RANDOMISATION AND RECURSION METHODS FOR MIXED-EXPONENTIAL LÉVY MODELS, WITH FINANCIAL APPLICATIONS

ALEKSANDAR MIJATOVIĆ, MARTIJN PISTORIUS, JOHANNES STOLTE

Abstract. We develop a new Monte Carlo variance reduction method to estimate the expectation of two commonly encountered path-dependent functionals: first-passage times and occupation times of sets. The method is based on a recursive approximation of the first-passage time probability and expected occupation time of sets of a Lévy bridge process that relies in part on a randomisation of the time parameter. We establish this recursion for general Lévy processes and derive its explicit form for mixed-exponential jump-diffusions, a dense subclass (in the sense of weak approximation) of Lévy processes, which includes Brownian motion with drift, Kou's double-exponential model and hyper-exponential jump-diffusion models. We present a highly accurate numerical realisation and derive error estimates. By way of illustration the method is applied to the valuation of range accruals and barrier options under exponential Lévy models and Bates-type stochastic volatility models with exponential jumps. Compared with standard Monte Carlo methods, we find that the method is significantly more efficient.
Keywords: Lévy bridge process, stochastic volatility model with jumps, first-passage time, occupation time, mixed-exponential jump-diffusion, Markov bridge sampling, continuous Euler-Maruyama scheme.
MSC 2010: 65C05, 91G60.

1. Introduction

Motivation and brief outline. The Markov bridge sampling method for the estimation of the expectation $\mathbb{E}[F(T,\xi)]$ of a given path-functional F of a Markov process ξ and the horizon $T>0$ consists of averaging conditional expectations $\widetilde{F}(\xi_{t_0},\ldots,\xi_{t_N})$ over M independent copies $(\xi^{(i)}_{t_0},\ldots,\xi^{(i)}_{t_N})$, $i=1,\ldots,M$, of the values $(\xi_{t_0},\ldots,\xi_{t_N})$ that ξ takes on the grid $\mathbb{T}_N=\{0=t_0<t_1<\ldots<t_N=T\}$:

$$\mathbb{E}[F(T,\xi)] \approx \frac{1}{M}\sum_{i=1}^{M}\widetilde{F}(\xi^{(i)}_{t_0},\ldots,\xi^{(i)}_{t_N}), \tag{1.1}$$

where $\widetilde{F}(\xi_{t_0},\ldots,\xi_{t_N})$ denotes the regular version of the conditional expectation $\mathbb{E}[F(T,\xi)|\xi_{t_0},\ldots,\xi_{t_N}]$. The name of the method derives from the fact that, conditional on the values $(\xi_{t_0},\ldots,\xi_{t_N})$, the stochastic processes $\{\xi_t, t\in[t_i,t_{i+1}]\}$, for $i=0,\ldots,N-1$, are equal in law to Markov bridge processes. The estimator in (1.1) is unbiased and has strictly smaller variance than the standard Monte Carlo estimator, as a consequence of the tower property of conditional expectation and the conditional variance formula. The Markov bridge sampling method has the advantage that it allows for refinements of the generated path to the required level of accuracy, and can be combined with importance sampling. Such a bridge method is especially suited for the evaluation of expectations of path-dependent functionals (see [12], for example). Since the function $\widetilde{F}$ is in general not available in closed or analytically tractable form, the viability of the Markov bridge method hinges on the ability to efficiently approximate the function $\widetilde{F}$. In this paper we derive an efficient approximation method for the

Date: October 28, 2014
Acknowledgements. We thank the Editor and an anonymous referee, and Dan Crisan, Lane Hughston, Antoine Jacquier, Felicity Pearce, Vladimir Piterbarg, Johannes Ruf, David Taylor and Josef Teichmann, the participants of the Global Derivatives Trading & Risk Management - Barcelona (2012), the Seventh World Congress of the Bachelier Finance Society - Sydney (2012), the Finance and Stochastics Seminar at Imperial College London (2011), and a satellite workshop at the University of the Witwatersrand - Johannesburg (2011), for useful comments. JS was supported by an EPSRC DTA grant and a doctoral grant (Ref. Nr. D/11/42213) from the German Academic Exchange Service (DAAD).

conditional expectations $\widetilde{F}$ of certain path-dependent functionals given in terms of occupation times of sets and first-passage times, which is achieved by approximating the law of the bridge process by the law of the process pinned down at an independent random time with small variance. Since the latter law is analytically tractable when ξ is a mixed-exponential Lévy process, this allows us to develop a Markov bridge Monte Carlo method for estimation of the corresponding expectation $\mathbb{E}[F(T,\xi)]$. To demonstrate the potential of the simulation method we extend the approach to a two-dimensional Markovian setting, and deploy the method to numerically approximate the values of two common path-dependent derivatives, barrier options and range accruals, under a version of the Bates model [7], which is an example of a stochastic volatility model with jumps that is widely used in financial modelling—we refer to [22, 16] for background.

Literature overview. In the literature [20, 39, 41] a number of bridge sampling methods exist dealing with cases in which ξ is a one-dimensional Lévy process. In [20] an adaptive bridge sampling method is developed for real-valued Lévy processes based on short-time asymptotics of stopped Lévy processes. By conditioning on the jump-skeleton and exploiting the explicit form of the distribution of the maximum of a Brownian bridge, a simulation method for pricing of barrier options under jump-diffusions is presented in [39], and a refinement of this algorithm and application to the pricing of corporate bonds is given in [41]. An exact simulation algorithm for generation of diffusion sample paths deploying Brownian bridges is designed and analysed in [9].

Several alternative methods have been developed for approximation of path-dependent functionals, often based on weak or strong (pathwise) approximations of the solution of the SDE. In the setting of diffusions, a classical treatment of various strong and weak approximation schemes is given in [31]. More recently, the problem of approximation of general path-dependent functionals has also received attention in the case of Lévy-driven SDEs. In [17] a multi-level Monte Carlo algorithm is developed for path-dependent functionals of Lévy driven SDEs that are Lipschitz continuous in the supremum norm, and identifies error bounds. This algorithm is based on an approximation of the driving Lévy process by a Lévy jump-diffusion constructed by replacing the small jumps by a Brownian motion, as was investigated in [4]. Adopting an alternative approach that does not rely on the Brownian small-jump approximation, a multi-level extension is presented in [19] of the Monte Carlo method developed in [33] for estimation of Lipschitz functions of the final value and running maximum of a real-valued Lévy process. Some functionals that are of interest in various applications are not included in the analysis of [17, 19], as these fail to satisfy the Lipschitz condition. The bridge method that we present in the current paper provides approximations in two such cases, namely, the distribution of the running maximum and the expected occupation time of sets.

Approximation of bridge functionals. As mentioned above, a key-step in the development of the Markov bridge method is the availability of an efficient approximation of the conditional expectations $\widetilde{F}$. As in general the transition probabilities of the Markov processes considered here are not explicitly available, the first step is to approximate the Markov process in question by its continuous-time Euler-Maruyama (EM) scheme. The approximation of expectations of path-dependent functionals under stochastic volatility models with jumps using the continuous-time EM-scheme is based on the *harness property* of a Markov process which states that, for any two epochs t_1 and t_2 the collections of values of the Markov process at times in between t_1 and t_2 is independent of the values for t outside this interval, conditional on the values of the process at t_1 and t_2. Noting that a Lévy process that is conditioned to start from position x and to take the value y at the horizon T is equal in law to a Lévy bridge process from $(0,x)$ to (T,y), we are led to the problem of evaluating the expectations of path-dependent functionals of Lévy bridges.

Randomisation method and recursions. The approximation method of the Lévy bridge quantities that we present is based in part on a randomisation of the time-parameter. This randomisation method was originally developed in [14] for the valuation of American put options, and is known as Erlangisation in risk theory [1, Ch. IX.8]. The method has been deployed in [2] for the efficient computation of ruin probabilities and in [5, 11, 30, 33, 35, 36] for the valuation of American-type and barrier options. This randomisation method is based on the fact that, according to the law of large numbers, the average of independent exponential random variables with mean t converges to t. An average of n such exponential random variables is equal in distribution to a Gamma$(n, n/t)$ random variable $\Gamma_{n,n/t}$, which has mean t and variance t^2/n. As observed in [18, Ch. VII.6], the approximation of the value $f(t)$ of a continuous bounded function f at $t > 0$ by the expectation $\mathbb{E}[f(\Gamma_{n,n/t})]$ of f evaluated at the random time $\Gamma_{n,n/t}$ is asymptotically exact: since $\Gamma_{n,n/t}$ converges to a point mass at t, it follows that the expectation $\mathbb{E}[f(\Gamma_{n,n/t})]$ converges to $f(t)$ as n tends to infinity. As regards the rate of convergence, the form of the PDF of $\Gamma_{n,n/t}$ implies that, in the case that f is C^2 at t, the decay of the error $\mathbb{E}[f(\Gamma_{n,n/t})] - f(t)$ is linear in $1/n$, in line with [2, Theorem 6], and that, moreover, $\mathbb{E}[f(\Gamma_{n,n/t})]$ admits the following expansion if the function f is C^{2k} at t:

$$\mathbb{E}[f(\Gamma_{n,n/t})] - f(t) = \sum_{m=1}^{k} b_m(t) \left(\frac{1}{n}\right)^m + o(n^{-k}) \qquad \text{as } n \to \infty,$$

for certain functions $b_1, \ldots, b_k$ (given in Theorem 3.1 below). We apply this expansion to functions $f(t)$ that are equal to the expectations of path-dependent functionals of Lévy bridges living on the time-interval $[0, t]$. We note that $\mathbb{E}[f(\Gamma_{n,n/t})]$ is equal to the expectation of the corresponding path-functional of the Lévy process X pinned down at an independent random time that is equal in distribution to $\Gamma_{n,n/t}$. For the path-dependent functionals that we consider (namely, first-passage times and occupation times of sets) the corresponding functions f are sufficiently smooth, so that the use of the Richardson extrapolation is fully justified. It holds furthermore (see Theorem A.4) that the density functions $D_n(x, y)$ and $\Omega_n(x, y)$, $n \in \mathbb{N}$, given by $D_{n,q}(x, y)\mathrm{d}y = \mathbb{P}(\overline{X}_{\Gamma_{n,q}} \leq x, X_{\Gamma_{n,q}} \in \mathrm{d}y)$ and $\Omega_{n,q}(x, y)\mathrm{d}x\,\mathrm{d}y = \mathbb{E}\left[\int_0^{\Gamma_{n,q}} I_{\{X_u \in \mathrm{d}x, X_{\Gamma_{n,q}} \in \mathrm{d}y\}}\,\mathrm{d}u\right]$ corresponding to a random horizon $\Gamma_{n,n/t}$ satisfy the following recursions for $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$D_{n+1,q}(x, y) = \int_{-\infty}^{x} D_{n,q}(x - w, y - w)D_{1,q}(x, w)\mathrm{d}w, \qquad \max\{y, 0\} \leq x, \tag{1.2}$$

$$\Omega_{n+1,q}(x, y) = \int_{-\infty}^{\infty} \left[\Omega_{1,q}(x, w)u_{n,q}(y - w) + \Omega_{n,q}(x - w, y - w)u_{1,q}(w)\right] \mathrm{d}w, \tag{1.3}$$

where $u_{n,q}$ is the probability density function of the random variable $X_{\Gamma_{n,q}}$. For the dense class of mixed-exponential Lévy processes (see Definition 2.1 below) we present explicit solutions to these recursions. By way of numerical illustration the method was implemented for a number of models in this class, and the numerical outcomes are reported in Section 4, confirming the theoretically predicted rates of decay of the error. We observed that the Richardson extrapolation based on a small number (about ten) recursive steps already yields highly accurate approximations.

Markov bridge method. We combine subsequently these approximations with a continuous-time EM scheme to estimate the conditional expectations $\widetilde{F}$ corresponding to the first-passage times and occupation times of sets of a stochastic volatility process with jumps. To illustrate the effectiveness of the method we evaluated a barrier option and a range note under a Bates-type model using the proposed Markov bridge Monte Carlo scheme, and report the results in Section 5. The rates of decay of the error that we find numerically in the case of barrier options are in line with the corresponding error estimates that were established in [24] for the case of killed diffusion processes.

Contents. The remainder of this paper is organized as follows. In Section 2 explicit expressions are derived for the first-passage probabilities and expected occupation times of a mixed-exponential Lévy process. Section 3 is devoted to error estimates and numerical illustrations are presented in Section 4. Section 5 contains a Markov bridge sampling method based on the randomisation method and numerical illustrations. The proof of the recursions (1.2) and (1.3) is deferred to Appendix A.

2. Maximum and occupation time of mixed-exponential Lévy models

We show in this section that the recursions in (1.2) and (1.3) admit explicit solutions in the case that the Lévy process X is a mixed-exponential jump-diffusion, the definition of which we recall next.

Definition 2.1. (i) A random variable has a *mixed-exponential density* if it has PDF f given by

$$f(x) = \sum_{i=1}^{m^+} p_i^+ \alpha_i^+ \mathrm{e}^{-\alpha_i^+ x} I_{(0,\infty)}(x) + \sum_{j=1}^{m^-} p_j^- \alpha_j^- \mathrm{e}^{-\alpha_j^- |x|} I_{(-\infty,0)}(x), \text{ where} \tag{2.1}$$

$$\sum_{k=1}^{m^\pm} p_k^\pm = q^\pm, \quad q^+ + q^- = 1 \quad \text{and} \quad -\alpha_{m^-}^- < \cdots < -\alpha_1^- < 0 < \alpha_1^+ < \cdots < \alpha_{m^+}^+.$$

(ii) A Lévy process $X = \{X_t, t \in \mathbb{R}_+\}$ is a *mixed-exponential jump-diffusion* (MEJD) if it is of the form

$$X_t \;=\; \mu t + \sigma W_t + \sum_{i=1}^{N_t} U_i, \tag{2.2}$$

where μ is a real number and σ is strictly positive, W is a standard Brownian motion, N is a Poisson process with intensity λ, and the jump-sizes $\{U_i, i \in \mathbb{N}\}$ are IID with mixed-exponential density. Here, the collections $W = \{W_t, t \in \mathbb{R}_+\}$, $N = \{N_t, t \in \mathbb{R}_+\}$ and $\{U_i, i \in \mathbb{N}\}$ are independent.

Remark 2.2. (i) Including in Def. 2.1 the additional restriction that the weights $p_k^\pm$ are nonnegative, the Lévy process is a hyper-exponential jump-diffusion (HEJD). While HEJD processes are dense in the class of all Lévy processes with a completely monotone Lévy density, the collection of mixed-exponential jump-diffusions is dense in the class of all Lévy processes, in the sense of weak convergence of probability measures (see [10]).

(ii) The parameters $\{p_k^\pm, k = 1, \ldots, m^\pm\}$ cannot be chosen arbitrarily but need to satisfy a restriction to guarantee that f is a PDF. Necessary and sufficient conditions for f to be a PDF are

$$p_1^\pm > 0, \qquad \sum_{k=1}^{m^\pm} p_k^\pm \alpha_k^\pm \geq 0, \qquad \text{and} \qquad \forall l = 1, ..., m^\pm: \qquad \sum_{k=1}^{l} p_k^\pm \alpha_k^\pm \geq 0,$$

respectively. For a proof of these results and alternative conditions see [6]. In Section 5 we will impose the additional condition $\alpha_1^+ > 1$, which ensures that the expectation $\mathbb{E}[S_t]$ of the exponential Lévy process $S_t = \exp\{X_t\}$ is finite for any non-negative t.

(iii) Samples can be drawn from the mixed-exponential distribution by using the acceptance-rejection method (see[40]) and taking as the instrumental distribution a double-exponential distribution. The double-exponential density multiplied by a constant will dominate the original mixed-exponential density. In the next section this method was used to obtain the Monte Carlo results.

(iv) Since σ is strictly positive, Assumption A.1 is satisfied for the MEJD process X, and $X_{\Gamma_{n,q}}$, $n \in \mathbb{N}, q > 0$, has a density by Lemma A.3.

From the definition of the MEJD process X it is straightforward to verify that the characteristic exponent $\Psi(s) = -\log \mathbb{E}[\mathrm{e}^{\mathbf{i}sX_1}]$ is a rational function of the form

$$\Psi(s) = -\mathbf{i}\mu s + \frac{\sigma^2 s^2}{2} - \lambda \left(\sum_{i=1}^{m^+} p_i^+ \frac{\alpha_i^+}{\alpha_i^+ - \mathbf{i}s} + \sum_{j=1}^{m^-} p_j^- \frac{\alpha_j^-}{\alpha_j^- + \mathbf{i}s} - 1 \right), \qquad s \in \mathbb{R}.$$

The distributions of X, the running supremum $\overline{X}$ and the running infimum $\underline{X}$ at the random time $\Gamma_{1,q}$ and also the functions $D_{1,q}$ and $\Omega_{1,q}$ can be expressed, as we shall see below, in terms of the roots $\{\rho_k^+, k = 1, \ldots, m^+ + 1\}$ and $\{\rho_k^-, k = 1, \ldots, m^- + 1\}$ with positive and negative real parts of the Cramér-Lundberg equation

$$q + \Psi(-\mathbf{i}s) = 0, \quad q > 0. \tag{2.3}$$

For the MEJD X the Wiener-Hopf factors Ψ_q^+ and Ψ_q^- can be identified explicitly. It is well-known that $\Psi_q^+(\theta)$ and $\Psi_q^-(\theta)$ have neither zeros nor poles on the half-planes $\{\Im(z) > 0\}$ and $\{\Im(z) < 0\}$ respectively, as a consequence of the fact that Ψ_q^+ and Ψ_q^- are the characteristic functions of infinitely divisible distributions supported on the positive and negative half-lines respectively (see [42, Ch. 9]). In particular, using that $\Psi_q^+(\theta)$ and $\Psi_q^-(\theta)$ satisfy $q/(q + \Psi(\theta)) = \Psi_q^+(\theta)\Psi_q^-(\theta)$ for $\theta \in \mathbb{R}$, the Wiener-Hopf factors of a mixed-exponential jump-diffusion can be identified as certain rational functions (see [37]):

Lemma 2.3. *Let $q > 0$ be given. The functions Ψ_q^+ and Ψ_q^- are given explicitly by*

$$\Psi_q^+(s) \quad := \quad \prod_{i=1}^{m^+} \left(1 - \mathbf{i}s/\alpha_i^+\right) \prod_{i=1}^{m^++1} \left(1 - \mathbf{i}s/\rho_i^+(q)\right)^{-1}, \tag{2.4}$$

$$\Psi_q^-(s) \quad := \quad \prod_{j=1}^{m^-} \left(1 + \mathbf{i}s/\alpha_j^-\right) \prod_{j=1}^{m^-+1} \left(1 - \mathbf{i}s/\rho_j^-(q)\right)^{-1}. \tag{2.5}$$

The fact that the Wiener-Hopf factors Ψ_q^+ and Ψ_q^- are rational functions implies that, when the roots of the Cramér-Lundberg equation are distinct, the running supremum $\overline{X}_{\Gamma_{1,q}}$ and infimum $\underline{X}_{\Gamma_{1,q}}$ of X at $\Gamma_{1,q}$, where $\overline{X}_t := \sup_{s \le t} X_s$ and $\underline{X}_t := \inf_{s \le t} X_s$ denote the running supremum and infimum of X at $t \in \mathbb{R}_+$, also follow mixed-exponential distributions.

Lemma 2.4. *Let $q > 0$ be given and suppose that the roots of* (2.3) *are distinct. The random variables $\overline{X}_{\Gamma_{1,q}}$, $-\underline{X}_{\Gamma_{1,q}}$ and $X_{\Gamma_{1,q}}$ have mixed-exponential distributions with densities $\overline{u}_{1,q}$, $\underline{u}_{1,q}$ and $u_{1,q}$ given by*

$$\overline{u}_{1,q}(x) = \sum_{i=1}^{m^++1} A_i^+(q)\rho_i^+(q)\mathrm{e}^{-\rho_i^+(q)x}, \qquad \underline{u}_{1,q}(x) = \sum_{j=1}^{m^-+1} A_j^-(q)(-\rho_j^-(q))\mathrm{e}^{\rho_j^-(q)x}, \qquad x > 0, \tag{2.6}$$

$$u_{1,q}(x) = \sum_{i=1}^{m^++1} B_i(q)\mathrm{e}^{-\rho_i^+(q)x} I_{(0,\infty)}(x) + \sum_{j=1}^{m^-+1} C_j(q)\mathrm{e}^{-\rho_j^-(q)x} I_{(-\infty,0)}(x), \qquad x \in \mathbb{R}, \tag{2.7}$$

with, for $i = 1, \ldots, m^+ + 1$ and $j = 1, \ldots, m^- + 1$,

$$A_i^+(q) := \frac{\prod_{k=1}^{m^+}(1 - \rho_i^+(q)/\alpha_k^+)}{\prod_{k \neq i}(1 - \rho_i^+(q)/\rho_k^+(q))}, \qquad A_j^-(q) := \frac{\prod_{k=1}^{m^-}(1 + \rho_j^-(q)/\alpha_k^-)}{\prod_{k \neq j}(1 - \rho_j^-(q)/\rho_k^-(q))}, \tag{2.8}$$

$$B_i(q) := A_i^+(q)\Psi_q^-(\rho_i^+(q))\rho_i^+(q), \qquad C_j(q) := A_j^-(q)\Psi_q^+(\rho_j^-(q))(-\rho_j^-(q)), \tag{2.9}$$

where we define $A_k^{\pm} \equiv 1$ in the case $m^{\pm} = 0$ (i.e. if there are no positive/negative jumps).

Proof. It is straightforward to verify that the coefficients of the function $(1 - \mathbf{i}s/\rho_i^+(q))^{-1}$ in the partial-fraction decompositions of the functions $q/(q + \Psi(s))$ and $\Psi_q^+(s)$ are given by $C_i(q)$ and $A_i^+(q)$, respectively, while the coefficients of the function $(1 - \mathbf{i}s/\rho_j^-(q))^{-1}$ in the partial-fraction decompositions of the functions $q/(q + \Psi(s))$ and $\Psi_q^-(s)$ are given by $B_j(q)$ and $A_j^-(q)$ respectively. Subsequently inverting the Fourier transforms $(1 - \mathbf{i}s/\rho_i^+(q))^{-1}$ and $(1 - \mathbf{i}s/\rho_j^-(q))^{-1}$ yields the stated expressions for the densities of $\overline{X}_{\Gamma_{1,q}}$, $-\underline{X}_{\Gamma_{1,q}}$ and $X_{\Gamma_{1,q}}$. □

The functions $\Omega_{n,q}$ and $D_{n,q}$ and the density $u_{n,q}$ can be explicitly identified by combining the forms of the functions $\Omega_{1,q}$ and $D_{1,q}$ (identified below) with the recursive relations in (1.2) and (1.3). From the form of these recursive relations it follows that the functions $\Omega_{n,q}$, $D_{n,q}$ and $u_{n,q}$ can be expressed as linear combinations of exponentials with the weights given by certain polynomials—the explicit expressions are given in the following result.

Consider the polynomials $\widetilde{P}^{\pm}_{k,i,n}$, $\widetilde{P}^{\pm}_{i,j,k,n}$ and real numbers $\widetilde{c}^{\pm}_{i,j,n}$ defined by

$$\int_0^x P^+_{k,n}(y)\mathrm{e}^{-\rho_k^+ y-\rho_i^+(x-y)}\mathrm{d}y = \mathrm{e}^{-\rho_k^+ x}\widetilde{P}^+_{k,i,n}(x) - \mathrm{e}^{-\rho_i^+ x}\widetilde{c}^+_{k,i,n},$$

$$\int_x^0 P^-_{k,n}(y)\mathrm{e}^{-\rho_k^- y-\rho_i^-(x-y)}\mathrm{d}y = \mathrm{e}^{-\rho_k^- x}\widetilde{P}^-_{k,i,n}(x) - \mathrm{e}^{-\rho_i^- x}\widetilde{c}^-_{k,i,n},$$

$$\int_0^x \mathrm{e}^{\rho_i^+(z-x)}P_{i,j,n}(x-z,y-z)u_1(z)\mathrm{d}z = \sum_{k=1}^{m^++1}\widetilde{P}^+_{i,j,k,n}(x,y)\mathrm{e}^{-\rho_k^+ x},$$

$$\int_0^x \mathrm{e}^{-\rho_j^-(x-z)}u_n(z)\mathrm{d}z = \sum_{k=1}^{m^-+1}\widetilde{P}^-_{i,j,k,n}(x)\mathrm{e}^{-\rho_k^- x},$$

where we denoted $\rho_h^+ = \rho_h^+(q)$ and $\rho_h^- = \rho_h^-(q)$, and $P^+_{k,n}$ and $P^-_{k,n}$ are the polynomials to be defined shortly. The fact that there exist polynomials and constants satisfying the above relations follows by repeated integration by parts. By induction the following expressions for the functions $u_{n,q}$, $D_{n,q}$ and $\Omega_{n,q}$ can be derived:

Proposition 2.5. *For any $n \in \mathbb{N}\cup\{0\}$ we have*

$$\begin{aligned}
u_{n+1,q}(x) &= \sum_{k=1}^{m^++1} P^+_{k,n+1}(x)\mathrm{e}^{-\rho_k^+ x}I_{(0,\infty)}(x) + \sum_{k=1}^{m^-+1} P^-_{k,n+1}(x)\mathrm{e}^{-\rho_k^- x}I_{(-\infty,0)}(x), \qquad x\in\mathbb{R},\\
D_{n+1,q}(x,y) &= u_{n+1,q}(y) - \sum_{i=1}^{m^++1}\sum_{j=1}^{m^-+1} P_{i,j,n+1}(x,y)\mathrm{e}^{-\rho_j^-(y-x)-\rho_i^+ x}, \qquad x\in\mathbb{R}_+, x\geq y,\\
\Omega_{n+1,q}(x,y) &= q^{-(n+1)}\cdot\sum_{k=1}^{n+1} u_{n+2-k,q}(x)u_{k,q}(y-x), \qquad x,y\in\mathbb{R},
\end{aligned}$$

with as before $\rho_j^- = \rho_j^-(q)$ and $\rho_i^+ = \rho_i^+(q)$, and with $P^+_{k,1}\equiv B_k(q)$, $P^-_{k,1}\equiv C_k(q)$ and $P_{i,j,1}\equiv \frac{E_{ij}(q)}{\rho_j^- - \rho_i^+} := \frac{A_i^+(q)A_j^-(q)\rho_i^+(q)\rho_j^-(q)}{\rho_j^- - \rho_i^+}$, and where $P^{\pm}_{k,n+1}$ and $P_{i,j,n+1}$ are polynomials and $c^{\pm}_{k,i,n}$ are real numbers that are defined recursively for $n\in\mathbb{N}$, as follows:

$$P^+_{k,n+1}(x) = \sum_{r=1}^{m^-+1}\left(C_r(q)\int_0^\infty \mathrm{e}^{(\rho_r^- - \rho_k^+)z}P^+_{k,n}(x+z)\mathrm{d}z + B_k(q)c^-_{k,r,n}\right) + \sum_{r=1}^{m^++1} B_r(q)\left(\widetilde{P}^+_{k,r,n}(x) - \widetilde{c}^+_{r,k,n}\right),$$

$$P^-_{k,n+1}(x) = \sum_{r=1}^{m^++1}\left(B_r(q)\int_{-\infty}^0 \mathrm{e}^{(\rho_r^+ - \rho_k^-)z}P^-_{k,n}(x+z)\mathrm{d}z + C_k(q)c^+_{k,r,n}\right) + \sum_{r=1}^{m^-+1} C_r(q)\left(\widetilde{P}^-_{k,r,n}(x) - \widetilde{c}^-_{r,k,n}\right),$$

$$\begin{aligned}
P_{i,j,n+1}(x,y) &= \int_{-\infty}^0 P_{i,j,n}(x-z,y-z)\mathrm{e}^{\rho_i^+ z}u_1(z)\mathrm{d}z + \sum_{k=1}^{m^++1}\widetilde{P}^+_{k,j,i,n}(x,y) - \sum_{k=1}^{m^-+1}\widetilde{P}^-_{i,k,j,n}(y-x)\\
&+ B_i(q)\int_0^\infty P^-_{j,n}(y-x-z)\mathrm{e}^{(\rho_j^- - \rho_i^+)z}\mathrm{d}z + \frac{E_{i,j}(q)}{\rho_j^- - \rho_i^+}\int_0^\infty u_{n,q}(z)\mathrm{e}^{\rho_j^- z}\mathrm{d}z\\
&- \sum_{k=1}^{m^++1}\sum_{l=1}^{m^-+1}\frac{E_{k,l}(q)}{\rho_l^- - \rho_k^+}\int_{-\infty}^0 P_{i,j,n}(-z,y-x-z)\mathrm{e}^{\rho_i^+ z - \rho_l^- z}\mathrm{d}z,\\
c^-_{k,r,n} &= \int_{-\infty}^0 \mathrm{e}^{(\rho_k^+ - \rho_r^-)z}P^-_{r,n}(z)\mathrm{d}z, \qquad c^+_{k,r,n} = \int_0^\infty \mathrm{e}^{(\rho_k^- - \rho_r^+)z}P^+_{r,n}(z)\mathrm{d}z.
\end{aligned}$$

Proof. By combining the identity $\mathbb{P}[\overline{X}_{\Gamma_{1,q}}\in \mathrm{d}x, x - X_{\Gamma_{1,q}}\in\mathrm{d}z] = \mathbb{P}[\overline{X}_{\Gamma_{1,q}}\in\mathrm{d}x]\mathbb{P}[-\underline{X}_{\Gamma_{1,q}}\in\mathrm{d}z]$, $x,z\in\mathbb{R}_+$, (which follows from the Wiener-Hopf factorisation of X) with Lemma 2.4 and performing a one-dimensional integration, we get the expression for the function $D_{1,q}$. The Markov property and stationarity of increments yields $\Omega_{1,q}(x,y) = q^{-1}u_{1,q}(y-x)\,u_{1,q}(x)$, whence we have the form of the function $\Omega_{1,q}$ by inserting the expression (2.7) for u^q . The expressions for $u_{n+1,q}$, $D_{n+1,q}$ and $\Omega_{n+1,q}$ follow by induction with respect to n, utilising (i) the fact that $u_{n+1,q}$ is equal to the convolution of $u_{n,q}$ and $u_{1,q}$, as a consequence of the independence and stationarity of the increments of X, (ii) the form of $D_{1,q}$ and the recursive relation in (1.2), and (iii) the form of $\Omega_{1,q}$ and the recursive relation in (1.3). □

3. Convergence and error-estimates

The randomisation method consists in approximating the value $f(t)$ of a function f at time $t > 0$ by the expectation $\mathbb{E}[f(\Gamma_{n,n/t})]$ of f evaluated at a random time $\Gamma_{n,n/t}$ that follows a Gamma distribution with expectation $\mathbb{E}[\Gamma_{n,n/t}] = t$ and variance $\mathbb{E}[(\Gamma_{n,n/t} - t)^2] = t^2/n$. Since the random variables $\Gamma_{n,n/t}$ converges in distribution to t as n tends to infinity, the error $\mathbb{E}[f(\Gamma_{n,n/t})] - f(t)$ converges to zero for any bounded and continuous function f. The error can be expanded in terms of powers of $1/n$ provided that f is sufficiently smooth, as shown in the following result:

Theorem 3.1. *Let k be a given non-negative integer and consider $f \in C^{2k+2}(\mathbb{R}_+)$. There exist functions $b_1, \ldots, b_{k+1} : \mathbb{R}_+ \to \mathbb{R}$ such that we have, for any $t \in \mathbb{R}_+$,*

$$n^{k+1}\left[\mathbb{E}[f(\Gamma_{n,n/t})] - f(t) - \sum_{m=1}^{k} b_m(t)\left(\frac{1}{n}\right)^m\right] = b_{k+1}(t) + o(1) \qquad \text{as } n \to \infty. \tag{3.1}$$

In particular, denoting by $f^{(m)}$ the mth derivative of f, we have

$$b_1(t) = \frac{t^2}{2} f^{(2)}(t), \qquad b_2(t) = \frac{t^4}{8} f^{(4)}(t) + \frac{t^3}{3} f^{(3)}(t), \qquad b_3(t) = \frac{t^6}{48} f^{(6)}(t) + \frac{t^5}{6} f^{(5)}(t) + \frac{t^4}{4} f^{(4)}(t),$$
$$b_4(t) = \frac{t^8}{384} f^{(8)}(t) + \frac{t^7}{24} f^{(7)}(t) + \frac{13t^6}{72} f^{(6)}(t) + \frac{t^5}{5} f^{(5)}(t).$$

Remark 3.2. (i) Theorem 3.1 implies that for $f \in C^2(\mathbb{R}_+)$ the error of the approximation of $f(t)$ by $\mathbb{E}[f(\Gamma_{n,n/t})]$ decays linearly, that is, $\mathbb{E}[f(\Gamma_{n,n/t})] - f(t) = \frac{b_1(t)}{n} + o(\frac{1}{n})$ as n tends to infinity.

(ii) Theorem 3.1 also provides a justification of the use of the Richardson extrapolation to increase the speed of convergence if the function f is sufficiently smooth. Since the error of the approximation is given in terms of positive integer powers of $1/n$, the Richardson extrapolation that utilises the first N values $\mathbb{E}[f(\Gamma_{1,1/t})], \ldots, \mathbb{E}[f(\Gamma_{N,N/t})]$ is explicitly given by

$$P_{1:N} \quad = \quad \sum_{k=1}^{N} \frac{(-1)^{N-k} k^N}{k!(N-k)!} \mathbb{E}[f(\Gamma_{k,k/t})], \tag{3.2}$$

(see [38, §1.3] for a derivation of this formula). Note in particular that in order to deploy the extrapolation (3.2) it suffices to know the existence of functions b_m such that (3.1) holds and it is not required to find their explicit form. In the case $f \in C^{2k+2}(\mathbb{R}_+)$, $k < N$ Theorem 3.1 implies that the error $P_{1:N} - f(t)$ of the interpolation $P_{1:N}$ is $o(N^{-k-1})$. In particular, if f is C^∞ then the error $P_{1:N} - f(t)$ is $O(N^{-k-1})$ for every k, as N tends to infinity. Refer to [43] for background on the theory of extra- and interpolation.

Proof of Theorem 3.1. While we expect this result to be known in the literature, we have not been able to find a reference and provide a brief proof. Taylor's theorem and the fact that $f \in C^{2k+2}$ imply

$$f(s) - f(t) = \sum_{m=1}^{2k+1} \frac{(s-t)^m}{m!} f^{(m)}(t) + R(s,t)$$

where the remainder term is given by $R(s,t) = \frac{(s-t)^{2k+2}}{(2k+2)!} f^{(2k+2)}(\xi)$ for some ξ between s and t. Replacing s by the independent Gamma random variable $\Gamma_{n,n/t}$ we get

$$\mathbb{E}[f(\Gamma_{n,n/t}) - f(t)] = \sum_{m=2}^{2k+1} \frac{a_{m,n}}{m!} f^{(m)}(t) + \mathbb{E}[R(\Gamma_{n,n/t}, t)]$$

with $a_{m,n} = \mathbb{E}[(\Gamma_{n,n/t} - t)^m]$, where we have $a_{1,n} = 0$ as the expectation $\mathbb{E}[\Gamma_{n,n/t}]$ is equal to t. The numbers $a_{m,n}$ are equal to $a_{m,n} = \frac{\mathrm{d}^m}{\mathrm{d}u^m}\big|_{u=0} M(u)$ where M denotes the moment-generating function of the random variable $\Gamma_{n,n/t} - t$ which is given by

$$M(u) = \left(1 - \frac{ut}{n}\right)^{-n} \exp\{-ut\}, \qquad u \leq \frac{n}{t}.$$

In particular, it follows from the form of M that the $a_{m,n}$ are linear combinations of positive integer powers of $1/n$. Reordering of terms and straightforward manipulations result in the identity in (3.1). □

We next turn to the problem of approximation of the distribution of the supremum and the expected occupation time of the set $(-\infty, x]$ of the *Lévy bridge process* $X^{(0,0)\to(t,y)}$ from $(0,0)$ to (t,y) (its definition is recalled in Appendix A):

$$\vec{d}_t(x,y) := \mathbb{P}\left(\overline{X}^{(0,0)\to(t,y)} \leq x\right), \qquad \vec{\omega}_t(x,y) := \mathbb{E}\left[\int_0^t I_{\left\{X_u^{(0,0)\to(t,y)} \leq x\right\}} \mathrm{d}u\right], \qquad \text{with} \tag{3.3}$$

$$\overline{X}^{(0,0)\to(t,y)} := \sup_{u\in[0,t]} X_u^{(0,0)\to(t,y)}.$$

By spatial and temporal homogeneity of X, the corresponding quantities in the case of a general starting point (s,z) are given in terms of $\vec{d}$ and $\vec{\omega}$ by $\vec{d}_{t-s}(x-z, y-z)$ and $\vec{\omega}_{t-s}(x-z, y-z)$. The approximations of $\vec{d}$ and $\vec{\omega}$ are given in terms of the randomised bridge process $X^{(0,0)\to(\Gamma_{n,q},y)}$ (see Appendix A) as follows:

$$\vec{D}_q^{(n)}(x,y) := \mathbb{P}\left(\overline{X}^{(0,0)\to(\Gamma_{n,q},y)} \leq x\right), \qquad \vec{\Omega}_q^{(n)}(x,y) := \mathbb{E}\left[\int_0^{\Gamma_{n,q}} I_{\left\{X_u^{(0,0)\to(\Gamma_{n,q},y)} \leq x\right\}} \mathrm{d}u\right].$$

We derive next error estimates for these randomised bridge approximations.

Corollary 3.3. *Let x, $y \in \mathbb{R}$ and $t > 0$. For some constants C^d and C^ω we have, for all positive integers n,*

$$\left|\vec{D}_{n/t}^{(n)}(x,y) - \vec{d}_t(x,y)\right| \leq \frac{C^d}{n}, \qquad \left|\vec{\Omega}_{n/t}^{(n)}(x,y) - \vec{\omega}_t(x,y)\right| \leq \frac{C^\omega}{n}. \tag{3.4}$$

Proof. Since the distribution of X_t has a continuous density $p_t(y)$ and $s \mapsto \vec{d}_s(x,y)$, $s \mapsto \vec{\omega}_s(x,y)$ and $s \mapsto p_s(y)$ are C^2 at $s = t$ with $p_t(y) > 0$, the estimates in (3.4) follow by applying Theorem 3.1 to the functions $t \mapsto \vec{d}_t(x,y)p_t(y)$, $t \mapsto \vec{\omega}_t(x,y)p_t(y)$ and $t \mapsto p_t(y)$. □

4. Numerical illustration: first-passage time probabilities and occupation times

To provide a numerical illustration of the randomisation method, we implemented the recursive formulas (given in Proposition 2.5) to approximate the following expectations of path-dependent functionals:

$$\mathbb{P}\left(\sup_{u\in[0,t]} X_u^{(0,x)\to(t,y)} \leq z\right), \qquad \mathbb{E}\left[\int_0^t I_{\left\{X_u^{(0,x)\to(t,y)} \in (a,b)\right\}} \mathrm{d}u\right] \qquad \begin{array}{l} x = 1, y = 1.1, z = 1.2, \\ t = 1, a = 1.05, b = 1.25, \end{array}$$

for the case[1] that the underlying Lévy process X is equal to a HEJD process with typical parameters, which are detailed in Table 1. The outcomes are reported in Table 2 and Figure 1. In Table 2 the values are listed of the first-passage probabilities and the expected occupation times of the randomised Lévy bridges corresponding to a $\Gamma(n,n)$-randomisation of the fixed time $T = 1$ for a number of values of n. We also reported the results obtained by applying a Richardson extrapolation $P_{1:n}$ of order n, using the first n outcomes (defined in (3.2)). The logarithms of the corresponding absolute errors are plotted in Figure 1. The errors were computed with respect to the value $P_{1:11}$ that was obtained after Richardson's extrapolation with $n = 11$ stages.

Empirically we observe that the rate of decay of the error of the un-extrapolated outcomes to be (approximately) linear for both different functionals, in line with the theoretical error bound given in Corollary 3.3: indeed, the ordinary least squares (OLS) regression lines (dark grey) in the log-log plots had slopes equal to

[1] See [44, Chapter 3] for additional numerical examples.

TABLE 1. The model parameters used throughout the paper. The parameters for the Kou model are taken from [32], the ones for the HEJD model from [27], and the ones for the MEJD model from [13] (which for the latter two models have been re-expressed using our notation).

	KOU	HEJD	MEJD
σ	0.2	$\sqrt{0.042}$	0.2
λ	3.0	11.5	1.0
α^+	50	(5, 10, 15, 25, 30, 60, 80)	(213.0215, 236.0406, 237.1139, 939.7441, 939.8021)
α^-	25	(5, 10, 15, 25, 30, 60, 80)	(213.0215, 236.0406, 237.1139, 939.7441, 939.8021)
p^+	0.3	$(0.05, 0.05, 0.1, 0.6, 1.2, 1.9, 6.1) * 0.51/\lambda$	(4.36515, 1.0833, -5, 0.0311, 0.02045)
p^-	0.7	$(0.5, 0.3, 1.1, 0.8, 1, 4, 2.3) * 0.64/\lambda$	(4.36515, 1.0833, -5, 0.0311, 0.02045)

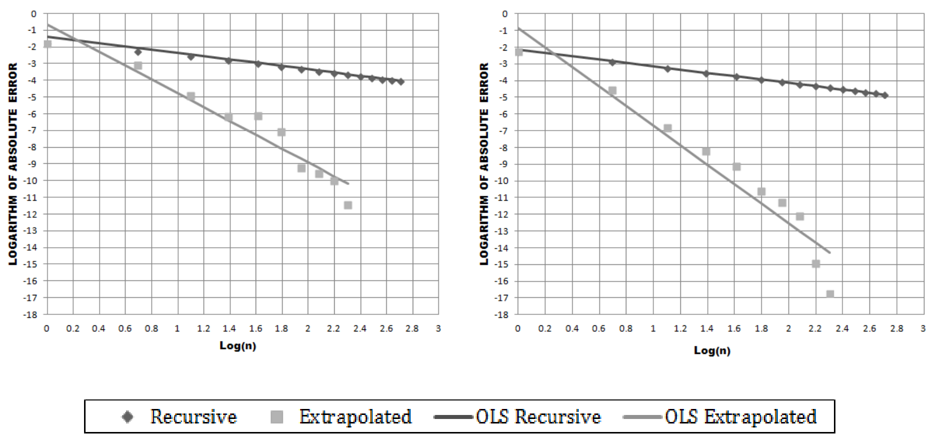


FIGURE 1. The logarithms of the absolute errors of the outcomes generated by the recursive algorithm for (a) the one-sided first-passage probabilities and (b) the expected occupation time under the HEJD processes as a function of n, where n is the number of steps in the recursions. In each sub-figure the errors of the recursive values and the Richardson extrapolated values are displayed. Also ordinary least square estimations of either series of results are plotted (in the case of the un-extrapolated values the OLS line was estimated using the last six values only). The slopes of the dark lines in sub-figures (a) and (b) are given by -0.98 and -0.99, respectively. The starting point of the bridge is 1.0, the end point is 1.1 and the barrier level is 1.2 and the range is $(1.05, 1.25)$. In all cases the Lévy bridge process is assumed to start at time 0 and to end at time 1. The model parameters that were used are given in Table 1.

-0.94 (-0.98) and -0.98 (-0.99) in the case of the first-passage probabilities (and expected occupation times) of the Lévy bridges corresponding to the HEJD model. Moreover, in line with the theoretical error estimates given in Theorem 3.1, we observe that the application of the Richardson extrapolation leads to a significantly faster decay of the error. By comparing the error plots of the expectations of the two path-dependent functionals we note that the logarithmic errors for the expected occupation times (for a given n) are consistently and significantly the smaller of the two, suggesting that the randomisation method converges faster in this case. This feature is likely to be related to the higher degree of smoothness in the case of the expected occupation time. Finally, we mention that we computed the roots the Cramér-Lundberg equation featuring in the solutions $D_{n,q}$ and $\Omega_{n,q}$ by deploying the Newton-Raphson method.[2,3]

[2] We investigated the round-off error resulting from the computation of the roots based on single precision arithmetic, and found that in that case the computed roots were accurate up to an error of $1.0e^{-11}$.

[3] In order to efficiently approximate the first-passage time probability and the expected occupation time of the Lévy bridge process, one could combine the procedure described in this section with interpolation: One would then compute these quantities for a grid of points and construct subsequently functions on the real line $\mathbb{R}$ by using (linear) interpolation.

TABLE 2. Approximations of one-sided first-passage time (FPT) probabilities and expected occupation times obtained recursively (P_n) and with Richardson extrapolation ($P_{1:n}$) for the HEJD model as a function of n, where n is the number of recursions. The starting point of the bridge is assumed to be 1.0, the end point is 1.1, the barrier level is 1.2 and the range is $(1.05, 1.25)$. In all cases the Lévy bridge is assumed to start at time 0 and to end at time 1. The model parameters r that were used are given in Table 1.

FPT probability			Expected occupation time	
P_n HEJD	$P_{1:n}$ HEJD	n	P_n HEJD	$P_{1:n}$ HEJD
0.3006853	0.3006853	1	0.3680801	0.3680801
0.3617512	0.4228170	2	0.4142655	0.4604509
0.3911554	0.4635372	3	0.4322124	0.4719338
0.4084846	0.4734619	4	0.4415893	0.4711338
0.4198448	0.4735378	5	0.4473202	0.4707490
0.4278257	0.4720958	6	0.4511786	0.4708328
0.4337174	0.4713210	7	0.4539517	0.4708704
0.4382332	0.4711443	8	0.4560403	0.4708630
0.4417979	0.4711707	9	0.4576699	0.4708578
0.4446794	0.4712065	10	0.4589767	0.4708575
0.4470546	0.4712177	11	0.4600480	0.4708575

5. Illustration: Option valuation using the bridge sampling method

By way of illustration we next present the numerical results that were obtained by valuing an up-and-in barrier option and a range note under a number of models by using a Markov bridge algorithm described in Table 3 below (the recursive method for approximation of first-passage time probabilities and expected occupation times from Section 4 is applied).

We assume that the stock price process $S = \{S_t, t \in \mathbb{R}_+\}$ evolves according to a Bates-type stochastic volatility model with mixed-exponential jumps. The process S is thus specified by the exponential model

$$S_t = \exp\{Y_t\}, \qquad t \in \mathbb{R}_+,$$

where the log-price process $Y = \{Y_t, t \in \mathbb{R}_+\}$ satisfies the stochastic differential equation

$$\mathrm{d}Y_t = \left(\mu - \frac{Z_t}{2}\right)\mathrm{d}t + \sqrt{|Z_t|}\mathrm{d}B_t + \mathrm{d}J_t, \qquad Y_0 = x, \tag{5.1}$$

$$\mathrm{d}Z_t = \kappa(\delta - Z_t)\mathrm{d}t + \xi\sqrt{|Z_t|}\mathrm{d}W_t, \qquad t \in \mathbb{R}_+, \qquad Z_0 = v, \tag{5.2}$$

where x and v are strictly positive, (B, W) is a two-dimensional Brownian motion with correlation-parameter ρ and J_t is an independent compound Poisson process with intensity λ and jump-sizes distributed according to a mixed-exponential distribution F with mean m. The parameters κ, δ, and ξ of the model are positive and represent the speed of mean-reversion of the volatility, the long term volatility level and the volatility of volatility parameter. The parameter μ is set equal to $\mu = r - q - \lambda m$ which ensures that the moment condition $\mathbb{E}[\exp\{Y_t\}] = \exp\{(r-q)t + Y_0\}$ is satisfied for all non-negative t, where the constants r and q are non-negative constants representing the risk-free rate of return and the dividend yield. Under this moment condition it holds that the process $\{\mathrm{e}^{-(r-q)t}S_t, t \in \mathbb{R}_+\}$ is a martingale. Note that choosing κ and ξ equal to zero yields the mixed-exponential jump-diffusion process.

By way of example we consider an up-and-in call (UIC) option and a range note (RN). By arbitrage pricing theory, the UIC option and the RN have values at time 0 given by

$$UIC(K,H) = \mathbb{E}\left[\mathrm{e}^{-rT}(S_T - K)^+ I_{\{\sup_{0\le t\le T} S_t > H\}}\right], \quad RN(a_1, a_2) = \mathbb{E}\left[\mathrm{e}^{-rT}\cdot\frac{C}{T}\int_0^T I_{\{a_1 \le S_u \le a_2\}}\mathrm{d}u\right],$$

where K is the strike price, H is the barrier level, C is the nominal, and a_1 and a_2 are the lower and upper bound of the range respectively.

5.1. **Markov Bridge sampling method.** The first step is to approximate the log-price process Y by a process that has piecewise constant drift and volatility deploying the Euler-Maruyama approximation of the process (Y, Z) on the equidistant partition $\mathbb{T}_N$ which can be expressed as

$$Y'_{\tau_{n+1}} = Y'_{\tau_n} + \left(\mu - \frac{Z'_{\tau_n}}{2}\right)\Delta_n + \sqrt{|Z'_{\tau_n}|}\Delta W_n + \Delta J_n, \qquad Y'_0 = x, \tag{5.3}$$

$$Z'_{\tau_{n+1}} = Z'_{\tau_n} + \kappa(\delta - Z'_{\tau_n})\Delta_n + \xi\sqrt{|Z'_{\tau_n}|}\Delta B_n, \qquad Z'_0 = v, \tag{5.4}$$

for $n \in \mathbb{N}\backslash\{0\}$, with $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$, $\Delta B_n = B_{\tau_{n+1}} - B_{\tau_n}$, $\Delta J_n = J_{\tau_{n+1}} - J_{\tau_n}$, and $\Delta_n = \tau_{n+1} - \tau_n = T/N$. See [26, 29] for results on strong and weak-convergence of this scheme. The Markov bridge-sampling method is based on the continuous-time Euler-Maruyama approximation Y' leaving the (piecewise constant) approximation $(Z'_{\tau_n})_{n\in\mathbb{N}}$ for Z given in (5.4) unchanged. We arrive at the approximation

$$Y'_t = Y'_{\tau_n} + \left(\mu - \frac{Z_{\tau_n}}{2}\right)(t - \tau_n) + \sqrt{|Z'_{\tau_n}|}(W_t - W_{\tau_n}) + (J_t - J_{\tau_n}), \tag{5.5}$$

$$Z'_t = Z'_{\tau_n}, \tag{5.6}$$

for $t \in [\tau_n, \tau_{n+1}]$. Observe that with this choice of interpolation it holds that, conditional on the values of the random variable Z'_{τ_n}, the process $\{Y'_{t-\tau_n}, t \in [\tau_n, \tau_{n+1}]\}$ is a Lévy process, for each $n = 0, \ldots, N-1$. The bridge sampling algorithm is summarised in Table 3.

TABLE 3. Bridge sampling algorithm for approximating $\mathbb{E}[F(T, Y, Z)]$.

0. Fix $M, N \in \mathbb{N}$ sufficiently large.
1. Sample M IID copies $\xi^{(1)}, \ldots, \xi^{(M)}$ from the law of $\left(Y'_{\tau_1}, Z'_{\tau_1}, \ldots, Y'_{\tau_N}, Z'_{\tau_N}\right)$,
2. Evaluate the estimator $\frac{1}{M}\sum_{i=1}^{M} \widetilde{F}^{(N)}\left(\xi^{(i)}\right)$,

with $\widetilde{F}^{(N)}(y_0, z_0, \ldots, y_N, z_N) = \mathbb{E}\left[F(T, Y', Z')\middle| Y'_{\tau_0} = y_0, Z'_{\tau_0} = z_0, \ldots, Y'_{\tau_N} = y_N, Z'_{\tau_N} = z_N\right]$.

Remark 5.1. The choice $N = 1$ in the above algorithm corresponds to the case of a *single* large step bridge sampling, which is the version of the algorithm that was implemented to produce the results reported in Section 4.

Next we focus on the application of the bridge sampling method to the approximation of the expectation of two path-dependent functionals that are given in terms of the running maximum and the occupation time of Y as follows:

$$F_S(T, Y, Z) := g(Y_T) I_{\left\{\overline{Y}_T \leq a\right\}}, \quad a > 0, \text{ with } \overline{Y}_t := \sup\{Y_s \,:\, s \leq t\},$$

$$F_O(T, Y, Z) := \int_0^T g(Y_s)\mathrm{d}s,$$

for some function $g : \mathbb{R}_+ \to \mathbb{R}$. The functionals F_S and F_O admit the following multiplicative and additive decompositions into parts that only involve the processes $Y^{i-1,i} := \{Y_{t+\tau_{i-1}}, t \in [0, \tau_i - \tau_{i-1}]\}$, for $i = 1, \ldots, N$:

$$F_S(T, Y, Z) = g(Y_T)\prod_{i=1}^{N} F_S^{(i)}(Y, Z), \qquad F_S^{(i)}(Y, Z) = I_{\left\{\sup_{s\in[\tau_{i-1},\tau_i]} Y_s \leq a\right\}},$$

$$F_O(T, Y, Z) = \sum_{i=1}^{N} F_O^{(i)}(Y, Z), \qquad F_O^{(i)}(Y, Z) = \int_{\tau_{i-1}}^{\tau_i} g(Y_s)\mathrm{d}s.$$

TABLE 4. Model parameters of the generalised Bates model, the maturity, strike, barrier and spot levels and range of the up-and-in call option and range note to be used in Figure 2 and Table 5 (with jump-parameters as given in Table 1).

κ	δ	ξ	ρ	V_0	K	H	(a_1, a_2)	S_0	r	d	T
1.0	0.1	0.2	-0.5	0.07	100	120	(1.15,1.35)	100	0.05	0.0	1.0

These decompositions in turn imply that the conditional expectations

$$\widetilde{F}_S^{(N)}(y_0, z_0, \dots, y_N, z_N) := \mathbb{E}\left[F_S(T, Y', Z')\middle| Y'_{\tau_0} = y_0, Z'_{\tau_0} = z_0, \dots, Y'_{\tau_N} = y_N, Z'_{\tau_N} = z_N\right], \tag{5.7}$$

$$\widetilde{F}_O^{(N)}(y_0, z_0, \dots, y_N, z_N) := \mathbb{E}\left[F_O(T, Y', Z')\middle| Y'_{\tau_0} = y_0, Z'_{\tau_0} = z_0, \dots, Y'_{\tau_N} = y_N, Z'_{\tau_N} = z_N\right] \tag{5.8}$$

can be expressed in terms of Lévy bridge processes, as shown next.

Proposition 5.2. *For any $N \in \mathbb{N}$ the following decompositions hold true:*

$$\widetilde{F}_S^{(N)}((y_0, z_0), \dots, (y_N, z_N)) = g(y_N) \prod_{i=1}^{N} \widetilde{F}_S^{(i)}(y_{i-1}, y_i, z_{i-1}), \tag{5.9}$$

$$\widetilde{F}_O^{(N)}((y_0, z_0), \dots, (y_N, z_N)) = \sum_{i=1}^{N} \widetilde{F}_O^{(i)}(y_{i-1}, y_i, z_{i-1}), \tag{5.10}$$

where the functions $x \mapsto \widetilde{F}_S^{(i)}(x, y, z)$ and $x \mapsto \widetilde{F}_O^{(i)}(x, y, z)$ are given by

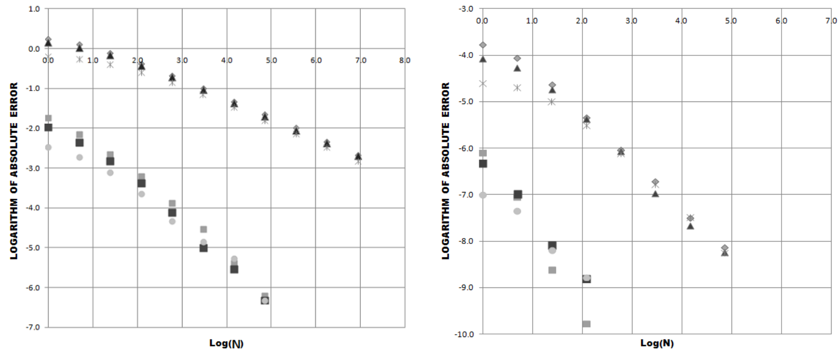
$$\widetilde{F}_S^{(i)}(x, y, z) = \mathbb{E}\left[I_{\left(\sup_{s \le \Delta} L_s^{(0,x)\to(\Delta,y),i} \le a\right)}\right], \qquad \widetilde{F}_O^{(i)}(x, y, z) = \mathbb{E}\left[\int_0^\Delta g\left(L_s^{(0,x)\to(\Delta,y),i}\right) \mathrm{d}s\right],$$

with $\Delta = T/N$, where $L^{(0,x)\to(\Delta,y),i}$ denotes the Lévy bridge process from $(0, x)$ to (Δ, y), with underlying Lévy process $L^{(i)}$ that is equal in law to $Y^{i-1,i}$ conditional on $Z_{\tau_{i-1}} = z$ and $Y_{\tau_i} = x$.

Proof. The decompositions hold true as a consequence of the harness property of a Lévy process, the definition of a Lévy bridge and the fact that a Lévy process is temporally homogeneous. □

5.2. **Bates-type stochastic volatility model with jumps.** By approximating the log-price process Y of the Bates-type model by the EM scheme in (5.3)–(5.6), and computing first-passage time probabilities and expected occupation times of the process Y' as before using the recursive algorithm (as in Section 4), we obtained the approximate values of an up-and-in call option and a range note under the Heston model and Bates-type models with double-exponential and hyper exponential jumps. We ran the algorithm in Table 3 with 10 million paths ($M = 10^7$) on a uniform grid Υ with $N = 2^i$ steps for $i = 0, 1, ..., 10$. We used the recursions with $n = 7$ steps and approximated the functions $\widehat{F}_S^{(i)}(x, y, z)$ by evaluating these on a grid of points and using (tri-linear) interpolation to obtain approximations of the values of the function outside the grid. By way of comparison, we also report the results obtained by a standard (discrete-time) Euler-Maruyama approximation with 10 million paths and a varying number of (equidistant) time-steps.

For the results displayed in Figure 2 we take the value corresponding to $N = 1024$ as true value and compute the logarithm of the absolute errors for all other outcomes with respect to this value. In order to estimate the rates of decay of the error we added ordinary least-square regression lines to the figures. The slopes of the OLS lines for the Heston model and the Bates-type model with double-exponential and hyper-exponential jumps that we found are -1.03, -1.02, and -1.04 in the case of the up-and-in call option and -1.36, -0.96 and -1.02, in the case of the range note, which suggests a rate of decay of the error that is linear in the reciprocal of the number of steps.



(a) Up-and-in call option (b) Range note

FIGURE 2. The absolute error of the values of an up-and-in barrier option and range note under the Heston and Bates-type models plotted on a log-log scale against the number of time-steps N. Parameters are as given in Tables 1 and 4.

By way of comparison we also implemented the standard (discrete-time) Euler-Maruyama scheme for each of the three models, and found the corresponding three slopes of the OLS lines to be equal to -0.48 in the case of the values of the up-and-in call options and to -1.00 in the case of values of the range notes. These results suggest that, in the case of an UIC option, only a square-root rate holds for the decay of the error as function of the reciprocal of the number of time-steps rather than a linear rate, which is in line with the well-known fact that the strong order of the discrete-time EM scheme is 0.5, and that, furthermore, for killed diffusion models the weak error of the discrete-time EM scheme has been shown to be bounded by a constant times $N^{-1/2}$ in the number of time-steps N under suitable regularity assumptions on the coefficients and the pay-off function (see [24, Thms. 2.3, 2.4]).

APPENDIX A. PROOF OF RECURSIONS FOR MAXIMA AND OCCUPATION TIMES OF A LÉVY BRIDGE

Let $X = \{X_t, t \in \mathbb{R}_+\}$ be a Lévy process (a stochastic process with stationary and independent increments and right-continuous paths with left limits such that $X_0 = 0$) that is defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$, where $\mathbf{F} = \{\mathcal{F}_t, t \in \mathbb{R}_+\}$ denotes the completed right-continuous filtration generated by X. We refer to [34, 42] for general treatments of the theory of Lévy processes. To avoid degeneracies we exclude in the sequel the case that $|X|$ is a subordinator. The bridge method under consideration involves randomised bridge processes that can informally be described as processes that are equal in law to X conditioned to take a given value at certain independent random times.

Formally, such a process can be constructed by invoking general results on existence of conditional distributions and disintegration (see Kallenberg [28, Thms. 6.3, 6.4]). More specifically, let the triplet (X, τ_1, τ_2) of the Lévy process X and independent random times τ_1, τ_2 with $\tau_1 \leq \tau_2$ be defined on the Borel space $D \times U$ that is the product of the Skorokhod space D of rcll functions and the space $U = \mathbb{R}_+^2$. Then, by disintegration, we obtain a family of conditional laws conditional on different values of $(\eta_1, \eta_2) := (X_{\tau_1}, X_{\tau_2})$ that may be used to define the randomised bridge process with starting point (τ_1, y_1) and end point (τ_2, y_2) by $\{X_{(s+\tau_1)\wedge\tau_2}, s \in \mathbb{R}_+\}$ for almost all realisations (y_1, y_2) of (η_1, η_2).

Under regularity assumptions on the Lévy process X and for specific choices of the random times the construction in the previous paragraph may be extended to all realisations of (η_1, η_2), drawing on results in [15] where weak-continuity results and pathwise constructions of a Markov bridges have been recently provided (see also [45] for the case of Lévy processes conditioned to stay positive).

TABLE 5. A comparison of different Monte Carlo methods (all ran with antithetic variates and 1 million paths) for (i) an up-and-in call option and (ii) a range note. The stochastic volatility parameters and parameters of the derivative contracts are as given in Table 4, and the jump parameters as given in Table 1. In the column 'Time' the run times are reported in seconds (which include in particular the time to find the roots of the Cramér-Lundberg equation). The continuous-time EM schemes were run using the first-passage time probabilities of the corresponding randomised bridge processes computed using $n = 7$ recursive steps (for the barrier option) and using the expected occupation time of the corresponding randomised bridge process computed using $n = 5$ recursive steps (for the range note). To obtain the values in the table marked with † and * we used the exact Brownian bridge probability and numerical integration, respectively.

		Heston		**Bates (Kou)**		**Bates (HEJD)**	
	Steps	Midpoint (Error)	Time	Midpoint (Error)	Time	Midpoint (Error)	Time
Barrier option							
Discrete-time EM	100	12.755 (±0.0389)	7.9	13.333 (±0.0407)	9.1	15.358 (±0.0483)	9.8
Discrete-time EM	1,000	12.866 (±0.0388)	80	13.432 (±0.0406)	88	15.387 (±0.0481)	94
Discrete-time EM	10,000	12.935 (±0.0387)	789	13.467 (±0.0406)	888	15.413 (±0.0482)	958
Continuous-time EM	100	12.948 (±0.0387)	18	13.468 (±0.0406)	20	15.457 (±0.0482)	82
Continuous-time EM	1,000	12.956 (±0.0388)	163	13.534 (±0.0408)	165	15.478 (±0.0482)	233
Continuous-time EM†	1,000	12.951 (±0.0388)	125				
Range note							
Discrete-time EM	100	15.352 (±0.0373)	8.4	15.374 (±0.0367)	9.1	15.387 (±0.0354)	10
Discrete-time EM	1,000	15.288 (±0.0371)	81	15.315 (±0.0365)	93	15.309 (±0.0352)	98
Discrete-time EM	10,000	15.288 (±0.0371)	793	15.304 (±0.0365)	928	15.286 (±0.0351)	1079
Continuous-time EM	10	15.177 (±0.0367)	54	15.237 (±0.0362)	68	15.255 (±-0.035)	132
Continuous-time EM	100	15.288 (±0.0371)	114	15.294 (±0.0365)	126	15.327 (±0.0352)	364
Continuous-time EM*	100	15.288 (±0.0371)	1491				

Assumption A.1. The Lévy process X satisfies the integrability condition

$$\int_{\mathbb{R}\setminus(-1,1)} \frac{\mathrm{d}\theta}{|\Psi(\theta)|} < \infty, \tag{A.1}$$

where Ψ is the characteristic exponent of X, which is the function $\Psi : \mathbb{R} \to \mathbb{C}$ that satisfies the identity $\mathbb{E}[\exp(\mathbf{i}\theta X_t)] = \exp(-t\Psi(\theta))$ for all $\theta \in \mathbb{R}$ and $t \in \mathbb{R}_+$.

As random times we consider Gamma random variables $\Gamma_{n,q}$, $n \in \mathbb{N}, q > 0$, with mean n/q and variance n/q^2 that are independent of X. We suppose that the pair $(X, \Gamma_{n,q})$ is defined on the product space $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+), \mathbf{P} \times P)$. To simplify notation we use in the sequel $\mathbb{P}$ to denote the product-measure $\mathbf{P} \times P$. It follows from Sato [42, Prop. 28.1] that under Assumption A.1 the distributions under $\mathbb{P}$ of both $X_{\Gamma_{n,q}}$ and X_t, $t > 0$, admit continuous densities:

Lemma A.2. *Let Assumption A.1 hold.* (i) *Then for any $q > 0$ and $n \in \mathbb{N}$ the random variable $X_{\Gamma_{n,q}}$ has a density $u_{n,q}$ that is continuous and bounded.*

(ii) *For any $t > 0$, X_t admits a bounded density $p(t, x)$ that is continuous in $(t, x) \in (0, \infty) \times \mathbb{R}$.*

Under Assumption A.1 one may define the randomised Lévy bridge process starting at $(0, x)$ and pinned down at $(\Gamma_{n,q}, y)$ for any $x, y \in \mathbb{R}$. We recall first from [15, Theorem 1] that, under Assumption A.1 and for any $t > 0$ and $x, y \in \mathbb{R}$ such that $p(t, y - x) > 0$, there exists a Markov process on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, denoted by $X^{(0,x)\to(t,y)} = \{X_u^{(0,x)\to(t,y)}, u \in [0, t]\}$, that starts at time 0 at x a.s., is equal to y at time t a.s., and satisfies the disintegration property. The process $X^{(0,x)\to(t,y)} = \{X_u^{(0,x)\to(t,y)}, u \in [0, t]\}$ is referred to as the *Lévy bridge process* from $(0, 0)$ to (t, y).

We next specify the definition a Lévy bridge process pinned down at a Gamma random time and a given fixed end point. For any pair $x, y \in \mathbb{R}$ with $u_{n,q}(y - x) > 0$, the *randomised Lévy bridge process* $X^{(0,x)\to(\Gamma_{n,q},y)} =$

$\{X_t^{(0,x)\to(\Gamma_{n,q},y)}, t \in \mathbb{R}_+\}$ starting from $(0,x)$ and pinned down at $(\Gamma_{n,q}, y)$ is the stochastic process with sample paths $t \mapsto X_{t\wedge s}^{(0,x)\to(s,y)}(\omega)\Big|_{s=\Gamma_{n,q}(\gamma)}$ for given realisations (ω,γ) in the sample space $\Omega \times \mathbb{R}_+$. The process $X^{(0,x)\to(\Gamma_{n,q},y)}$ satisfies the disintegration property (which can be shown by a similar line of reasoning as was given in the proof of [15, Theorem 1]), and is hence equal in law to the corresponding process obtained by the construction described in the second paragraph of this section. The derivation of the expressions for the functions $\vec{D}_q^{(1)}(x,y)$ and $\vec{\Omega}_q^{(1)}(x,y)$ is based in part on the following auxiliary result concerning the differentiability of two related functions under Assumption A.1 (the proof of which is omitted as it follows by standard arguments).

Lemma A.3. *Let Assumption A.1 hold and let q be any strictly positive number.*

(i) For any fixed $x \in \mathbb{R}_+$, the function $y \mapsto \mathbb{P}(\overline{X}_{\Gamma_{1,q}} \leq x, X_{\Gamma_{1,q}} \leq y)$ is continuously differentiable on $\mathbb{R}$ and its derivative $y \mapsto D_{1,q}(x,y)$ is bounded.

(ii) The map $(x,y) \mapsto \mathbb{E}\left[\int_0^{\Gamma_{1,q}} I_{\{X_u \leq x\}} \mathrm{d}u\, I_{\{X_{\Gamma_{1,q}} \leq y\}}\right]$ is continuously differentiable with respect to x and y in $\mathbb{R}$. The mixed derivative with respect to x and y is given by $\Omega_{1,q}(x,y)$ for $x, y \in \mathbb{R}$.

The functions $D_{1,q}$ and $\Omega_{1,q}$ admit semi-analytical expressions, which can be derived using the Markov property and the Wiener-Hopf factorisation of X. We recall (see *e.g.* Bertoin [8, Ch. VI]) that the probabilistic form of the Wiener-Hopf factorisation of X states that (a) the running supremum $\overline{X}_{\Gamma_{1,q}}$ and the *drawdown* $\overline{X}_{\Gamma_{1,q}} - X_{\Gamma_{1,q}}$ of X at the random time $\Gamma_{1,q}$ are independent, and (b) the drawdown $\overline{X}_{\Gamma_{1,q}} - X_{\Gamma_{1,q}}$ has the same law as the negative of the running infimum $-\underline{X}_{\Gamma_{1,q}}$. The probabilistic form of the Wiener-Hopf factorisation implies that the characteristic function of the random variable $X_{\Gamma_{1,q}}$ is equal to the product of the characteristic functions Ψ_q^+ and Ψ_q^- of $\overline{X}_{\Gamma_{1,q}}$ and $\underline{X}_{\Gamma_{1,q}}$,

$$\Psi_q^+(\theta) = \mathbb{E}[\exp(\mathbf{i}\theta\overline{X}_{\Gamma_{1,q}})], \qquad \Psi_q^-(\theta) = \mathbb{E}[\exp(\mathbf{i}\theta\underline{X}_{\Gamma_{1,q}})].$$

In the following result we establish that the functions $D_{n,q}$, $\Omega_{n,q}$ are well-defined and satisfy the recursions (1.2)—(1.3):

Theorem A.4. *Let $q > 0$, $n \in \mathbb{N}$ and let Assumption A.1 hold.*

(i) *For any $x \in \mathbb{R}_+$, the function $y \mapsto \mathbb{P}(\overline{X}_{\Gamma_{n,q}} \leq x, X_{\Gamma_{n,q}} \leq y)$ admits a continuous bounded density denoted by $D_{n,q}$. Moreover, the function $(x,y) \mapsto \mathbb{E}\left[\int_0^{\Gamma_{n,q}} I_{\{X_u \leq x\}} \mathrm{d}u\, I_{\{X_{\Gamma_{n,q}} \leq y\}}\right]$ is continuously differentiable on $\mathbb{R}^2$ with bounded mixed-derivative denoted by $\Omega_{n,q}$.*

(ii) *The functions $D_{n,q}$ and $\Omega_{n,q}$ satisfy the recursions* (1.2)—(1.3).

Remark A.5. Since the pinned process $X^{(0,0)\to(\Gamma_{n,q},y)}$ is equal in law to the process $X^{\Gamma_{n,q}} = \{X_u, u \in [0, \Gamma_{n,q}]\}$ stopped at the random time $\Gamma_{n,q}$ and conditioned on $\{X_{\Gamma_{n,q}} = y\}$, it follows that the functions $\vec{D}_q^{(n)}$ ($\vec{\Omega}_q^{(n)}$) are equal to the ratio of $D_{n,q}$ ($\Omega_{n,q}$, respectively) and $u_{n,q}$, that is,

$$D_{n,q}(x,y) = \vec{D}_q^{(n)}(x,y)u_{n,q}(y), \qquad \Omega_{n,q}(x,y) = \frac{\mathrm{d}}{\mathrm{d}x}\vec{\Omega}_q^{(n)}(x,y)u_{n,q}(y), \qquad x \in \mathbb{R}_+,\ y \in \mathbb{R}.$$

Proof of Theorem A.4. (i) Several applications of the strong Markov property of X and the lack of memory property of the exponential distribution yield

$$\begin{aligned}
\mathbb{P}\left[\overline{X}_{\Gamma_{n,q}} \leq x, X_{\Gamma_{n,q}} \in \mathrm{d}y\right] &= \mathbb{P}\left[\tau_x^+ \geq \Gamma_{n,q}, X_{\Gamma_{n,q}} \in \mathrm{d}y\right] \\
&= \mathbb{P}\left[X_{\Gamma_{n,q}} \in \mathrm{d}y\right] - \sum_{k=1}^n \mathbb{P}\left[\Gamma_{k-1,q} \leq \tau_x^+ < \Gamma_{k,q}, X_{\Gamma_{n,q}} \in \mathrm{d}y\right] \\
&= \mathbb{P}\left[X_{\Gamma_{n,q}} \in \mathrm{d}y\right] - \sum_{k=1}^n \int_{\mathbb{R}_+} \mathbb{E}\left[I_{\{\Gamma_{k-1,q} \leq \tau_x^+ < \Gamma_{k,q}\}} I_{\{X_{\tau_x^+} \in \mathrm{d}z\}}\right] \mathbb{P}[z + X_{\Gamma_{n-k+1,q}} \in \mathrm{d}y],
\end{aligned}$$

with $\Gamma_{0,q} := 0$. Taking the Fourier transform of the measure $r_x^{n,q}(\mathrm{d}y) := \mathbb{P}\left[\overline{X}_{\Gamma_{n,q}} \leq x, X_{\Gamma_{n,q}} \in \mathrm{d}y\right]$ we find

$$\mathcal{F}r_x(s) = \mathbb{E}[\exp\{\mathbf{i}sX_{\Gamma_{n,q}}\}] - \sum_{k=1}^n \mathbb{E}[\exp\{\mathbf{i}sX_{\Gamma_{n-k+1,q}}\}]\mathbb{E}[\exp\{\mathbf{i}sX_{\tau_x^+}\}I_{\{\Gamma_{k-1,q} \leq \tau_x^+ < \Gamma_{k,q}\}}], \qquad s \in \mathbb{R}. \tag{A.2}$$

Since the second factors in the sum in (A.2) are bounded by one and

$$\mathbb{E}[\exp(\mathbf{i}\theta X_{\Gamma_{n,q}})] = \left(\frac{q}{q+\Psi(\theta)}\right)^n, \tag{A.3}$$

we have $|\mathcal{F}r_x(s)| \leq \sum_{k=1}^n \int q^k |q+\Psi(s)|^{-k} \mathrm{d}s$, for any $x \in \mathbb{R}_+$, $q>0$ and $n \in \mathbb{N}$, which is finite by Assumption A.1 and the bound $|q/(q+\Psi(s))| \leq 1$ that holds for all $s \in \mathbb{R}$. We conclude that, for any $x \in \mathbb{R}_+$, the measure $r_x^{n,q}(\mathrm{d}y)$ admits a continuous bounded density (by Sato [42, Prop. 28.1]).

We show the required differentiability of $\mathbb{E}\left[\int_0^{\Gamma_{n,q}} I_{\{X_u \leq x\}} \mathrm{d}u \, I_{\{X_{\Gamma_{n,q}} \leq y\}}\right]$ by induction with respect to n. Noting that the case $n=1$ follows from Lemma A.3(ii), we next turn to the induction step. Assume thus that the assertion is valid for given $n \in \mathbb{N}$. We have by an application of the Markov property

$$\begin{aligned}\mathbb{E}\left[\int_0^{t+u} I_{\{X_s \leq x\}} \mathrm{d}s \;\, I_{\{X_{t+u} \in \mathrm{d}b\}}\right] &= \int_{w \in \mathbb{R}} \mathbb{E}\left[\int_0^t I_{\{X_s \leq x\}} \mathrm{d}s \;\, I_{\{X_t \in \mathrm{d}w\}}\right] \mathbb{P}\left[w + X_u \in \mathrm{d}b\right] \\ &+ \int_{w \in \mathbb{R}} \mathbb{E}\left[\int_0^u I_{\{w+X_s \leq x\}} \mathrm{d}s \;\, I_{\{w+X_u \in \mathrm{d}b\}}\right] \mathbb{P}\left[X_t \in \mathrm{d}w\right],\end{aligned} \tag{A.4}$$

for any real x. Replacing in (A.4) t and u by the independent random times $\Gamma_{1,q}$ and $\Gamma_{n-1,q}$, using the fact that their sum is equal in distribution to $\Gamma_{n,q}$ and that the random variables $X_{\Gamma_{n,q}}$ and $X_{\Gamma_{1,q}}$ have continuous densities $u_{n,q}$ and $u_{1,q}$ (by Lemma A.3), it follows from the induction assumption that the assertion is valid for $n+1$. It follows thus by induction that we have the required differentiability for all $n \in \mathbb{N}$.

(ii) Since we may write

$$\overline{X}_t = \max\left\{X_s + \sup_{0 \leq u \leq t-s} (X_{u+s} - X_s), \overline{X}_s\right\}, \qquad \text{for any } s,t \text{ with } 0 \leq s \leq t,$$

it follows as a consequence of the stationarity and independence of increments of X, and the fact that a $\Gamma_{n,q}$ random variable is equal in distribution to the sum of independent $\Gamma_{n-1,q}$ and $\Gamma_{1,q}$ random variables that we have

$$\begin{aligned}\mathbb{P}\left(\overline{X}_{\Gamma_{n,q}} \leq x, X_{\Gamma_{n,q}} \in \mathrm{d}w\right) &= \mathbb{P}\left(\max\left\{X_{\Gamma_{1,q}} + \overline{X}'_{\Gamma_{n-1,q}}, \overline{X}_{\Gamma_{1,q}}\right\} \leq x, X_{\Gamma_{1,q}} + X'_{\Gamma_{n-1,q}} \in \mathrm{d}w\right) \\ &= \int_{(-\infty,x]} \mathbb{P}(\overline{X}_{\Gamma_{1,q}} \leq x, X_{\Gamma_{1,q}} \in \mathrm{d}z)\mathbb{P}(z + \overline{X}_{\Gamma_{n-1,q}} \leq x, z + X_{\Gamma_{n-1,q}} \in \mathrm{d}w),\end{aligned} \tag{A.5}$$

where the random variables $\overline{X}'_{\Gamma_{n-1,q}}$ and $X'_{\Gamma_{n-1,q}}$ are independent of X. We arrive at the identity in (1.2) since the Lévy process X is spatially homogeneous.

The recursion follows from (A.4) replacing as before t and u by the independent random times $\Gamma_{1,q}$ and $\Gamma_{n-1,q}$ and using the fact that their sum is equal in distribution to a $\Gamma_{n,q}$ random variable. □

References

[1] S. Asmussen and H. Albrecher. *Ruin Probabilities.* 2nd ed., World Scientific, 2010.

[2] S. Asmussen, F. Avram, and M. Usabel. The Erlang approximation of finite time ruin probabilities. *Astin Bull.*, 32: 267-281, 2002.

[3] S. Asmussen, F. Avram, and M. Pistorius. Russian and American put options under exponential phase-type Lévy models. *Stoch. Proc. Appl.*, 109:79–111, 2004.

[4] S. Asmussen and J. Rosínski. Approximations of small jumps of Lévy Processes with a view towards simulation. *J. Appl. Probab.* 38:482–493, 2001.

[5] F. Avram, T. Chan, and M. Usabel. On the valuation of constant barrier options under spectrally negative exponential Levy models and Carr's approximation for American puts. *Stoch. Proc. Appl.*, 101: 75-107, 2002.

[6] D.J. Bartholomew. Sufficient conditions for a mixture of exponentials to be a probability density function. *Ann. Math. Statist.*, 40:2183–2188, 1969.

[7] D.S. Bates. Jumps and stochastic volatility: exchange rate processes implicit in Deutsche Mark options. *Rev. Fin. Studies*, 9:69-107, 1996.

[8] J. Bertoin. *Lévy Processes.* Cambride Universtiy Press, Cambridge, 1996.

[9] A. Beskos, O. Papaspiliopoulos, and G. O. Roberts. Retrospective exact simulation of diffusion sample paths with applications. *Bernoulli*, 12(6):1077–1098, 12 2006.

[10] R.F. Botta and C.M. Harris. Approximation with generalized hyperexponential distributions: Weak convergence results. *Queueing Systems*, 1:169–190, 1986.

[11] M. Boyarchenko and S. Levendorskiĭ. Valuation of continuously monitored double barrier options and related securities. *Math. Finance* 22:419-444, 2012.

[12] P.M. Boyle, M. Broadie, and P. Glasserman. Monte Carlo methods for security pricing. *J. Econom. Dynam. Control*, 21:1267–1321, 1997.

[13] N. Cai and S. Kou. Option pricing under a mixed-exponential jump diffusion model. *Manag. Science*, 57:2067–2081, 2011.

[14] P. Carr. Randomization and the American put. *Rev. Fin. Studies*, 11:597–626, 1998.

[15] L. Chaumont and G. Uribe Bravo. Markovian bridges: weak continuity and pathwise constructions. *Ann. Probab.*, 39(2):609–647, 2011.

[16] R. Cont and P. Tankov. *Financial Modelling With Jump Processes.* Chapman & Hall/CRC, 2004.

[17] S. Dereich. Multi-level Monte Carlo algorithms for Lévy-driven SDES with Gaussian correction. *Ann. Appl. Probab.*, 21:283–311, 2011.

[18] W. Feller. *An Introduction to Probability Theory and Its Applications.* Wiley, 1966.

[19] A. Ferreiro-Castilla, A. E. Kyprianou, R. Scheichl and G. Suryanarayana Multilevel Monte Carlo simulation for Lévy processes based on the Wiener-Hopf factorisation. *Stoch. Proc. Appl.*, 124: 985-1010, 2014.

[20] J. E. Figueroa-Lopez and P. Tankov. Small-time asymptotics of stopped Lévy bridges and simulation schemes with controlled bias. *Bernoulli* (to appear), 2013.

[21] G. Fusai and A. Tagliani. Pricing of occupation time derivatives: Continuous and discrete monitoring. *J. Comp. Finance*, 5:1–37, 2001.

[22] J. Gatheral *The Volatility Surface: A Practitioner's Guide.* Wiley Finance, 2006.

[23] P. Glasserman. *Monte Carlo Methods in Financial Engineering.* New York: Springer, 2004.

[24] E. Gobet. Weak approximation of killed diffusion using Euler schemes. *Stoch. Proc. Appl.*, 87:167–197, 2000.

[25] E. G. Haug. *The Complete Guide to Option Pricing Formulas.* McGraw-Hill, 2006.

[26] D. Higham and X. Mao. Convergence of Monte Carlo simulations involving the mean-reverting square root process. *J. Comp. Finance* 8:35-62, 2005.

[27] M. Jeannin and M. Pistorius. A transform approach to compute prices and Greeks of barrier options driven by a class of Lévy processes. *Quant. Finance*, 10:629–644, 2010.

[28] O. Kallenberg. *Foundations of modern probability.* Springer-Verlag, New York, second edition, 2002.

[29] P. E. Kloeden and A. Neuenkirch. Convergence of numerical methods for stochastic differential equations in finance. `arXiv:1204:6620`.

[30] F. Kleinert and K. van Schaik A variation of the Canadisation algorithm for the pricing of American options driven by Lévy processes. `arXiv:1304.4534`

[31] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations.* Springer, Berlin, 1992.

[32] S. G. Kou and H. Wang. Option pricing under a double exponential jump diffusion model. *Manag. Science*, 50:1178–1192, 2004.

[33] A. Kuznetsov, A.E. Kyprianou, J.C. Pardo, and K. van Schaik. A Wiener-Hopf Monte-Carlo simulation technique for Lévy processes. *Ann. Appl. Probab.*, 21:2171-2190, 2011.

[34] A. E. Kyprianou. *Introductory lectures on fluctuations of Lévy processes with applications.* Springer, 2006.

[35] A. E. Kyprianou and M. R. Pistorius. Perpetual options and Canadization through fluctuation theory. *Ann. Appl. Probab.*, 13:1077-1098, 2003.

[36] S. Levendorskiĭ. Convergence of Price and Sensitivities in Carr's Randomization Approximation Globally and Near Barrier. *SIAM J. Fin. Math.*, 2:79–111, 2011.

[37] Lewis, A.L. and Mordecki, E. Wiener-Hopf factorisation for Lévy processes having positive jumps with rational transforms. *J. Appl. Probab.* 6:118–134, 2008.

[38] G. Marchuk and V. Shaidurov. *Difference Methods and Their Extrapolations.* Springer Verlag, New York, 1983.

[39] S. Metwally and A. Atiya. Using Brownian bridge for fast simulation of jump-diffusion processes and barrier options. *J. Derivatives* 10:43–54, 2002.

[40] W. Press, S. A. Teukolsky, W. T. Vetterling, and B. Flannery. *Numerical recipes in C++.* Cambridge University Press, 2002.

[41] J. Ruf and M. Scherer. Pricing corporate bonds in an arbitrary jump-diffusion model based on an improved Brownian-bridge algorithm. *J. Comp. Finance* 14, Issue 3, 2011.

[42] K. Sato. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, 1999.

[43] A. Sidi. *Practical Extrapolation Methods: Theory and Applications.* Cambridge University Press, Cambridge, 2003.

[44] J. Stolte. *On accurate and efficient valuation of financial contracts under models with jumps*. PhD thesis, Imperial College London, 2013.

[45] G. Uribe Bravo. Bridges of Lévy processes conditioned to stay positive. *Bernoulli*, 20(1):190–206, 02 2014.

Department of Mathematics, Imperial College London

E-mail address: {a.mijatovic, m.pistorius, j.stolte09}@imperial.ac.uk