W. Schmidt. On the behaviour of certain Bessel functionals. An application to a class of stochastic differential equations. *Math. Nachr.*, 131:219–234, 1987.

[10] H.-J. Engelbert and W. Schmidt. Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations. I. *Math. Nachr.*, 143:167–184, 1989.

[11] H.-J. Engelbert and W. Schmidt. Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations. III. *Math. Nachr.*, 151:149–197, 1991.

[12] H.-J. Engelbert and G. Tittel. Integral functionals of strong Markov continuous local martingales. In R. Buckdahn, H.-J. Engelbert, and M. Yor, editors, *Stochastic Processes and Related Topics*, Proceedings of the 12th Winter School, Siegmundsburg (Germany), February 27-March 4, 2000, pages 111–132. CRC Press, 2002.

[13] T. Jeulin. *Semi-martingales et grossissement d'une Filtration*, volume 833 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1980.

[14] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[15] A. Mijatović and M. Urusov. On the martingale property of certain local martingales. To appear in *Probability Theory and Related Fields*, 2010.

[16] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, third edition, 1999.

[17] L. A. Shepp. Radon-Nikodym derivatives of Gaussian measures. *Ann. Math. Stat.*, 37:321–354, 1966.

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