CONTINUOUSLY MONITORED BARRIER OPTIONS UNDER MARKOV PROCESSES

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ABSTRACT. In this paper we present an algorithm for pricing barrier options in one-dimensional Markov models. The approach rests on the construction of an approximating continuous-time Markov chain that closely follows the dynamics of the given Markov model. We illustrate the method by implementing it for a range of models, including a local Lévy process and a local volatility jump-diffusion. We also provide a convergence proof and error estimates for this algorithm.

1. INTRODUCTION

1.1. **Background and motivation.** Barrier options are among the most popular exotic derivatives. Such contracts form effective risk management tools, and are liquidly traded in the Foreign Exchange markets. The most liquid barrier options in FX markets are continuously monitored single- or double-no-touch options and knock-in or knock-out calls and puts (see e.g. Hakala and Wystup [28], Lipton [45], [46], Wystup [56]). The main challenge in the risk management of large portfolios of barrier options faced by trading desks that make a market in these securities is to be able to price and hedge the barrier products in models that are flexible enough to describe the observed option prices (i.e. calibrate to the vanilla market price quotes).

It is by now well established that the classical Black-Scholes model lacks the flexibility to fit accurately to observed option price data (see e.g. Gatheral [23] and the references therein). A variety of models have been proposed to provide an improved description of the dynamics of the price of the underlying that can more accurately describe the option surface. Parametric diffusion models like the CEV process [14] have additional flexibility to fit the vanilla skew at a single maturity for as many options as there are free parameters in the model. The seminal idea (developed by Dupire [18] and Gyöngy [26]) that allows one to construct a model that can describe the entire implied volatility surface (across all strikes and maturities) is that of local volatility models, where a non-parametric form of the local volatility function is constructed from the option price data. It has been shown that in practice such models imply unrealistic dynamics of the option prices (see the formula for the implied volatility in a local volatility model given in [27]). The ramification is an unrealistic amount of vega risk, which is expensive to hedge. Therefore, even though in a local volatility model barrier options can be priced using a PDE solver, this modelling framework alone is not suitable for the risk management of a large portfolio of barrier options.

At the other end of the spectrum are the jump processes with stationary and independent increments, which can fit very well the volatility smile at a single maturity (see e.g. [12] and the references therein). A variety of models in the exponential Lévy class have been proposed in the literature: CGMY [8], KoBoL [6], generalised hyperbolic [19], NIG [4] and Kou [37]. Exponential Lévy processes are simple examples of Markov processes whose law is uniquely determined by the distribution of the process at a single time. Since the set of call option prices at a fixed maturity for all strikes uniquely determines the marginal risk-neutral distribution at that maturity, calibration to option prices at multiple maturities in principle fixes the corresponding marginals. It has been reported (see e.g. [12]) that Lévy processes lack the flexibility of calibrating simultaneously across a range of strikes and maturities. Several generalisations within the one-dimensional Markov framework have been proposed.

If the stationarity assumption is relaxed while the property of independence of increments is retained, one arrives at the class of exponential additive processes, which have recently been shown to calibrate well to several maturities in equity markets. The Sato process introduced in Carr et al. [10] is an example of such an additive model used in financial modelling.

The independent increments property of a process implies that its transition probabilities are translation invariant in space, so that they only depend on the difference between the end and starting value of the process. It is well known that the distribution of a log-asset price depends in a non-linear way on the starting point (e.g. in equity markets it has been observed that if the current price is high, then the volatility is low and vice versa). To capture this effect one is led to consider Markov jump-processes whose increments are not independent. As a generalisation of local volatility models, the class of local Lévy processes introduced by Carr et al. [9] allows the modeller to modulate the intensity of the jumps as well as their distribution depending on where the underlying asset is trading. A local volatility jump-diffusion with similar structural properties was calibrated to the implied volatility surface in Andersen and Andreasen [2] and He et al. [29]. Due to the presence of jumps and the absence of stationarity and independence of increments, the problem of obtaining the first-passage probabilities for such a general class of processes is computationally less tractable.

There exists currently a good deal of literature on numerical methods for the pricing of barrier-type options. It is well known that in this case a straightforward Monte Carlo simulation algorithm will be time-consuming and yield unstable results for the prices and especially the sensitivities. The knock-in/out features in the barrier option payoffs lead to slower convergence of the Monte Carlo algorithm. To address this problem the following (semi-)analytical approaches have been developed for specific models:

(a) spectral expansions for several parametric diffusion models (Davydov and Linetsky [15], Lipton [45]),

(b) transform based approaches for exponential Lévy models (Boyarchenko and Levendorskii [5], Geman and Yor [24], Jeannin and Pistorius [35], Kou and Wang [38], Sepp [54]),

The method (a) employs the explicit spectral decompositions for this class of diffusion models, whereas the approach (b) exploits the independence and stationarity of the increments of the Lévy process, and the so-called Wiener-Hopf factorisation. Since both of these approaches hinge on special structural
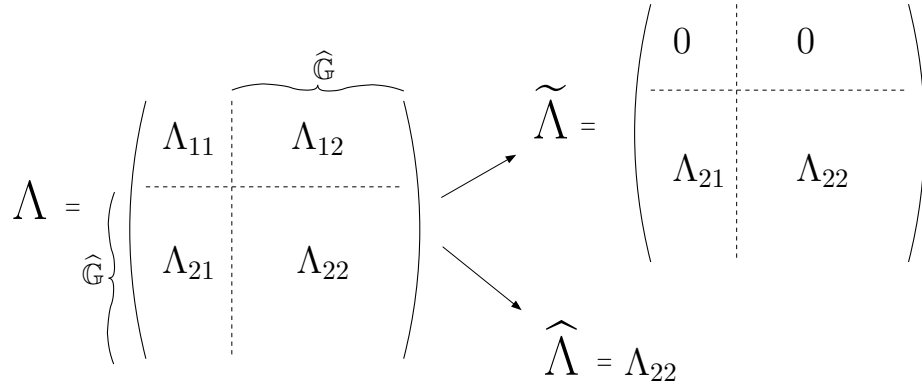
$$\Lambda \ = \ \begin{matrix} \\ \widehat{\mathbb{G}} \Big\{ \end{matrix} \left(\begin{array}{c|c} \Lambda_{11} & \overbrace{\Lambda_{12}}^{\widehat{\mathbb{G}}} \\ \hline \Lambda_{21} & \Lambda_{22} \end{array} \right) \qquad \widetilde{\Lambda} \ = \ \left(\begin{array}{c|c} 0 & 0 \\ \hline \Lambda_{21} & \Lambda_{22} \end{array} \right)$$

$$\widehat{\Lambda} \ = \ \Lambda_{22}$$

FIGURE 1. This is a schematic picture in bloc notation that demonstrates how the matrices $\widehat{\Lambda}$ and $\widetilde{\Lambda} = \widetilde{\Lambda}_0$ are obtained from the generator Λ. The matrix Λ is the generator matrix of an approximating continuous-time Markov chain X on the state-space $\mathbb{G}$. The subset $\widehat{\mathbb{G}} \subset \mathbb{G}$ consists of the elements of $\mathbb{G}$ that lie between the barriers. The matrix $\widehat{\Lambda}$ contains all the information necessary to price any contract that knocks out (or knocks in) when a barrier is breached. Similarly the matrix $\widetilde{\Lambda}$ is what is required to compute the distribution of any function that depends on the first-passage and overshoot of the chain X into the region on the other side of the barriers.

properties of the underlying processes, it is not clear if and how they can be extended to more general Markovian models.

A different approach, pioneered by Kushner (see e.g. [41]), is the discrete time Markov chain approximation method. Originally developed for the numerical solution of stochastic optimal control problems in continuous time, this method consists of approximating the system of interest by a discrete time chain that closely follows its dynamics, and solving the problem of interest for this chain. An application to the pricing of American type options is given in Kushner [40]. Bally et al. [3] develop a quantization method to efficiently value American options on baskets of assets under a local volatility model. Using a discrete time Markov chain, Duan et al. [16] price a discretely monitored barrier option in the Black-Scholes and NGARCH models. Rogers and Stapleton [51] investigate an efficient binomial tree method for barrier option pricing (see also references therein for related methods). Related are PDE and PIDE finite difference discretization methods that have been investigated by various authors; Zvan et al. [57] consider barrier and related options in the Black-Scholes model; Tavella and Randall [55] present an overview of PDE finite difference methods for the pricing of financial instruments; Wang et al. [32] develop a robust scheme for American options under a CGMY model, and Cont and Voltchkova [13] follow a viscosity approach for the PIDEs connected to European and barrier option under Lévy models. Markov chains have also been employed to directly model the evolution of price processes; Albanese and Mijatović [1] model the stochasticity of risk reversals and carried out a calibration study in FX markets under a certain continuous-time Markov chain constructed to model the FX spot process.

1.2. **Contribution of the current paper.** In this paper we consider the problem of pricing barrier options in the setting of one-dimensional Markov processes, which in particular includes the case of local Lévy models as well as local volatility jump-diffusions and additive processes. The presented approach is probabilistic in nature and is based on the following two elementary observations: (i) given a Markov asset price process S it is straightforward to construct a continuous-time Markov chain model X whose

law is close to that of S, by approximating the generator of the process S with an intensity matrix; (ii) the corresponding first-passage problem for a continuous-time Markov chain can be solved explicitly via a closed-form formula that only involves the generator matrix of the chain X. More precisely, for a given Markov asset price model S on the state-space $\mathbb{E} = [0, \infty)$ with corresponding generator $\mathcal{L}$ the algorithm for the pricing of any barrier product (including rebate options, which depend on the position at the moment of first-passage) consists of the following two steps:

(i) Construct a finite state-space $\mathbb{G} \subset \mathbb{E}$ and a generator matrix Λ for the chain X that approximates the operator $\mathcal{L}$ on $\mathbb{G}$.

(ii) To value knock-out and rebate options, obtain the matrices $\widehat{\Lambda}$ and $\widetilde{\Lambda}$ by the procedure in Figure 1 and apply closed-form formulas in terms of these matrices (given in equations (3.9) and (3.11) below).

The form of the generators of Markov processes that commonly arise in pricing theory (including the local Lévy class) is well known from general theory (and are reviewed in Section 2). The state-space $\mathbb{G}$ in step (i) is taken to be non-uniform with a higher density of points in the relevant areas such as spot and barrier levels. Subsequently, the generator matrix Λ is defined by matching the instantaneous moments of the Markov processes S and the chain X on the state-space $\mathbb{G}$. This criterion implies in particular that the chain X has the same average drift as the asset price process S. Step (ii) of the algorithm consists of the evaluation of the closed-form formulas for the first-passage probabilities that can be derived employing continuous-time Markov chain theory (see Theorem 1 below). The evaluation of this formula consists of exponentiation of either the matrix $\widehat{\Lambda}$ or $\widetilde{\Lambda}$, which can be performed using the Padé approximation algorithm that is implemented in standard packages such as Matlab (see also [30]). The outputs of the algorithm yield arbitrage-free prices in a certain continuous time Markov chain model for the risky asset price process. This feature is a consequence of the fact that the algorithm is based on the construction of an approximate stochastic model. This property is not shared by many other numerical methods used in practice that are based on purely analytical considerations.

We implemented this algorithm for a number of models that include the features of local volatility as well as jumps. We obtained an accurate match with the numerical results under diffusion and Lévy models that were considered elsewhere in the literature (see Section 6). We prove that by refining the grid the prices generated by this approach converge to those of the limiting model. We also establish, under additional regularity assumptions, an error bound that is linear in the spatial mesh size and the truncation error (see Section 5). We showed that an additional logarithmic factor may arise in this error bound when the Lévy density has a pole of order two at the origin. Numerical experiments (reported in Section 6) appear to suggest that, for a number of models, the error actually decays quadratically in the spatial mesh size. An extension to the case of time-dependent characteristics is presented in an unabridged version of this paper [49], which also includes extra numerical examples, as well as some sample code.

There is good deal of literature devoted to the study of the (weak) convergence of Markov chains to limiting processes. However the (sharp) rates of convergence of prices generated by the Markov chain approximation to those of the limiting model are rarely available, especially for barrier options. For discrete time Markov chains some explicit rates have been established (see e.g. Broadie et al. [7] and

Gobet and Menozzi [25]). Establishing the sharp rates of convergence for specific models remains an open question, left for future research.

The remainder of the paper is organized as follows. In Section 2 we define the class of models and barrier option contracts that is considered, and state some preliminary results about Markov processes. Section 3 presents the formulas for the first-passage quantities of the continuous-time Markov chains. In Section 4 we describe the discretization algorithm to construct the intensity matrix Λ of the chain X. Section 5 states the convergence results, which are proved in Appendix A. Numerical results are presented in Section 6 and Section 7 concludes the paper.

2. Problem setting: Barrier options for Markov processes

The problem under consideration is that of the valuation of general barrier options, which can be formulated as follows. Given a random process $S = \{S_t\}_{t \geq 0}$ modelling the price evolution of a risky asset, non-negative payoff and rebate functions g and h, and a set A specifying the range of values for which the contract 'knocks out', it is of interest to evaluate the expected discounted value of the random cash flow associated to a general barrier option contract

$$(2.1) \qquad g(S_T)\mathbf{I}_{\{\tau_A > T\}} + h(S_{\tau_A})\mathbf{I}_{\{\tau_A \leq T\}},$$

where $\mathbf{I}_C$ denotes the indicator of a set C and

$$\tau_A = \inf\{t \geq 0 : S_t \in A\}$$

is the first time that S enters the set A. Furthermore, it is relevant to quantify the sensitivities of this value with respect to different parameters such as the spot value $S_0 = x$. The cash flow in (2.1) consists of a payment $g(S_T)$ in the case the contract has not knocked out by the time T, and a rebate $h(S_{\tau_A})$ if it has. Examples of commonly traded options included in this setting are the down-and-out, up-and-out and double knock-out options. In particular, by taking $A = \emptyset$ we retrieve the case of a standard European claim with payoff $g(S_T)$ at maturity T.

We will consider this valuation problem in a Markovian setting, assuming that the underlying S is a Markov process with state-space $\mathbb{E} := [0, \infty)$ defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$, where $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ denotes the standard filtration generated by S. Thus, S takes values in $\mathbb{E}$ and satisfies the Markov property:

$$(2.2) \qquad \mathbf{E}[f(S_{t+s})|\mathcal{F}_t] = P_s f(S_t),$$

for all $s, t \geq 0$ and bounded Borel functions f, where $\mathbf{E}$ denotes the expectation under the probability measure $\mathbf{P}$ and $P_s f$ is given by

$$(2.3) \qquad P_s f(x) := \mathbf{E}_x[f(S_s)] := \mathbf{E}[f(S_s)|S_0 = x].$$

By taking expectations in (2.2) we see that the family $(P_t)_{t \geq 0}$ forms a semigroup:

$$P_{t+s}f = P_t(P_s f), \quad \text{for all } s, t \geq 0, \text{ and } P_0 f = f.$$

Informally, these conditions state that the expected value of the random cash flow $f(S_{t+s})$ occurring at time $t + s$ conditional on the available information up to time t depends on the past via the value S_t only. Setting the rate of discounting equal to a non-negative constant r, for any pair of non-negative

Borel functions g and h the expected discounted value of the barrier cash flow (2.1) at the epoch $\tau_A \wedge T$, the earlier of maturity T and the first entrance time τ_A, is given by

$$(2.4) \qquad \mathbf{E}_x\left[e^{-rT}g(S_T)\mathbf{I}_{\{\tau_A > T\}}\right] + \mathbf{E}_x\left[e^{-r\tau_A}h(S_{\tau_A})\mathbf{I}_{\{\tau_A \leq T\}}\right].$$

If S represents the price of a tradeable asset, r is the risk-free rate, d is the dividend yield and the process $\{e^{-(r-d)t}S_t\}_{t\geq 0}$ is a martingale, standard arbitrage arguments imply that no arbitrage is introduced if expression (2.4) is used as the current price of the option with payoff (2.1).

Before proceeding we review some key concepts of the standard Markovian setup that will be needed in the sequel. For background on the (general) theory of Markov processes we refer to the classical works Chung and Walsh [11], Ethier and Kurtz [20], Itô and McKean [33] and Rogers and Williams [52] (the latter two in particular treat the case of Markov processes with continuous sample paths). In what follows we will restrict S to be in a subclass of Markov processes for which, if the function f is continuous and tends to zero at infinity, the expected payoff $P_t f(x)$ has the following properties: it depends continuously on the spot $S_0 = x$ and on expiry t and also decays to zero when x tends to infinity. More precisely, denoting by $C_0(\mathbb{E})$ the set of continuous functions f on $\mathbb{E}$ that tend to zero at infinity, we make the following assumption:

Assumption 1. *S is a Feller process on $\mathbb{E}$, that is, for any $f \in C_0(\mathbb{E})$, the family $(P_t f)_{t\geq 0}$, with $P_t f$ defined in (2.3), satisfies:*

   (i) *$P_t f \in C_0(\mathbb{E})$ for any $t > 0$;*
   (ii) *$\lim_{t\to 0} P_t f(x) = f(x)$ for any $x \in \mathbb{E}$.*

The Feller property guarantees that there exists a version of the process S with càdlàg paths satisfying the strong Markov property. In particular, a Feller process is a Hunt process.

Throughout the paper we will take the knock-out set A to be of the form

$$(2.5) \qquad A = [0,\ell] \cup [u,\infty), \qquad 0 \leq \ell < u \leq \infty,$$

which includes the cases of double and single barrier options—the latter by taking $\ell = 0$ or $u = \infty$. To rule out degeneracies we will make the following assumption on the behaviour of S at the boundary points ℓ and u:

Assumption 2. *For every $x \in \mathbb{E}$ we have $\mathbf{P}_x(\tau_A = \tau_{A^\circ}) = 1$ where $A^\circ = [0,\ell) \cup (u,\infty)$.*

This assumption states that the first entrance times into A and its interior coincide almost surely, if the spot S_0 is equal to x. If $u < \infty$, a sufficient condition for Assumption 2 to be satisfied is $\mathbf{P}_x(\tau_{A^\circ} = 0) = 1$ for $x \in \{\ell, u\}$; that is, when started at ℓ or u, the process S immediately enters A°.

The class of Feller processes satisfying Assumption 2 includes many of the models employed in quantitative finance such as (Feller-)diffusions, jump-diffusions with non-generate diffusion coefficient and Lévy processes whose Lévy measure admits a density.

The family $(P_t)_{t\geq 0}$ is determined by its infinitesimal generator $\mathcal{L}$ that is defined as

$$(2.6) \qquad \mathcal{L}f(x) := \lim_{t\downarrow 0} \tfrac{1}{t}(P_t f - f)(x)$$

for any function $f \in C_0(\mathbb{E})$ for which the right-hand side of (2.6) converges in the strong sense.[1] The set $\mathcal{D}$ of such functions is called the domain of the operator $\mathcal{L}$ and is dense in $C_0(\mathbb{E})$. These fundamental facts about semigroups and their generators can be found in [20, Ch. 1].

We next give a few examples of Feller processes with their generators.

Example 1. A diffusion asset price model $S = \{S_t\}_{t \geq 0}$ evolves under a risk-neutral measure according to the stochastic differential equation (SDE)

$$(2.7) \qquad \frac{\mathrm{d}S_t}{S_t} = \gamma \mathrm{d}t + \sigma\left(S_t\right)\mathrm{d}W_t,$$

where $S_0 > 0$ is the initial price, $\gamma \in \mathbb{R}$ and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a given measurable function. To guarantee the absence of arbitrage we assume that σ is chosen such that the discounted process $\{\mathrm{e}^{-\gamma t}S_t\}_{t \geq 0}$ is a martingale, If, in addition, infinity is not entrance[2] for S and σ is a continuous function, then S is a Feller process, and its infinitesimal generator $\mathcal{L}_D$ acts on $f \in C_c^2(\mathbb{E})$[3] as

$$(2.8) \qquad \mathcal{L}_D f(x) = \frac{\sigma(x)^2 x^2}{2} f''(x) + \gamma x f'(x),$$

where f' denotes the derivative of f with respect to x (see [20, Sec. 8.1]).

Example 2. The price process in an exponential Lévy model S given by

$$(2.9) \qquad S_t := S_0 \mathrm{e}^{(r-d)t} \frac{\mathrm{e}^{L_t}}{\mathbf{E}[\mathrm{e}^{L_t}]}$$

where r and d are constants representing the interest rate and dividend yield and $L = \{L_t\}_{t \geq 0}$ is a Lévy process, such that $\mathbf{E}[\mathrm{e}^{L_t}] < \infty$ for all $t > 0$. By construction, $\{\mathrm{e}^{-(r-d)t}S_t\}_{t \geq 0}$ is a martingale. Further, $\mathbf{E}[\mathrm{e}^{L_t}] < \infty$ if and only if the Lévy measure ν integrates $\exp(y)$ at infinity, that is,

$$(2.10) \qquad \int_{(1,\infty)} \mathrm{e}^y \nu(\mathrm{d}y) < \infty.$$

The law of L is determined by its characteristic exponent Ψ, which is related to the characteristic function Φ_t of L_t by $\Phi_t(s) = \exp(t\Psi(s))$ and which, under condition (2.10), has the Lévy-Khintchine representation

$$\Psi(s) = \mathtt{i}cs - \frac{\sigma^2 s^2}{2} + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathtt{i}sy} - 1 - \mathtt{i}sy\right)\nu(\mathrm{d}y),$$

where (c, σ^2, ν) is the characteristic triplet, with $\sigma, c \in \mathbb{R}$ and ν the Lévy measure, which satisfies the integrability condition $\int_{(-1,1)} |y|^2 \nu(\mathrm{d}y) < \infty$. The process S is a Feller process with an infinitesimal generator acting on $f \in C_c^2(\mathbb{E})$ as (cf. Sato [53, Thm. 31.5])

$$\mathcal{L}f(x) = \frac{\sigma^2 x^2}{2} f''(x) + \gamma x f'(x) + \int_{\mathbb{R}} [f(x\mathrm{e}^y) - f(x) - x f'(x)(\mathrm{e}^y - 1)]\nu(\mathrm{d}y),$$

where $\gamma = r - d$.

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[1] That is, the convergence is with respect to the norm $\|f\| := \sup_{x \in \mathbb{E}} |f(x)|$ of the Banach space $(C_0(\mathbb{E}), \|\cdot\|)$.

[2] See Itô and McKean [33] for an explicit criterion in terms of γ and σ for this to be the case.

[3] $C_c^2(\mathbb{E})$ denotes the set of C^2 functions with compact support in $\mathbb{E}^o = (0, \infty)$.

Example 3. More generally, one may specify the price process S by directly prescribing its generator $\mathcal{L}$ to act on sufficiently regular functions f as

$$(2.11) \qquad\qquad \mathcal{L}f(x) = \mathcal{L}_D f(x) + \mathcal{L}_J f(x),$$

where $\mathcal{L}_D f$ is given in (2.8) and

$$\mathcal{L}_J f(x) = \int_{(-1,\infty)} [f(x(1+y)) - f(x) - f'(x)xy]\nu(x,\mathrm{d}y),$$

where for every $x \in \mathbb{E}$, $\nu(x,\mathrm{d}y)$ is a (Lévy) measure with support in $(-1,\infty)$ such that

$$\int_{(-1,\infty)} \min\{y^2, |y|\}\nu(x,\mathrm{d}y) < \infty.$$

The discounted process $\{\mathrm{e}^{-\gamma t} S_t\}_{t\geq 0}$ is a local martingale. Sufficient conditions on σ and ν to guarantee the existence of a Feller process S corresponding to this generator were established in Kolokoltsov [36, Thm. 1.1].

A key step in solving the barrier valuation problems is the observation that the expected values of knock-out options and general barrier options can be expressed in terms of the *marginal* distributions of two Markov processes associated to S. Given a Markov process S the processes that have the same dynamics as S before entering A, but are stopped or jump to the graveyard state ∂ upon entering the set A, respectively, are themselves Markov processes. Let $\widehat{S}^A = \{S_t \mathbf{I}_{\{t<\tau_A\}} + \partial \mathbf{I}_{\{t\geq\tau_A\}}\}_{t\geq 0}$ denote the killed process and $\widetilde{S}^A$ the process killed at rate r that is stopped upon entering the set A. Then the value of the barrier options can be expressed as

$$\mathbf{E}_x \left[g(S_T)\mathbf{I}_{\{\tau_A>T\}}\right] = \mathbf{E}_x \left[g(\widehat{S}_T^A)\right] \quad =: \quad \widehat{P}_T^A g(x),$$

$$\mathbf{E}_x \left[\mathrm{e}^{-rT} g(S_T)\mathbf{I}_{\{\tau_A>T\}}\right] + \mathbf{E}_x \left[\mathrm{e}^{-r\tau_A} h(S_{\tau_A})\mathbf{I}_{\{\tau_A\leq T\}}\right] = \quad \mathbf{E}_x[\mathrm{e}^{-r(T\wedge\tau_A)} f(S_{T\wedge\tau_A})]$$

$$= \quad \mathbf{E}_x[f(\widetilde{S}_T^A)] =: \widetilde{P}_T^A f(x),$$

where we assume that $g(\partial) = 0$ and the function f is defined as $f(x) = \mathbf{I}_A(x)h(x) + \mathbf{I}_{\mathbb{R}_+\setminus A}(x)g(x)$. To calculate the value-functions of barrier options written on the underlying price process S we thus need to identify $\widehat{P}_T^A g(x)$ and $\widetilde{P}_T^A f(x)$. This can be achieved by employing the infinitesimal generators of the semigroups $(\widetilde{P}_t^A)_{t\geq 0}$ and $(\widehat{P}_t^A)_{t\geq 0}$ associated to the Markov processes $\widetilde{S}^A$ and $\widehat{S}^A$ which are explicitly expressed in terms of the generator $\mathcal{L}$ as follows:

Lemma 1. *(i) For any $f \in \mathcal{D}$, where $\mathcal{D}$ is the domain of the generator $\mathcal{L}$, we have*

$$(2.12) \qquad \lim_{t\downarrow 0} t^{-1}(\widetilde{P}_t^A f(x) - f(x)) \;=\; \widetilde{k}(x) := \begin{cases} 0, & x \in A, \\ (\mathcal{L} - r)f(x), & x \notin A, \end{cases}$$

where the convergence is pointwise. If $\widetilde{S}^A$ is a Feller process, and

$$(2.13) \qquad \lim_{x\to\partial A} \{\mathcal{L}f(x) - rf(x)\} = 0, \qquad \text{where } \partial A \text{ is the boundary of } A,$$

then $\widetilde{\mathcal{L}}^A f = \widetilde{k}$, where $\widetilde{\mathcal{L}}^A$ is the infinitesimal generator of the semigroup $(\widetilde{P}_t^A)_{t\geq 0}$.

(ii) Let $g \in \mathcal{D}$ and assume that x satisfies

$$(2.14) \qquad \mathbf{P}_x(\tau_A \leq t) = o(t) \text{ as } t \searrow 0 \qquad \text{or} \qquad g(S_{\tau_A}) = 0 \qquad \mathbf{P}_x\text{-a.s.}$$

Then

$$(2.15) \qquad \lim_{t \downarrow 0} t^{-1}(\widehat{P}_t^A g(x) - g(x)) \;\; = \;\; \mathcal{L}g(x),$$

where the convergence is pointwise. If $\widehat{S}^A$ is a Feller process and

$$g|_A = 0 \qquad and \qquad \lim_{x \to \partial A} \mathcal{L}g(x) = 0,$$

then $\widehat{\mathcal{L}}^A g = \mathcal{L}g$, where $\widehat{\mathcal{L}}^A$ is the infinitesimal generator of the semigroup $(\widehat{P}_t^A)_{t \geq 0}$.

Lemma 1 is a straightforward consequence of the definition of the infinitesimal generator and the Hille-Yosida theorem, see e.g. [53, Lemma 31.7] (see [49] for the complete proof of Lemma 1). If $\widetilde{S}^A$ and $\widehat{S}^A$ are themselves Feller processes, the relations between $\widetilde{P}^A$ and $\widetilde{\mathcal{L}}^A$, and between $\widehat{P}^A$ and $\widehat{\mathcal{L}}^A$ can formally be expressed as follows:

$$(2.16) \qquad \widetilde{P}_t^A = \exp\left(t \widetilde{\mathcal{L}}^A\right), \qquad\qquad \widehat{P}_t^A = \exp\left(t \widehat{\mathcal{L}}^A\right).$$

Equation (2.16) can be given a precise meaning if, for example, $\widetilde{P}_t^A$ and $\widehat{P}_t^A$ can be defined as a self-adjoint operator on a separable Hilbert space (see e.g. Ch. XII in Dunford and Schwarz [17], or Hille and Philips [31]). By determining the spectral decompositions of $\widetilde{\mathcal{L}}^A$ and $\widehat{\mathcal{L}}^A$ one can construct spectral expansions of $\widetilde{P}_t^A f(x)$ and $\widehat{P}_t^A f(x)$, which in the case of a discrete spectrum reduces to a series expansion. See Linetsky [42, 43, 44] for a development of this spectral expansion approach for one-dimensional diffusion models in finance, and an overview of related literature.

When (asymmetric) jumps are present, the operator is non-local and not self-adjoint, and the spectral theory has been less well developed, with fewer explicit results. Here we will follow a different approach: we will approximate S by a finite state Markov chain, and show that for the approximating chain a matrix analog of the identities (2.12)–(2.16) holds true, where the infinitesimal generators $\widetilde{\mathcal{L}}^A$ and $\widehat{\mathcal{L}}^A$ can be easily obtained from $\mathcal{L}$. We give a self-contained development of this approach in Section 3.

3. Exit probabilities for continuous-time Markov chains

Given a Markov price process S of interest, the idea is to construct a continuous-time finite state Markov chain X whose dynamics are "close" to those of S, and to calculate the relevant expectations for this approximating chain. In this section we will focus on the latter; we will return to the question of how to construct such a chain in Section 4. Assume therefore we are given a finite state continuous-time Markov chain $X = \{X_t\}_{t \geq 0}$. From Markov chain theory it is well known that the chain is completely specified by its state-space (or grid) $\mathbb{G} \subset \mathbb{E}$ and its generator matrix Λ, which is an $N \times N$ square matrix with zero row sums and non-positive diagonal elements, if $\mathbb{G}$ has N elements. Given the generator matrix Λ the family of transition matrices $(P_t)_{t \geq 0}$ of X, defined by $P_t(x, y) := \mathbf{P}_x[X_t = y]$ for $x, y \in \mathbb{G}$, is given by

$$P_t = \exp(t\Lambda).$$

In particular, the expexted discounted pay-off $\phi(X_T)$ at maturity T is then given by

$$(3.1) \qquad \mathbf{E}_x[\mathrm{e}^{-rT}\phi(X_T)] \;\; = \;\; \mathrm{e}^{-rT} \cdot (\exp(T\Lambda)\phi)(x)$$

for $x \in \mathbb{G}$ and any function $\phi : \mathbb{G} \to \mathbb{R}$. Here and throughout the paper we will identify any square matrix $\mathcal{A} \in \mathbb{R}^{N \times N}$ and any vector ϕ in $\mathbb{R}^N$ with functions

$$\mathcal{A} : \mathbb{G} \times \mathbb{G} \to \mathbb{R}, \qquad \mathcal{A}(x, y) := e'_x \mathcal{A} e_y, \qquad x, y \in \mathbb{G}, \quad \text{and}$$

$$\phi : \mathbb{G} \to \mathbb{R}, \qquad \phi(x) := e'_x \phi, \qquad x \in \mathbb{G},$$

where the vectors e_x, e_y denote the corresponding standard basis vectors of $\mathbb{R}^N$ and $'$ stands for transposition.

The generator Λ can be retrieved from the family of matrices $(P_t)_{t \geq 0}$ defined above by differentiation at $t = 0$, that is,

$$P_t = I + t\Lambda + o(t) \quad \text{as} \ t \searrow 0.$$

In order for X itself to form a pricing model, defined under a martingale measure, we will suppose in addition that X is non-negative and that the discounted process is a martingale; more precisely we assume that $\{M_t\}_{t \geq 0}$ is a martingale where

$$(3.2) \qquad M_t = \begin{cases} \mathrm{e}^{-\gamma t} X_t, & t < \zeta, \\ \mathrm{e}^{-\gamma \zeta} X_\zeta, & t \geq \zeta, \end{cases}$$

with $\zeta = \inf\{t \geq 0 : X_t \in \partial \mathbb{G}\}$ the hitting time of the "boundary" $\partial \mathbb{G}$ which consists of the smallest and largest elements of the state space $\mathbb{G}$, and $\gamma = r - d$. The Markov property of X implies that the process M given in (3.2) is a martingale precisely if

$$(3.3) \qquad (\Lambda \eta)(x) = \gamma\, \eta(x) \qquad\qquad \text{for } x \in \mathbb{G} \backslash \partial \mathbb{G},$$

where the function $\eta : \mathbb{G} \to \mathbb{E}$ is given by $\eta(z) = z$. Below we will show how to express the exit probabilities of the chain using matrix exponentiation in a way that is identical in form to the expected value (3.1) of a European pay-off. To that end, we partition $\mathbb{G}$ into a 'continuation' set $\widehat{\mathbb{G}}$ and a 'knock-out' set $\widehat{\mathbb{G}}^c := \mathbb{G} \backslash \widehat{\mathbb{G}}$, where

$$(3.4) \qquad\qquad \widehat{\mathbb{G}} \ := \ \{x \in \mathbb{G} \ : x \in A^c\},$$

and define the first exit time of X from $\widehat{\mathbb{G}}$ by

$$(3.5) \qquad\qquad \tau := \inf\{t \in \mathbb{R}_+ \ : \ X_t \notin \widehat{\mathbb{G}}\},$$

where we use the convention $\inf \emptyset := \infty$ and where we will take the set A as in (2.5).

The value of a general barrier knock-out option with a rebate depends on the joint distribution of the exit time τ_A from A and the positions of the underlying at maturity and at the moment of exit. The corresponding quantities for the chain X can be expressed in terms of two transformations of X, namely the chain $\widehat{X}$ that is killed upon exiting $\widehat{\mathbb{G}}$ and the chain $\widetilde{X}$ that is absorbed at that instance, respectively. Correspondingly, we associate to the generator matrix Λ, two matrices: the $\widehat{N} \times \widehat{N}$ matrix

$\widehat{\Lambda}$, where $\widehat{N} := |\widehat{\mathbb{G}}|$, and the $N \times N$ matrix $\widetilde{\Lambda}_r$, defined by

$$(3.6) \qquad \widetilde{\Lambda}_r(x, y) := \begin{cases} \Lambda(x, y) - r & \text{if } x \in \widehat{\mathbb{G}},\ x = y, \\ \Lambda(x, y) & \text{if } x \in \widehat{\mathbb{G}},\ y \in \mathbb{G},\ x \neq y, \\ 0 & \text{if } x \in \widehat{\mathbb{G}}^c,\ y \in \mathbb{G}, \end{cases}$$

$$(3.7) \qquad \widehat{\Lambda}(x, y) := \widehat{\Lambda}_0(x, y) := \Lambda(x, y) \qquad \text{if } x \in \widehat{\mathbb{G}},\ y \in \widehat{\mathbb{G}}.$$

We can now state the key result of this section:

Theorem 1. *For any $T > 0$, $x \in \mathbb{G}$ and $r \geq 0$ and any function $\phi : \mathbb{G} \to \mathbb{R}$ it holds that*

$$(3.8) \qquad \mathbf{E}_x\left[\mathrm{e}^{-r(T \wedge \tau)} \phi(X_{T \wedge \tau}) \right] = \left(\exp\left(T \widetilde{\Lambda}_r \right) \phi \right)(x).$$

In particular, for $\psi : \widehat{\mathbb{G}} \to \mathbb{R}$ and $\xi : \mathbb{G} \to \mathbb{R}$ with $\xi(x) = 0$ for $x \in \widehat{\mathbb{G}}$ we have that

$$(3.9) \qquad \mathbf{E}_x\big[\psi(X_T)\mathbf{I}_{\{\tau > T\}}\big] = \left(\exp\left(T \widehat{\Lambda} \right) \psi \right)(x) \qquad \text{for any} \quad x \in \widehat{\mathbb{G}},$$

$$(3.10) \qquad \mathbf{E}_x\big[\phi(X_T)\mathbf{I}_{\{\tau \leq T\}}\big] = \left(\exp\left(\Lambda T \right) \phi \right)(x) - \left(\exp\left(T \widehat{\Lambda} \right) \widehat{\phi} \right)(x)\, \mathbf{I}_{\widehat{\mathbb{G}}}(x) \quad \text{for any} \quad x \in \mathbb{G},$$

$$(3.11) \qquad \mathbf{E}_x\big[\mathrm{e}^{-r\tau}\xi(X_\tau)\mathbf{I}_{\{\tau \leq T\}}\big] = \left(\exp\left(T \widetilde{\Lambda}_r \right) \xi \right)(x) \qquad \text{for any} \quad x \in \mathbb{G},$$

where $\widehat{\phi} = \phi|_{\widehat{\mathbb{G}}}$, the restriction of ϕ to $\widehat{\mathbb{G}}$.

Formulas (3.9)–(3.11) give a simple way of computing barrier option prices by a single matrix exponentiation. The expectation in (3.9) can be obtained by computing the spectral decomposition of the matrix $\widehat{\Lambda} = UDU^{-1}$ and applying the formula $\exp\left(T \widehat{\Lambda} \right) = U \exp(TD)U^{-1}$. The powerful Padé approximation method for matrix exponentiation, described in [30], can also be used to compute efficiently the matrix exponentials in Theorem 1. Since the state-space is finite, Theorem 1 is a corollary of Lemma 1. We present next a direct probabilistic derivation.

Proof. To prove equation (3.8), we will verify that the expected value of an Arrow-Debreu barrier security that pays 1 precisely if X is in the state y at the earlier of the maturity T and the knock-out time τ is given by

$$(3.12) \qquad \mathbf{E}_x[\mathrm{e}^{-r(T \wedge \tau)}\mathbf{I}_{\{X_{T \wedge \tau} = y\}}] = \left(\exp(T \widetilde{\Lambda}_r) \right)(x, y) \qquad \text{for all } x, y \in \mathbb{G}.$$

For a given time grid $\mathbb{T}_n = (k\Delta t,\ k = 0, 1, 2, \ldots, n)$ with $\Delta t = T/n$ denote by $\widetilde{P}_T^{(n)}(x, y)$ the expected value of the corresponding discretely monitored Arrow-Debreu security and let

$$\tau_n = \inf\{s \in \mathbb{T}_n : X_s \notin \widehat{\mathbb{G}}\}$$

be the corresponding time at which the barrier is crossed. Since the paths of the chain X are piecewise constant, it follows that $\tau_n \downarrow \tau$ and $X_{\tau_n} \to X_\tau$ as n tends to infinity. Hence the expected values of the discretely monitored Arrow-Debreu securities converge to the expected value of the continuously monitored one,

$$\widetilde{P}_T^{(n)}(x, y) = \mathbf{E}_x[\mathrm{e}^{-r(T \wedge \tau_n)}\mathbf{I}_{\{X_{T \wedge \tau_n} = y\}}] \longrightarrow \mathbf{E}_x[\mathrm{e}^{-r(T \wedge \tau)}\mathbf{I}_{\{X_{T \wedge \tau} = y\}}].$$

Clearly, since $\widehat{\mathbb{G}}^c$ is the knock-out set, it holds for all $t \geq 0$ that

$$\widetilde{P}_t^{(n)}(x,y) = \begin{cases} 1 & \text{if } x \in \widehat{\mathbb{G}}^c, \ x = y \\ 0 & \text{if } x \in \widehat{\mathbb{G}}^c, \ x \neq y \end{cases}$$

$$= \left(I - \overline{I} \right)(x,y) \qquad \text{for all } x \in \widehat{\mathbb{G}}^c, \ y \in \mathbb{G},$$

where $\overline{I}$ is a square matrix of size N with $\overline{I}(x,x) = 1$ if $x \in \widehat{\mathbb{G}}$ and zero else. Further, for $x \in \widehat{\mathbb{G}}$ an application of the Markov property of X shows that

$$\widetilde{P}_T^{(n)}(x,y) = \mathrm{e}^{-r\Delta t} \sum_{z \in \mathbb{G}} P_{\Delta t}(x,z) \widetilde{P}_{T-\Delta t}^{(n)}(z,y)$$

$$= \left(\overline{I} \left(\mathrm{e}^{-r\Delta t} P_{\Delta t} \right) \widetilde{P}_{T-\Delta t}^{(n)} \right)(x,y).$$

Combining the two cases, iterating the argument and using the differentiability of P_t at $t = 0$ shows that

$$\widetilde{P}_T^{(n)}(x,y) = \left(\left(I - \overline{I} + \overline{I}\,\mathrm{e}^{-r\Delta t} P_{\Delta t} \right) \widetilde{P}_{T-\Delta t}^{(n)} \right)(x,y)$$

$$= \left(\left(I - \overline{I} + \overline{I}\,\mathrm{e}^{-r\Delta t} P_{\Delta t} \right)^{T/\Delta t} \right)(x,y)$$

$$= \left(\left(I + \overline{I}\,\mathrm{e}^{-r\Delta t}(P_{\Delta t} - I) + \overline{I}(\mathrm{e}^{-r\Delta t} - 1) \right)^{T/\Delta t} \right)(x,y)$$

$$= \left(\left(I + \Delta t(\widetilde{\Lambda}_0 - r\overline{I}) + o(\Delta t) \right)^{T/\Delta t} \right)(x,y),$$

since $\widetilde{\Lambda}_0 = \overline{I}\Lambda$. When $\Delta t = T/n$ tends to zero, this expression converges to $\left(\exp\left(T\widetilde{\Lambda}_r \right) \right)(x,y)$, which completes the proof of (3.12) and hence implies (3.8). Equation (3.11) follows then directly by applying (3.8) to ξ. Noting that for any $\psi : \widehat{\mathbb{G}} \to \mathbb{R}$ and any $x \in \widehat{\mathbb{G}}$ we have

$$\left(\widehat{\Lambda}\psi \right)(x) = \left(\widetilde{\Lambda}_0 \psi_0 \right)(x), \qquad \text{where } \psi_0 : \mathbb{G} \to \mathbb{R} \text{ is given by } \psi_0(y) := \psi(y)\mathbf{I}_{\widehat{\mathbb{G}}}(y),$$

we get that

$$\left(\exp(T\widehat{\Lambda})\psi \right)(x) = \left(\exp(T\widetilde{\Lambda}_0)\psi_0 \right)(x),$$

which yields (3.9). Finally, (3.10) is a direct consequence of (3.1), (3.9) and the fact that a European option with pay-off $\phi(X_T)$ is equal to the sum of a knock-out barrier option and a knock-in barrier option with the same pay-off $\phi(X_T)$ and same knock-out/knock-in levels. $\qquad\square$

4. Construction of the Markov chain

The Markov chain approximation algorithm for homogeneous Feller processes can now be described as follows:

(1) Construct an approximating Markov chain:
  (a) specify a (non-uniform) grid $\mathbb{G}$;
  (b) define a generator matrix Λ of a Markov chain with state-space $\mathbb{G}$.
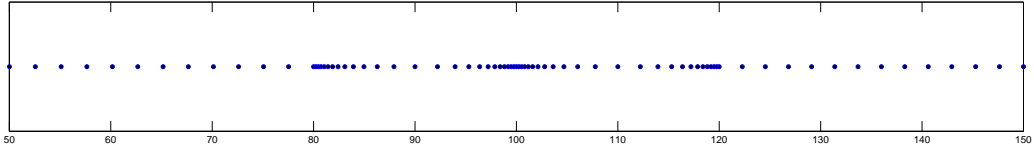(2) Compute the barrier option prices using formulae (3.8)–(3.11).

FIGURE 2. The figure shows a non-uniform grid that was generated using the algorithm described in Section 4. The grid consists of $N = 75$ points that are densely distributed around the spot $S_0 = 100$ and the upper and lower barriers $\ell = 80$ and $u = 120$; its smallest and largest values were chosen as $x_1 = 50$ and $x_N = 150$, and $N_i = 25$, $i = 1, 2, 3$, points were placed in the areas $[x_1, \ell]$, $[\ell, u]$ and $[u, x_N]$ with respective density parameters $d_1^- = 100$, $d_1^+ = 1$, $d_2^- = 1$, $d_2^+ = 1$, and $d_3^- = 1$, $d_3^+ = 100$.

A suitable choice of the grid in step (1a) is essential for the effectiveness of the above pricing algorithm. The construction of an optimal grid (according to some criterion) is a topic of separate study, which will not be pursued further in this paper. One of the features of a good grid is that it has sufficient resolution in regions of interest, such as the current spot value and the barrier levels, which is a necessary condition for constructing a Markov chain market model that approximates well the dynamics of the given price process. Another desirable feature is that the grid "covers" a sufficiently large part of the state space, which is needed to control the truncation error that arises when approximating an infinite state space by a finite state space. To employ a uniform grid that satisfies these conditions would be computationally expensive. The use of adaptive meshes for option pricing was proposed in Figlewski and Gao [21]. Here we employ the following procedure for generating a suitable non-uniform grid $\mathbb{G}$, based on an algorithm from [55]:

(1) Pick $N_i \in \mathbb{N}$ and the density parameters $d_i^{\pm} \in (0, \infty)$, $i = 1, 2, 3$, and the smallest and largest values x_1, x_N of the grid $\mathbb{G}$, where $N = N_1 + N_2 + N_3$.
(2) Define $\mathbb{G}_i := \text{GENERATESUBGRID}(a_i, s_i, b_i, N_i, d_i^-, d_i^+)$, for $i = 1, 2, 3$, where $a_1 = x_1, s_1 = \ell, b_1 = (S_0 + \ell)/2 = a_2, s_2 = S_0, b_2 = (u + S_0)/2 = a_3, s_3 = u, b_3 = x_N$.
(3) Set $\mathbb{G} := \mathbb{G}_1 \cup \mathbb{G}_2 \cup \mathbb{G}_3$,

where the subgrid is generated by the following procedure:

GENERATESUBGRID(a, s, b, M, g_1, g_2)
(1) Compute $c_1 = \text{arcsinh}\left(\frac{a-s}{g_1}\right)$, $c_2 = \text{arcsinh}\left(\frac{b-s}{g_2}\right)$.
(2) Define the lower part of the grid by the formula $x_k := s + g_1 \sinh(c_1(1 - (k-1)/(M/2 - 1)))$ for $k \in \{1, \ldots, M/2\}$. Note that $x_1 = a, x_{M/2} = s$.
(3) Define the upper part of the grid using the formula $x_{k+M/2} := s + g_2 \sinh(c_2 2k/M)$ for $k \in \{1, \ldots, M/2\}$. Note that $x_M = b$.
RETURN $\{x_1, \ldots, x_M\}$

The non-uniform state-space $\mathbb{G} \subset \mathbb{E}$ for the Markov chain X is constructed by concatenating the three subgrids that are generated by specifying lower, middle and upper points $a < s < b$ and density parameters g_1 and g_2. A smaller density corresponds to a grid that is more concentrated around the middle point s. Observe that the algorithm above places the current spot S_0 and the barrier levels $\ell < u$ on the grid $\mathbb{G}$ and that the resolution of the grid around ℓ, S_0 and u can be controlled by the density

parameters $d_i^{\pm}, i = 1, 2, 3$. A Matlab implementation of this grid generator can be downloaded from [48]. The remainder of this section will be devoted to step (1b) in the algorithm described above.

4.1. **Jump processes with state-dependent characteristics.** The construction of the generator matrix Λ of the approximating Markov chain X is now carried out in two steps: we first define the jump matrix Λ_J, which corresponds to the discretization of the jump measure ν, and then characterize a tri-diagonal generator matrix Λ_D by stipulating that the Markov chain X with the generator $\Lambda = \Lambda_J + \Lambda_D$ has the same instantaneous moments as the process S.

We start by building the state-space $\mathbb{G} \subset \mathbb{E}$ with N elements using the algorithm described in the beginnig of this section. Define the sets

$$\partial\mathbb{G} := \{x_1, x_N\} \qquad \text{and} \qquad \mathbb{G}^o := \mathbb{G}\backslash\partial\mathbb{G},$$

where the "boundary" $\partial\mathbb{G}$ consist of the smallest (i.e. x_1) and largest (i.e. x_N) elements in $\mathbb{G}$ and the "interior" $\mathbb{G}^o$ is the complement of the boundary. For any given $x \in \mathbb{G}^o$ we associate to $\mathbb{G}$ the set $\mathbb{G}_x \subset (-1, \infty)$ defined by

$$\mathbb{G}_x \quad := \quad \left\{\frac{z}{x} - 1 \ : \ z \in \mathbb{G}\right\}.$$

The set $\mathbb{G}_x$ consists of the relative jump sizes of jumps starting from x and arriving at any other point in $\mathbb{G}$. The x-th row of the jump part Λ_J of the generator of X is obtained by discretizing the jump measure $\nu(x, \mathrm{d}y)$ on the set $\mathbb{G}_x$. In particular let $\mathbb{G}_x = \{y_i \ : \ y_1 < \ldots < y_{N-1}\}$ and define a function $\alpha_x : \mathbb{G}_x \to [-1, \infty]$ such that

$$(4.1) \qquad\qquad\qquad \alpha_x(y_i) \in (y_i, y_{i+1}) \quad \text{for} \quad i \in \{1, \ldots, N-1\}$$

where $\alpha_x(y_0) = -1, \alpha_x(y_N) = \infty$. A possible natural choice for $\alpha_x(y_i)$ would be the mid-point of the interval

$$\alpha_x(y_i) = \frac{y_i + y_{i+1}}{2}.$$

We can now define the jump part of the generator as

$$(4.2) \qquad\qquad\qquad \Lambda_J(x, x(1+y_i)) \quad := \quad \int_{\alpha_x(y_{i-1})}^{\alpha_x(y_i)} \nu(x, \mathrm{d}y)$$

where $i \in \{1, \ldots, N\}$ and $y_i \neq 0$ and

$$(4.3) \qquad\qquad\qquad \Lambda_J(x, x) \quad := \quad -\sum_{z \in \mathbb{G}\backslash\{x\}} \Lambda_J(x, z).$$

Note that the function α_x generates a partititon of $(-1, \infty)$. The jump intensities of the chain, defined in formula (4.2), are obtaiend by integrating the Lévy measure over the corresponding part of the partition. For $x \in \partial\mathbb{G}$ we set $\Lambda_J(x, y) := 0$ for all $y \in \mathbb{G}$. It is clear that the matrix Λ_J constructed in this way is a generator matrix.

In the second step we match the first and second instantaneous moments of the asset price process S. In other words the chain X must satisfy conditions

$$(4.4) \qquad \mathbf{E}_x\left[(S_{\Delta t} - S_0)^j\right] \quad = \quad \mathbf{E}_x\left[(X_{\Delta t} - X_0)^j\right] + o(\Delta t), \qquad \text{for} \quad x \in \mathbb{G}^o, \ j \in \{1, 2\}$$

for all starting states $x \in \mathbb{G}^o$. Note that condition (4.4) implicitly assumes that the second instantaneous moment of S exists. This is the case if the jump measure satisfies the following condition

$$(4.5) \qquad \int_{(-1,\infty)} y^2 \nu(x,\mathrm{d}y) < \infty \quad \text{for all} \quad x \in \mathbb{E}$$

which we now assume to hold.

The task now is to find a tri-diagonal generator matrix Λ_D such that the chain generated by the sum $\Lambda_D + \Lambda_J$ satisfes (4.4). The tri-diagonal matrix Λ_D therefore has to satisfy the following conditions

$$(4.6) \qquad \sum_{z \in \mathbb{G}} \Lambda_D(x,z) = 0 \quad \text{and} \quad \Lambda_D(x,z) + \Lambda_J(x,z) \geq 0 \quad \forall z \in \mathbb{G} \backslash \{x\},$$

$$(4.7) \qquad \sum_{z \in \mathbb{G}} \Lambda_D(x,z)(z-x) = (r-d)x - \sum_{z' \in \mathbb{G}} \Lambda_J(x,z')(z'-x),$$

$$(4.8) \qquad \sum_{z \in \mathbb{G}} \Lambda_D(x,z)(z-x)^2 = x^2 \left[\sigma(x)^2 + \int_{-1}^{\infty} y^2 \nu(x,\mathrm{d}y) \right] - \sum_{z' \in \mathbb{G}} \Lambda_J(x,z')(z'-x)^2$$

for all $x \in \mathbb{G}^o$, where r, d are the instantaneous interest rate and dividend yield respectively and σ is the local volatility function in (2.11). The right-hand side of equation (4.7) is the difference of the risk-neutral drift and the drift induced by the presence of jumps. Similarly the right-hand side of the linear equation in (4.8) consists of the difference of the instantaneous second moments of the asset price process S (computed directly from its generator (2.11)) and the chain that corresponds to the jump generator Λ_J. As usual we assume the absorbing boundary condition $\Lambda_D(x,y) = 0$ for all $x \in \partial\mathbb{G}, y \in \mathbb{G}$.

The linear system in (4.6)–(4.8) can typically be satisfied by a tri-diagonal generator matrix Λ_D if $\sigma(x)$ is strictly positive. Once we find Λ_D we define the generator matrix of the approximating chain X by

$$\Lambda := \Lambda_D + \Lambda_J.$$

Remark 1. In the case that S is a diffusion time-changed by an independent Lévy subordinator, there is an alternative approach to constructing the approximating continuous-time Markov chain X, based on the Phillips theorem. For the details of this construction see [49].

5. CONVERGENCE AND ERROR ESTIMATES

5.1. **Convergence of barrier option prices.** Consider a sequence of finite-state continuous-time Markov chains $X^{(n)}$ approximating a given Feller price process $S = \{S_t\}_{t \geq 0}$. For $X^{(n)}$ to replicate as closely as possible the dynamics of S one chooses the generator matrix $\Lambda^{(n)}$ with the corresponding state-space $\mathbb{G}^{(n)}$ such that it is uniformly close to the infinitesimal generator $\mathcal{L}$ of S, in the sense that the distance $\epsilon_n(f)$ between the generators is small for a sufficiently large class $\widetilde{\mathcal{D}}$ of regular test functions f, where

$$\epsilon_n(f) := \max_{x \in (\mathbb{G}^{(n)})^o} \left| \Lambda^{(n)} f_n(x) - \mathcal{L}f(x) \right|,$$

where $(\mathbb{G}^{(n)})^o$ equals $\mathbb{G}^{(n)}$ without the smallest and the largest elements and $f_n = f|_{(\mathbb{G}^{(n)})^o}$ is the restriction of f to $(\mathbb{G}^{(n)})^o$. More specifically, if $\epsilon_n(f)$ tends to zero as n tends to infinity for f in the class $\widetilde{\mathcal{D}}$ and the probability that the chain exits $(\mathbb{G}^{(n)})^o$ before time T tends to zero, then the sequence of processes $(X^{(n)})_{n \in \mathbb{N}}$ converges weakly to the process S. This weak convergence on the level of the

process implies in particular that the marginal distributions of $X^{(n)}$ will converge to those of S, and therefore the values of European options converge, that is,

$$\mathbf{E}_x[f(X_T^{(n)})] \to \mathbf{E}_x[f(S_T)]$$

for $x \in \mathbb{E}$, maturity $T > 0$, and continuous bounded functions f.

The payoff of the barrier option can be described in terms of the first-passage time of S and the position of S at that moment, which are both functionals of the path $\{S_t\}_{t \geq 0}$. For the weak convergence of $X^{(n)}$ to S to carry over to convergence of barrier-type payoffs, continuity (in the Skorokhod topology) is required of these two functionals, which is guaranteed to hold under Assumption 2. In view of the fact that the payoff of a barrier option is typically a discontinuous function, an additional condition is needed to ensure the convergence of the barrier option prices; we will assume that ℓ and u are such that

$$(5.1) \qquad \mathbf{P}_x\left(S_T \in \{\ell, u\}\right) = 0.$$

Most models used in mathematical finance satisfy this condition. Even if (5.1) is not satisfied, this does not constitute a limitation in practice, since for any given process S the condition is satisfied for all but countably many pairs (ℓ, u). The statement of the convergence is made precise in the following theorem.

Theorem 2. *Let S be a Feller process with state-space $\mathbb{E}$ and infinitesimal generator $\mathcal{L}$ that does not vanish at zero and infinity.[4] Let $\left(X^{(n)}\right)_{n \in \mathbb{N}}$ be a sequence of Markov chains with generator matrices $\Lambda^{(n)}$ such that the rows corresponding to the smallest and the largest elements in $\mathbb{G}^{(n)}$ are equal to zero. Assume further that the following two conditions are satisfied for any function f in a core of $\mathcal{L}$:[5]*

$$(5.2) \qquad \epsilon_n(f) \to 0 \qquad \text{as } n \to \infty,$$

$$(5.3) \qquad \text{either (i) } \lim_{y \searrow 0} \mathcal{L}f(y) = 0 \text{ or (ii) } \lim_{n \to \infty} \mathbb{P}_x\left[\tau_{(\mathbb{G}^{(n)})^o}^{(n)} > T\right] = 1,$$

where for any set $G \subset \mathbb{E}$ we define $\tau_G^{(n)} = \inf\{t \geq 0 : X_t^{(n)} \notin G\}$. If (5.1) holds, then, as $n \to \infty$,

$$\mathbf{E}_x\left[g\left(X_T^{(n)}\right)\mathbf{I}_{\{\tau_A^{(n)} > T\}}\right] \quad \longrightarrow \quad \mathbf{E}_x\left[g(S_T)\mathbf{I}_{\{\tau_A > T\}}\right],$$

$$\mathbf{E}_x\left[e^{-r\tau_A^{(n)}}h\left(X_{\tau_A^{(n)}}^{(n)}\right)\mathbf{I}_{\{\tau_A^{(n)} \leq T\}}\right] \quad \longrightarrow \quad \mathbf{E}_x\left[e^{-r\tau_A}h(S_{\tau_A})\mathbf{I}_{\{\tau_A \leq T\}}\right],$$

for any bounded continuous functions $g, h : \mathbb{E} \to \mathbb{R}$.

Remark 2. Since condition (5.2) is required to hold for all f in a core of $\mathcal{L}$, it follows that $\mathbb{G}^{(n)}$ will eventually 'fill up' the part of the state-space $\mathbb{E}$ where the generator does not vanish. Indeed, if there were to exist an open interval that does not intersect $\mathbb{G}^{(n)}$ for large n, then condition (5.2) would not hold for functions $f \in C^\infty(\mathbb{E})$ with compact support in this open interval satisfying $\mathcal{L}f \neq 0$, as $\Lambda^{(n)}f_n \equiv 0$. In particular, condition (5.2) implies that $x_1^{(n)} \to 0$ and $x_n^{(n)} \to \infty$ when $\mathbb{G}^{(n)} = \{x_1^{(n)}, \ldots, x_n^{(n)}\}$ with $x_1^{(n)} < \ldots < x_n^{(n)}$ if the generator does not vanish in a neighbourhood of zero and infinity.

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[4]The set where the generator does not vanish is defined to be the set of all $x \in \mathbb{E} \cup \{\infty\}$ with the following property: for every open interval I in $\mathbb{E}$ that contains x (if $x = \infty$ then I takes the form (M, ∞) for some $M \in \mathbb{E}$) there exists a function $f \in \mathcal{D}$ with compact support in I such that the function $\mathcal{L}f$ is not identically equal to zero.

[5]A core $\mathcal{C}$ of the operator $\mathcal{L}$ is a subspace of the domain of $\mathcal{L}$ that is (i) dense in $C_0(\mathbb{E})$ and (ii) there exists $\lambda > 0$ such that the set $\{(\lambda - \mathcal{L})f : f \in \mathcal{C}\}$ is dense in $C_0(\mathbb{E})$.

Remark 3. Note that, since $\mathcal{L}f$ is by definition an element in $C_0(\mathbb{E})$ for any f in the domain of $\mathcal{L}$, it holds that $\mathcal{L}f(y) \to 0$ as $y \nearrow \infty$ and that $\mathcal{L}f(0)$ is well defined and equal to $\lim_{y \searrow 0} \mathcal{L}f(y)$. Assume that $\mathcal{L}$ does not vanish at zero and infinity and that the rows corresponding to the smallest and the largest elements in $\mathbb{G}^{(n)}$ of the generator matrices $\Lambda^{(n)}$ are equal to zero. Then, if conditions (5.2) and (5.3)(i) are satisfied, we have

$$(5.4) \qquad \max_{x \in \mathbb{G}^{(n)}} \left| \Lambda^{(n)} f_n(x) - \mathcal{L}f(x) \right| \to 0 \qquad \text{as} \quad n \to \infty.$$

Remark 4. In practice the condition of boundedness of the payoff f is not restrictive as it is always possible to consider the truncation $f \wedge M$ for large constants M without losing noticeable accuracy. Further, under additional regularity properties on the parameters of the process S, the convergence in Theorem 2 also holds true for barrier call options. To see why this is the case note that

$$(5.5) \qquad \mathbf{E}_x \left[(S_T - K)^+ \mathbf{I}_{\{\tau_A < T\}} \right] = x\, \overline{\mathbf{E}}_x \left[(1 - K/S_T)^+ \mathbf{I}_{\{\tau_A < T\}} \right]$$

where $\overline{\mathbf{E}}$ denotes the expectation under the measure $\overline{\mathbf{P}}$ given by $d\overline{\mathbf{P}}_x|_{\mathcal{F}_t} = x^{-1} S_t d\mathbf{P}_x|_{\mathcal{F}_t}$. Under $\overline{\mathbf{P}}$ the process S remains a Markov process, and, under additional regularity properties S is still a Feller process. The convergence then follows from Theorem 2 as the pay-off function on the right-hand side of (5.5) is bounded.

Remark 5. Theorem 2 follows by combining Ethier and Kurtz [20, Theorem 4.2.11] and results in Section VI.2 in Jacod and Shiryaev [34]. The complete proof of Theorem 2 can be found in [49].

5.2. **Error estimates.** In this section we quantify the speed at which the algorithm converges by providing error estimates for a specific choice of a sequence of approximating Markov chains, assuming sufficient smoothness of the value-function of the barrier option. Consider a Feller process S with infinitesimal generator acting on $v \in C^2(\mathbb{E})$ with compact support as

$$(5.6) \qquad \mathcal{L}v(x) \;=\; \mathcal{L}_D v(x) + \mathcal{L}_J v(x) \qquad \text{with}$$

$$(5.7) \qquad \mathcal{L}_D v(x) \;:=\; \frac{\sigma^2(x)x^2}{2} v''(x) + \gamma x v'(x),$$

$$(5.8) \qquad \mathcal{L}_J v(x) \;:=\; \int_{(-1,\infty)} [v(x(1+y)) - v(x) - v'(x)xy] g(x,y) dy,$$

where $'$ denotes differentiation with respect to x, $\gamma = r - d$ (r and d are the instantaneous interest rate and dividend yield respectively) and $g : \mathbb{E} \times ((-1,\infty)\backslash\{0\}) \to \mathbb{R}_+$ is a nonnegative locally bounded function such that

$$\int_{(-1,\infty)\backslash\{0\}} g(x,y)(|y| \wedge y^2) dy < \infty \qquad \text{for all } x \in \mathbb{E},$$

where $x \wedge y = \min\{x, y\}$.

We next describe the sub-class of Markov processes that we will consider.

Definition 1. The process S is called *uniformly of bounded jump-variation* if the jump-density g satisfies the integrability condition

$$(5.9) \qquad C_0 := \sup_{x \in [\ell,u]} \mathcal{I}(x) < \infty,$$

where

$$(5.10) \qquad \mathcal{I}(x) := \int_{(-1,1)\setminus\{0\}} |y| g(x,y) \mathrm{d}y.$$

Definition 2. The process S is called *locally of stable type* on $[\ell, u]$ if there exist constants $\overline{\kappa}_\pm, \underline{\kappa}_\pm > 0$ and $\alpha_\pm \in (0,2)$ such that for all $x \in [\ell, u]$

$$(5.11) \qquad \frac{\underline{\kappa}_+}{|y|^{1+\alpha_+}} \quad \le \quad g(x,y) \le \frac{\overline{\kappa}_+}{|y|^{1+\alpha_+}}, \qquad y \in (0,1),$$

$$(5.12) \qquad \frac{\underline{\kappa}_-}{|y|^{1+\alpha_-}} \quad \le \quad g(x,y) \le \frac{\overline{\kappa}_-}{|y|^{1+\alpha_-}}, \qquad y \in (-1,0).$$

We will consider three different cases:

- **Case O**: S is uniformly of bounded jump-variation.
- **Case I**: S is locally of stable type with $\alpha_- = 1$ or $\alpha_+ = 1$.
- **Case II**: S is locally of stable type with $\alpha_\pm \in (1,2)$.

Case I can be considered to be a boundary case separating cases O and II, as S is uniformly of bounded jump-variation if it is of stable type with $\alpha_\pm < 1$. The class of processes of uniformly bounded jump-variation (case O) contains processes S whose jump-part forms a compound Poisson process (such as the Kou model [37]), as well as the Lévy models of bounded variation such as for example the VG model. Examples of processes satisfying the condition in cases I and II are the Generalised Hyperbolic Lévy models and CGMY models with $Y > 1$, respectively. More generally, the class of processes that are locally of stable type contains the class of *regular Lévy processes of exponential type* studied in Boyarchenko and Levendorskii [6].

We will impose in addition the following regularity conditions:

Assumption 3. *There exists a constant C' such that for all $x \in [\ell, u]$*

$$\sigma^2(x) x^2 + |\gamma| x + \int_{(-1,\infty)\setminus\{0\}} (y^2 \wedge |y|) g(x,y) \mathrm{d}y \le C'.$$

Assumption 4. *The pay-off function f is Lipschitz-continuous[6] and has compact support. The barrier option value-function $\widetilde{P}_t^A f(x) = \mathbf{E}_x[e^{-r(t \wedge \tau_A)} f(S_{t \wedge \tau_A})]$ is $C^{1,3}([0,T] \times \mathbb{E})$.*

Remark 6. In the strictly elliptic case ($\sigma(x) > 0$ for $x \in [\ell, u]$) with a jump-part that is uniformly of bounded variation, explicit sufficient conditions on the data f, g and σ^2 guaranteeing the smoothness of $v_A : (t,x) \mapsto \widetilde{P}_t^A f(x)$ required in Assumption 4 follow from classical existence and uniqueness results for Cauchy-Dirichlet problems associated to second order partial-integro differential operators. Specifically, Theorem II.3.3 in Garroni & Menaldi [22] implies that $v_A \in C^{1,3}([0,T] \times [\ell, u])$ if, for some $0 < \epsilon < 1$, the following conditions are satisfied: (a) f has compact support in (ℓ, u) and $f \in C^{2+\epsilon}([\ell, u])$[7], (b) $\sigma^2 \in C^1([\ell, u])$ and (c) $x \mapsto g(x,y)|y|^{2+\epsilon}$ is $C^1([\ell, u])$ with bounded derivatives, uniformly over $(x,y) \in [\ell, u] \times (-1, \infty)$. Note that (c) implies that the $\mathcal{L}_J v$ in (5.8) is $C^1([\ell, u])$ if v is $C^2([\ell, u])$.

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[6] A function $f : \mathbb{E} \to \mathbb{R}$ is Lipschitz continuous if there exists a constant $L > 0$ such that $|f(x) - f(x)| \le L|x - y|$ for all $x, y \in \mathbb{E}$.

[7] $C^{n+\epsilon}([\ell, u])$ is the set of functions $f : [\ell, u] \to \mathbb{R}$ that are n times continuously differentiable with the nth derivative ϵ-Hölder continuous.

Throughout the rest of this section we take

$$0 < \ell < u < \infty$$

and consider the spatial grid given by

$$(5.13) \qquad \mathbb{G}^{(n)} = \{x_i^{(n)} : 0 = x_1^{(n)} < \ldots < x_n^{(n)}\}.$$

We will denote by

$$(5.14) \qquad h(n) = \max_i \left| x_{i+1}^{(n)} - x_i^{(n)} \right|$$

the mesh of the spatial grid, and by

$$(5.15) \qquad k(n) = \max_{x_k^{(n)} \in \widehat{\mathbb{G}}^{(n)}} \int_{(-1,\infty) \setminus [L(x_k^{(n)}), U(x_k^{(n)})]} g(x_k^{(n)}, y) \mathrm{d}y,$$

the tail mass of the jump-measure, where $\widehat{\mathbb{G}}^{(n)} = \mathbb{G}^{(n)} \cap [\ell, u]$ and

$$L(x_k^{(n)}) := \frac{x_2^{(n)}}{x_k^{(n)}} - 1 \qquad \text{and} \qquad U(x_k^{(n)}) := \frac{x_{n-1}^{(n)}}{x_k^{(n)}} - 1$$

are the relative jump-sizes from $x_k^{(n)}$ to the one-but-largest and one-but-smallest elements $x_2^{(n)}$ and $x_{n-1}^{(n)}$ of the state-space $\mathbb{G}^{(n)}$. We approximate S by a sequence of Markov chains $(X^{(n)})_{n \in \mathbb{N}}$ with infinitesimal generators $\Lambda^{(n)}$. An explicit description of $\Lambda^{(n)}$ is given in Appendix A.1.

Theorem 3. *Let Assumptions 3 and 4 hold, and consider cases O, I or II. Assume also that the sequences of grids $\mathbb{G}^{(n)}$ satisfies*

$$\lim_{n \to \infty} h(n) = \lim_{n \to \infty} k(n) = 0.$$

Then there exist constants C_1, C_2, independent of $h(n)$ and $k(n)$, given in (5.14) and (5.15), such that for all n sufficiently large and $x \in \mathbb{G}^{(n)}$

$$\left| \mathbf{E}_{0,x} \left[\mathrm{e}^{-\int_0^{T \wedge \tau_A^{(n)}} r^{(n)}(t) \mathrm{d}t} f\left(X_{T \wedge \tau_A^{(n)}}^{(n)}\right) \right] - \mathbf{E}_{0,x} \left[\mathrm{e}^{-\int_0^{T \wedge \tau_A} r(t) \mathrm{d}t} f\left(S_{T \wedge \tau_A}\right) \right] \right| \leq C_1 E(h) + C_2 k,$$

where $h = h(n)$, $k = k(n)$ and

$$(5.16) \qquad E(h) = \begin{cases} h, & \text{in cases O and II,} \\ -h \log h, & \text{in case I.} \end{cases}$$

The proof of Theorem 3 is given in Appendix A.2. The logarithmic factor in the error bound (5.16) in case I is due to the form (5.11)–(5.12) of the singularity of the jump measure at zero in this case (see the proof for details). Note that the discontinuous part of a process satisfying case I is of infinite variation, but has zero $(1 + \epsilon)$-variation for every $\epsilon > 0$.

Numerical experiments, reported in Section 6, suggest that in several pricing models of interest the error of the Markov generator (MG) algorithm described in Section 4 is actually of order h^2. A further theoretical investigation of error bounds and sharp rates of convergence of the algorithm, is left for future research.

| $\sigma_0 = 0.2$, $r = 0.02$ | | | $\sigma_0 = 0.5$, $r = 0.05$ | | | $\sigma_0 = 0.5$, $r = 0.05$ | | |
| $K = 2$, $\ell = 1.5$, $u = 2.5$ | | | $K = 2$, $\ell = 1.5$, $u = 3$ | | | $K = 1.75$, $\ell = 1$, $u = 3$ | | |
GY	KI	MG	GY	KI	MG	GY	KI	MG
0.0411	0.041089	0.041082	0.0178	0.017856	0.017856	0.07615	0.076172	0.076165

TABLE 1. The comparison of double barrier call option prices with strike K obtained in [24] and [39] in the case of geometric Brownian motion. The model is given by SDE (2.7) with the constant volatility function $\sigma(x) := \sigma_0$ and drift $\gamma = r - d$, where the interest rate r is given in the table and the dividend yield equals $d = 0$. The asset price process S starts at $S_0 = 2$ and the maturity in all the cases is $T = 1$ year. The state-space of the approximating chain is defined by the algorithm in Section 4 and the parameters $N = 200$, $x_1 = 0.2, x_N = 10$, $d_1^- = 100, d_1^+ = 1, d_2^- = 10, d_2^+ = 10, d_3^- = 1, d_3^+ = 100$. The computation for the pricing of the barrier products takes about 0.03 seconds (using Matlab(R) 7.8 on Intel(R) Xeon(R) CPU E5430 @ 2.66GHz) for each of the parameter choices in this table.

6. Numerical examples

In this section the Markov chain algorithm is examined numerically in a variety of contexts. Subsection 6.1 contains a numerical examples for the geometric Brownian motion model and a comparison with a binomial tree method for the pricing of barrier options. In subsection 6.2 the algorithm is applied to two cases of Lévy driven SDEs. Further numerical examples, including comparisons of computational times of the various algorithms, can be found in [49].

6.1. **Geometric Brownian motion.** The model is given by SDE (2.7) where the volatility function $\sigma(x) = \sigma_0$ is constant and the drift equals $\gamma = r - d$, where r is the risk free rate and d is the dividend yield. We now compare our algorithm (MG), based on the Markov generator of the approximating chain X, with the results obtained in Geman and Yor [24] and Kunitomo and Ikeda [39]. The numerical results are contained in Table 1.

In Figure 3 the errors and computation times of the Markov generator algorithm (MG) and a trinomial tree are plotted against the number of points, for the computation of the price of a double knock-out barrier call option. Note that the graph of the logarithm of the absolute pricing error for the MG algorithm given in Figure 3 is approximately linear in $\log(N)$ with slope -2, which implies that the error itself is approximately quadratic in $1/N$. The pricing algorithm, given in Ritchken [50, Ex. 7], is based on the trinomial tree and it appears to converge at the rate $1/N$. Figure 3 shows that when the trinomial tree algorithm of Ritchken [50, Ex. 7] is combined with the second order Richardson extrapolation, the convergence appears to be of higher order than in the MG algorithm.

6.2. **Lévy and local Lévy models.**

6.2.1. *The CGMY/KoBoL process.* In this section we assume that the price S is again an exponential Lévy process given by (2.9), where the Lévy processes L is a CGMY process [8] with Lévy density given by the formula

$$(6.1) \qquad k(y) := C \left(\mathbf{I}_{(-\infty,0)} \frac{\mathrm{e}^{-G|y|}}{|y|^{Y+1}} + \mathbf{I}_{(0,\infty)} \frac{\mathrm{e}^{-My}}{|y|^{Y+1}} \right), \qquad \text{where} \quad M > 1, G \geq 0, C > 0, Y < 2.$$
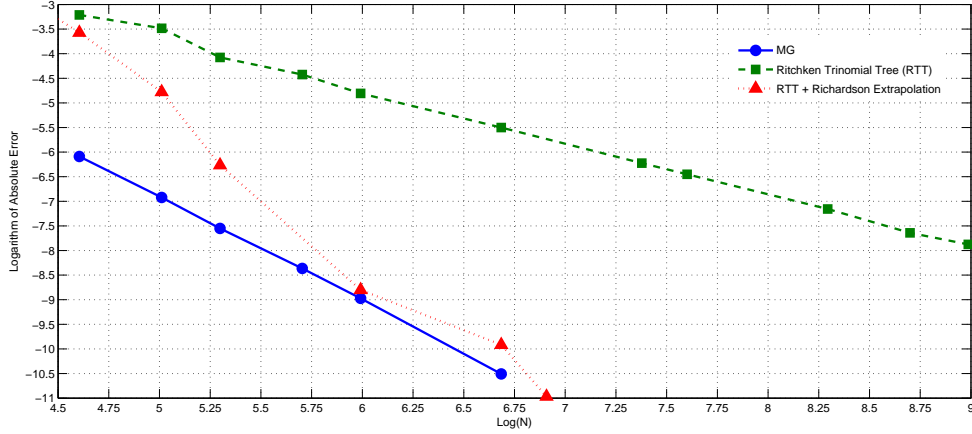
FIGURE 3. This figure depicts the pricing errors for the double barrier knock-out call option ($S_0 = 95$, $K = 100$, $\ell = 90$, $u = 140$, $T = 1$, $r = 0.1$, $\sigma_0 = 0.25$) of the MG algorithm, the trinomial tree method described in Ritchken [50, Ex. 7] and its second order Richardson extrapolation. The true value of this contract used to compute the pricing errors equals 1.4583798 (this was obtained by the MG algorithm with $N = 3000$ states). The figure is on a "log-log" scale where the vertical axes contains the logarithm of the absolute value of the error and the horizontal axis depicts the logarithm of the number of states N in the grid.

The inequality $Y < 2$ is induced by the integrability condition on the Lévy measure at zero and the condition $M > 1$ implies the exponential moment condition (2.10).

Madan and Yor [47] show that the CGMY process L has the same law as the time-changed Brownian motion $\{W_{Z_t} + \theta Z_t\}_{t \geq 0}$ with $\theta = (G - M)/2$ and Lévy subordinator Z that is independent of W and that has Laplace exponent ψ_Z given by

$$(6.2) \qquad \mathbf{E}_0[\mathrm{e}^{-uZ_t}] = \mathrm{e}^{t\psi_Z(u)} = \exp\left(tC\Gamma(-Y)[2r(u)^Y \cos(\eta(u)Y) - M^Y - G^Y]\right), \quad u \geq -\frac{GM}{2},$$

where Γ denotes the Gamma function and the functions r, η are given by the formulae

$$r(u) := \sqrt{2u + GM} \quad \text{and} \quad \eta(u) := \arctan\left(\frac{2\sqrt{2u - \theta^2}}{G + M}\right).$$

We compare our numerical results with those obtained in Boyarchenko and Levendorskii [5] using a Fourier method (see Table 2).

In Figure 4 the error and computation times are plotted of the prices of double-no-touch and double knock-out put options as a function of the grid size. As the true values we took the outcomes of the algorithm for $N = 6400$. The figure appears to suggest that the error is approximately proportional to $N^{-1.2}$ and N^{-2}, respectively, where N is the number of point in the grid.
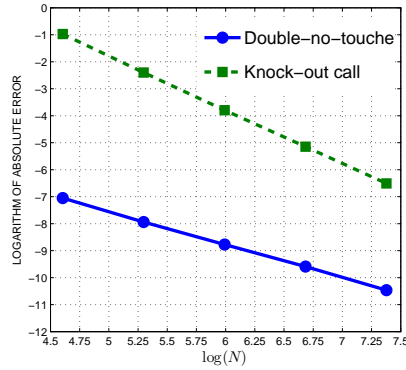
FIGURE 4. The figure displays the logarithm of the absolute pricing errors for the MG algorithm applied to the CGMY model as a function of $\log(N)$, where N is the number of states in the grid. Under consideration are the double no touch and double knock out call options with $K = S_0 = 3500$, lower barrier $\ell = 2800$ and upper barrier $u = 4200$, driven by a CGMY process with parameters as in Table 2. As true values we took the values 78.752 and 0.9508 obtained by running the algorithm with $N = 6400$ points.

Spot	Knock-out put		Double-no touch		Spot	Knock-out put		Double-no touch	
%	BL	MG	BL	MG	%	BL	MG	BL	MG
82	302.28	301.07	0.5778	0.5757	101	64.07	64.53	0.9481	0.9483
85	370.38	370.38	0.8009	0.8004	104	36.96	37.18	0.9351	0.9352
88	341.35	341.78	0.8881	0.8880	107	22.73	22.84	0.9112	0.9113
91	279.86	280.41	0.9280	0.9280	110	14.60	14.65	0.8708	0.8709
94	207.71	208.30	0.9464	0.9465	113	9.61	9.64	0.8020	0.8019
97	136.63	137.24	0.9527	0.9529	116	6.30	6.32	0.6771	0.6767
100	78.19	78.74	0.9506	0.9507	119	3.52	3.54	0.4049	0.4049

TABLE 2. Barrier option prices under the CGMY model. The first column contains the spot price as percentage of 3500. The CGMY parameters are $C = 1$, $G = 9$, $M = 8$, $Y = 0.5$. The resulting risk-neutral drift is $\mu = r - d - \psi_Z(-\theta - 1/2) \approx -0.0423$. Option parameters $K = 3500$ (strike of the put), $\ell = 2800$, $u = 4200$, $r = 0.03$, $d = 0$, $T = 0.1$. The columns BL and MG report the results obtained by Boyarchenko and Levendorskii [5] and by the Markov generator algorithm (with $N = 800$ points) respectively. It takes about 22 seconds to run the MG algorithm for each starting spot price

6.2.2. *A local Lévy model.* The following Lévy driven SDE specifies a Markov process S with local volatility and double-exponential jumps,

$$(6.3) \qquad \frac{\mathrm{d}S_t}{S_{t-}} = (r - d - \lambda\zeta(S_{t-}/S_0)^\beta)\mathrm{d}t + (S_{t-}/S_0)^\beta\mathrm{d}L_t, \qquad t > 0, \qquad \text{where}$$

$$(6.4) \qquad L_t := \sigma_0 W_t + \sum_{i=1}^{N_t}\left(\mathrm{e}^{K_i} - 1\right), \quad S_0, \sigma_0 \in (0, \infty) \quad \text{and} \quad \beta \in \mathbb{R}.$$

The special case of this model for $\beta = 0$ was introduced into the mathematical finance literature by Kou [37]. The random variables K_i, $i \in \mathbb{N}$, are independent of both the Brownian motion W and the

Local Lévy model		MG: $N = 400$	MG: $N = 800$	MG: $N = 1200$	KW
$\beta = 0$	$\lambda = 3$	10.0528	10.0530	10.0530	10.05307
	$\lambda = 0.01$	9.2768	9.2771	9.2772	9.27724
$\beta = -1$	$\lambda = 3$	9.7685	9.7688	9.7688	N/A
	$\lambda = 0.01$	8.9572	8.9575	8.9575	N/A
$\beta = -3$	$\lambda = 3$	9.0185	9.0187	9.0188	N/A
	$\lambda = 0.01$	8.0855	8.0858	8.0858	N/A

TABLE 3.   Up-and-in call option prices under the model S defined in (6.3) and (6.4). The parameters are given by $S_0 = 100$, $r = 5\%$, $d = 0$, $\sigma_0 = 0.2$, $p = 0.3$, $1/\eta_1 = 0.02$, $1/\eta_2 = 0.04$, $\beta \in \{0, -1, -3\}$ and $\lambda \in \{0.01, 3\}$. The strike is $K = 100$ and the upper barrier $u = 120$ while time to maturity $T = 1$. The column KW denotes the results of Kou and Wang (see [38], Table 3) while MG denotes the algorithm based on the Markov generator. The state-space of the approximating chain is defined by the algorithm analogous to the one in Section 4, adapted in an obvious way to a single barrier contract. Its size is $N = n \cdot 400$ for $n = 1, 2, 3$. The computation time is about half a second for $n = 1$, five seconds for $n = 2$ and seventeen seconds for $n = 3$ on the same hardware as in Table 1. We also ran the algorithm for $N = 1600$ and obtained identical results (up to four decimals) as the ones in the column $N = 1200$.

Poisson process N with intensity $\lambda > 0$ and are distributed according to the double exponential density

$$(6.5) \qquad f_K(k) \quad = \quad p\eta_1 e^{-\eta_1 k} \mathbf{I}_{(0,\infty)} + (1-p)\eta_2 e^{\eta_2 k} \mathbf{I}_{(-\infty,0)}, \qquad \text{where}$$

$$\eta_1 > 1, \ \eta_2 > 0 \quad \text{and} \quad p \in [0,1].$$

The parameter ζ is given by

$$\zeta := \mathbf{E}\left[e^{K_1} - 1 \right] = \frac{p\eta_1}{\eta_1 - 1} + \frac{(1-p)\eta_2}{\eta_2 + 1} - 1.$$

If $\beta < 0$ the process S has positive probability of hitting zero in finite time, in which case we take 0 to be absorbing.

It is clear that the model described by (6.3) and (6.4) has a generator of the form given in (2.11) with $\sigma(x) = \sigma_0(x/S_0)^\beta$ and $\mu(x) = \lambda\zeta(x/S_0)^\beta$. The jump measure $\nu(x, \mathrm{d}y)$ in representation (2.11) is supported in $(-1, \infty)$ and in our case by (6.5) takes the explicit form

$$(6.6) \qquad \nu(x, \mathrm{d}y) \quad = \quad (x/S_0)^\beta \lambda \left[p\eta_1(y+1)^{-1-\eta_1} \mathbf{I}_{(0,\infty)} + (1-p)\eta_2(y+1)^{\eta_2-1} \mathbf{I}_{(-1,0)} \right] \mathrm{d}y.$$

Representation (6.6) of the jump measure $\nu(x, \mathrm{d}y)$ of the asset price process S can now be used to construct the "jump" generator Λ_J defined in equations (4.2) and (4.3). Furthermore it is clear from (6.6) that the instantaneous variance term caused by the jumps of the process S in (4.8) is of the form

$$\int_{-1}^{\infty} y^2 \nu(x, \mathrm{d}y) = (x/S_0)^\beta 2\lambda \left(\frac{p}{(\eta_1 - 1)(\eta_1 - 2)} + \frac{1-p}{(\eta_2 + 1)(\eta_2 + 2)} \right) \qquad \text{if} \quad \eta_1 > 2.$$

Note that in the present model the instantaneous variance is finite (cf. condition (4.5)) if $\eta_1 > 2$ which is a condition that is typically satisfied in applications. The numerical results of the MG algorithm applied to an up-and-in call option are contained in Table 3, and are compared in the case $\beta = 0$ with the corresponding results of Kou and Wang [38] (see Table 3 in [38]).
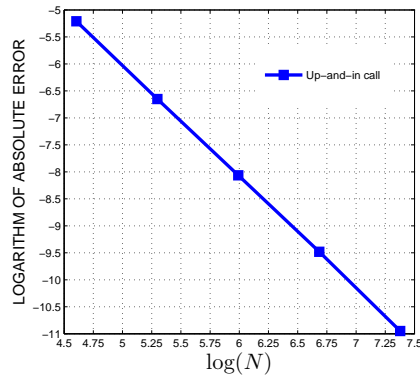
FIGURE 5. The figure displays the logarithm of the absolute pricing errors for the MG algorithm applied to the model in (6.3) as a function of $\log(N)$, where N is the number of states in the grid. The option under consideration is the up and in call option specified in Table 3. We take the value 9.768837, obtained by running the MG algorithm for $N = 5000$, to be the correct prices in model (6.3) given by the parameter values $\lambda = 3$ and $\beta = -1$. In order to ensure that the truncation error is under control we computed the option value with the largest grid point equal to 300 and 600 (for $N = 5000$) and obtained the same price as above in the first six decimal points.

Figure 5 presents the pricing error for the up and in call option described in Table 3. The error of the pricing algorithm is plotted on a log-log-scale against the number of points in the state-space, for the parameter values $\lambda = 3$ and $\beta = -1$. Unlike in the case of double knock-out options where a truncation error can be avoided, a truncation error will be present when pricing a single up-and-in option, as in the example above. To ensure that the truncation error is sufficiently small we ran the algorithm for an increasing sequence of values of x_N, the largest point in the state-space $\mathbb{G}$ of the chain, and observed that the outcomes did not change up to the required accuracy. The slope of the line in Figure 5 is equal to -2, which suggests that also in this case the algorithm is of order h^2 in the mesh size.

7. CONCLUSION

In this paper we presented an algorithm for pricing barrier options in one-dimensional Markovian models based on an approximation by continuous-time Markov chains. The generator of the approximating chain is constructed by the matching of instantaneous moments of the infinitesimal generator of the Markov process in question, on a suitable non-uniform grid. The approximate barrier option prices are then obtained by calculating the corresponding first-passage distributions for the approximating Markov chain.

To illustrate the flexibility of the method we implemented the algorithm for a number of models, including local volatility models with jumps and models with time-dependent jump-distributions (see [49]). In the cases of the diffusion and jump-diffusion models where results had been obtained before in the literature, the algorithm produced outcomes that accurately matched those results, and we numerically investigated the order of decay of the error.

We provided a mathematical proof of the convergence of the outcomes of the algorithm to the true prices and derived error estimates under additional regularity assumptions. We derived a theoretical upper bound for the error of the outcomes produced by the algorithm that is linear in the spatial mesh

size and the truncation error. We showed that an additional logarithmic factor may arise in this error bound when the Lévy density has a pole of order two at the origin. In addition, this bound is also linear in the time mesh size if the model is time-inhomogeneous (see [49]). Numerical experiments suggest that for a number of models the error of the outcomes generated by the algorithm actually decays quadratically in the spatial mesh-size. It would be of interest to establish error bounds under weaker regularity assumptions, and obtain sharp rates of convergence for the specific models, which is a topic left for future research.

Although in principle the method also applies to higher-dimensional Markov processes, the size of the generator matrix would make straightforward application of the algorithm computationally infeasible. The investigation of efficient extensions of the approach to Markov processes of moderate dimension is another topic left for future research.

APPENDIX A. PROOFS

A.1. **Explicit construction of the generator matrices $\Lambda^{(n)}$.** Recall that the generator matrix $\Lambda^{(n)}$ is of the form $\Lambda_D^{(n)} + \Lambda_J^{(n)}$. The action of $\Lambda_D^{(n)}$, which discretizes the operator (5.7), on a function $f : \mathbb{G}^{(n)} \to \mathbb{R}$ is described by

$$(A.1) \qquad \Lambda_D^{(n)} f(x_i) = \begin{cases} \frac{\sigma_i^2}{2} \Delta f(x_i) + (\gamma_i)_+ \nabla^+ f(x_i) + (\gamma_i)_- \nabla^- f(x_i) & i \in \{2, \dots, n-1\}, \\ 0 & i \in \{1, n\}, \end{cases}$$

where $y_\pm = \max\{\pm y, 0\}$ and $\mathbb{G}^{(n)}$ is defined in (5.13). We write x_i instead of $x_i^{(n)}$ and will do so in all that follows. We denote by $\nabla^\pm$ and Δ the first and second order difference operators given by

$$(A.2) \qquad \nabla^- f(x_i) \;=\; \frac{1}{x_i - x_{i-1}}[f(x_i) - f(x_{i-1})],$$

$$(A.3) \qquad \nabla^+ f(x_i) \;=\; \frac{1}{x_{i+1} - x_i}[f(x_{i+1}) - f(x_i)],$$

$$(A.4) \qquad \Delta f(x_i) \;=\; \frac{2}{x_{i+1} - x_{i-1}}[\nabla^+ f - \nabla^- f](x_i),$$

with σ_i^2 and γ_i defined by

$$\sigma_i^2 := \sigma^2(x_i) x_i^2, \qquad \gamma_i := \gamma x_i,$$

where $\sigma^2(x)$ and γ are as in (5.6). Note that the discretization in (A.1) defines a birth-death process also in the case that $\sigma_i = 0$.

The jump part of the generator, given in (5.8), can be rewritten as follows

$$(A.5) \qquad \mathcal{L}_J f(x) = \int_1^\infty [f(x(1+y)) - f(x) - f'(x)xy] g(x,y) \mathrm{d}y$$

$$+ \begin{cases} \int_{-1}^1 \frac{1}{|y|}[f(x(1+y)) - f(x)]|y|g(x,y)\mathrm{d}y - f'(x)x \int_{-1}^1 yg(x,y)\mathrm{d}y, & \mathcal{I}(x) < \infty, \\ \int_{-1}^1 \frac{1}{y^2}[f(x(1+y)) - f(x) - f'(x)xy](y^2 g(x,y))\mathrm{d}y, & \mathcal{I}(x) = \infty. \end{cases}$$

Note that $\mathcal{I}(x)$, defined in (5.10), is either finite for all x or infinite for all x because we are assuming that the process S we are approximating falls into one of the categories O, I, II. The representation (A.5) has a natural discretization, which we will now describe, because the second integral in (A.5) is an integral of a continuous function against a finite measure.

For different levels of activity of the purely discontinuous part of S, quantified by whether or not $\mathcal{I}_i = \mathcal{I}(x_i)$ defined in (5.10) is finite, the matrix $\Lambda_J^{(n)}$ is defined in the following way. Let

$$A_i^0(h) = (-1 + x_{\underline{b}_i}/x_i, -1 + x_{\bar{b}_i}/x_i)$$

where the levels $\underline{b}_i$ and $\bar{b}_i$ are given by

(A.6) $\bar{b}_i = \min\{j > i : x_j \geq x_i + C_+(h)h\}$, $\underline{b}_i = \max\{j < i : x_j \leq x_i - C_-(h)h\}$,

with $h = h(n)$ the spatial mesh size and

(A.7) $C_\pm(h) = d_\pm \cdot c(\alpha_\pm, h)$,   for some constants $d_\pm > \dfrac{\overline{\kappa}_\pm}{\underline{\kappa}_\pm}$,

where $\alpha_\pm$, $\overline{\kappa}_\pm$ and $\underline{\kappa}_\pm$ are the constants given in (5.11) and (5.12), and

(A.8) $c(\alpha, h) = \begin{cases} \dfrac{2 - \alpha}{\alpha - 1}, & \alpha \in (1, 2), \\ -\log(h), & \alpha = 1. \end{cases}$

We can then define

$$c_{ik}^{(p)} = \begin{cases} \int_{\mathbb{R}} |y|^p g(x_i, y) \mathbf{I}_{\left[\frac{x_{k-1}}{x_i} - 1, \frac{x_k}{x_i} - 1\right]}(y) \mathrm{d}y, & k > i, \\ \int_{\mathbb{R}} |y|^p g(x_i, y) \mathbf{I}_{A_i^0(h)}(y) \mathrm{d}y, & k = i, \\ \int_{\mathbb{R}} |y|^p g(x_i, y) \mathbf{I}_{\left[\frac{x_k}{x_i} - 1, \frac{x_{k+1}}{x_i} - 1\right]}(y) \mathrm{d}y, & k < i. \end{cases}$$

Finally we define the discretized version of the generator as

(A.9) $\Lambda_J^{(n)} f(x_i) = \displaystyle\sum_{k: x_k > 2x_i} \left[c_{ik}^{(0)}(f(x_k) - f(x_i)) - x_i \nabla^- f(x_i) c_{ik}^{(1)} \right] + \overline{\Lambda}_J^{(n)} f(x_i), \ i \notin \{1, n\}$,

       $\Lambda_J^{(n)} f(x_i) = 0, \qquad i \in \{1, n\}$.

If $\mathcal{I}_i$ is finite $\overline{\Lambda}_J^{(n)} f(x_i)$ is given by

(A.10) $\overline{\Lambda}_J^{(n)} f(x_i) := \displaystyle\sum_{k \neq i : x_k \leq 2x_i} \frac{x_i}{|x_k - x_i|} \left[f(x_k) - f(x_i) \right] c_{ik}^{(1)} - x_i \alpha_i \widetilde{\nabla} f(x_i)$

where

(A.11) $\widetilde{\nabla} f(x_i) := \begin{cases} \nabla^- f(x_i) & \alpha_i \geq 0 \\ \nabla^+ f(x_i) & \alpha_i < 0 \end{cases}$ where $\alpha_i := \displaystyle\int_{-1}^1 y g(x_i, y) \mathrm{d}y$.

If $\mathcal{I}_i$ is equal to infinity we have

$$\overline{\Lambda}_J^{(n)} f(x_i) := \frac{1}{2} c_{ii}^{(2)} x_i^2 \Delta f(x_i)$$

(A.12) $+ \displaystyle\sum_{k: (x_k - x_i)/x_i \in (-1,1) \backslash A_i^0(h)} \left(\frac{x_i}{x_k - x_i} \right)^2 \left[f(x_k) - f(x_i) - (x_k - x_i) \nabla^0 f(x_i) \right] c_{ik}^{(2)}$.

Here $\nabla^0 f$ is the (central) difference operator given by

(A.13) $\nabla^0 f(x_i) = \dfrac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} \nabla^+ f(x_i) + \dfrac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}} \nabla^- f(x_i)$.

Clearly, the first part in the expression in (A.9) defines the generator matrix of a Markov chain taking values in $\mathbb{G}^{(n)}$. Similarly the expression in (A.10) yields a generator of a Markov chain. In (A.12) the levels $\bar{b}_i$ and $\underline{b}_i$ are chosen so as to ensure that $\Lambda_J^{(n)}$ defined in (A.9) is in fact a generator.

To verify that (A.12) defines a Markov jump process we need to show that in the matrix $\Lambda_J^{(n)}$ defined in (A.9) all off-diagonal elements are positive. It follows from the definitions of Δf and $\nabla^0 f$ that this is the case if the following condition holds for all $x_i \in \widehat{\mathbb{G}}^{(n)}$:

$$(A.14) \qquad x_i c_{ii}^{(2)} \quad \geq \quad \max\{(x_i - x_{i-1})\nu_i, (x_i - x_{i+1})\nu_i\}, \qquad \text{where}$$

$$(A.15) \qquad \nu_i \quad := \quad \sum_{k:(x_k-x_i)/x_i\in(-1,1)\backslash A_i^0(h)} \frac{x_i}{x_k - x_i} c_{ik}^{(2)}.$$

For h sufficiently small it is easily checked from the definition of ν_i that

$$\nu_i \leq \int_{(-1,1)\backslash A_i^0(h)} |y|g(x_i, y)\mathrm{d}y,$$

which implies that (A.14) is satisfied if, for any $x \in [\ell, u]$,

$$x \int_{A_i^0(h)} y^2 g(x,y)\mathrm{d}y \geq h \int_{(-1,1)\backslash A_i^0(h)} |y|g(x,y)\mathrm{d}y.$$

This condition is satisfied if, for any $x \in [\ell, u]$, the following two conditions hold for h sufficiently small

$$(A.16) \qquad x \int_{\left(0, \frac{C_+(h)h}{x}\right)} y^2 g(x,y)\mathrm{d}y \quad \geq \quad h \int_{\left(\frac{C_+(h)h}{x}, 1\right)} yg(x,y)\mathrm{d}y,$$

$$(A.17) \qquad x \int_{\left(-\frac{C_-(h)h}{x}, 0\right)} y^2 g(x,y)\mathrm{d}y \quad \geq \quad h \int_{\left(-1, -\frac{C_-(h)h}{x}\right)} |y|g(x,y)\mathrm{d}y.$$

In view of (5.11) it follows that (A.16) holds if the following inequality is satisfied

$$\underline{\kappa}_+ \int_{(0,C_+(h)h/x)} y^{1-\alpha_+}\mathrm{d}y \geq h\overline{\kappa}_+ \int_{(C_+(h)h/x,1)} y^{-\alpha_+}\mathrm{d}y$$

$$\Leftrightarrow \begin{cases} \dfrac{\underline{\kappa}_+ x}{2 - \alpha_+}\left(\dfrac{C_+(h)h}{x}\right)^{2-\alpha_+} \geq \dfrac{\overline{\kappa}_+ h}{\alpha_+ - 1}\left(\left(\dfrac{C_+(h)h}{x}\right)^{1-\alpha_+} - 1\right) & \text{in case II,} \\[4mm] \underline{\kappa}_+ x\left(\dfrac{C_+(h)h}{x}\right) \geq -\overline{\kappa}_+ h\left(\log(C_+(h)) + \log(h) - \log(x)\right) & \text{in case I.} \end{cases}$$

The latter holds for all h sufficiently small if

$$\begin{cases} C_+(h) \cdot \dfrac{\underline{\kappa}_+}{2 - \alpha_+} \geq \dfrac{\overline{\kappa}_+}{\alpha_+ - 1}, & \text{in case II,} \\[4mm] \liminf_{h\to\infty} C_+(h)/|\log h| > \overline{\kappa}_+/\underline{\kappa}_+, & \text{in case I.} \end{cases}$$

Clearly, $C_+(h)$, defined in (A.7), satisfies this condition. The fact that Condition (A.17) is satisfied follows by a similar line of reasoning.

A.2. Proof of error estimates.

The following lemma is an important auxiliary result for the development of error-estimates. It formalises the intuition that two semi-groups should be "close" if the corresponding infinitesimal generators are "close".

Suppose that S is a Feller process on the state-space $\mathbb{E}$ with associated semigroup P and corresponding infinitesimal generator $\mathcal{L}$, and let $(X^{(n)})_{n\in\mathbb{N}}$ be a sequence of Markov chains with state-spaces $\mathbb{G}^{(n)}$,

semigroups $P^{(n)}$ and corresponding generator matrices $\Lambda^{(n)}$. For any $f \in \mathcal{D}(\mathcal{L})$, the domain of $\mathcal{L}$, consider the following error measure

$$(\text{A.18}) \qquad\qquad \epsilon_n(f) = \left\| \Lambda^{(n)} f - \mathcal{L} f \right\|_n, \qquad s \geq 0,$$

where we write $\Lambda^{(n)} f = \Lambda^{(n)} f_n$ and $P^{(n)} f = P^{(n)} f_n$ with $f_n = f|_{\mathbb{G}^{(n)}}$ and

$$(\text{A.19}) \qquad\qquad \|g\|_n = \sup_{x \in \mathbb{G}^{(n)}} |g(x)|$$

for any $g : \mathbb{E} \to \mathbb{R}$. Then the following estimate holds true:

Lemma 2. *Let $f \in \mathcal{D}(\mathcal{L})$ and $T > 0$, and suppose that there exists a function $h : \mathbb{N} \to \mathbb{R}_+$ and a function $c_f : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $n \in \mathbb{N}$ and $s \geq 0$*

$$\epsilon_n(P_s f) \leq c_f(s) \cdot h(n).$$

Then it holds that

$$\sup_{t \in [0,T]} \left\| P_t f - P_t^{(n)} f \right\|_n \leq \int_0^T c_f(s) \mathrm{d}s \cdot h(n).$$

Proof. The proof closely follows that of Lemma 6.2 in Ethier and Kurtz [20]. Note first that

$$\frac{\partial}{\partial s} \left(P_{t-s}^{(n)} P_s f \right) = \Lambda^{(n)} P_{t-s}^{(n)} P_s f - P_{t-s}^{(n)} \mathcal{L} P_s f$$

$$(\text{A.20}) \qquad\qquad = P_{t-s}^{(n)} \left(\Lambda^{(n)} P_s f - \mathcal{L} P_s f \right),$$

as $P_{t-s}^{(n)}$ and $\Lambda^{(n)}$ commute. Since the function in (A.20) is continuous in s, the fundamental theorem of calculus and the fact that $P_0^{(n)} f = f$ imply that

$$P_t f - P_t^{(n)} f = \int_0^t P_{t-s}^{(n)} \left(\Lambda^{(n)} P_s f - \mathcal{L} P_s f \right) \mathrm{d}s.$$

Thus, the triangle inequality yields that

$$\left\| P_t f - P_t^{(n)} f \right\|_n \leq \int_0^t \left\| P_{t-s}^{(n)} \left(\Lambda^{(n)} P_s f - \mathcal{L} P_s f \right) \right\|_n \mathrm{d}s$$

$$\leq \int_0^t \left\| \Lambda^{(n)} P_s f - \mathcal{L} P_s f \right\|_n \mathrm{d}s$$

$$\leq \int_0^T c_f(s) \mathrm{d}s \cdot h(n)$$

for any $t \in [0,T]$, as $c_f(s)$ is non-negative. $\square$

Before we proceed we introduce some further notation. The integral part of the infinitesimal generator, and its approximation will be denoted by

$$(\text{A.21}) \qquad I f(x) = \int_{(1,\infty)} [f(x(1+y)) - f(x) - xyf'(x)]g(x,y)\mathrm{d}y,$$

$$(\text{A.22}) \qquad I_n f(x_i) = \int_{(1,x_n/x_i-1)} [f_n(x_i(1+y)) - f_n(x_i)]g(x_i,y)\mathrm{d}y - \mu_i x_i \nabla^- f(x_i),$$

where $\mu_i = \int_{(1,\infty)} yg(x_i,y)\mathrm{d}y$, $'$ denotes the derivative with respect to x and

$$f_n(x) = \sum_{x_k \in (2x_i,\infty)} f(x_{k+1}) \mathbf{I}_{\left[\frac{x_k}{x_i}-1, \frac{x_{k+1}}{x_i}-1\right]}(x).$$

Furthermore, we define

$$(A.23) \qquad I^{(k)}f(x) = \int_{(-1,1]} F^{(k)}(x,y)g(x,y)|y|^k \mathrm{d}y,$$

$$(A.24) \qquad I_n^{(k)}f(x_i) = \int_{(-1,1]} F_{n,i}^{(k)}(y)g(x_i,y)|y|^k \mathrm{d}y,$$

where, for $k = 1, 2$,

$$
\begin{aligned}
F^{(k)}(x,y) &= |y|^{-k}[f(x(1+y)) - f(x) - f'(x)xy], \\
F_{n,i}^{(1)}(y) &= -x_i\widetilde{\nabla}f(x_i)\mathrm{sgn}(y) + \sum_{x_j \in [0,2x_i]\setminus\{x_i\}} F_{n,ij}^{(1)}\mathbf{I}_{\left[\frac{x_j}{x_i}-1, \frac{x_{j+1}}{x_i}-1\right]}(y), \\
F_{n,i}^{(2)}(y) &= \sum_{(x_j/x_i-1)\in(-1,1)\setminus A_i^0(h)} \left[F_{n,ij}^{(2)} - \frac{x_i^2}{x_j - x_i}\nabla^0 f(x_i)\right] \mathbf{I}_{\left[\frac{x_j}{x_i}-1, \frac{x_{j+1}}{x_i}-1\right]}(y) \\
&\quad + \frac{x_i^2}{2}\Delta f(x_i)\mathbf{I}_{A_i^0(h)}(y),
\end{aligned}
$$

where $\widetilde{\nabla}$ is defined in (A.11), with

$$
F_{n,ij}^{(1)} = \frac{x_i}{|x_j - x_i|}
\begin{cases}
[f(x_{j+1}) - f(x_i)], & j > i, \\
[f(x_j) - f(x_i)], & j < i,
\end{cases}
$$

and

$$
F_{n,ij}^{(2)} = \left(\frac{x_i}{x_j - x_i}\right)^2
\begin{cases}
[f(x_j) - f(x_i)], & \frac{x_j}{x_i} - 1 \in (-1,0)\setminus A_i^0(h), \\
[f(x_{j+1}) - f(x_i)], & \frac{x_j}{x_i} - 1 \in (0,1)\setminus A_i^0(h).
\end{cases}
$$

Proof of Theorem 3. In view of Assumption 4, it follows that the stopped and discounted process $\widetilde{S}^A = \{\mathrm{e}^{-r(t\wedge\tau_A)}S_{t\wedge\tau_A}\}_{t\geq 0}$ is a Feller process. The infinitesimal generator of the stopped semi-group $\widetilde{P}^A$ corresponding to $\widetilde{S}^A$ is denoted by $\widetilde{\mathcal{L}}^A$. The process $\widetilde{S}^A$ is approximated by the Markov chains with infinitesimal generators $\widetilde{\Lambda}_r^{(n)}$, where $\Lambda^{(n)}$ is described above and $\widetilde{\Lambda}_r^{(n)}$ is obtained from $\Lambda^{(n)}$ by setting the rows of $\Lambda^{(n)}$ corresponding to A equal to zero and subtracting r on the diagonal elements of the rows corresponding to A^c, as in (3.6).

For any f satisfying Assumption 4, $\widetilde{P}_t f$ has compact support as $\widetilde{P}_t f(x) = f(x)$ for all $x \in A$ (recall we take $u < \infty$). Furthermore also by Assumptions 3 and 4 and the form of the Feller process S under consideration $\widetilde{P}_t f$ is contained in the domain $\mathcal{D}$ of the infinitesimal generator $\mathcal{L}$. Since

$$\mathrm{e}^{-r(s\wedge\tau_A)}\widetilde{P}_t f(S_{s\wedge\tau_A}) = E[\mathrm{e}^{-r((s+t)\wedge\tau_A)}f(S_{(s+t)\wedge\tau_A})|\mathcal{F}_s]$$

is a martingale, Dynkin's lemma implies that

$$(\mathcal{L} - r)(\widetilde{P}_t f)(x) = 0, \qquad x \in A^c,$$

so that in particular condition (2.13) holds true with f replaced by $\widetilde{P}_t f$. From Lemma 1 and Assumption 4 we find that $\widetilde{\mathcal{L}}^A(\widetilde{P}_t f)$ is given by (2.12). Recall that, in the case under consideration, $\mathcal{L}f$ is of the form (2.11).

The form of the approximation and the triangle inequality imply that, for $H = \widetilde{P}_s f$, $s \in [0, T]$, the distance $\epsilon'_n(H)$ between $\mathcal{L}$ and $\Lambda^{(n)}$ can be estimated as

$$\epsilon'_n(H) := \left\| \widetilde{\Lambda}_r^{(n)} H - (\widetilde{\mathcal{L}}^A - r) H \right\|_n \leq \frac{1}{2} \|\Sigma^2\| \|\Delta_n H - H''\| + \|\Gamma\| \|\nabla_n H - H'\| + \|IH - I_n H\|$$

$$+ \begin{cases} \|I^{(1)} H - I_n^{(1)} H\| & \text{if} \quad \mathcal{I}_{i,j} < \infty, \\ \|I^{(2)} H - I_n^{(2)} H\| & \text{if} \quad \mathcal{I}_{i,j} = \infty, \end{cases}$$

where $\mathcal{I}_{i,j}$ is defined in (5.10). The norm $\|\cdot\|_n$ was defined in (A.19) and we wrote $\|\cdot\| = \sup_{x \in \widehat{\mathbb{G}}^{(n)}} |\cdot|$ for the supremum over all the points in the grid that lie between the barriers. The functions Σ and Γ are given by $\Sigma^2(x) = \sigma^2(x) x^2$ and $\Gamma(x) = \gamma x$ where $\gamma = r - d$. The operator Δ_n is equal to Δ defined in (A.4) and ∇_n is given by ∇^+ or ∇^- in (A.3) and (A.2) according to whether the constant γ is positive or negative, as in (A.1). We denote by H' the derivative of H with respect to x.

Writing

$$\|\cdot\|_\infty = \sup_{x \in \mathbb{E}} |\cdot|,$$

Assumptions 3 and 4 and second and third order Taylor expansions yield that

$$\|\nabla_n H - H'\| \leq \frac{h(n)}{2} \|H''\|_\infty$$

$$\|\Delta_n H - H''\| \leq \frac{2h(n)}{3} \|H'''\|_\infty.$$

Furthermore, the triangle inequality and Assumptions 3 and 4 yield that, for $h(n)$ sufficiently small, we have

$$\|IH - I_n H\| \leq \max_{x_i \in \widehat{\mathbb{G}}^{(n)}} \left\{ \sum_{x_k \in (2x_i, x_n]} \int_{\left(\frac{x_k}{x_i} - 1, \frac{x_{k+1}}{x_i} - 1 \right)} |H(x_i(1+y)) - H(x_{k+1})| \, g(x_i, y) \mathrm{d}y \right.$$

$$+ \left. \int_{\left(1, 1 + \frac{h(n)}{x_i} \right] \cup \left[\frac{x_n}{x_i} - 1, \infty \right)} H(x_n(1+y)) g(x_i, y) \mathrm{d}y + |\mu_i| x_i \left| H'(x_i) - \nabla^- H(x_i) \right| \right\}$$

$$\leq h(n) \cdot \|H'\|_\infty \cdot \max_{x_i \in \widehat{\mathbb{G}}^{(n)}} \int_{(1, \infty)} g(x_i, y) \mathrm{d}y$$

$$+ \|H\|_\infty \max_{x_i \in \widehat{\mathbb{G}}^{(n)}} \int_{\left(1, 1 + \frac{h(n)}{x_i} \right] \cup \left[\frac{x_n}{x_i} - 1, \infty \right)} g(x_i, y) \mathrm{d}y + \frac{h(n)}{2} \|H''\|_\infty \max_{x_i \in \widehat{\mathbb{G}}^{(n)}} |\mu_i| x_i$$

$$\leq C' \|H'\|_\infty \cdot h(n) + \|H\|_\infty \cdot \left(k(n) + \frac{h(n)}{\ell} \cdot \mathrm{const} \right) + \frac{h(n)}{2} \|H''\|_\infty C' u,$$

where C' is given in Assumption 3, $\mu_i = \int_{(1, \infty)} y g(x_i, y) \mathrm{d}y$, and $h(n)$ and $k(n)$ are the spatial mesh size and tail-mass defined in (5.14) and (5.15), and $\mathtt{const}$ is a bound for g on $[\ell, u] \times [1, 2]$. In case O the triangle inequality yields that, for n sufficiently large,

$$\|I^{(1)} H - I_n^{(1)} H\| \leq \widetilde{C}_0 \left(\|H''\|_\infty + \|H'\|_\infty \right) h(n)$$

for some positive constant $\widetilde{C}_0$. Indeed,

$$(A.25) \quad \|I^{(1)}H - I_n^{(1)}H\|$$

$$\leq \max_{x_i \in \widehat{\mathbb{G}}^{(n)}} \left\{ \sum_{x_k \in [0,2x_i] \setminus \{x_i\}} \int_{\left(\frac{x_k}{x_i}-1, \frac{x_{k+1}}{x_i}-1\right)} \left| F^{(1)}(x_i, y) - F^{(1)}\left(x_i, \frac{x_k}{x_i} - 1\right) \right| g(x_i, y) y \mathrm{d}y \right.$$

$$+ \quad x_i \left| \int_{(-1,1]} y g(x_i, y) \mathrm{d}y \right| \cdot \left| H'(x_i) - \widetilde{\nabla} H(x_i) \right|$$

$$+ \quad \left. \int_{\left(0, \frac{x_{i+1}}{x_i}-1\right)} \left| F^{(1)}(x_i, y) - x_i \widetilde{\nabla} H(x_i) \right| g(x_i, y) y \mathrm{d}y \right\}$$

$$=: \quad \max_{x_i \in \widehat{\mathbb{G}}^{(n)}} \left\{ J_1(x_i) + J_2(x_i) + J_3(x_i) \right\}.$$

In view of Assumptions 3 and 4 the three terms in (A.26) can be estimated as follows:

$$J_1(x_i) \leq \sum_{x_k \in (0,2x_i], k \neq i} \int_{\left(\frac{x_k}{x_i}-1, \frac{x_{k+1}}{x_i}-1\right)} \left[\sup_{y \in \left(\frac{x_k}{x_i}-1, \frac{x_{k+1}}{x_i}-1\right)} \left| \frac{\partial F}{\partial y}(x_i, y) \right| \frac{x_{k+1} - x_k}{x_i} \right] g(x_i, y) y \mathrm{d}y$$

$$\leq \sum_{x_k \in (0,2x_i], k \neq i} \int_{\left(\frac{x_k}{x_i}-1, \frac{x_{k+1}}{x_i}-1\right)} g(x_i, y) y \mathrm{d}y \cdot \|H''\|_\infty \cdot x_i \cdot h(n)$$

$$\leq \frac{3u}{2} \cdot C' \cdot \|H''\|_\infty \cdot h(n),$$

using that $|\frac{\partial F^{(1)}}{\partial y}(x, y)| \leq \frac{3x^2}{2}\|H''\|_\infty$, by a Taylor expansion. Furthermore, for $h(n)$ sufficiently small, we have

$$J_2(x_i) \leq u \cdot \frac{h(n)}{2} \cdot \|H''\|_\infty \int_{(-1,1)} g(x_i, y)|y| \mathrm{d}y$$

$$\leq \frac{u}{2} \cdot \|H''\|_\infty C_0 \cdot h(n),$$

$$J_3(x_i) \leq \frac{x_{i+1} - x_i}{x_i} \left(\frac{x_i^2}{2} \|H''\|_\infty + x_i \|H'\|_\infty \right) \int_{\left(0, \frac{x_{i+1}}{x_i}-1\right)} y g(x_i, y) \mathrm{d}y$$

$$\leq h(n) \cdot \left(\frac{u}{2} \|H''\|_\infty + \|H'\|_\infty \right) \cdot C_0,$$

where we used that for $y \in (x_i, x_{i+1})$, $|F^{(1)}(x_i, y)| \leq \frac{x_i^2}{2}\|H''\|_\infty$, by a Taylor expansion.

In cases I and II,

$$\|I^{(2)}H - I_n^{(2)}H\|$$

$$\leq \max_{x_i \in \widehat{\mathbb{G}}^{(n)}} \left\{ \sum_{x_k \in (-1,1) \setminus A_i^0(h)} \int_{\left(\frac{x_k}{x_i}-1, \frac{x_{k+1}}{x_i}-1\right)} \left| F^{(2)}(x_i, y) - F^{(2)}(x_i, \tfrac{x_k}{x_i}-1) \right| g(x_i, y) y^2 \mathrm{d}y \right.$$

$$+ \sum_{x_k \in (-1,1) \setminus A_i^0(h)} \frac{x_i^2}{|x_k - x_i|} \left| H'(x_i) - \nabla^0 H(x_i) \right| \cdot \int_{\left(\frac{x_k}{x_i}-1, \frac{x_{k+1}}{x_i}-1\right)} y^2 g(x_i, y) \mathrm{d}y$$

$$+ \left. \int_{A_i^0(h)} \left| F^{(2)}(x_i, y) - \tfrac{x_i^2}{2} \Delta H(x_i) \right| g(x_i, y) y^2 \mathrm{d}y \right\}$$

$$(A.26) \quad =: \max_{x_i \in \widehat{\mathbb{G}}^{(n)}} \left\{ K_1(x_i) + K_2(x_i) + K_3(x_i) \right\}.$$

In view of Assumptions 3 and 4 the three terms in (A.26) can be estimated as follows:

$$K_1(x_i)$$

$$\leq \sum_{(x_k/x_i-1) \in (-1,1) \setminus A_i^0(h)} \frac{x_{k+1} - x_k}{x_i} \cdot \max_{y \in \left(\frac{x_k}{x_i}-1, \frac{x_{k+1}}{x_i}-1\right)} \left| \frac{\partial F^{(2)}}{\partial y}(x_i, y) \right| \cdot \int_{\left(\frac{x_k}{x_i}-1, \frac{x_{k+1}}{x_i}-1\right)} g(x_i, y) y^2 \mathrm{d}y$$

$$\leq \frac{5}{6\ell} \cdot C'' \cdot \|H'''\|_\infty \cdot h(n),$$

where C'' is a constant that satisfies $C'' \leq 3C'$ for h sufficiently small and the bound

$$\frac{\partial F^{(2)}}{\partial y}(x, y) = \frac{\partial}{\partial y} \frac{H(x(1+y)) - H(x) - H'(x)xy}{y^2} \leq \frac{5x^3}{6} \cdot \max_{z \in [1 \wedge (1+y), 1 \vee (1+y)]} |H'''(xz)|$$

holds. Furthermore we find

$$K_2(x_i) \leq x_i^2 \cdot \|H'''\|_\infty \cdot \sum_{(x_k/x_i-1) \in (-1,1) \setminus A_i^0(h)} \left| \frac{(x_{i+1}-x_i)(x_i-x_{i-1})}{x_k-x_i} \right| \int_{\left(\frac{x_k}{x_i}-1, \frac{x_{k+1}}{x_i}-1\right)} g(x_i, y) y^2 \mathrm{d}y$$

$$\leq u^2 \|H'''\|_\infty \cdot C'' \cdot h(n).$$

Note that the equality

$$F^{(2)}(x_i, y) - \tfrac{x_i^2}{2} \Delta H(x_i) = \frac{x_i^2}{2} \left[H''(x_i) - \Delta H(x_i) \right] + \frac{x_i^3 y}{6} H'''(\xi_y x_i),$$

for some $\xi_y \in [1 \wedge (1+y), 1 \vee (1+y)]$, implies that

$$K_3(x_i) \leq \frac{x_i^3}{3!} \cdot \|H'''\|_\infty \cdot \int_{A_i^0(h)} |y|^3 g(x_i, y) \mathrm{d}y \leq \frac{u^2}{3!} \cdot C' \cdot \|H'''\|_\infty \cdot h(n)(C_+(h(n)) + C_-(h(n)) + 2).$$

Putting everything together we find that

$$(A.27) \quad \epsilon_n'(H) \leq h(n) A_0 \left[\|H'''\|_\infty + \|H''\|_\infty + \|H'\|_\infty + \|H\|_\infty \right] + 2k(n) \|H\|_\infty$$

$$+ h(n) \begin{cases} A_1, & \text{cases 0 and II,} \\ A_2 + A_3 |\log(h(n))|, & \text{case I,} \end{cases}$$

for some positive constants A_i, $i = 0, \ldots, 3$. All supremum norms in (A.27) are finite, since they are equal to the maxima of continuous functions over compact sets. Combining (A.27) with Lemma 2, which is applicable in view of the assumptions, the theorem follows. $\qquad \square$

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