COUPLING AND TRACKING OF REGIME-SWITCHING MARTINGALES

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Abstract. This paper describes two explicit couplings of standard Brownian motions B and V, which naturally extend the mirror coupling and the synchronous coupling and respectively maximise and minimise (uniformly over all time horizons) the coupling time and the tracking error of two regime-switching martingales. The generalised mirror coupling minimises the coupling time of the two martingales while simultaneously maximising the tracking error for all time horizons. The generalised synchronous coupling maximises the coupling time and minimises the tracking error over all co-adapted couplings. The proofs are based on the Bellman principle. We give counterexamples to the conjectured optimality of the two couplings amongst a wider classes of stochastic integrals.

1. Introduction

Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathsf{P})$ be a filtered probability space that supports a standard $(\mathcal{F}_t)$-Brownian motion $B = (B_t)_{t\geq 0}$ and let

$$\mathcal{V} := \{V = (V_t)_{t\geq 0} \,:\, V \text{ is an } (\mathcal{F}_t)\text{-Brownian motion with } V_0 = 0\}$$

be the set of all $(\mathcal{F}_t)$-Brownian motions on this probability space. It is well-known that for any time horizon $T > 0$ the Brownian motion in $\mathcal{V}$ which minimises the probability that the processes $X = x+B$ and $Y(V) = y+V$ couple after time T (for any starting points $x, y \in \mathbb{R}$), i.e. the Brownian motion that solves the problem

$$\text{minimise} \quad \mathsf{P}\left[\tau_0(X - Y(V)) > T\right] \qquad \text{over } V \in \mathcal{V},$$

where $\tau_0(X-Y(V)) := \inf\{t \geq 0 : X_t = Y_t(V)\}$, is given by the *mirror coupling* $V = -B$ (see e.g. [5]). Furthermore it is easy to see that the Brownian motion which minimises the tracking error of $Y(V)$ with respect to the target X at time T, i.e. solves

$$\text{minimise} \quad \mathsf{E}\left[(X_T - Y_T(V))^2\right] \qquad \text{over } V \in \mathcal{V},$$

is given by the *synchronous coupling* $V = B$. This paper investigates the following generalisations of these questions.

2010 *Mathematics Subject Classification.* 60H05, 60J27, 93E20.

Key words and phrases. generalised mirror and synchronous coupling of Brownian motion, coupling time and tracking error of regime-switching martingales, Bellman principle, continuous-time Markov chains, stochastic integrals.

We would like to thank Peter Bank, David Hobson and Dimitry Kramkov for useful comments. We are grateful to two anonymous referees whose suggestions and comments greatly improved the paper.

1.1. **Problems.** Let $Z = (Z_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$-Feller process, i.e. a Feller process on our probability space, which is $(\mathcal{F}_t)$-Markov. Let the state space $\mathbb{E}$ of Z be a subset of a Euclidean space $\mathbb{R}^d$ for some $d \in \mathbb{N}$. For real Borel measurable functions $\sigma_i : \mathbb{E} \to \mathbb{R}$, $i = 1, 2$, define the stochastic integrals $X = (X_t)_{t\geq 0}$ and $Y(V) = (Y_t(V))_{t\geq 0}$ by

$$X_t := x + \int_0^t \sigma_1(Z_s)\, \mathrm{d}B_s \quad \text{and} \quad Y_t(V) := y + \int_0^t \sigma_2(Z_s)\, \mathrm{d}V_s, \tag{1.1}$$

where $x, y \in \mathbb{R}$ and $V \in \mathcal{V}$. Throughout the paper we assume that for each starting point the process Z is a semimartingale (in particular, it is non-explosive and has cádlág paths) and

$$\mathsf{E} \int_0^t \sigma_i^2(Z_s)\mathrm{d}s < \infty \qquad \text{for all} \qquad t > 0,\ i = 1, 2. \tag{1.2}$$

This implies that the processes X and $Y(V)$ in (1.1) are well-defined true martingales (e.g. see [11, Cor IV.1.25]). In the case the state space $\mathbb{E}$ of Z is embedded in a multidimensional space, a natural choice for the volatility functions σ_1 and σ_2 are the projections resulting in $\sigma_1(Z)$ and $\sigma_2(Z)$ being coordinate processes of Z in $\mathbb{R}^d$. Furthermore, to avoid degenerate situations, we assume throughout the paper that $(|\sigma_1| + |\sigma_2|)(z) > 0$ for all $z \in \mathbb{E}$. The class of stochastic integrals in (1.1), with the integrand Z typically a jump-diffusion (i.e. a Feller process), arises frequently and is of interest in the theory and practice of mathematical finance in the guise of stochastic volatility models (see e.g. [3]).

We are interested in the "distance" between the two processes X and $Y(V)$ for any $V \in \mathcal{V}$. In other words we seek to understand how large and small the following quantities can be

$$\mathsf{E}\left[\phi(X_T - Y_T(V))\right] \qquad \text{and} \qquad \mathsf{P}\left[\tau_0(X - Y(V)) > T\right], \tag{1.3}$$

for $T > 0$ a fixed time horizon,

(1.4) $\phi : \mathbb{R} \to \mathbb{R}$ a convex function satisfying $|\phi(x)| \leq a|x|^p + b$ for some $a, b > 0$, $p \geq 2$ and $\forall x \in \mathbb{R}$,

and $\tau_0(X - Y(V)) := \inf\{t \geq 0 : X_t = Y_t(V)\}$ the *coupling time* of the processes X and $Y(V)$. Since V is an arbitrary $(\mathcal{F}_t)$-Brownian motion, the law of the difference $X - Y(V)$ is in general not easy to describe. Therefore we cannot expect to be able to identify the quantities in (1.3) explicitly. Our goal is to establish sharp upper and lower bounds for the expectations in (1.3), which hold for any choice of Brownian motion $V \in \mathcal{V}$ *and* are based on a natural generalisations of the mirror and synchronous couplings of Brownian motions described in Section 1.2. More precisely, we are looking for Brownian motions $V^M, V^S \in \mathcal{V}$ such that the following inequalities hold for all $V \in \mathcal{V}$:

$$\textbf{(T)} \qquad \mathsf{E}\left[\phi(X_T - Y_T(V^S))\right] \leq \mathsf{E}\left[\phi(X_T - Y_T(V))\right] \leq \mathsf{E}\Big[\phi(X_T - Y_T(V^M))\Big],$$

$$\textbf{(C)} \quad \mathsf{P}\left[\tau_0(X - Y(V^M)) > T\right] \leq \mathsf{P}\left[\tau_0(X - Y(V)) > T\right] \leq \mathsf{P}\Big[\tau_0(X - Y(V^S)) > T\Big],$$

where the generalised mirror (resp. synchronous) coupling holds for B and V^M (resp. V^S).

In Problems **(T)** and **(C)**, the goal is not merely to prove the existence in an abstract sense of the integrators $V^M, V^S \in \mathcal{V}$, *but* primarily to understand for which classes of $(\mathcal{F}_t)$-Feller processes Z are the generalised mirror and synchronous couplings of Brownian motions, described in Section 1.2, extremal in the inequalities of Problems **(T)** and **(C)**. In particular, for the volatility processes Z with the property that the generalised mirror and synchronous couplings satisfy the inequalities above for all Brownian motions $V \in \mathcal{V}$, the following holds: maximising the coupling time of the stochastic integrals minimises the "convex distance" of the two processes and vice versa uniformly over all time horizons $T > 0$.

1.2. **Results.** In the setting of processes (1.1), it is natural to define generalised synchronous and mirror couplings of Brownian motions in the following way. Let the functions $\hat{c}_I, \hat{c}_{II} : \mathbb{E} \to \mathbb{R}$ be given by the formulae

$$\hat{c}_I(z) := \operatorname{sgn}(\sigma_1(z)\sigma_2(z)), \qquad \hat{c}_{II}(z) := -\operatorname{sgn}(\sigma_1(z)\sigma_2(z))$$

for any $z \in \mathbb{E}$, and define the Brownian motions $V^I = (V_t^I)_{t\geq 0}$ and $V^{II} = (V_t^{II})_{t\geq 0}$ in $\mathcal{V}$ by

$$V_t^I := \int_0^t \hat{c}_I(Z_s)\,\mathrm{d}B_s \qquad \text{and} \qquad V_t^{II} := \int_0^t \hat{c}_{II}(Z_s)\,\mathrm{d}B_s. \tag{1.5}$$

Note that $\hat{c}_{II} = -\hat{c}_I$ and hence $V^{II} = -V^I$. It is clear from (1.5) that B and V^I generalise the synchronous coupling of Brownian motions, while the pair B and V^{II} extends the notion of the mirror coupling. A natural conjecture, based on the case where X and $Y(V)$ are Brownian motions, goes as follows.

Conjecture. For any $(\mathcal{F}_t)$-Feller process Z and $V \in \mathcal{V}$, the inequalities in **(T)** and **(C)** are satisfied by $V^S = V^I$ and $V^M = V^{II} = -V^I$.

1.2.1. *The conjecture fails in the class of general $(\mathcal{F}_t)$-Feller processes.* Let the Feller process Z, with state space $\mathbb{E} := (0,\infty)$, be defined as

$$Z_t := z_0 M_t, \qquad \text{where } M_t := \exp(B_t - t/2) \text{ and } z_0 > 0, \tag{1.6}$$

and the volatility functions $\sigma_1, \sigma_2 : \mathbb{E} \to \mathbb{R}$ given by $\sigma_i(z) := -iz$ for any $z \in \mathbb{E}$ and $i = 1, 2$. The corresponding candidate extremal Brownian motions V^I and V^{II}, defined in (1.5), are in this case given by the classical synchronous $V^I = B$ and mirror $V^{II} = -B$ couplings. The fact that $M_t = 1 + \int_0^t M_s \mathrm{d}B_s$ yields $\int_0^t \sigma_i(Z_s)\mathrm{d}B_s = -iz_0(M_t - 1)$, for $i = 1, 2$, which in particular implies the following for all $t \geq 0$:

$$X_t - Y_t(V^I) = x - y + z_0(M_t - 1) \quad \text{and} \quad X_t - Y_t(V^{II}) = x - y - 3z_0(M_t - 1). \tag{1.7}$$

Fix a time horizon $T > 0$ and note that, since (1.7) implies the supports of the random variables $X_T - Y_T(V^I)$ and $X_T - Y_T(V^{II})$ are given by

$$\operatorname{supp}\left(X_T - Y_T(V^I)\right) = (x - y - z_0, \infty) \quad \text{and} \quad \operatorname{supp}\left(X_T - Y_T(V^{II})\right) = (-\infty, x - y + 3z_0),$$

any non-negative non-zero convex function $\phi : \mathbb{R} \to \mathbb{R}$ that satisfies the assumptions in (1.4), with support (i.e. the closure of $\phi^{-1}(0,\infty)$) contained in the half-line $(x - y + 3z_0, \infty)$, clearly yields

$$0 = \mathsf{E}\left[\phi\left(X_T - Y_T(V^{II})\right)\right] < \mathsf{E}\left[\phi\left(X_T - Y_T(V^I)\right)\right].$$

Hence the tracking part of the conjecture fails for $Z = z_0 M$.

Assume that the starting points in (1.1) satisfy $x - y < -3z_0$ and define the stopping time $\tau := \inf\{t \geq 0 : B_t - t/2 = \log(1 - (x - y)/z_0)\}$. Note that the representations in (1.7) imply $\mathsf{P}\left[\tau_0(X - Y(V^{II})) = \infty\right] = 1$ and $\mathsf{P}\left[\tau_0(X - Y(V^I)) > T\right] = \mathsf{P}[\tau > T] < 1$ for any time horizon $T > 0$. Therefore the coupling part of the conjecture also fails:

$$\mathsf{P}\left[\tau_0(X - Y(V^I)) > T\right] < \mathsf{P}\left[\tau_0(X - Y(V^{II})) > T\right] = 1.$$

1.2.2. *The generalised mirror and synchronous couplings are optimal if Z is a continuous-time Markov chain.* Unless otherwise stated, in the rest of the paper Z denotes an $(\mathcal{F}_t)$-Markov semimartingale with a countable state space. More precisely, we assume that

(1.8) Z is a non-explosive, irreducible, cádlág $(\mathcal{F}_t)$-Markov process on a discrete space $\mathbb{E} \subset \mathbb{R}^d$.

Assumption (1.8) makes $\mathbb{E}$ a countable set (i.e. the cardinality of $\mathbb{E}$ is at most that of $\mathbb{N}$) and Z a continuous-time $(\mathcal{F}_t)$-Markov chain on $\mathbb{E}$. The following assumptions on the semigroup P of the volatility chain Z implies that the expectations in the inequalities of **(T)** are finite (see Section 3):

$$\forall z \in \mathbb{E} : \quad (P_T(|\sigma_1|^p + |\sigma_2|^p))(z) < \infty. \tag{1.9}$$

Theorem 1.1. *Let a Markov chain Z satisfy* (1.2), (1.8) *and* (1.9) *and ϕ be as in* (1.4). *Then*

$$\mathsf{E}\left[\phi(X_T - Y_T(V^I))\right] \quad \leq \quad \mathsf{E}\left[\phi(X_T - Y_T(V))\right] \quad \leq \quad \mathsf{E}\left[\phi(X_T - Y_T(V^{II}))\right] \quad \text{for any } V \in \mathcal{V}.$$

The integrability condition in (1.9) is not necessary for the solution of Problem **(C)**.

Theorem 1.2. *Let an $(\mathcal{F}_t)$-Markov chain Z satisfy* (1.2) *and* (1.8). *Then*

$$\mathsf{P}\left[\tau_0(X - Y(V^{II})) > T\right] \leq \mathsf{P}\left[\tau_0(X - Y(V)) > T\right] \leq \mathsf{P}\left[\tau_0(X - Y(V^I)) > T\right] \quad \text{for any } V \in \mathcal{V}.$$

Remarks.
- (i) The function $\hat{c}_I = -\hat{c}_{II}$, and hence the Brownian motions $V^I = -V^{II}$, that feature in the solution of Problems **(T)** and **(C)** depend neither on the maturity T nor on the precise form of the convex cost function ϕ. No local regularity (e.g. differentiability) of ϕ is required for Theorem 1.1 to hold. Note also that essentially no restriction on the volatility functions σ_1 and σ_2 in the stochastic integrals in (1.1) is necessary, for the two theorems to hold. Furthermore, the assumptions in Theorems 1.1 and 1.2 place no restrictions on the filtration $(\mathcal{F}_t)_{t\geq 0}$; in particular $(\mathcal{F}_t)_{t\geq 0}$ need not be generated by the processes B and Z.
- (ii) Brownian motion V^I (resp. V^{II}) is chosen to minimise (resp. maximise) at each moment in time the Radon-Nikodym derivative of the quadratic variation of the process $X - Y(V)$ over the set $\mathcal{V}$. It is clear that V^I and V^{II} can also be defined for much more general integrands than the ones considered in (1.1) and that the generalisations will still be locally extremal.
- (iii) Section 3.2 shows that local maximisation/minimisation of the Radon-Nikodym derivative mentioned in item (ii) is also globally optimal in a non-Markovian setting in the special case of the quadratic tracking (i.e. where the cost function is $\phi(x) = x^2$). Section 4.3 establishes a coupling result, analogous to Theorem 1.2, in the case where the volatility processes are time-inhomogeneous but deterministic. However, Sections 1.2.1 and 5.3 show that the generalisations of Theorems 1.1 and 1.2 do not hold for general $(\mathcal{F}_t)$-Feller processes.
- (iv) The key fact, established in Lemma 2.3, that enables us to prove Theorems 1.1 and 1.2 is that the chain Z is in fact independent of the driving Brownian motion B (see Section 2.3). It is therefore natural to ask whether the results in the theorems above hold for a general $(\mathcal{F}_t)$-Feller process Z, which is independent of B. The example in Section 5.3 shows that Theorem 1.1 cannot be generalised even if such independence is assumed.
- (v) The results in Theorems 1.1 and 1.2 are likely to remain valid in the generalised setting given by the filtered space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathsf{P})$ supporting an additional filtration $(\mathcal{G}_t)_{t\geq 0}$, such that $\mathcal{F}_t \subseteq \mathcal{G}_t$

for $t \geq 0$, with properties that every Brownian motion in $V \in \mathcal{V}$ is also a $(\mathcal{G}_t)$-Brownian motion and the continuous $(\mathcal{G}_t)$-Feller process Z is independent of any $V \in \mathcal{V}$. These conditions are satisfied for example by $\mathcal{G}_t := \mathcal{F}_t \otimes \mathcal{H}_t$, where the filtration $(\mathcal{H}_t)_{t\geq 0}$ is independent of $(\mathcal{F}_t)_{t\geq 0}$ and supports a continuous $(\mathcal{H}_t)$-Feller (and hence $(\mathcal{G}_t)$-Feller) process Z, e.g. Z is a stochastic volatility process (i.e. a solution of an SDE) driven by an $(\mathcal{H}_t)$-Brownian motion. The reason why such a generalisation is likely to remain true lies in the fact that the representation in (2.3) still holds in this setting and the continuity of the paths of the process Z could be used to perform the necessary localisations in the proofs of Theorems 1.1 and 1.2. Note that by Lemma 2.3 the setting of the paper is given by $\mathcal{G}_t := \mathcal{F}_t$ and Z a continuous-time $(\mathcal{F}_t)$-Markov chain.[1]

(vi) The volatility functions σ_1 and σ_2 are typically distinct, which makes the maximal coupling time $\tau_0(X - Y(V^I))$ finite. Hence the upper bound in Theorem 1.2 is non-trivial (i.e. smaller than 1).

(vii) Recall that $\text{sgn}(x)$ is 1 if $x > 0$ and -1 if $x < 0$. In the setting of Theorems 1.1 and 1.2, the choice of $\text{sgn}(0)$ in $\{1, -1\}$ can be arbitrary, since by [11, Prop IV.1.13] it influences neither the laws of the processes $\phi(X - Y(V^I))$, $\phi(X - Y(V^{II}))$ nor of the variables $\tau_0(X - Y(V^I))$, $\tau_0(X - Y(V^{II}))$.

(viii) In [1] the authors establish an inequality, analogous to the first inequality of Theorem 1.1, in the case X and $Y(V)$ are solutions of driftless SDEs. A related inverse question to the tracking problem is studied in [9]. A general reference on the theory of coupling is given in [8].

(ix) The seminal paper [4] introduced regime-switching models to economics and finance. Since then, regime-switching models have found a plethora of applications in areas as diverse as macroeconomics, term-structure modelling and option pricing (see e.g. [7] and the references therein).

1.3. **Structure of the paper.** Sections 2.1 and 2.2 state two well-known lemmas that allow us to relate the coupling inequalities above to problems in stochastic control. Section 2.3 proves that the $(\mathcal{F}_t)$-Markov chain Z and the Brownian motion B are independent. Sections 3 and 3.1 prove Theorem 1.1. Section 3.2 discusses Problem **(T)** in a non-Markovian setting and establishes a generalisation of Theorem 1.1 in the case of a quadratic cost function. In Sections 4, 4.1 and 4.2, we establish Theorem 1.2. Section 4.3 proves an analogue of Theorem 1.2 in the case the volatility processes are time-inhomogeneous but deterministic. Section 5 discusses four counterexamples to the Conjecture above in the case where certain assumptions of Theorems 1.1 and 1.2 are violated. Appendix A contains the proofs of Lemmas 2.1 and 2.2. of Section 2.

2. Preliminaries

2.1. **The set of Brownian motions on a probability space.** Without loss of generality we may assume that the probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathsf{P})$, where the $(\mathcal{F}_t)$-Brownian motion B and the chain Z in (1.1) are defined, supports a further $(\mathcal{F}_t)$-Brownian motion $B^\perp \in \mathcal{V}$, which is independent of B. If this were not the case, we could enlarge the probability space and note that this only increases the set $\mathcal{V}$ of all $(\mathcal{F}_t)$-Brownian motions. Since the extremal Brownian motions V^I, V^{II} in Problems **(T)** and **(C)** are constructed from B and Z alone, they must also be extremal in the original problem. We shall henceforth assume that $B^\perp \in \mathcal{V}$ exists. Any $V \in \mathcal{V}$ and the process $X - Y(V)$, which plays a key role in all that follows, therefore possess the following representation.

[1] We thank one of the referees for this remark.

Lemma 2.1. *For any $V \in \mathcal{V}$ there exist $(\mathcal{F}_t)$-Brownian motion $W \in \mathcal{V}$ and $C = (C_t)_{t\geq 0}$, such that W and B are independent, C is progressively measurable with $-1 \leq C_t \leq 1$ for all $t \geq 0$ P-a.s., and the following representations hold:*

$$V_t = \int_0^t C_s \, \mathrm{d}B_s + \int_0^t (1 - C_s^2)^{1/2} \, \mathrm{d}W_s, \tag{2.1}$$

and $X - Y(V) = R(V)$, where $R(V) = (R_t(V))_{t\geq 0}$ is given by $R_0(V) = r := x - y$ and

$$R_t(V) := r + \int_0^t (\sigma_1(Z_s) - C_s \sigma_2(Z_s)) \, \mathrm{d}B_s - \int_0^t (1 - C_s^2)^{1/2} \sigma_2(Z_s) \, \mathrm{d}W_s. \tag{2.2}$$

Remarks. (i) Equality (2.1) in Lemma 2.1 is a well-known representation for a Brownian motion $V \in \mathcal{V}$ in terms of B (see e.g. [1] and the references therein). For completeness and because of the importance of the representation in (2.2), which follows directly from (2.1), the proof of Lemma 2.1 is given in the appendix (see Section A.1); it is this proof that requires the existence of $B^\perp \in \mathcal{V}$ independent of B.

(ii) Note that W and B in Lemma 2.1 are independent, but the process C may depend on either (or both) Brownian motions B, W.

2.2. **Q-matrices, related operators and martingales.** Let Q denote the Q-matrix of the continuous-time $(\mathcal{F}_t)$-Markov chain Z. We define the action of Q on the space of bounded functions on $\mathbb{E}$ in the standard way: for a bounded $g : \mathbb{E} \to \mathbb{R}$, let

$$Qg : \mathbb{E} \to \mathbb{R} \qquad \text{be given by the formula} \quad (Qg)(z) := \sum_{z' \in \mathbb{E}} Q(z, z') g(z'),$$

since the series converges absolutely for every $z \in \mathbb{E}$.

Let the function $H : \mathbb{E} \times \mathbb{R} \to \mathbb{R}$ satisfy the assumptions: $H(\cdot, z) \in \mathcal{C}^2(\mathbb{R})$ and $H(r, \cdot) : \mathbb{E} \to \mathbb{R}$ is bounded for any $r \in \mathbb{R}$. Then, for any $c \in [-1, 1]$, we define $\mathcal{L}^c H : \mathbb{E} \times \mathbb{R} \to \mathbb{R}$ by the formula:

$$(\mathcal{L}^c H)(r, z) \quad := \quad \frac{1}{2} \left(\sigma_1^2 - c 2 \sigma_1 \sigma_2 + \sigma_2^2 \right)(z) \frac{\partial^2 H}{\partial r^2}(r, z) + (QH(r, \cdot))(z). \tag{2.3}$$

The operator $\mathcal{L}^c$ is closely related to a generator of the process $(R(V), Z)$ and will play an important role in the solution of the stochastic control problems.

The next lemma describes a class of martingales related to the chain Z.

Lemma 2.2. *Let $F : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{E} \to \mathbb{R}$ be a bounded function, such that for any $z \in \mathbb{E}$ the restriction to the first two coordinates $F(\cdot, \cdot, z) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume that the generator Q satisfies*

$$\sup\{-Q(z, z) \, : \, z \in \mathbb{E}\} < \infty. \tag{2.4}$$

Let $U = (U_t)_{t\geq 0}$ be any continuous semimartingale, adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Then the process $M^U = (M_t^U)_{t\geq 0}$, given by

$$M_t^U \quad := \quad \sum_{0 < s \leq t} [F(s, U_s, Z_s) - F(s, U_s, Z_{s-})] - \int_0^t (QF(s, U_s, \cdot))(Z_{s-}) \, \mathrm{d}s,$$

is a true $(\mathcal{F}_t, \mathsf{P}_z)$-martingale for any starting point $z \in \mathbb{E}$.

Remarks. (i) The key point in Lemma 2.2 is that we do not assume that the process (U, Z) is Markov, since all that is required of U is that it has continuous paths and is adapted to the underlying filtration on the original probability space. This fact plays a crucial role in the solution of our optimisation problems, as it allows us to eliminate all the (suboptimal) non-Markovian couplings of the Brownian motions V and B, the laws of which are not tractable.

(ii) Assumption (2.4) on Q is equivalent to stipulating that Q is a bounded linear operator. This is clearly satisfied when the state space $\mathbb{E}$ is finite.

(iii) The result in Lemma 2.2 is well-known but a precise reference appears difficult to find. For this reason, and because of its importance in the proofs of Theorems 1.1 and 1.2, a proof of Lemma 2.2 is given in Appendix A.2.

2.3. $(\mathcal{F}_t)$**-Brownian motion and continuous-time** $(\mathcal{F}_t)$**-Markov chain are independent.** Intuitively, the independence of the chain Z and a Brownian motion $W \in \mathcal{V}$ follows from the fact that any $(\mathcal{F}_t)$-martingale of the form $(\psi(Z_t, t))_{t\geq 0}$, where ψ is a real function defined on the product $\mathbb{E} \times \mathbb{R}_+$, is equal to the sum of its jumps minus an absolutely continuous compensator and therefore has constant covariation with any continuous semimartingale adapted to $(\mathcal{F}_t)_{t\geq 0}$. The key fact underpinning this argument is that Z is a Markov process on the filtration $(\mathcal{F}_t)_{t\geq 0}$ (see Section 5.2 for counterexamples to Theorems 1.1 and 1.2 when this assumption is relaxed).

Lemma 2.3. *An $(\mathcal{F}_t)$-Markov chain Z is independent of any $(\mathcal{F}_t)$-Brownian motion W in $\mathcal{V}$.*

Proof. We first show that the random variables W_T and Z_T are independent for any $T > 0$. Let the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{E} \to \mathbb{R}$ be bounded and measurable with f suitably smooth. We need to establish the equality $\mathsf{E}[f(W_T)g(Z_T)] = \mathsf{E}[f(W_T)]\mathsf{E}[g(Z_T)]$. Define the $(\mathcal{F}_t)$-martingales $M^f = (M^f_t)_{t\in[0,T]}$ and $N^g = (N^g_t)_{t\in[0,T]}$ by

$$M^f_t := \mathsf{E}[f(W_T)|\mathcal{F}_t] \qquad \text{and} \qquad N^g_t := \mathsf{E}[g(Z_T)|\mathcal{F}_t].$$

Note that it is sufficient to prove that the product $M^f N^g = (M^f_t N^g_t)_{t\in[0,T]}$ is a martingale since in that case we have

$$\mathsf{E}[f(W_T)]\mathsf{E}[g(Z_T)] = M^f_0 N^g_0 = \mathsf{E}[M^f_T N^g_T] = \mathsf{E}[f(W_T)g(Z_T)]. \tag{2.5}$$

Now $M^f_t = (P^W_{T-t}f)(W_t)$, where P^W is the Brownian semigroup, and hence M^f is a continuous martingale. Similarly we have $N^g_t = (P_{T-t}g)(Z_t)$, where P denotes the semigroup for Z, and hence Itô's lemma for general semimartingales [10, Sec II.7, Thm. 33] and the Kolmogorov backward equation imply $\mathrm{d}N^g_t = (P_{T-t}g)(Z_t) - (P_{T-t}g)(Z_{t-}) - (Q(P_{T-t}g))(Z_t)\mathrm{d}t$ (Q denotes the generator matrix for Z). In particular, the quadratic variation of N^g is equal to the sum of its jumps, i.e. the continuous part of the process $[N^g, N^g]$ is almost surely zero. Hence the continuity of M^f and [10, Sec II.6, Thm. 28] imply that the covariation satisfies $\mathrm{d}[M^f, N^g]_t = 0$. Therefore, by the product rule, the infinitesimal increment of the process $M^f N^g$ equals

$$\mathrm{d}(M^f_t N^g_t) = N^g_{t-}\mathrm{d}M^f_t + M^f_{t-}\mathrm{d}N^g_t + \mathrm{d}[M^f, N^g]_t = N^g_t\mathrm{d}M^f_t + M^f_t\mathrm{d}N^g_t$$

(the subscripts $t-$ can be change to t since M^f is continuous), making $M^f N^g$ a martingale, since both M^f and N^g are bounded martingales, and equality (2.5) follows. By an approximation argument and the Dominated Convergence Theorem we conclude that (2.5) holds for arbitrary bounded measurable functions f and g and the independence of W_T and Z_T follows.

To prove independence of random vectors $(W_{t_1},\ldots,W_{t_n})$ and $(Z_{t_1},\ldots,Z_{t_n})$ for any $n\in\mathbb{N}$ and a sequence of times $0=t_0<t_1<\cdots<t_n$, pick any bounded measurable functions $f:\mathbb{R}^n\to\mathbb{R}$ and $g:\mathbb{E}^n\to\mathbb{R}$ and define recursively the functions $f_k:\mathbb{R}^{k\vee 1}\to\mathbb{R}$ and $g_k:\mathbb{E}^{k\vee 1}\to\mathbb{R}$ for $k=n,\ldots,0$, which are again bounded and measurable, by $f_n:=f, g_n:=g$ and

$$f_{k-1}(W_{t_1},\ldots,W_{t_{k-1}}):=\mathsf{E}[f_k(W_{t_1},\ldots,W_{t_k})|\mathcal{F}_{t_{k-1}}],\quad g_{k-1}(Z_{t_1},\ldots,Z_{t_{k-1}}):=\mathsf{E}[g_k(Z_{t_1},\ldots,Z_{t_k})|\mathcal{F}_{t_{k-1}}].$$

Note that f_0 and g_0 are constant functions. Equality (2.5) applied to the bounded measurable functions $x\mapsto f(W_{t_1},\ldots,W_{t_{n-1}},x)$ and $z\mapsto g(Z_{t_1},\ldots,Z_{t_{n-1}},z)$ shows that the following conditional expectation factorises:

$$\mathsf{E}[f(W_{t_1},\ldots,W_{t_n})g(Z_{t_1},\ldots,Z_{t_n})|\mathcal{F}_{t_{n-1}}]=f_{n-1}(W_{t_1},\ldots,W_{t_{n-1}})g_{n-1}(Z_{t_1},\ldots,Z_{t_{n-1}}).$$

Therefore, by iteration and the tower property, we see that the following holds

$$\mathsf{E}[f(W_{t_1},\ldots,W_{t_n})g(Z_{t_1},\ldots,Z_{t_n})]=f_0g_0=\mathsf{E}[f(W_{t_1},\ldots,W_{t_n})]\mathsf{E}[g(Z_{t_1},\ldots,Z_{t_n})].$$

Since f and g were arbitrary, the processes W and Z are independent. □

It follows from Lemma 2.3 that an $(\mathcal{F}_t)$-adapted volatility process, given by a strong solution of an SDE, cannot be approximated pathwise by a continuous-time $(\mathcal{F}_t)$-Markov chain.

Corollary 2.4. *Let Z' be an $(\mathcal{F}_t)$-adapted Feller semimartingale, which solves a scalar SDE with Lipschitz drift and diffusion coefficients μ,σ such that $\sigma>c>0$. Then there exists no sequence of continuous-time $(\mathcal{F}_t)$-Markov chains that converges to Z' almost surely on compacts.*

Proof. The process $W=(W_t)_{t\geq 0}$, where $W_t:=\int_0^t(\mathrm{d}Z'_s-\mu(Z'_s)\mathrm{d}t)/\sigma(Z'_s)$, is an $(\mathcal{F}_t)$-adapted continuous local martingale with $[W,W]_t=t$. W is therefore an $(\mathcal{F}_t)$-Brownian motion (by Lévy's characterisation theorem) and Z' is a strong solution of the SDE $\mathrm{d}Z'_t=\mu(Z'_t)\mathrm{d}t+\sigma(Z'_t)\mathrm{d}W_t$. By Lemma 2.3, any sequence of continuous-time $(\mathcal{F}_t)$-Markov chains is independent of W and therefore also independent of Z'. Therefore, since Z' is non-deterministic, the sequence cannot converge to Z' almost surely on compacts. □

3. Tracking

In this section we consider the problem of *tracking* X by the process $Y(V)$, defined in (1.1), where the control is being exercised solely by choosing the driving Brownian motion V. Recall that the tracking criterion, stated for a convex function ϕ in (1.4) and a time horizon $T>0$, can be equivalently expressed in terms of the following problems:

$$\begin{aligned}&\text{minimise}\quad \mathsf{E}\left[\phi(X_T-Y_T(V))\right]\quad\text{over}\quad V\in\mathcal{V},\\&\text{maximise}\quad \mathsf{E}\left[\phi(X_T-Y_T(V))\right]\quad\text{over}\quad V\in\mathcal{V}.\end{aligned}$$

Theorem 3.1. *Let the Brownian motions V^I and V^{II} be as in* (1.5). *Assume Z satisfies* (1.2), (1.8) *and* (1.9) *and that the function ϕ is as in* (1.4). *Then for any positive T we have*

$$\inf_{V\in\mathcal{V}} \mathsf{E}\left[\phi(X_T - Y_T(V))\right] = \mathsf{E}\left[\phi(X_T - Y_T(V^I))\right], \tag{3.1}$$

$$\sup_{V\in\mathcal{V}} \mathsf{E}\left[\phi(X_T - Y_T(V))\right] = \mathsf{E}\left[\phi(X_T - Y_T(V^{II}))\right]. \tag{3.2}$$

In this section we prove Theorem 3.1, which clearly implies Theorem 1.1, and hence solves Problem **(T)**. The proof of Theorem 3.1 is based on Bellman's principle, a martingale verification argument and an approximation scheme. The first stage consists of "approximating" Problems (3.1)-(3.2). More precisely, we proceed in two steps: we first introduce a stopped chain Z^n and, in the second step, the stopped process $R^{K,n}(V)$.

To this end let $U_n \subset \mathbb{R}^d$, $n \in \mathbb{N}$, be a family of compact subsets such that $\cup_{n\in\mathbb{N}} U_n = \mathbb{R}^d$ and $U_n \subset U^\circ_{n+1}$, for all $n \in \mathbb{N}$, where U°_{n+1} denotes the interior of U_{n+1} in $\mathbb{R}^d$. For each $n \in \mathbb{N}$, define a stopping time τ_n and the stopped $(\mathcal{F}_t)$-Markov chain Z^n by

$$Z^n_t := Z_{t\wedge\tau_n}, \qquad \text{where} \qquad \tau_n := \inf\{t \geq 0 \,:\, Z_t \in \mathbb{E}\setminus U_n\} \qquad (\inf\emptyset = \infty). \tag{3.3}$$

Hence, Z^n is an $(\mathcal{F}_t)$-Markov chain with the state space $\mathbb{E}$ and a Q-matrix Q_n given by

$$Q_n(z,z') = I_{U_n}(z)Q(z,z'), \qquad z,z' \in \mathbb{E}, \tag{3.4}$$

where $I_{\{\cdot\}}$ denotes the indicator function. In particular, since U_n is compact and hence $U_n \cap \mathbb{E}$ must be finite by (1.8), Q_n satisfies assumption (2.4) in Lemma 2.2. Since the chain Z has cádlág paths, the sequence of positive random variables $(\tau_n)_{n\in\mathbb{N}}$ is non-decreasing and the following holds

$$\tau_\infty := \lim_{n\to\infty} \tau_n = \infty \qquad \mathsf{P}_z\text{-a.s.} \qquad \text{for any } z \in \mathbb{E}.$$

Hence, we can extend the definition in (3.3) in a natural way to the case $n = \infty$ by $Z^\infty := Z$.

Fix a large $K > 0$ and define, for any $V \in \mathcal{V}$, the stopping time

$$\tau^K(V) := \inf\{t \geq 0 : |R_t(V)| \geq K\} \qquad (\inf\emptyset = \infty),$$

where $R(V)$ is given in (2.2). The stopped process of interest $R^{K,n}(V) = (R^{K,n}_t(V))_{t\geq 0}$ can now be defined by

$$R^{K,n}_t(V) := R_{t\wedge\tau_n\wedge\tau^K(V)}(V). \tag{3.5}$$

For given ϕ satisfying (1.4), $T > 0$ and any $K \in (0,\infty)$ and $n \in \mathbb{N}\cup\{\infty\}$, consider the problems

$$\text{minimise} \quad \mathsf{E}\left[\phi(R^{K,n}_T(V))\right] \quad \text{over} \quad V \in \mathcal{V}, \tag{3.6}$$

$$\text{maximise} \quad \mathsf{E}\left[\phi(R^{K,n}_T(V))\right] \quad \text{over} \quad V \in \mathcal{V}. \tag{3.7}$$

By Lemma 2.3, the processes $(R(V^I), Z)$ and $(R(V^{II}), Z)$ are Markov. Therefore we can define the candidate value functions $\psi^{(I)}_{K,n}, \psi^{(II)}_{K,n} : \mathbb{R}\times\mathbb{E}\times[0,T] \to \mathbb{R}_+$ for Problems (3.6) and (3.7) by

$$\psi^{(I)}_{K,n}(r,z,t) := \mathsf{E}_{r,z}\left[\phi(R^{K,n}_t(V^I))\right] \qquad \text{and} \qquad \psi^{(II)}_{K,n}(r,z,t) := \mathsf{E}_{r,z}\left[\phi(R^{K,n}_t(V^{II}))\right], \tag{3.8}$$

respectively. Note that by definition we have $\psi^{(I)}_{K,n}(r,z,t) = \psi^{(II)}_{K,n}(r,z,t) = \phi(r)$ if $r \in \mathbb{R}\setminus(-K,K)$ or $z \in \mathbb{R}\setminus U_n$.

Lemma 3.2. *Assume that ϕ, given in (1.4), is bounded from below and $\phi \in \mathcal{C}^2(\mathbb{R})$. For any $K \in (0,\infty)$ and $n \in \mathbb{N} \cup \{\infty\}$, the functions $\psi^{(I)}_{K,n}$ and $\psi^{(II)}_{K,n}$, defined in (3.8), have the following properties.*

(i) For all $r \in \mathbb{R}$, $z \in \mathbb{E}$ and $t \in [0,T]$, there exists a constant $\ell \in \mathbb{R}$, such that

$$\ell \leq \psi^{(I)}_{K,n}(r,z,t), \psi^{(II)}_{K,n}(r,z,t) \leq \max\{\phi(\max\{K,r\}), \phi(\min\{-K,r\})\}.$$

(ii) For each $z \in \mathbb{E}$ we have $\psi^{(I)}_{K,n}(\cdot,z,\cdot), \psi^{(II)}_{K,n}(\cdot,z,\cdot) \in \mathcal{C}^{2,1}(\mathbb{R} \times (0,T])$.

(iii) For any $r \in \mathbb{R}$, $z \in \mathbb{E}$ and $t \in (0,T]$, the derivatives satisfy the following inequalities:

$$\left|\frac{\partial \psi^{(I)}_{K,n}}{\partial r}\right|(r,z,t),\ \left|\frac{\partial \psi^{(II)}_{K,n}}{\partial r}\right|(r,z,t) \quad \leq \quad \max\{\phi'(\max\{K,r\}), -\phi'(\min\{-K,r\})\}, \tag{3.9}$$

$$\frac{\partial^2 \psi^{(I)}_{K,n}}{\partial r^2}(r,z,t),\ \frac{\partial^2 \psi^{(II)}_{K,n}}{\partial r^2}(r,z,t) \quad \geq \quad 0. \tag{3.10}$$

Proof. Part (i) follows from (3.8) and the properties of ϕ. To prove that $\psi^{(I)}_{K,n}$ is differentiable in r, define $S := R^{K,n}_t(V^I) - R^{K,n}_0(V^I)$ and note that its distribution does not depend on the starting point of $R^{K,n}(V^I)$. Since $\phi \in \mathcal{C}^2(\mathbb{R})$, Lagrange's mean value theorem implies that, for any small $h > 0$, there exists a random variable $\xi_{S,h}$ such that

$$\phi(r+h+S) - \phi(r+S) = h\phi'(r+\xi_{S,h}) \qquad \text{and} \qquad \xi_{S,h} \in (S, h+S). \tag{3.11}$$

Since $|S| \leq K$ almost surely and r is fixed, the continuity of ϕ' yields that the random variable $|\phi'(r+\xi_{S,h})|$ is bounded above by a constant. Equation (3.11), almost sure convergence of $\xi_{S,h}$ to S, as $h \to 0$, and the Dominated Convergence Theorem imply that $\psi^{(I)}_{K,n}(\cdot,z,t)$ is differentiable in r and

$$\frac{\partial \psi^{(I)}_{K,n}}{\partial r}(r,z,t) = \mathsf{E}_{r,z}\left[\phi'(R^{K,n}_t(V^I))\right]. \tag{3.12}$$

Furthermore, the convexity of ϕ and (3.12) yield the first inequality in (3.9). An identical argument applied to the function $\psi^{(II)}_{K,n}(\cdot,z,t)$ implies its differentiability in r and yields (3.9).

Since ϕ'' is continuous by assumption, we can apply an analogous argument to the one above, now using formula (3.12) instead of (3.8), to conclude that the functions $\psi^{(I)}_{K,n}(\cdot,z,t)$ and $\psi^{(II)}_{K,n}(\cdot,z,t)$ are in $\mathcal{C}^2(\mathbb{R})$ with

$$\frac{\partial^2 \psi^{(I)}_{K,n}}{\partial r^2}(r,z,t) = \mathsf{E}_{r,z}\left[\phi''(R^{K,n}_t(V^I))\right], \qquad \frac{\partial^2 \psi^{(II)}_{K,n}}{\partial r^2}(r,z,t) = \mathsf{E}_{r,z}\left[\phi''(R^{K,n}_t(V^{II}))\right].$$

The convexity of ϕ now implies part (iii) of the lemma. Differentiability of $\psi^{(I)}_{K,n}(r,z,\cdot)$ in t follows from the smoothness of ϕ and the standard properties of Itô integrals. □

Pick a function $F : \mathbb{R} \times \mathbb{E} \times [0,T) \to \mathbb{R}$ such that $F(\cdot,z,\cdot) \in \mathcal{C}^{2,1}(\mathbb{R} \times [0,T))$ for each $z \in \mathbb{E}$, and for each $r \in \mathbb{R}$, $t \in [0,T)$ the restriction to the second coordinate $F(r,\cdot,t) : \mathbb{E} \to \mathbb{R}$ is bounded. Then for any constant $c \in [-1,1]$ we define the function $\mathcal{K}^cF : \mathbb{R} \times \mathbb{E} \times [0,T) \to \mathbb{R}$ by the formula:

$$(\mathcal{K}^cF)(r,z,t) \quad = \quad (\mathcal{L}^cF(\cdot,\cdot,t))(r,z) + \frac{\partial F}{\partial t}(r,z,t),$$

where the operator $\mathcal{L}^c$ is as defined in (2.3).

Lemma 3.3 (HJB equation)**.** *Let ϕ in* (1.4) *be bounded from below and satisfy $\phi \in \mathcal{C}^2(\mathbb{R})$. Let $n \in \mathbb{N}$ and $K \in (0,\infty)$. Then the functions*

$$F^{(I)}(r,z,t) := \psi^{(I)}_{K,n}(r,z,T-t) \qquad \textit{and} \qquad F^{(II)}(r,z,t) := \psi^{(II)}_{K,n}(r,z,T-t),$$

(see (3.8) *for the definition of $\psi^{(I)}_{K,n}$ and $\psi^{(II)}_{K,n}$) satisfy the HJB equations:*

for any triplet $(r,z,t) \in (-K,K) \times (\mathbb{E} \cap U_n) \times [0,T)$ (see (3.3) *for the role of the set U_n) we have*

$$\inf_{c\in[-1,1]} \left(\mathcal{K}^c F^{(I)}\right)(r,z,t) = 0, \tag{3.13}$$

$$\sup_{c\in[-1,1]} \left(\mathcal{K}^c F^{(II)}\right)(r,z,t) = 0. \tag{3.14}$$

Furthermore, if at least one of the conditions $|r| \geq K$ or $z \in \mathbb{E} \setminus U_n$ or $t = T$ is satisfied, it holds

$$F^{(I)}(r,z,t) = F^{(II)}(r,z,t) = \phi(r). \tag{3.15}$$

Remark. Unlike Lemma 3.2, the proof of Lemma 3.3 depends on Lemma 2.2 and so requires the assumption $n < \infty$.

Proof. Note first that the definitions in (3.8) imply the boundary behaviour stated in (3.15).

We now focus on the proof of (3.13). Recall that for any starting point $z \in \mathbb{E}$ and $t \in [0,T)$, on the event $\{\tau_n \geq t\}$ we have $Z^n_t = Z_t$. The Markov property of the process $(R(V^I), Z)$ and the equality in (3.15) now imply

$$\begin{aligned}
\mathsf{E}\left[\phi(R^{K,n}_T(V^I))|\mathcal{F}_t\right] &= \mathsf{E}\left[\phi(R^{K,n}_T(V^I))I_{\{\tau_n<t\}}|\mathcal{F}_t\right] + \mathsf{E}\left[\phi(R^{K,n}_T(V^I))I_{\{\tau_n\geq t\}}|\mathcal{F}_t\right] \\
&= \phi(R^{K,n}_{\tau_n}(V^I))I_{\{\tau_n<t\}} + \psi^{(I)}_{K,n}(R^{K,n}_t(V^I), Z^n_t, T-t)I_{\{\tau_n\geq t\}} \\
&= \psi^{(I)}_{K,n}(R^{K,n}_t(V^I), Z^n_t, T-t).
\end{aligned}$$

The following observations are key:

- the quadratic covariation $[R^{K,n}(V^I), Z^{n,i}]_t$ vanishes for all $t \geq 0$ and $i = 1,\ldots,d$, where $Z^{n,i}$ is the i-th component of Z^n (recall that we are assuming $\mathbb{E} \subset \mathbb{R}^d$);
- the chain Z^n satisfies the assumptions of Lemma 2.2 and hence the process $M^U = (M^U_t)_{t\in[0,T]}$, given by

  $$\begin{aligned}
  M^U_t &:= \sum_{0<s\leq t}\left[\psi^{(I)}_{K,n}(R^{K,n}_s(V^I), Z^n_s, T-s) - \psi^{(I)}_{K,n}(R^{K,n}_s(V^I), Z^n_{s-}, T-s)\right] \\
  &\quad - \int_0^t (Q_n\psi^{(I)}_{K,n}(R^{K,n}_s(V^I), \cdot, T-s))(Z^n_{s-})\,\mathrm{d}s,
  \end{aligned}$$

  where Q_n is the generator of the chain Z^n given in (3.4), is a true $(\mathcal{F}_t, \mathsf{P}_z)$-martingale for any starting point $z \in \mathbb{E}$.

By Lemma 3.2, the function $\psi^{(I)}_{K,n}$ possesses the necessary smoothness so that Itô's lemma for general semimartingales [10, Sec II.7, Thm. 33] can be applied to the process $(\psi^{(I)}_{K,n}(R^{K,n}_t(V^I), Z^n_t, T-t))_{t\in[0,T]}$, which is itself a bounded martingale. Since $Q_n(z,z') = Q(z,z')$ for any $z \in \mathbb{E} \cap U_n$, $z' \in \mathbb{E}$ and on the event $\{t \leq \tau_n\}$ we have $Z_t = Z_t^n \in U_n$, the pathwise representation of this bounded martingale implies that the following process $N = (N_t)_{t\in[0,T]}$,

$$N_t = \int_0^{t\wedge\tau_n\wedge\tau^K(V)} \left[\frac{1}{2}(|\sigma_1| - |\sigma_2|)^2(Z_s)\frac{\partial^2\psi_{K,n}^{(I)}}{\partial r^2}(R_s^{K,n}(V^I), Z_s, T-s) + (Q\psi_{K,n}^{(I)}(R_s^{K,n}(V^I), \cdot, T-s))(Z_s) - \frac{\partial\psi_{K,n}^{(I)}}{\partial t}(R_s^{K,n}(V^I), Z_s, T-s)\right] \mathrm{d}s,$$

is a continuous martingale. The quadratic variation of N is clearly equal to zero and hence $N_t = 0$ for all $t \in [0, T]$ and starting points (r, z). Since by (1.8) the process $(R^{K,n}(V^I), Z)$ visits a neighbourhood of any point in the product $(-K, K) \times (\mathbb{E} \cap U_n)$ on the stochastic interval $[0, T \wedge \tau_n \wedge \tau^K(V)]$ with positive probability, for all $(r, z, t) \in (-K, K) \times (\mathbb{E} \cap U_n) \times [0, T)$ we must have:

$$\frac{1}{2}(|\sigma_1| - |\sigma_2|)^2(z)\frac{\partial^2\psi_{K,n}^{(I)}}{\partial r^2}(r, z, T-t) + (Q\psi_{K,n}^{(I)}(r, \cdot, T-t))(z) - \frac{\partial\psi_{K,n}^{(I)}}{\partial t}(r, z, T-t) = 0. \tag{3.16}$$

To prove (3.13), observe that $(|\sigma_1|-|\sigma_2|)^2 = \inf_{c\in[-1,1]}(\sigma_1^2-2c\sigma_1\sigma_2+\sigma_2^2)$. Then (3.10) of Lemma 3.2 implies that

$$(\sigma_1^2 - 2c\sigma_1\sigma_2 + \sigma_2^2)(z)\frac{\partial^2\psi_{K,n}^{(I)}}{\partial r^2}(r, z, T-t) \geq (|\sigma_1| - |\sigma_2|)^2(z)\frac{\partial^2\psi_{K,n}^{(I)}}{\partial r^2}(r, z, T-t)$$

for any $c \in [-1, 1]$ and each $(r, z, t) \in (-K, K) \times (\mathbb{E} \cap U_n) \times [0, T)$. This inequality and identity (3.16) imply (3.13). The proof of (3.14) is analogous and therefore left to the reader. □

3.1. **Proof of Theorem 3.1.** Assume that ϕ satisfies condition (1.4) as well as

$$\ell \leq \phi(x)\ \forall x \in \mathbb{R},\ \ell \in \mathbb{R}, \quad \text{and} \quad \phi \in \mathcal{C}^2(\mathbb{R}). \tag{3.17}$$

Pick $V \in \mathcal{V}$ and, for any $t \in [0, T]$, define Brownian motions $V^{It} = (V_s^{It})_{s\geq0} \in \mathcal{V}$ and $V^{IIt} = (V_s^{IIt})_{s\geq0} \in \mathcal{V}$ by

$$V_s^{It} := \begin{cases} V_s & \text{if } s \leq t, \\ V_t + V_s^I - V_t^I & \text{if } s > t, \end{cases} \quad \text{and} \quad V_s^{IIt} := \begin{cases} V_s & \text{if } s \leq t, \\ V_t + V_s^{II} - V_t^{II} & \text{if } s > t, \end{cases} \tag{3.18}$$

where V^I, V^{II} are given in (1.5). In other words, for each $t \geq 0$, the Brownian motions V^{It} and V^{IIt} are arbitrary (but fixed) up to time t and have increments equal to those of the candidate optimal Brownian motions after this time. We now consider two Bellman processes $(B_t^I(V))_{t\in[0,T]}$ and $(B_t^{II}(V))_{t\in[0,T]}$, associated to Problems (3.6)-(3.7), given by

$$B_t^I(V) := \psi_{K,n}^{(I)}(R_t^{K,n}(V), Z_t^n, T-t) \quad \text{and} \quad B_t^{II}(V) := \psi_{K,n}^{(II)}(R_t^{K,n}(V), Z_t^n, T-t). \tag{3.19}$$

The definitions in (1.5) of V^I, V^{II}, together with Lemma 2.3, imply that the processes $(R(V^I), Z)$ and $(R(V^{II}), Z)$ are Markov. The definition of the Brownian motion V^{It} in (3.18) and the properties of the function $\psi_{K,n}^{(I)}$ therefore imply

$$\begin{aligned} \mathsf{E}\left[\phi(R_T^{K,n}(V^{It}))|\mathcal{F}_t\right] &= \mathsf{E}\left[\phi(R_T^{K,n}(V^{It}))I_{\{\tau_n<t\}}|\mathcal{F}_t\right] + \mathsf{E}\left[\phi(R_T^{K,n}(V^{It}))I_{\{\tau_n\geq t\}}|\mathcal{F}_t\right] \\ &= \phi(R_{\tau_n}^{K,n}(V))I_{\{\tau_n<t\}} + \psi_{K,n}^{(I)}(R_t^{K,n}(V), Z_t^n, T-t)I_{\{\tau_n\geq t\}} \\ &= \psi_{K,n}^{(I)}(R_t^{K,n}(V), Z_t^n, T-t). \end{aligned}$$

This equality, together with a similar argument based on the definitions of V^{IIt} and $\psi^{(II)}_{K,n}$, yields the following representations for the Bellman processes

$$B_t^I(V) = \mathsf{E}\left[\phi(R_T^{K,n}(V^{It}))|\mathcal{F}_t\right] \quad \text{and} \quad B_t^{II}(V) = \mathsf{E}\left[\phi(R_T^{K,n}(V^{IIt}))|\mathcal{F}_t\right].$$

By Lemma 3.2 we can apply Itô's formula for general semimartingales (see [10, Sec II.7, Thm. 33]) to $B^I(V)$ and $B^{II}(V)$. Lemma 2.2 and inequalities (3.9) imply that the local martingale parts of these path decompositions of processes $B^I(V)$ and $B^{II}(V)$ are true martingales. Therefore, the fact that the quadratic covariation $[R^{K,n}(V^{It}), Z^{n,i}]_t$ vanishes for all $t \geq 0$ for each component $Z^{n,i}$ of Z^n, together with Lemma 3.3, implies that, for any $V \in \mathcal{V}$, $B^I(V)$ is a submartingale and $B^{II}(V)$ a supermartingale. Furthermore it follows from the discussion above and Lemma 3.3 that $B^I(V^I)$ and $B^{II}(V^{II})$ are martingales. This establishes the Bellman principle and solves the optimisation problems in (3.6) and (3.7). Put differently, we have established the following inequalities for any starting points $r \in \mathbb{R}$, $z \in \mathbb{E}$, any $K \in (0,\infty)$, $n \in \mathbb{N}$ and all Brownian motions $V \in \mathcal{V}$:

$$\mathsf{E}_{r,z}\left[\phi(R_T^{K,n}(V^I))\right] \leq \mathsf{E}_{r,z}\left[\phi(R_T^{K,n}(V))\right] \leq \mathsf{E}_{r,z}\left[\phi(R_T^{K,n}(V^{II}))\right] \tag{3.20}$$

The next step in the proof of Theorem 3.1 requires two limiting arguments. First, note that for any Brownian motion $V \in \mathcal{V}$ the definition of the process $R_T^{K,n}(V)$ in (3.5) implies

$$R_T^{K,\infty}(V) = \lim_{n\uparrow\infty} R_T^{K,n}(V) \quad \mathsf{P}_{r,z}\text{-a.s.}$$

for any starting points $r \in \mathbb{R}$ and $z \in \mathbb{E}$. Furthermore, by Lemma 3.2 (i), the random variables $\phi(R_T^{K,n}(V))$ are bounded in modulus by a constant uniformly in $n \in \mathbb{N}$. Therefore, the Dominated Convergence Theorem implies that the inequalities in (3.20) hold for $n = \infty$.

For the second limiting argument, note that assumption (1.9) and the semigroup property imply $P_t f(z) < \infty$ for all $t \in [0,T]$, $z \in \mathbb{E}$, where $f := |\sigma_1|^p + |\sigma_2|^p : \mathbb{E} \to [0,\infty)$ and $p \in [2,\infty)$. The Kolmogorov backward equation implies

$$(P_T f)(z) - f(z) = \int_0^T \sum_{z'\in\mathbb{E}} Q(z,z')(P_t f)(z')\mathrm{d}t = \sum_{z'\in\mathbb{E}} Q(z,z') \int_0^T (P_t f)(z')\mathrm{d}t \quad \text{for } z \in \mathbb{E},$$

where the second equality follows from the following facts: $Q(z,z') \in [0,\infty)$ for all $z' \in \mathbb{E}\setminus\{z\}$, $P_t f(z) \in [0,\infty)$ for all $t \in [0,T]$, $z \in \mathbb{E}$, and Fubini's theorem for non-negative functions. This, together with assumption (1.9) and another application of Fubini's theorem, yields

$$\mathsf{E}_z \int_0^T \left(|\sigma_1|^p + |\sigma_2|^p\right)(Z_t)\,\mathrm{d}t < \infty \qquad \text{for } z \in \mathbb{E}. \tag{3.21}$$

Furthermore, it is clear from the definition of $R^{K,\infty}(V)$, for any $V \in \mathcal{V}$, that

$$\lim_{K\to\infty} \phi(R_T^{K,\infty}(V)) = \phi(R_T(V)) \quad \mathsf{P}_{r,z}\text{-a.s.}$$

The following almost sure inequality is a direct consequence of the definition in (3.5)

$$-S \leq R_T^{K,\infty}(V) \leq S \qquad \text{for all } K > 0, \text{ where } S := \sup_{t\in[0,T]} |R_t(V)|. \tag{3.22}$$

By assumptions (1.4) and (3.17) the following inequalities hold for some constants $a, b > 0$ and $\ell \in \mathbb{R}$:

$$|\phi(R_T^{K,\infty}(V))| \leq \max\{|\ell|, |\phi(S)|, |\phi(-S)|\} \leq \max\{|\ell|, a|S|^p + b\} \leq a|S|^p + b + |\ell|.$$

The Burkholder-Davis-Gundy inequality [11, Thm IV.4.1] applied to the martingale $R(V)$ at time T, together with inequality (3.21), implies that $|S|^p$ is an integrable random variable. The Dominated Convergence Theorem therefore yields the L^1-convergence for $\phi(R_T^{K,\infty}(V)) \to \phi(R_T(V))$ as $K \to \infty$. By (3.20) for $n = \infty$, we obtain the following inequalities for any $V \in \mathcal{V}$:

$$
\begin{aligned}
\mathsf{E}_{r,z}[\phi(R_T(V^{II}))] = \lim_{K\to\infty} \mathsf{E}_{r,z}[\phi(R_T^{K,\infty}(V^{II}))] &\geq \lim_{K\to\infty} \mathsf{E}_{r,z}[\phi(R_T^{K,\infty}(V))] = \mathsf{E}_{r,z}[\phi(R_T(V))] \\
&\geq \lim_{K\to\infty} \mathsf{E}_{r,z}[\phi(R_T^{K,\infty}(V^{I}))] = \mathsf{E}_{r,z}[\phi(R_T(V^{I}))],
\end{aligned} \tag{3.23}
$$

implying Theorem 3.1. under the additional assumption in (3.17).

In order to relax the assumption $\phi \in \mathcal{C}^2(\mathbb{R})$, fix a non-negative $g \in \mathcal{C}^\infty(\mathbb{R})$ with support in $[M, 0]$, for some $M \in (-\infty, 0)$, satisfying $\int_{-\infty}^0 g(y)\,\mathrm{d}y = 1$. For each $n \in \mathbb{N}$, define the convolution

$$\phi_n(x) := \int_{-\infty}^0 \phi(x + y/n) g(y)\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

Note that $\phi_n : \mathbb{R} \to \mathbb{R}$ is a convex function, which satisfies both (1.4) and (3.17) (here we still assume that ϕ is bounded from below), and the sequence $(\phi_n)_{n\in\mathbb{N}}$ converges point-wise to ϕ as $n \uparrow \infty$ (see e.g. [11], proof of Theorem VI.1.1 and Appendix 3).[2] Since ϕ satisfies (1.4), for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\ell \leq \phi_n(x) \leq \max\{\phi(x + M/n), \phi(x)\} \leq a \max\{|x + M/n|^p, |x|^p\} + b \leq A|x|^p + B,$$

where the constants $A, B > 0$ are independent of both n and x. Since the random variable $|S|^p$ is integrable (see previous paragraph), where S is defined in (3.22), so is $|R_T(V)|^p$ for any $V \in \mathcal{V}$. The inequality above and the Dominated Convergence Theorem imply

$$\lim_{n\to\infty} \mathsf{E}[\phi_n(R_T(V))] = \mathsf{E}[\phi(R_T(V))] \qquad \text{for any } V \in \mathcal{V},$$

which together with the inequalities in (3.23), establishes Theorem 3.1 for ϕ that are bounded from below and satisfy (1.4).

Since for any $V \in \mathcal{V}$ the processes X and $Y(V)$ are true martingales by (1.2), we may substitute ϕ with a function $\phi^c(x) := \phi(x) + cx$, $x \in \mathbb{R}$, for any constant $c \in \mathbb{R}$, without altering the solution of Problems (3.1)-(3.2). For any ϕ satisfying (1.4) there exists some $c \in \mathbb{R}$ such that ϕ^c is bounded from below and hence Theorem 3.1 follows. □

3.2. **Non-Markovian Tracking.** The Markovian structure of Z does not feature explicitly in the conclusion of Theorem 3.1, but only in its assumptions. It is therefore natural to ask whether, under some additional hypothesis, Theorem 3.1 can be generalised to a non-Markov volatility process Z. In this section we argue intuitively that, for such a generalisation to hold for a large class of convex cost functions ϕ, an underlying Markovian structure is in fact necessary but show that it is possible in the special case $\phi(x) = x^2$ (see Section 5.2.1 for an explicit example of a process Z, with a countable discrete state space $\mathbb{E}$ in $\mathbb{R}$, which is not $(\mathcal{F}_t)$-Markov and the conclusion of Theorem 3.1 fails).

[2] We thank one of the referees for observing that Theorems 1.1 and 3.1 require neither smoothness nor boundedness from below of the function ϕ and suggesting the argument presented here.

Assume (in this section only) that the stochastic integrals X and $Y(V)$ are given by

$$X_t = x + \int_0^t H_s \, \mathrm{d}B_s \qquad \text{and} \qquad Y_t(V) = y + \int_0^t J_s \, \mathrm{d}V_s, \tag{3.24}$$

for some progressively measurable integrands $H = (H_t)_{t\geq 0}$ and $J = (J_t)_{t\geq 0}$ on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathsf{P})$ and any $V \in \mathcal{V}$. As usual, we denote the difference of X and $Y(V)$ by $R(V) = X - Y(V)$. The extremal Brownian motions V^I and V^{II}, defined in (1.5), can be generalised naturally by $V_t^I = \int_0^t \operatorname{sgn}(H_s J_s) \, \mathrm{d}B_s$ and $V_t^{II} = -V_t^I$. Hence, for any fixed $V \in \mathcal{V}$, we can define the Brownian motions V^{It} and V^{IIt} as in (3.18). If the generalisation of Theorem 3.1 were to hold in this setting, the Bellman processes $B^I(V)$ and $B^{II}(V)$, defined in (3.19), would be a submartingale and a supermartingale, respectively, for any $V \in \mathcal{V}$. We will focus on $B^I(V)$, as the issues with $B^{II}(V)$ are completely analogous. Representation (2.1) of V in Lemma 2.1 and Itô's formula yield

$$\begin{aligned} \phi\left(R_T(V^{It})\right) &= \phi\left(R_0(V^{It})\right) + M_T^I + \frac{1}{2}\int_0^t \phi''\left(R_s(V)\right)\left(H_s^2 - 2C_s H_s J_s + J_s^2\right) \mathrm{d}s \\ &+ \frac{1}{2}\int_t^T \phi''\left(R_s(V^I) - R_t(V^I) + R_t(V)\right)\left(|H_s| - |J_s|\right)^2 \mathrm{d}s, \end{aligned}$$

where M^I is a local martingale, which we assume to be a true martingale. The process $B_t^I(V) = \mathsf{E}\left[\phi(R_T(V^{It}))|\mathcal{F}_t\right]$ is a submartingale if and only if the conditional expectation $\mathsf{E}[B_{t'}^I(V) - B_t^I(V)|\mathcal{F}_t]$, proportional to

$$\begin{aligned} &\mathsf{E}\Big[\textstyle\int_{t'}^T \left[\phi''\left(R_s(V^I) - R_{t'}(V^I) + R_{t'}(V)\right) - \phi''\left(R_s(V^I) - R_t(V^I) + R_t(V)\right)\right]\left(|H_s| - |J_s|\right)^2 \mathrm{d}s \\ &+ \textstyle\int_t^{t'} \phi''\left(R_s(V)\right)\left(H_s^2 - 2C_s H_s J_s + J_s^2\right) - \phi''\left(R_s(V^I) - R_t(V^I) + R_t(V)\right)\left(|H_s| - |J_s|\right)^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] \end{aligned}$$

by the formula above, is non-negative for all $0 \leq t < t' \leq T$. Hence $B^I(V)$ is a submartingale for general integrands J and H if ϕ'' does not depend on the state, i.e. when the cost criterion ϕ is quadratic, and we obtain:

Proposition 3.4. *Let $R(V) = X - Y(V)$, where $X, Y(V)$ are as in* (3.24)*, and $T > 0$. Then we have*

$$\mathsf{E}\left[(X_T - Y_T(V^I))^2\right] \quad \leq \quad \mathsf{E}\left[(X_T - Y_T(V))^2\right] \quad \leq \quad \mathsf{E}\left[(X_T - Y_T(V^{II}))^2\right] \quad \textit{for any } V \in \mathcal{V}.$$

This proposition is consistent with an argument based on Itô's isometry: the variance of a stochastic integral is equal to the expectation of its quadratic variation and hence minimising/maximising its variance is equivalent to locally minimising/maximising the Radon-Nikodym derivative of its quadratic variation. Furthermore, it is also clear from the representation above that in the absence of an underlying Markovian structure, for a general convex ϕ, the process $B^I(V)$ may fail to be a submartingale and hence the strategy in Theorem 3.1 is not optimal for general non-Markovian integrands (see Section 5.2.1 for an explicit example demonstrating this phenomenon).

4. Coupling

In this section we consider the problems of *minimising* and *maximising* the coupling time of the processes X and $Y(V)$ defined in (1.1), where the controller is free to choose the driving Brownian motion V in the integral $Y(V)$ and the volatility is driven by a continuous-time $(\mathcal{F}_t)$-Markov chain Z. Put differently, we seek sharp upper and lower bounds for the probability of the event that the coupling of X and $Y(V)$ occurs after a fixed time T. The couplings are characterised by the stochastic extrema of the stopping time $\tau_0(X-Y(V)) := \inf\{t \geq 0 : X_t = Y_t(V)\}$ (with convention $\inf\emptyset = \infty$). More precisely, for any fixed $T > 0$, we consider the following problems:

$$\text{minimise} \quad \mathsf{P}\left[\tau_0(X-Y(V)) > T\right] \quad \text{over} \quad V \in \mathcal{V},$$
$$\text{maximise} \quad \mathsf{P}\left[\tau_0(X-Y(V)) > T\right] \quad \text{over} \quad V \in \mathcal{V}.$$

Theorem 4.1. *Let V^I and V^{II} be as given by (1.5) and Z satisfy (1.2) and (1.8). Then for any $T > 0$ we have*

$$\inf_{V\in\mathcal{V}} \mathsf{P}\left[\tau_0(X-Y(V)) > T\right] = \mathsf{P}\left[\tau_0(X-Y(V^{II})) > T\right], \tag{4.1}$$
$$\sup_{V\in\mathcal{V}} \mathsf{P}\left[\tau_0(X-Y(V)) > T\right] = \mathsf{P}\left[\tau_0(X-Y(V^{I})) > T\right]. \tag{4.2}$$

In this section we prove Theorem 4.1, which clearly implies Theorem 1.2, and hence solves Problem **(C)** for a continuous-time $(\mathcal{F}_t)$-Markov chain Z. The aim is to minimise and maximise the coupling time of the martingales X and $Y(V)$ given in (1.1). Due to the symmetry in Problem **(C)**, we may therefore assume without loss of generality that the starting points of the processes $X_0 = x$ and $Y_0(V) = y$ satisfy the inequality

$$x \leq y. \tag{4.3}$$

The candidate value functions in Problems (4.1) and (4.2) will be functionals of the law of the Markov processes $(R(V^{II}), Z)$ and $(R(V^I), Z)$, respectively, where $R(V)$ is given in (2.2) and the Brownian motions V^{II} and V^I are defined in (1.5). The first step in the proof of Theorem 4.1 is to localise Problems (4.1) and (4.2). With this in mind, for any $n \in \mathbb{N}$ recall definition (3.3) of the stopping time τ_n and the stopped chain Z^n. Unlike in Section 3, in the case of coupling it is important to localise the process $R(V)$ by stopping only the integrand. The process $R^n(V) := (R^n_t(V))_{t\geq 0}$ is therefore given by

$$R^n_t(V) := r + \int_0^t \sigma_1(Z^n_s)\,\mathrm{d}B_s - \int_0^t \sigma_2(Z^n_s)\,\mathrm{d}V_s, \qquad r \leq 0, \tag{4.4}$$

where B is the fixed Brownian motion and $V \in \mathcal{V}$ any Brownian motion on our probability space. As in the previous section, in this circumstance it is also natural to identify the limit $(R^\infty(V), Z^\infty)$ with the process $(R(V), Z)$. For $n \in \mathbb{N} \cup \{\infty\}$, we define the first entry time of the process $R^n(V)$ into the positive half-line by

$$\tau_0^+(R^n(V)) := \inf\{t \geq 0 : R^n_t(V) > 0\} \qquad (\text{with } \inf\emptyset = \infty). \tag{4.5}$$

The localisation procedure will allow us to reduce the problem to the case where the generator of the volatility chain Z is bounded, which will in turn make it possible to establish sufficient regularity of the candidate value functions and conclude that certain processes are true martingales (see Section 4.1). The two Markov processes (R^{IIn}, Z^n) and (R^{In}, Z^n), which play a key role in the solution of Problems (4.1) and (4.2), are defined by

$$R^{IIn}_t := r + \int_0^t \Sigma_{II}(Z^n_s)\,\mathrm{d}B_s \qquad \text{and} \qquad R^{In}_t := r + \int_0^t \Sigma_{I}(Z^n_s)\,\mathrm{d}B_s, \tag{4.6}$$

for any $r \leq 0$, where B and Z^n are as above and the functions $\Sigma_{II}, \Sigma_I : \mathbb{E} \to \mathbb{R}$ are given by

$$\Sigma_{II}(z) := \sigma_1(z) + \operatorname{sgn}(\sigma_1(z)\sigma_2(z))\sigma_2(z) \quad \forall z \in \mathbb{E}, \tag{4.7}$$

$$\Sigma_I(z) := \sigma_1(z) - \operatorname{sgn}(\sigma_1(z)\sigma_2(z))\sigma_2(z) \quad \forall z \in \mathbb{E}. \tag{4.8}$$

Note that, according to our definitions, we have $R^n(V^{II}) \neq R^{IIn}$ and $R^n(V^I) \neq R^{In}$ for any $n \in \mathbb{N}$, since the Brownian motions V^I and V^{II}, defined in (1.5), are given in terms of Z and not Z^n. However, if we define the Brownian motions V^{In} and V^{IIn} by (1.5) with Z replaced by Z^n, then the equalities $R^n(V^{IIn}) = R^{IIn}$ and $R^n(V^{In}) = R^{In}$ hold.

The proof of Theorem 4.1 can now be carried out in three steps. First, we formulate a pair of "approximate" coupling problems (for each $n \in \mathbb{N}$):

$$\text{minimise} \quad \mathsf{P}_{r,z}\left[\tau_0^+(R^n(V)) > T\right] \quad \text{over} \quad V \in \mathcal{V}, \tag{4.9}$$

$$\text{maximise} \quad \mathsf{P}_{r,z}\left[\tau_0^+(R^n(V)) > T\right] \quad \text{over} \quad V \in \mathcal{V}, \tag{4.10}$$

for a fixed $T > 0$ and any starting points $r \leq 0$, $z \in \mathbb{E}$. The following probabilistic representations for the candidate value functions of Problems (4.9) and (4.10) play an important role in their solutions:

$$\zeta_n^{(II)}(r,z,t) := \mathsf{P}_{r,z}\left[\tau_0^+\left(R^{IIn}\right) > t\right], \tag{4.11}$$

$$\zeta_n^{(I)}(r,z,t) := \mathsf{P}_{r,z}\left[\tau_0^+\left(R^{In}\right) > t\right]. \tag{4.12}$$

The second step, described in Section 4.1, solves Problems (4.9) and (4.10). Lemmas 4.2 and 4.3 establish the necessary analytical properties of the candidate value functions $\zeta_n^{(II)}$ and $\zeta_n^{(I)}$, which enable us to prove (see Lemma 4.4) the optimality of the Brownian motions V^{IIn} and V^{In}. More precisely, the representations in (4.11)-(4.12) are used to establish the required differentiability of the functions $\zeta_n^{(II)}$ and $\zeta_n^{(I)}$, which allows us to study the pathwise evolution of the corresponding Bellman processes. The optimality of V^{IIn} and V^{In}, established in Lemma 4.4, is a consequence of the non-positivity of the second derivatives $\frac{\partial^2 \zeta_n^{(II)}}{\partial r^2}$ and $\frac{\partial^2 \zeta_n^{(I)}}{\partial r^2}$ proved in Lemma 4.3.

The third step in the proof of Theorem 4.1, given in Section 4.2, applies approximation arguments, which establish the Brownian motions V^{II} and V^I as the solutions of Problems (4.1) and (4.2).

Finally, Section 4.3 discusses the issues that arise with a direct approach, based on the Dambis, Dubins-Schwarz theorem (see e.g. [11, Thm V.1.6]), to the coupling problems in (4.1) and (4.2).

4.1. **The stochastic time-change.** Throughout this section we fix $n \in \mathbb{N}$. Let $\Sigma_{II} : \mathbb{E} \to \mathbb{R}$ be as in (4.7) and note that our standing assumption $(|\sigma_1| + |\sigma_2|)(z) > 0$ implies $\Sigma_{II}^2(z) > 0$ for all $z \in \mathbb{E}$. Therefore, the stochastic time-change $A^{II} = (A_t^{II})_{t \geq 0}$, given by

$$A_t^{II} := \int_0^t \Sigma_{II}^2(Z_s^n)\, \mathrm{d}s, \tag{4.13}$$

is a differentiable, strictly increasing process. Furthermore, the definition of Z^n and (4.13) imply that the almost sure limit $\lim_{t\uparrow\infty} A_t^{II} = \infty$ holds. Hence, the inverse $E^{II} = (E_s^{II})_{s \geq 0}$, defined as the unique solution of

$$A^{II}_{E_s^{II}} = s, \quad s \geq 0, \qquad \text{also satisfies} \qquad E^{II}_{A_t^{II}} = t \qquad \text{for all } t \geq 0,$$

and is a strictly increasing process with differentiable trajectories. Since Z^n is an $(\mathcal{F}_t)$-Markov chain, it is by Lemma 2.3 independent of the $(\mathcal{F}_t)$-Brownian motion B in (4.6). Therefore the laws of the processes (R^{IIn}, Z^n) and $(r + B_{A^{II}}, Z^n)$ coincide, where $B_{A^{II}}$ denotes the Brownian motion B time-changed by the increasing process A^{II}.

Let $\Sigma_I : \mathbb{E} \to \mathbb{R}$ be as in (4.8) and assume further that $|\sigma_1|(z) \neq |\sigma_2|(z)$ for all $z \in \mathbb{E}$. This implies the inequality $\Sigma_I^2(z) > 0$ for all $z \in \mathbb{E}$. Define, in an analogous way to (4.13), the strictly increasing continuous time-change $A^I = (A_t^I)_{t\geq 0}$ and its inverse $E^I = (E_s^I)_{s\geq 0}$, and note that the processes (R^{In}, Z^n) and $(r + B_{A^I}, Z^n)$ have the same law. We can now state and prove Lemma 4.2.

Lemma 4.2. *Fix $n \in \mathbb{N}$ and pick any $r < 0$. Define the stopping time $\tau_r^B := \inf\{s : B_s = -r\}$ (with $\inf \emptyset = \infty$) and recall that the function $G(r,t) := \mathsf{P}\left[\tau_r^B > t\right]$, for any $t \geq 0$, takes the form*

$$G(r,t) = 2N\left(-\frac{r}{\sqrt{t}}\right) - 1,$$

where $N(\cdot)$ denotes the standard normal cdf.

(a) For any $z \in \mathbb{E}$ the following representation holds:

$$\zeta_n^{(II)}(r,z,t) = \mathsf{E}_z\left[G(r, A_t^{II})\right].$$

Hence the partial derivatives $\frac{\partial \zeta_n^{(II)}}{\partial r}(r,z,t)$, $\frac{\partial^2 \zeta_n^{(II)}}{\partial r^2}(r,z,t)$, $\frac{\partial \zeta_n^{(II)}}{\partial t}(r,z,t)$ exist for $r < 0, t > 0$.

(b) Assume further that $|\sigma_1|(z') \neq |\sigma_2|(z')$ for all $z' \in \mathbb{E}$. Then for any $z \in \mathbb{E}$ we have

$$\zeta_n^{(I)}(r,z,t) = \mathsf{E}_z\left[G(r, A_t^{I})\right]$$

and the partial derivatives $\frac{\partial \zeta_n^{(I)}}{\partial r}(r,z,t)$, $\frac{\partial^2 \zeta_n^{(I)}}{\partial r^2}(r,z,t)$, $\frac{\partial \zeta_n^{(I)}}{\partial t}(r,z,t)$ exist for any $r < 0, t > 0$.

Proof. We first establish (a). Recall the definition of the time-change process A^{II} and its inverse E^{II} introduced above and note that the following equalities hold almost surely by the definition of the stopping time τ_r^B:

$$E_{\tau_r^B}^{II} = \inf\{E_s^{II} : B_s = -r\} = \inf\{t : B_{A_t^{II}} = -r\} \qquad \text{(with } \inf\emptyset = \infty).$$

Therefore, since the processes (R^{IIn}, Z^n) and $(r + B_{A^{II}}, Z^n)$ are equal in law, so are the random variables $\tau_0^+(R^{IIn})$ and $E_{\tau_r^B}^{II}$. Since E^{II} is a strictly increasing continuous inverse of A^{II}, we have

$$\begin{aligned} \mathsf{P}_{r,z}\left[t < \tau_0^+(R^{IIn})\right] &= \mathsf{P}_z\left[A_t^{II} < \tau_r^B\right] \\ &= \mathsf{E}_z\left[G(r, A_t^{II})\right]. \end{aligned} \tag{4.14}$$

This, together with definition (4.11), implies the representation of $\zeta_n^{(II)}$ in part (a) of the lemma.

The required differentiability of $\zeta_n^{(II)}$ in r follows from (4.14), along the same lines as in the proof of Lemma 3.2. An application of the Dominated Convergence Theorem, the mean value theorem and the smoothness and boundedness of the functions $\frac{\partial G}{\partial r}$ and $\frac{\partial^2 G}{\partial r^2}$ on a rectangle $(r-\varepsilon, r+\varepsilon) \times (0,\infty)$ for any fixed $r < 0$ and small $\varepsilon > 0$, such that $\varepsilon + r < 0$, together imply the existence of $\frac{\partial \zeta_n^{(II)}}{\partial r}(r,z,t)$ and $\frac{\partial^2 \zeta_n^{(II)}}{\partial r^2}(r,z,t)$.

The differentiability of $\zeta_n^{(II)}$ in t is more delicate as it is intimately related to the integrability of the chain Z^n and the unboundedness of the function Σ_{II}. We start with the following observation.

Claim. The stopping time τ_n, defined in (3.3), is a continuous random variable and

$$\mathsf{E}_z\left[I_{\{\tau_n \leq s\}} \Sigma_{II}^2(Z_{\tau_n})\right] < \infty \qquad \text{for any } z \in \mathbb{E} \text{ and } s \geq 0. \tag{4.15}$$

Since $\mathsf{P}_z[\tau_n > t] = \mathsf{P}_z[Z_t^n \in U_n \cap \mathbb{E}]$, the continuity of τ_n follows (the definition of the sets U_n is given above equation (3.3)). To prove (4.15), note first that the assumption in (1.2), the irreducibility of the chain Z assumed in (1.8) and definition (4.7) imply $\mathsf{E}_z \int_0^t \Sigma_{II}^2(Z_s)\mathrm{d}s < \infty$ for all $z \in \mathbb{E}$ and $t \geq 0$. If $\mathsf{E}_z\Sigma_{II}^2(Z_{t_0}) = \infty$ for some $t_0 \geq 0$ and $z \in \mathbb{E}$, then the Markov property, the irreducibility of the chain Z and the positivity of $\Sigma_{II}^2 > 0$ together imply the following equalities for any $u > 0$: $\mathsf{E}_z\left[\Sigma_{II}^2(Z_{t_0+u})\right] = \mathsf{E}_z\left[\mathsf{E}[\Sigma_{II}^2(Z_{t_0+u})|\mathcal{F}_u]\right] = \mathsf{E}_z\left[\mathsf{E}[\Sigma_{II}^2(Z_{t_0+u})|Z_u]\right] = \infty$. Hence $\mathsf{E}_z \int_0^t \Sigma_{II}^2(Z_s)\mathrm{d}s = \infty$ for any $t \geq t_0$, which contradicts assumption (1.2). This contradiction implies

$$\mathsf{E}_z\left[\Sigma_{II}^2(Z_t)\right] < \infty \qquad \text{for any } z \in \mathbb{E} \text{ and } t \geq 0. \tag{4.16}$$

Since $\Sigma_{II}^2 > 0$, by definition we must have $(Q\Sigma_{II}^2)(z) \in (-\infty, \infty]$ for all $z \in \mathbb{E}$. Recall that P denotes the semigroup of Z. The backward Kolmogorov equation implies the equality

$$\begin{aligned}\mathsf{E}_z\left[\Sigma_{II}^2(Z_t)\right] - \Sigma_{II}^2(z) &= (P_t\Sigma_{II}^2)(z) - \Sigma_{II}^2(z) = \int_0^t ((P_uQ)\Sigma_{II}^2)(z)\,\mathrm{d}u \\ &= \int_0^t \mathsf{E}_z\left[(Q\Sigma_{II}^2)(Z_u)\right]\,\mathrm{d}u, \qquad \text{for all } z \in \mathbb{E} \text{ and } t \geq 0,\end{aligned}$$

which, together with (4.16) and the irreducibility of the chain Z, yields $|(Q\Sigma_{II}^2)(z)| < \infty$ for all $z \in \mathbb{E}$. We therefore get

$$\|Q_n\Sigma_{II}^2\|_\infty := \sup_{z\in\mathbb{E}} |(Q_n\Sigma_{II}^2)(z)| < \infty, \tag{4.17}$$

since, by definition (3.4), there are only finitely many states $z \in \mathbb{E}$, such that $(Q_n\Sigma_{II}^2)(z) \neq 0$, and for each of those states we have $(Q_n\Sigma_{II}^2)(z) = (Q\Sigma_{II}^2)(z)$.

Definition (3.3) implies the following inequalities

$$I_{\{\tau_n \leq s\}}\Sigma_{II}^2(Z_{\tau_n}) \leq \Sigma_{II}^2(Z_{s\wedge\tau_n}) = \Sigma_{II}^2(Z_s^n) \qquad \text{for any } s \geq 0.$$

Hence, to prove (4.15), we need to show $\mathsf{E}_z\Sigma_{II}^2(Z_s^n) < \infty$ for all states $z \in \mathbb{E}$ and times $s \geq 0$. Recall, from the definition of Q_n in (3.4), that Q_n is a bounded operator on the Banach space $\ell_\infty(\mathbb{E})$ of bounded real functions mapping $\mathbb{E}$ into $\mathbb{R}$. Let $\|Q_n\|_\infty < \infty$ denote its norm and recall that the norm satisfies $\|Q_n^k\|_\infty \leq \|Q_n\|_\infty^k$ for all $k \in \mathbb{N}$. We can therefore use the exponential series to define a bounded operator $\exp(sQ_n)$ and express the semigroup of Z^n as follows: $\mathsf{E}_z\Sigma_{II}^2(Z_s^n) = \left(\exp(sQ_n)\Sigma_{II}^2\right)(z)$. Hence, by (4.17), we find

$$\mathsf{E}_z\left[\Sigma_{II}^2(Z_s^n)\right] \leq \Sigma_{II}^2(z) + s\sum_{k=0}^\infty \frac{(s\|Q_n\|_\infty)^k}{(k+1)!}\|Q_n\Sigma_{II}^2\|_\infty < \infty,$$

for all $z \in \mathbb{E}$ and $s \geq 0$. This implies (4.15) and proves the claim.

In order to prove that $\zeta_n^{(II)}$ is differentiable in time, fix $t > 0$, $r < 0$, $z \in \mathbb{E}$ and, for any $\Delta t > 0$, define the random variable

$$D_{\Delta t}(r, z, t) := \left[G(r, A_{t+\Delta t}^{II}) - G(r, A_t^{II})\right] / \left(A_{t+\Delta t}^{II} - A_t^{II}\right).$$

Since $t > 0$ (resp. $\Delta t > 0$), we have $A_t^{II} > 0$ (resp. $(A_{t+\Delta t}^{II} - A_t^{II}) > 0$) P_z-a.s. Note also that the random variable $|D_{\Delta t}(r, z, t)|$ is bounded by a constant uniformly in $\Delta t > 0$. This follows from the existence of a uniform bound on $\frac{\partial G}{\partial t}(r,\cdot)$ in the second variable for any fixed $r<0$ and the mean value theorem. Furthermore the following limits hold:

$$\lim_{\Delta t\to 0} D_{\Delta t}(r,z,t) = \frac{\partial G}{\partial t}(r, A_t^{II}) \quad \mathsf{P}_z\text{-a.s.}, \qquad \lim_{\Delta t\to 0} \frac{A_{t+\Delta t}^{II}-A_t^{II}}{\Delta t} = \Sigma_{II}^2(Z_t^n) \quad \mathsf{P}_z\text{-a.s..} \tag{4.18}$$

The quotient $(\zeta_n^{(II)}(r,z,t+\Delta t)-\zeta_n^{(II)}(r,z,t))/\Delta t$ now takes the form

$$\begin{aligned} \mathsf{E}_z\left[D_{\Delta t}(r,z,t)\left(A_{t+\Delta t}^{II}-A_t^{II}\right)/\Delta t\right] &= \mathsf{E}_z\left[D_{\Delta t}(r,z,t)I_{\{\tau_n\le t\}}\left(A_{t+\Delta t}^{II}-A_t^{II}\right)/\Delta t\right] \\ &+ \mathsf{E}_z\left[D_{\Delta t}(r,z,t)I_{\{\tau_n\ge t+\Delta t\}}\left(A_{t+\Delta t}^{II}-A_t^{II}\right)/\Delta t\right] \\ &+ \mathsf{E}_z\left[D_{\Delta t}(r,z,t)I_{\{t<\tau_n<t+\Delta t\}}\left(A_{t+\Delta t}^{II}-A_t^{II}\right)/\Delta t\right]. \end{aligned} \tag{4.19}$$

Since $I_{\{\tau_n\le t\}}\left(A_{t+\Delta t}^{II}-A_t^{II}\right)/\Delta t = I_{\{\tau_n\le t\}}\Sigma_{II}^2(Z_{\tau_n})$, inequality (4.15) in the claim above, the Dominated Convergence Theorem, boundedness of $D_{\Delta t}(r,z,t)$ and (4.18) imply that the first expectation on the right-hand side of (4.19) converges to $\mathsf{E}_z\left[\frac{\partial G}{\partial t}(r,A_t^{II})I_{\{\tau_n\le t\}}\Sigma_{II}^2(Z_{\tau_n})\right]$ as $\Delta t\to 0$.

The random variable $I_{\{\tau_n\ge t+\Delta t\}}\left(A_{t+\Delta t}^{II}-A_t^{II}\right)/\Delta t$ is bounded by a constant for all Δt, since, on the event $\{\tau_n\ge t+\Delta t\}$, the chain Z has not left the finite state space $U_n\cap\mathbb{E}$ by the time $t+\Delta t$. Therefore, by the Dominated Convergence Theorem, the second expectation on the right-hand side of (4.19) converges to $\mathsf{E}_z\left[\frac{\partial G}{\partial t}(r,A_t^{II})I_{\{\tau_n>t\}}\Sigma_{II}^2(Z_t)\right]$ as $\Delta t\to 0$.

We will now prove that the third expectation on the right-hand side of (4.19) converges to 0 as $\Delta t\to 0$. By decomposing the path of Z^n at τ_n on the event $\{t<\tau_n<t+\Delta t\}$ and applying the arguments used in the previous two paragraphs to each of the two parts of the trajectory of Z^n, there exists a constant $C^+>0$ such that

$$\begin{aligned} \mathsf{E}_z\left[\frac{|D_{\Delta t}(r,z,t)|}{C^+}I_{\{t<\tau_n<t+\Delta t\}}\frac{A_{t+\Delta t}^{II}-A_t^{II}}{\Delta t}\right] &\le \mathsf{E}_z\left[\frac{\tau_n-t}{\Delta t}I_{\{t<\tau_n<t+\Delta t\}}\right] \\ &+ \mathsf{E}_z\left[\frac{t+\Delta t-\tau_n}{\Delta t}\Sigma_{II}^2(Z_{\tau_n})I_{\{t<\tau_n<t+\Delta t\}}\right] \\ &\le \mathsf{P}_z\left[t<\tau_n<t+\Delta t\right]+\mathsf{E}_z\left[\Sigma_{II}^2(Z_{\tau_n})I_{\{t<\tau_n<t+\Delta t\}}\right]. \end{aligned}$$

The probability $\mathsf{P}_z[t<\tau_n<t+\Delta t]$ tends to zero as $\Delta t\to 0$ by the claim and $\Sigma_{II}^2(Z_{\tau_n})I_{\{t<\tau_n<t+\Delta t\}}$ is, for $\Delta t\in(0,1)$, bounded above by the random variable $\Sigma_{II}^2(Z_{\tau_n})I_{\{\tau_n<t+1\}}$, which is integrable by (4.15). Therefore, another application of the Dominated Convergence Theorem implies that the function $\zeta_n^{(II)}$ is right-differentiable in time. In the case $\Delta t<0$, analogous arguments to the ones described above yield the left-differentiability of $\zeta_n^{(II)}$. The limits in (4.18) and their counterparts for $\Delta t<0$ imply that the left- and right-derivatives in t of $\zeta_n^{(II)}$ coincide and part (a) follows.

For the proof of part (b), note that, under the assumption $|\sigma_1|(z)\ne|\sigma_2|(z)$ for all $z\in\mathbb{E}$, we have $\Sigma_I^2(z)=(|\sigma_1|-|\sigma_2|)^2(z)>0$ for all $z\in\mathbb{E}$. Therefore, a completely analogous argument to the one that established the equality in (4.14), based on the stochastic time-change A^I and the fact that the laws of the processes (R^{In},Z^n) and $(r+B_{A^I},Z^n)$ coincide, where B_{A^I} denotes the Brownian motion B time-changed by the increasing process A^I, implies the representation of $\zeta_n^{(I)}$ given in part (b) of the lemma. The differentiability of $\zeta_n^{(I)}$ follows along the same lines as in part (a). The details of the arguments are now straightforward and are left to the reader. □

Lemma 4.3 shows that the functions $\zeta_n^{(II)}$ and $\zeta_n^{(I)}$ solve the HJB equations that correspond to the Problems (4.9) and (4.10).

Lemma 4.3. *Let* $\zeta_n^{(II)}$ *and* $\zeta_n^{(I)}$ *be given by* (4.11)-(4.12).

(a) The modulus of the partial derivative $|\frac{\partial \zeta_n^{(II)}}{\partial r}|$ *is bounded on the set* $(-\infty,-\varepsilon)\times\mathbb{E}\times(0,\infty)$ *for any* $\varepsilon>0$ *and the second derivative in space of* $\zeta_n^{(II)}$ *satisfies*

$$\frac{\partial^2 \zeta_n^{(II)}}{\partial r^2}(r,z,t) \leq 0 \qquad \text{for all } (r,z,t)\in(-\infty,0)\times\mathbb{E}\times(0,\infty). \tag{4.20}$$

If $|\sigma_1|(z)\neq|\sigma_2|(z)$ *for all* $z\in\mathbb{E}$*, then the modulus* $|\frac{\partial \zeta_n^{(II)}}{\partial r}|$ *is bounded on* $(-\infty,-\varepsilon)\times\mathbb{E}\times(0,\infty)$*, for any* $\varepsilon>0$*, and we have*

$$\frac{\partial^2 \zeta_n^{(I)}}{\partial r^2}(r,z,t) \leq 0 \qquad \text{for all } (r,z,t)\in(-\infty,0)\times\mathbb{E}\times(0,\infty). \tag{4.21}$$

(b) For any $T>0$ *the following holds for all* $r<0, t\in[0,T)$ *and* $z\in\mathbb{E}$

$$\inf_{c\in[-1,1]}\left\{\left[\mathcal{L}^c\left(\zeta_n^{(II)}(\cdot,\cdot,T-t)\right)\right](r,z)-\frac{\partial \zeta_n^{(II)}}{\partial t}(r,z,T-t)\right\} = 0, \tag{4.22}$$

where the function $\mathcal{L}^c\left(\zeta_n^{(II)}(\cdot,\cdot,T-t)\right)$ *is defined in* (2.3)*. Furthermore, we have*

$$\begin{aligned}\zeta_n^{(II)}(r,z,0) &= 1 \qquad \text{for all } (r,z)\in(-\infty,0)\times\mathbb{E},\\ \zeta_n^{(II)}(0,z,t) &= 0 \qquad \text{for all } (z,t)\in\mathbb{E}\times(0,\infty).\end{aligned}$$

If $|\sigma_1|(z)\neq|\sigma_2|(z)$ *for all* $z\in\mathbb{E}$*, then for all* $r<0, t\in[0,T)$ *and* $z\in\mathbb{E}$ *we have*

$$\sup_{c\in[-1,1]}\left\{\left[\mathcal{L}^c\left(\zeta_n^{(I)}(\cdot,\cdot,T-t)\right)\right](r,z)-\frac{\partial \zeta_n^{(I)}}{\partial t}(r,z,T-t)\right\} = 0 \tag{4.23}$$

and

$$\begin{aligned}\zeta_n^{(I)}(r,z,0) &= 1 \qquad \text{for all } (r,z)\in(-\infty,0]\times\mathbb{E},\\ \zeta_n^{(I)}(0,z,t) &= 0 \qquad \text{for all } (z,t)\in\mathbb{E}\times(0,\infty).\end{aligned}$$

Proof. (a) Let $G(r,t)$ be as defined in Lemma 4.2. Since $n'(x)=-xn(x)$, where $n(\cdot)$ is the standard normal pdf, we have

$$\frac{\partial G}{\partial r}(r,t) = -\frac{2}{\sqrt{t}}n\left(-\frac{r}{\sqrt{t}}\right), \tag{4.24}$$

$$\frac{\partial^2 G}{\partial r^2}(r,t) = 2\frac{r}{t^{3/2}}n\left(-\frac{r}{\sqrt{t}}\right)\leq 0, \tag{4.25}$$

for all $r<0$, $t>0$. The derivatives $\frac{\partial^i G}{\partial r^i}$, $i=1,2$, are bounded on $(r-\varepsilon,r+\varepsilon)\times(0,\infty)$ for any $r<0$ and small enough $\varepsilon>0$ and hence, as in the proof of Lemma 4.2, the Dominated Convergence Theorem implies

$$\frac{\partial \zeta_n^{(II)}}{\partial r}(r,z,t)=\mathsf{E}_z\left[\frac{\partial G}{\partial r}(r,A_t^{II})\right] \quad\text{and}\quad \frac{\partial^2 \zeta_n^{(II)}}{\partial r^2}(r,z,t)=\mathsf{E}_z\left[\frac{\partial^2 G}{\partial r^2}(r,A_t^{II})\right]$$

for all $r<0$, $z\in\mathbb{E}$, $t>0$. Inequality (4.20) now follows from the inequality in (4.25) and the boundedness of $|\frac{\partial\zeta_n^{(II)}}{\partial r}|$ on the product $(-\infty,-\varepsilon)\times\mathbb{E}\times(0,\infty)$ is a consequence of (4.24). Under the assumption that $|\sigma_1|(z)\neq|\sigma_2|(z)$ for all $z\in\mathbb{E}$, the properties of the partial derivatives in space of $\zeta_n^{(I)}$ follow from Lemma 4.2 (b) and (4.24)-(4.25) along the same lines.

(b) In order to prove that $\zeta_n^{(II)}$ satisfies the HJB equation above, define a bounded martingale $M^{II}=(M_t^{II})_{t\in[0,T]}$, where

$$M_t^{II}:=\mathsf{P}_{r,z}\left[\tau_0^+(R^{IIn})>T|\mathcal{F}_t\right],\qquad r\leq 0,z\in\mathbb{E},t\in[0,T],$$

where the process R^{IIn}, started at $R_0^{IIn}=r$, is given in (4.6) and the corresponding first-passage time $\tau_0^+(R^{IIn})$ is defined in (4.5). The Markov property of the process (R^{IIn},Z^n) and the definition of $\zeta_n^{(II)}$ in (4.11) imply the equality

$$\zeta_n^{(II)}\left(R_t^{IIn},Z_t^n,T-t\right)\;=\;\mathsf{P}_{r,z}\left[\tau_0^+(R^{IIn})>T|\mathcal{F}_t\right]=M_t^{II},\tag{4.26}$$

for all $r<0,z\in\mathbb{E},t\in[0,T]$.

Note that, by (4.24), the modulus $|\frac{\partial G}{\partial r}|$ is globally bounded on the set $(-\infty,-\varepsilon]\times(0,\infty)$ for any $\varepsilon>0$. Let $r<0$, pick any $\varepsilon\in(0,-r)$ and consider the stopped martingale $M^\varepsilon=(M_t^\varepsilon)_{t\in[0,T]}$, defined by

$$M_t^\varepsilon:=M^{II}_{\tau^+_{-\varepsilon}\wedge t},\qquad\text{where }\tau^+_{-\varepsilon}:=\inf\{s\geq 0:R_s^{IIn}=-\varepsilon\}.$$

Itô's formula for general semimartingales [10, Sec II.7, Thm. 33] applied to the representation in (4.26) of the martingale M^ε, Lemma 4.2 (a), Lemma 2.2 applied for the process $U=(R^{IIn}_{t\wedge\tau^+_{-\varepsilon}})_{t\in[0,T]}$ and the bounded function $\zeta_n^{(II)}$, and the facts that the quadratic covariation $[R^{IIn},Z^{n,i}]_t=0$ vanishes for all times t and coordinates $Z^{n,i}$ of the chain Z^n (recall that $\mathbb{E}\subset\mathbb{R}^d$), $\frac{\partial\zeta_n^{(II)}}{\partial r}$ is bounded on $(-\infty,-\varepsilon]\times\mathbb{E}\times(0,\infty)$ and $\mathsf{P}_{r,z}\left[R^{IIn}_{t\wedge\tau^+_{-\varepsilon}}\leq-\varepsilon,\,\forall t\geq 0\right]=1$ together yield that the process $N^\varepsilon=(N_t^\varepsilon)_{t\in[0,T]}$, where

$$\begin{aligned}N_t^\varepsilon\;=\;&\int_0^{t\wedge\tau_{-\varepsilon}+}\Bigg[\frac{1}{2}(|\sigma_1|+|\sigma_2|)^2(Z_s^n)\frac{\partial^2\zeta_n^{(II)}}{\partial r^2}(R_s^{IIn},Z_s^n,T-s)\\&+(Q\zeta_n^{(II)}(R_s^{IIn},\cdot,T-s))(Z_s^n)-\frac{\partial\zeta_n^{(II)}}{\partial t}(R_s^{IIn},Z_s^n,T-s)\Bigg]\,\mathrm{d}s,\end{aligned}$$

is a continuous martingale. Hence, since the quadratic variation of N^ε vanishes, we have $N_t^\varepsilon=0$ for all times t and starting points (r,z) with $r<-\varepsilon$. Since $\varepsilon>0$ is arbitrarily small and, by assumption (1.8), the process (R^{IIn},Z^n) visits neighbourhoods of all points in its state space with positive probability, this implies that for all $r<0$, $z\in\mathbb{E}$ and $t\in[0,T)$:

$$\frac{1}{2}(|\sigma_1|+|\sigma_2|)^2(z)\frac{\partial^2\zeta_n^{(II)}}{\partial r^2}(r,z,T-t)+(Q\zeta_n^{(II)}(r,\cdot,T-t))(z)-\frac{\partial\zeta_n^{(II)}}{\partial t}(r,z,T-t)=0.\tag{4.27}$$

To prove the first HJB equation above, note that for any $c\in[-1,1]$ the following inequality holds

$$(\sigma_1^2-2c\sigma_1\sigma_2+\sigma_2^2)(z)\frac{\partial^2\zeta_n^{(II)}}{\partial r^2}(r,z,T-t)\;\;\geq\;\;(|\sigma_1|+|\sigma_2|)^2(z)\frac{\partial^2\zeta_n^{(II)}}{\partial r^2}(r,z,T-t),$$

for all $r<0$ and $z\in\mathbb{E}$, since $\frac{\partial^2\zeta_n^{(II)}}{\partial r^2}(r,z,T-t)\le 0$ by (4.20). This inequality, the definition of $\mathcal{L}^c\zeta_n^{(II)}$ in (2.3) and identity (4.27) imply (4.22). The boundary behaviour of the function $\zeta_n^{(II)}$, stated in the lemma, at $t=0$ and at $r=0$ follows directly from the representation of $\zeta_n^{(II)}$ given in (4.11).

In the case of the function $\zeta_n^{(I)}$, by (4.21) it follows that

$$\frac{1}{2}(|\sigma_1|-|\sigma_2|)^2(z)\frac{\partial^2\zeta_n^{(I)}}{\partial r^2}(r,z,T-t) \quad\le\quad \frac{1}{2}(\sigma_1^2-2c\sigma_1\sigma_2+\sigma_2^2)(z)\frac{\partial^2\zeta_n^{(I)}}{\partial r^2}(r,z,T-t)$$

for any $c\in[-1,1]$ and all $r<0$, $z\in\mathbb{E}$, $t\in[0,T)$. An analogous argument to the one in the case of $\zeta_n^{(II)}$ establishes the HJB equation in (4.23) and the required boundary behaviour. This concludes the proof of the lemma. □

We can now prove that $\zeta_n^{(II)}$ and $\zeta_n^{(I)}$ are the value functions for Problems (4.9) and (4.10).

Lemma 4.4. *Pick a time horizon $T>0$ and, for any $V\in\mathcal{V}$, let $R^n(V)$ and $\tau_0^+(R^n(V))$ be as in (4.4) and (4.5) respectively.*

(a) The function $\zeta_n^{(II)}$, defined in (4.11), satisfies the following:

$$\zeta_n^{(II)}(r,z,T)=\inf_{V\in\mathcal{V}}\mathsf{P}_{r,z}\left[\tau_0^+(R^n(V))>T\right]\qquad \textit{for any } r\le 0, z\in\mathbb{E}.$$

(b) Assume that $|\sigma_1|(z)\neq|\sigma_2|(z)$ for all $z\in\mathbb{E}$. Then the function $\zeta_n^{(I)}$, given in (4.12), satisfies

$$\zeta_n^{(I)}(r,z,T)=\sup_{V\in\mathcal{V}}\mathsf{P}_{r,z}\left[\tau_0^+(R^n(V))>T\right]\qquad \textit{for any } r\le 0, z\in\mathbb{E}.$$

Proof. (a) Pick any Brownian motion $V\in\mathcal{V}$ and, for any $t\in[0,T]$, define the corresponding Brownian motion $V^{IInt}=(V_s^{IInt})_{s\ge0}\in\mathcal{V}$ by

$$V_s^{IInt}:=\begin{cases} V_s & \text{if } s\le t,\\ V_t+V_s^{IIn}-V_t^{IIn} & \text{if } s>t,\end{cases}\tag{4.28}$$

where $V^{IIn}\in\mathcal{V}$ is given in (1.5) with Z substituted by the stopped chain Z^n. The Bellman process $S^{II}=(S_t^{II})_{t\in[0,T]}$ takes the form

$$S_t^{II}:=\zeta_n^{(II)}\left(R^n_{\tau_0^+\wedge t}(V),Z^n_{\tau_0^+\wedge t},T-t\right)=\mathsf{P}_{r,z}\left[\tau_0^+(R^n(V^{IInt}))>T|\mathcal{F}_t\right],$$

for any $r\le 0, z\in\mathbb{E}, t\in[0,T]$, where the second equality follows from the Markov property and definitions (4.11) and (4.28) of the candidate value function $\zeta_n^{(II)}$ and of the Brownian motion V^{IIn}. Note that the stopping time τ_0^+ on the right-hand side of the second equality is given by $\tau_0^+:=\tau_0^+(R^n(V))$, and hence does depend on the choice of V. This is not explicitly stated in the formula for brevity of notation.

To establish that the Bellman principle applies, it is sufficient to prove that the process S^{II} is a submartingale for any starting point (r,z) of the process $(R^n(V),Z^n)$. By Lemma 4.2 (a) and Itô's formula for general semimartingales [10, Sec II.7, Thm. 33] we obtain the following pathwise representation of the process S^{II}:

$$\begin{aligned}(4.29)\qquad S^{II}_t &= \zeta^{(II)}_n(r,z,T)+\int_0^{t\wedge\tau_0^+}\frac{\partial\zeta^{(II)}_n}{\partial r}(R^n_s(V),Z^n_s,T-s)\,\mathrm{d}R^n_s(V)\\ &+ \int_0^{t\wedge\tau_0^+}\left[\left(\mathcal{L}^{C_t}\zeta^{(II)}_n\right)(R^n_s(V),Z^n_s,T-s)-\frac{\partial\zeta^{(II)}_n}{\partial t}(R^n_s(V),Z^n_s,T-s)\right]\mathrm{d}s\\ &+ \sum_{0<s\le t\wedge\tau_0^+}\left[\zeta^{(II)}_n(R^n_s(V),Z^n_s,T-s)-\zeta^{(II)}_n(R^n_s(V),Z^n_{s-},T-s)\right]\\ &- \int_0^{t\wedge\tau_0^+}(Q\zeta^{(II)}_n(R^n_s(V),\cdot,T-s))(Z^n_{s-})\,\mathrm{d}s,\end{aligned}$$

where $C=(C_t)_{t\ge 0}$ is the stochastic correlation process from Lemma 2.1, which corresponds to the Brownian motion V, and $\mathcal{L}^c\zeta^{(II)}_n$ is defined in (2.3) for any constant $c\in[-1,1]$. This representation of S^{II} relies on the fact that the continuous part of the quadratic covariation $[R^n(V),Z^{n,i}]_t$ vanishes for all times t and coordinates $Z^{n,i}$ of the chain Z^n. Since $C_t\in[-1,1]$ for all $t\ge 0$, equality (4.22) implies that the first integral on the right-hand side is almost surely non-negative and hence we have:

$$\begin{aligned}(4.30)\qquad S^{II}_t &\ge \zeta^{(II)}_n(r,z,T)+\int_0^{t\wedge\tau_0^+}\frac{\partial\zeta^{(II)}_n}{\partial r}(R^n_s(V),Z^n_s,T-s)\,\mathrm{d}R^n_s(V)\\ &+ \sum_{0<s\le t\wedge\tau_0^+}\left[\zeta^{(II)}_n(R^n_s(V),Z^n_s,T-s)-\zeta^{(II)}_n(R^n_s(V),Z^n_{s-},T-s)\right]\\ &- \int_0^{t\wedge\tau_0^+}(Q\zeta^{(II)}_n(R^n_s(V),\cdot,T-s))(Z^n_{s-})\,\mathrm{d}s.\end{aligned}$$

Apply Lemma 2.2, with $F(s,r,z):=\zeta^{(II)}_n(r,z,T-s)$, $U:=R^n(V)$ and the chain Z^n (with bounded generator Q_n), to conclude that the process given by the second and third lines of the last display is a martingale.

The first integral on the right-hand side of (4.30) is clearly a local martingale since the integrator $R^n(V)$ is a martingale. Define a stopping time $\tau^+_{-\varepsilon}:=\inf\{t\ge 0: R^n_t(V)=-\varepsilon\}$ for any small $\varepsilon>0$ and note that $\tau^+_{-\varepsilon}<\tau^+_0$. The quadratic variation of the local martingale

$$\left(\int_0^{t\wedge\tau^+_{-\varepsilon}}\frac{\partial\zeta^{(II)}_n}{\partial r}(R^n_s(V),Z^n_s,T-s)\,\mathrm{d}R^n_s(V)\right)_{t\in[0,T]}$$

is, by Lemma 4.3 (a) and assumption (1.2), bounded above by an integrable random variable. Therefore this stochastic integral is a martingale and, by taking expectations on both sides of inequality (4.30), we obtain

$$\mathsf{E}_{r,z}\left[S^{II}_{T\wedge\tau^+_{-\varepsilon}}\right]\ge\zeta^{(II)}_n(r,z,T)\qquad\text{for all } \varepsilon>0.$$

Since the paths of $R^n(V)$ are continuous we have $\lim_{\varepsilon\to0}\tau^+_{-\varepsilon}=\tau^+_0$ $\mathsf{P}_{r,z}$-a.s. Hence the definition of S^{II} and its representations in (4.29) imply the following limit:

$$\lim_{\varepsilon\to0}S^{II}_{T\wedge\tau^+_{-\varepsilon}}=S^{II}_{T\wedge\tau^+_0}=I_{\{\tau^+_0(R^n(V))>T\}}.$$

Indeed, on the complement of the event $\{\tau^+_0=T\}$ the limit holds by the definition of S^{II}. On $\{\tau^+_0=T\}$, the limit is implied by representations (4.29) because the integrals on the right-hand side of (4.29) are continuous in the upper bound of the integration (recall that $R^n(V)$ is a continuous martingale) and $\tau^+_{-\varepsilon} \uparrow \tau^+_0 = T$, while the sum in (4.29) is $\mathsf{P}_{r,z}$-a.s. constant in ε as the chain Z^n does not jump $\mathsf{P}_{r,z}$-a.s. at time T. Since S^{II} is a bounded process, this almost sure limit and the Dominated Convergence Theorem imply the inequality

$$\mathsf{P}_{r,z}\left[\tau^+_0(R^n(V)) > T\right] \geq \zeta^{(II)}_n(r,z,T) \qquad \text{for any } V \in \mathcal{V}.$$

This inequality and the definition of $\zeta^{(II)}_n$ in (4.11) imply part (a) of the lemma.

(b) For any Brownian motion $V \in \mathcal{V}$ and $t \in [0,T]$, define $V^{Int} = (V^{Int}_s)_{s\geq 0} \in \mathcal{V}$ by

$$V^{Int}_s := \begin{cases} V_s & \text{if } s \leq t, \\ V_t + V^{In}_s - V^{In}_t & \text{if } s > t, \end{cases}$$

where $V^{In} \in \mathcal{V}$ is given in (1.5) with Z^n in the place of Z. The Bellman process $S^I = (S^I_t)_{t\in[0,T]}$ is given by $S^I_t := \zeta^{(I)}_n\left(R^n_{\tau^+_0\wedge t}(V), Z^n_{\tau^+_0\wedge t}, T-t\right) = \mathsf{P}_{r,z}\left[\tau^+_0(R^n(V^{Int})) > T|\mathcal{F}_t\right]$, for any $r \leq 0, z \in \mathbb{E}, t \in [0,T]$, where again $\tau^+_0 := \tau^+_0(R^n(V))$ and the second equality holds by the Markov property and (4.12). Analogous arguments to the ones used in the proof of part (a) can now be applied to conclude that the process S^I, appropriately stopped is a supermartingale. Then the optimality of the Brownian motion $V^{In} \in \mathcal{V}$ follows by a limiting argument as in part (a). This concludes the proof of the lemma. □

4.2. **Proof of Theorem 4.1.** We establish Theorem 4.1 in two steps. The first step consists of generalising the result of Lemma 4.4 (b),

$$\zeta^{(I)}_n(r,z,T) = \sup_{V\in\mathcal{V}} \mathsf{P}_{r,z}\left[\tau^+_0(R^n(V)) > T\right] \qquad \text{for any } r \leq 0, z \in \mathbb{E} \text{ and } n \in \mathbb{N}, \tag{4.31}$$

to the case where the assumption $|\sigma_1|(z) \neq |\sigma_2|(z)$ for all $z \in \mathbb{E}$ is not satisfied. The function $\zeta^{(I)}_n$ in this expression is given in (4.12) and $R^n(V)$ and $\tau^+_0(R^n(V))$ are defined in (4.4) and (4.5) respectively. The second step in the proof of Theorem 4.1 consists of a limiting argument that generalises Lemma 4.4 to volatility chains with possibly unbounded generator matrices.

Consider the case of general volatility functions $\sigma_1, \sigma_2 : \mathbb{E} \to \mathbb{R}$, which are only assumed to satisfy integrability condition (1.2). Then, for any $\epsilon > 0$, there exists a function $\sigma^\epsilon_1 : \mathbb{E} \to \mathbb{R}$ that satisfies (1.2), coincides with σ_1 on the set where the moduli of the original volatility functions are already distinct,

$$\{z \in \mathbb{E} : \sigma^\epsilon_1(z) = \sigma_1(z)\} \quad = \quad \{z \in \mathbb{E} : |\sigma_1|(z) \neq |\sigma_2|(z)\}$$

and has the following properties:

$$|\sigma^\epsilon_1|(z) \neq |\sigma_2|(z), \qquad |\sigma^\epsilon_1(z) - \sigma_1(z)| < \epsilon, \qquad \operatorname{sgn}(\sigma^\epsilon_1(z)\sigma_2(z)) = \operatorname{sgn}(\sigma_1(z)\sigma_2(z)) \qquad \text{for all } z \in \mathbb{E}.$$

Note that, in order to define σ^ϵ_1, we used the fact that $|\sigma_1| + |\sigma_2| > 0$, which implies that if $|\sigma_1|(z) = |\sigma_2|(z)$ for some $z \in \mathbb{E}$, then $|\sigma_1|(z) > 0$.

Define the process $R^{n,\epsilon}(V)$ by (4.4), but with σ_1 replaced by σ^ϵ_1, and note that for any $t \geq 0$ we have

$$R^{n,\epsilon}_t(V) - R^n_t(V) = \int_0^t \left[\sigma^\epsilon_1(Z^n_s) - \sigma_1(Z^n_s)\right] \mathrm{d}B_s. \tag{4.32}$$

The chain Z has cádlág paths in a state space with discrete topology by assumption (1.8) and hence Z^n, defined in (3.3), has only finitely many jumps, say $N_T(Z^n) \in \mathbb{N} \cup \{0\}$, during the time interval $[0,T]$.

Therefore identity (4.32) implies the inequality $|R_t^{n,\epsilon}(V) - R_t^n(V)| \leq \epsilon(1+N_T(Z^n))(\sup_{s\in[0,T]} B_s - \inf_{s'\in[0,T]} B_{s'})$ for all $t \in [0,T]$. Since the right-hand side of this inequality does not depend on $t\in[0,T]$, the random variables $S_T^\epsilon(V) := \sup_{t\in[0,T]} R_t^{n,\epsilon}(V)$ and $S_T(V) := \sup_{t\in[0,T]} R_t^n(V)$ satisfy

$$\left|S_T^\epsilon(V) - S_T(V)\right| \quad \leq \quad \epsilon(1+N_T(Z^n))\left(\sup_{s\in[0,T]} B_s - \inf_{s'\in[0,T]} B_{s'}\right) \quad \text{and} \quad \lim_{\epsilon\to 0} S_T^\epsilon(V) = S_T(V)\ \mathsf{P}_{r,z}\text{-a.s.}$$

This implies $I_{\{S_T(V)<0\}} \leq \liminf_{\epsilon\to 0} I_{\{S_T^\epsilon(V)<0\}}$. Fatou's lemma and the fact that $\{S_T(V)<0\} = \{\tau_0^+(R^n(V)) > T\}$ therefore imply

$$\begin{aligned}\mathsf{P}_{r,z}\left[\tau_0^+(R^n(V))>T\right] &\leq \liminf_{\epsilon\to 0}\mathsf{P}_{r,z}\left[S_T^\epsilon(V)<0\right] = \liminf_{\epsilon\to 0}\mathsf{P}_{r,z}\left[\tau_0^+(R^{n,\epsilon}(V))>T\right] \\ &\leq \liminf_{\epsilon\to 0}\mathsf{P}_{r,z}\left[\tau_0^+\left(R^{In,\epsilon}\right)>t\right], \end{aligned} \tag{4.33}$$

where the process $R^{In,\epsilon}$ is defined in (4.6) with σ_1 substituted by σ_1^ϵ and the last inequality follows by Lemma 4.4 (b).

Define a strictly increasing process $A^{I,\epsilon} = (A_t^{I,\epsilon})_{t\geq 0}$ and a non-decreasing process $A^I = (A_t^I)_{t\geq 0}$, analogous to (4.13), by

$$A_t^{I,\epsilon} := \int_0^t (|\sigma_1^\epsilon| - |\sigma_2|)^2(Z_s^n)\,\mathrm{d}s, \qquad A_t^I := \int_0^t (|\sigma_1| - |\sigma_2|)^2(Z_s^n)\,\mathrm{d}s.$$

The properties of σ_1^ϵ imply that $A_t^{I,\epsilon} \geq A_t^I$ P_z-a.s. for all $t\geq 0$. As in the proof of Lemma 4.2, the independence of B and Z (by Lemma 2.3) implies that the processes $(R^{In,\epsilon}, Z^n)$ and $(r+B_{A^{I,\epsilon}}, Z^n)$ are equal in law, where $B_{A^{I,\epsilon}}$ denotes the Brownian motion B time-changed by the precess $A^{I,\epsilon}$. Similarly, we have that the laws of (R^{In}, Z^n) and $(r+B_{A^I}, Z^n)$ coincide, where R^{In} is given in (4.6). These observations imply the almost sure inequality, $\inf\{t\geq 0 : B_{A_t^{I,\epsilon}} = -r\} \leq \inf\{t\geq 0 : B_{A_t^I} = -r\}$, and the fact that the random variable on the left-hand side of this inequality has the same law as $\tau_0^+(R^{n,\epsilon}(V))$ while the one on the right-hand side is distributed as $\tau_0^+(R^n(V))$. This therefore implies the inequality

$$\mathsf{P}_{r,z}\left[\tau_0^+(R^{In,\epsilon})>T\right] \quad \leq \quad \mathsf{P}_{r,z}\left[\tau_0^+(R^{In})>T\right]$$

which, together with (4.33) and the definition of $\zeta^{(I)}$ in (4.12), yields (4.31) and hence concludes step one of the proof of Theorem 4.1.

In the second step of the proof we assume that the volatility process Z is a general $(\mathcal{F}_t)$-Markov chain with state space $\mathbb{E}\subset\mathbb{R}^d$, defined in Section 1. For any $n\in\mathbb{N}$, in (3.3) we defined a stopping time τ_n and a chain Z^n, which is equal to Z up to the time τ_n. Lemma 4.4 (a), equality (4.31) and the definitions of the functions $\zeta_n^{(II)}$ and $\zeta_n^{(I)}$ in (4.11)-(4.12) imply the following inequalities for any Brownian motion $V\in\mathcal{V}$,

$$\mathsf{P}_{r,z}\left[\tau_0^+(R^{IIn})>T\right] \leq \mathsf{P}_{r,z}\left[\tau_0^+(R^n(V))>T\right] \leq \mathsf{P}_{r,z}\left[\tau_0^+(R^{In})>T\right], \tag{4.34}$$

where $R^n(V)$ is given in (4.4) and R^{In}, R^{IIn} are defined in (4.6). Furthermore, for any t in the stochastic interval $[0,\tau_n]$ the following equalities hold:

$$R_t^n(V) = R_t(V), \qquad R_t^{IIn} = R_t(V^{II}), \qquad R_t^{In} = R_t(V^I),$$

where the process $R(V)$ is defined in (2.2) and the Brownian motions V^I and V^{II} are given in (1.5). Therefore, we have that, on the event $\{\tau_n > T\}$, the random variables $I_{\{\tau_0^+(R^n(V))>T\}}$ and $I_{\{\tau_0^+(R(V))>T\}}$ coincide. The same holds true for the pairs $I_{\{\tau_0^+(R^{IIn})>T\}}$ and $I_{\{\tau_0^+(R(V^{II}))>T\}}$, and $I_{\{\tau_0^+(R^{In})>T\}}$ and $I_{\{\tau_0^+(R(V^I))>T\}}$. Since $(\tau_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence of stopping times, such that $\tau_n \nearrow \infty$ P_z-a.s. as $n\to\infty$, we obtain the following almost sure limits:

$$\lim_{n\to\infty} I_{\{\tau_0^+(R^{IIn})>T\}} = I_{\{\tau_0^+(R(V^{II}))>T\}}, \qquad \lim_{n\to\infty} I_{\{\tau_0^+(R^{In})>T\}} = I_{\{\tau_0^+(R(V^{I}))>T\}},$$
$$\lim_{n\to\infty} I_{\{\tau_0^+(R^{n}(V))>T\}} = I_{\{\tau_0^+(R(V))>T\}}.$$

These equalities, a final application of the Dominated Convergence Theorem and the inequalities in (4.34) imply (4.1)-(4.2). This concludes the proof. □

4.3. **Time-varying extremal couplings.** It is tempting to try to prove/generalise the result in Theorem 4.1 via a direct argument based on the Dambis, Dubins-Schwartz (DDS)-Brownian motion [11, Thm V.1.6], avoiding the Bellman principle. Let $\Sigma^{(1)} = (\Sigma^{(1)}_t)_{t\geq 0}$ and $\Sigma^{(2)} = (\Sigma^{(2)}_t)_{t\geq 0}$ be two progressively measurable processes on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathsf{P})$, such that $\int_0^t \mathsf{E}\left(\Sigma^{(i)}_s\right)^2 \mathrm{d}s < \infty$ for $i = 1, 2$ and any $t \geq 0$. As usual, for any $V \in \mathcal{V}$, define the difference process $R(V) = (R_t(V))_{t\geq 0}$ by $R_t(V) := r + \int_0^t \Sigma^{(1)}_s \,\mathrm{d}B_s - \int_0^t \Sigma^{(2)}_s \,\mathrm{d}V_s$, $r \leq 0$, $t \geq 0$. Let the candidate extremal Brownian motions $V^{II} = (V^{II}_t)_{t\geq 0}$ and $V^I = (V^I_t)_{t\geq 0}$ be given by

$$V^{II}_t := -\int_0^t \operatorname{sgn}\left(\Sigma^{(1)}_s\Sigma^{(2)}_s\right) \mathrm{d}B_s \qquad \text{and} \qquad V^{I}_t := \int_0^t \operatorname{sgn}\left(\Sigma^{(1)}_s\Sigma^{(2)}_s\right) \mathrm{d}B_s. \tag{4.35}$$

Under these assumptions the process $R(V)$ is a martingale for each $V \in \mathcal{V}$. Hence, by [11, Thm V.1.6], there exists a (DDS)-Brownian motion W^V, adapted to the filtration $(\mathcal{F}_{E_u(V)})_{u\geq 0}$, where the processes $A(V) = (A_t(V))_{t\geq 0}$ and $E(V) = (E_u(V))_{u\geq 0}$ are defined by

$$A_t(V) := \int_0^t \left((\Sigma^{(1)}_s)^2 - 2C_s\Sigma^{(1)}_s\Sigma^{(2)}_s + (\Sigma^{(2)}_s)^2\right) \mathrm{d}s \qquad \text{and} \qquad E_u(V) := \inf\{s\,:\, A_s(V) > u\}$$

and $C = (C_t)_{t\geq 0}$ is the stochastic correlation between the Brownian motions B and V from (2.1) in Lemma 2.1, and the following representation holds

$$R_t(V) \;=\; r + W^V_{A_t(V)} \qquad \text{for all } t \geq 0.$$

It is clear from these definitions that the following inequalities hold almost surely for all times $t \geq 0$:

$$A^I_t := \int_0^t \left(|\Sigma^{(1)}_s| - |\Sigma^{(2)}_s|\right)^2 \mathrm{d}s \leq A_t(V) \leq \int_0^t \left(|\Sigma^{(1)}_s| + |\Sigma^{(2)}_s|\right)^2 \mathrm{d}s =: A^{II}_t. \tag{4.36}$$

Let $\tau_0^+(R(V))$, $\tau_0^+(r + W^V_{A^{II}})$ and $\tau_0^+(r + W^V_{A^I})$ denote the first-passage times over zero of the processes $R(V)$, $r + W^V_{A^{II}}$ and $r + W^V_{A^I}$, respectively, and note that the inequalities in (4.36) imply

$$\tau_0^+(r + W^V_{A^{II}}) \leq \tau_0^+(R(V)) \leq \tau_0^+(r + W^V_{A^I}) \tag{4.37}$$

on the entire probability space Ω for every Brownian motion $V \in \mathcal{V}$.

It is tempting to conclude from this that the processes $r + W^V_{A^{II}}$ and $R(V^{II})$, where the Brownian motion V^{II} is defined in (4.35) have the same law (ditto for the pair $r + W^V_{A^I}$ and $R(V^I)$), which would together with (4.37), yield a generalisation or an alternative proof of Theorem 4.1. However, the counterexample in Section 1.2.1 demonstrates that the generalised mirror coupling in (4.35) can be suboptimal in this setting. The counterexamples to Theorem 4.1, based on the continuous-time Markov chains in Section 5.2, which are adapted non-Markovian processes with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, clearly show that this approach cannot be used as an alternative proof of Theorem 4.1, because it only requires the volatility processes to be $(\mathcal{F}_t)$-adapted. We should stress here however, that in the case of deterministic integrands $\Sigma^{(1)}$ and $\Sigma^{(2)}$, Proposition 4.5 can be established. [3]

Proposition 4.5. *Let $\Sigma^{(1)},\Sigma^{(2)}$ be deterministic processes (i.e. measurable functions of time) that satisfy the integrability condition above, $|\Sigma^{(1)}_s|,|\Sigma^{(2)}_s|>0$ for all $s\geq 0$ and $A^{II}_t, A^I_t\nearrow\infty$ as $t\nearrow\infty$. Then for any time horizon $T>0$ and Brownian motion $V\in\mathcal{V}$, the following inequalities hold:*

$$\mathsf{P}_r\left[\tau_0^+(R(V^{II}))>T\right]\leq \mathsf{P}_r\left[\tau_0^+(R(V))>T\right]\leq \mathsf{P}_r\left[\tau_0^+(R(V^{I}))>T\right].$$

Proof. The integrability assumption $\int_0^t(\Sigma^{(i)}_s)^2\mathrm{d}s<\infty$, $i=1,2$, from the beginning of Section 4.3 implies that A^{II} is a well-defined, finite, strictly increasing differentiable function. Its inverse E^{II}, which is defined on $[0,\infty)$ since the limit A^{II} tends to infinity with increasing time, is also strictly increasing and differentiable and satisfies the following ODE:

$$E^{II}_u=\int_0^u\left(|\Sigma^{(1)}_{E^{II}_s}|+|\Sigma^{(2)}_{E^{II}_s}|\right)^{-2}\mathrm{d}s. \tag{4.38}$$

Since the left-hand side of (4.38) is finite for all $u\geq 0$, for any $V\in\mathcal{V}$ the process $W^{IIV}=(W^{IIV}_t)_{t\geq 0}$,

$$W^{IIV}_t:=\int_0^{A^{II}_t}\left(|\Sigma^{(1)}_{E^{II}_u}|+|\Sigma^{(2)}_{E^{II}_u}|\right)^{-1}\mathrm{d}W^V_u, \tag{4.39}$$

is well-defiend for all $t\geq 0$, where W^V denotes the (DDS)-Brownian motion introduced above. The quadratic variation of the continuous local martingale W^{IIV} is by (4.38) equal to $[W^{IIV}]_t=E^{II}_{A^{II}_t}=t$, making W^{IIV} a Brownian motion by Lévy's characterisation theorem. By (4.39) we obtain $\mathrm{d}W^{IIV}_{E^{II}_s}=\mathrm{d}W^V_s/(|\Sigma^{(1)}_{E^{II}_s}|+|\Sigma^{(2)}_{E^{II}_s}|)$ and $W^V_u=\int_0^u(|\Sigma^{(1)}_{E^{II}_v}|+|\Sigma^{(2)}_{E^{II}_v}|)\,\mathrm{d}W^{IIV}_{E^{II}_v}=\int_0^{E^{II}_u}(|\Sigma^{(1)}_s|+|\Sigma^{(2)}_s|)\,\mathrm{d}W^{IIV}_s$, where the last equality follows by [11, Prop V.1.4]. Hence we find $W^V_{A^{II}_t}=\int_0^t(|\Sigma^{(1)}_s|+|\Sigma^{(2)}_s|)\,\mathrm{d}W^{IIV}_s$ for all $t\geq 0$. Since $\Sigma^{(1)}$ and $\Sigma^{(2)}$ are non-zero everywhere by assumption, the process $W=(W_t)_{t\geq 0}$, given by $W_t:=\int_0^t\mathrm{sgn}(\Sigma^{(1)}_s)\mathrm{d}W^{IIV}_s$, is a Brownian motion and the equalities $|\Sigma^{(1)}_s|=\mathrm{sgn}(\Sigma^{(1)}_s)\Sigma^{(1)}_s$ and $\mathrm{sgn}(\Sigma^{(1)}_s\Sigma^{(2)}_s)=\mathrm{sgn}(\Sigma^{(1)}_s)\,\mathrm{sgn}(\Sigma^{(2)}_s)$ hold. Therefore, the processes $R(V^{II})$, where V^{II} is given in (4.35), and $r+W^V_{A^{II}}$ are equal in law and hence (4.37) implies the first inequality in the proposition. The second inequality follows along the same lines. □

Remarks. (i) It is important to note that the Brownian motion W^{IIV}, introduced in (4.39), is not an element of the set $\mathcal{V}$ as it is in general not adapted to the original filtration $(\mathcal{F}_t)_{t\geq 0}$. In fact, W^{IIV} is an $(\mathcal{F}_t)$-Brownian motion only in the case $V=V^{II}$.

(ii) The final step in the proof of Proposition 4.5 relies on the fact that the stochastic integrals

$$\int_0^\cdot\left(\Sigma^{(1)}_s+\mathrm{sgn}\left(\Sigma^{(1)}_s\Sigma^{(2)}_s\right)\Sigma^{(2)}_s\right)\mathrm{d}B_s,\quad \int_0^\cdot\left(\Sigma^{(1)}_s+\mathrm{sgn}\left(\Sigma^{(1)}_s\Sigma^{(2)}_s\right)\Sigma^{(2)}_s\right)\mathrm{sgn}\left(\Sigma^{(1)}_s\right)\mathrm{d}W^{IIV}_s,$$

where B is a fixed Brownian motion and W^{IIV} is defined in (4.39), are equal in law, which holds since $\Sigma^{(1)}$ and $\Sigma^{(2)}$ are deterministic. Assume that both processes $\Sigma^{(1)},\Sigma^{(2)}$ non-deterministic, but adapted to $(\mathcal{F}_t)_{t\geq 0}$ and independent of the Brownian motion B. Then, it is not clear whether

[3] We would like to thank David Hobson for this observation.

one can define the second stochastic integral, since W^{IIV} is not $(\mathcal{F}_t)$-Brownian motion. Even if this were possible, the laws of the two integrals would in general not coincide since the integrand and the integrator are independent in the former and dependent in the latter integral.

5. Counterexamples

5.1. **The presence of drift.** If either of the processes X and $Y(V)$ in (1.1) can have drift, the conclusion of Theorem 1.2 fails as the following example demonstrates.

Let $R(V)$ be the difference of X and $Y(V)$ and assume that it takes the form

$$R_t(V) = r + \mu t + B_t - \bar{\sigma} V_t,$$

where B is the fixed $(\mathcal{F}_t)$-Brownian motion, $V \in \mathcal{V}$ an arbitrary $(\mathcal{F}_t)$-Brownian motion, $\bar{\sigma}$ a volatility parameter different from 1, r a strictly negative starting point and μ a constant positive drift. Then the candidate extremal Brownian motions in (1.5) are given by $V^I = B$ and $V^{II} = -B$ and the following lemma holds.

Lemma 5.1. *For any starting point $r < 0$, time horizon $T > 0$, volatility $\bar{\sigma} > 0$ and positive drift $\mu > 0$, the inequality $\mathsf{P}_r\left[\tau_0^+(R(V^I)) > T\right] < \mathsf{P}_r\left[\tau_0^+(R(V^{II})) > T\right]$ holds.*

Lemma 5.1 implies that Theorem 1.2 cannot hold for processes with drift. An intuitive explanation for this phenomenon is as follows: in the presence of a large drift upwards, it is better to reduce the volatility as much as possible (in this case to the level $|1-\bar{\sigma}|$), instead of increasing it to its maximal value (equal to $(1+\bar{\sigma})$), since the drift makes the processes X and $Y(V)$ couple before time T.

Proof. Fix $r < 0$, $T > 0$, $\bar{\sigma} > 0$, $\mu > 0$ and define the function $F : (0,\infty) \to [0,1]$ by

$$F(v) := N\left(-\frac{r+\mu T}{\sqrt{T}} v\right) - \mathrm{e}^{-2\mu r v^2} N\left(-\frac{r-\mu T}{\sqrt{T}} v\right), \qquad v > 0,$$

and recall that $\mathsf{P}_r\left[\tau_0^+(R(V^I)) > T\right] = F\left(1/|1-\bar{\sigma}|\right)$, $\mathsf{P}_r\left[\tau_0^+(R(V^{II})) > T\right] = F\left(1/(1+\bar{\sigma})\right)$ (see e.g. [2, II.2.1, Eq. 1.1.4]), where $N(\cdot)$ denotes the normal cdf. To establish the lemma it is sufficient to show that F is strictly decreasing on the bounded interval $[1/(1+\bar{\sigma}), 1/|1-\bar{\sigma}|]$. Since the derivative takes the form $F'(v) = -2\mu\sqrt{T} n\left(-\frac{r+\mu T}{\sqrt{T}} v\right) + 4\mu r v \mathrm{e}^{-2\mu r v^2} N\left(-\frac{r-\mu T}{\sqrt{T}} v\right)$ and clearly satisfies $F'(v) < 0$ for all $v > 0$, the lemma follows.[4] □

5.2. **$(\mathcal{F}_t)$-adapted non-$(\mathcal{F}_t)$-Markov processes on a discrete state space.** In this section we construct two continuous-time $(\mathcal{F}_t)$-adapted processes with a countable discrete state space, neither of which are $(\mathcal{F}_t)$-Markov, and show that in both cases the strategies in Theorems 1.1 and 1.2 are suboptimal. In the first (resp. second) example, Section 5.2.1 (resp. Section 5.2.2), the constructed process is semi-Markov (resp. Markov) with respect to its natural filtration. This demonstrates that the assumption that the chain Z is an $(\mathcal{F}_t)$-Markov process, not just a Markov process with respect to its "natural" filtration, is indeed necessary in Theorems 1.1 and 1.2.

[4] We thank one of the referees for this simplification of our original argument.

5.2.1. $(\mathcal{F}_t)$*-semi-Markov process.* Recall that B is $(\mathcal{F}_t)$-Brownian motion, fix $\epsilon \in (0,1)$ and then let the random times T_n, $n \in \mathbb{N}\cup\{0\}$, be given by $T_0 := 0$ and

$$T_n := \inf\{t \geq T_{n-1} \,:\, |B_t - B_{T_{n-1}}| = \epsilon\} \qquad \text{for } n \geq 1.$$

Define the processes $N = (N_t)_{t\geq 0}$ and $W = (W_t)_{t\geq 0}$ by

$$N_t := \max\{n \in \mathbb{N}\cup\{0\} \,:\, T_n \leq t\} \qquad \text{and} \qquad W_t := B_{T_{N_t}}.$$

For every $t > 0$ we have $\{T_n \leq t\} \in \mathcal{F}_t$ for all $n \in \mathbb{N}$ and hence the process W is $(\mathcal{F}_t)$-adapted. Furthermore W is a continuous-time semi-Markov process (i.e. the pair (W,B) is $(\mathcal{F}_t)$-Markov) with state space $\epsilon\mathbb{Z}$ and cádlág trajectories. In particular, W has only finitely many jumps on any compact interval. Let

$$Z := z_0\mathcal{E}(W), \qquad \text{for a fixed } z_0 > 0,$$

where $\mathcal{E}$ denotes the Doléans-Dade stochastic exponential [10, Sec II.7, Thm. 37]. Therefore, by definition, we have

$$Z_t = z_0 + \int_0^t Z_{s-}\,\mathrm{d}W_s = z_0 + \int_0^{T_{N_t}} Z_{s-}\,\mathrm{d}W_s = Z_{T_{N_t}},$$

where the second equality follows from the facts that $T_{N_t} \leq t$, and that there are no jumps of W during the time interval $(T_{N_t}, t]$. The process Z has a countable state space,[5] which can be expressed as $\mathbb{E} := \{z_0(1-\epsilon)^n(1+\epsilon)^m : m,n \in \mathbb{N}\} \subset (0,\infty)$ and is a continuous-time semi-Markov process (as before, (Z,B) is $(\mathcal{F}_t)$-Markov).

Consider the stochastic integral $\int_0^\cdot Z_s \mathrm{d}B_s$ and note that the equality $W_{T_n} - W_{T_n-} = B_{T_n} - B_{T_{n-1}}$ holds for all $n \in \mathbb{N}$. Hence the stochastic integral can be expresses as follows:

$$\begin{aligned}\int_0^t Z_s \mathrm{d}B_s &= \int_0^{T_{N_t}} Z_s \mathrm{d}B_s + \int_{T_{N_t}}^t Z_s \mathrm{d}B_s = \sum_{n=1}^{N_t} Z_{T_{n-1}}(B_{T_n} - B_{T_{n-1}}) + Z_{T_{N_t}}(B_t - B_{T_{N_t}}) \\ &= (Z_{T_{N_t}} - z_0) + Z_{T_{N_t}}(B_t - B_{T_{N_t}}) = Z_t(1 + (B_t - B_{T_{N_t}})) - z_0.\end{aligned}$$

Therefore, since by definition we have $|B_t - B_{T_{N_t}}| < \epsilon$ and $Z_t > 0$, the following inequalities hold:

$$-z_0 \leq (1-\epsilon)Z_t - z_0 \leq \int_0^t Z_s \mathrm{d}B_s \qquad \text{for all } t \geq 0. \tag{5.1}$$

As in Section 1.2.1, define $\sigma_i : \mathbb{E} \to \mathbb{R}$ by $\sigma_i(z) := -iz$ for any $z \in \mathbb{E}$ and $i = 1,2$, and note that by (1.5) we have $V^I = B$ and $V^{II} = -B$. Hence, for any starting points $x, y \in \mathbb{R}$, definition (1.1) and inequality (5.1) yield the following almost sure inequalities:

$$X_t - Y_t(V^I) = x - y + \int_0^t Z_s \mathrm{d}B_s \geq x - y - z_0, \quad X_t - Y_t(V^{II}) = x - y - 3\int_0^t Z_s \mathrm{d}B_s \leq x - y + 3z_0.$$

For any time horizon $T > 0$, counterexamples to the Conjecture in Section 1.2 (for both Problems **(T)** and **(C)**) can now be constructed in the same way as in Section 1.2.1.

[5] An additional bijection is needed to define a chain with a state space that is a discrete subspace of a Euclidean space.

5.2.2. *Non-($\mathcal{F}_t$)-Markov Markov chain.* In order to define a process Z, which is a time-homogeneous Markov chain in its own filtration and has properties analogous to the ones in the previous section, we sample the path of the Brownian motion B at a sequence of holding times of a Poisson process.

Fix a small $\epsilon > 0$ and let $(e_n(\epsilon))_{n\in\mathbb{N}}$ be a sequence of independent exponential random variables on the initial probability space with $\mathsf{E}[e_n(\epsilon)] = \epsilon$ for all $n \in \mathbb{N}$. Define a sequence of random times T'_n, $n \in \mathbb{N} \cup \{-1, 0\}$, and a Poisson process $N' = (N'_t)_{t\geq 0}$ by

$$T'_{-1} := T'_0 := 0, \quad T'_n := T'_{n-1} + e_n(\epsilon) \quad \text{for } n \in \mathbb{N}, \qquad \text{and} \qquad N'_t := \max\{n \in \mathbb{N}\cup\{0\} \,:\, T'_n \leq t\}.$$

Let $h(\epsilon) := \exp(-1/\epsilon^2)$ and define the function $g_\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$g_\epsilon(x) := \begin{cases} \max\{y \in h(\epsilon)\mathbb{Z} : y \leq x\} & \text{if } x \in (-1 + h(\epsilon), 1), \\ 0 & \text{if } x \in \mathbb{R} \setminus (-1 + h(\epsilon), 1). \end{cases}$$

Note that $g_\epsilon(x) = h(\epsilon)\lfloor x/h(\epsilon)\rfloor$ for all $x \in (-1 + h(\epsilon), 1)$, where $\lfloor y\rfloor$ denotes the integer part of $y \in \mathbb{R}$. The function g_ϵ is supported in $[-1, 1]$ and satisfies

$$x - h(\epsilon) < g_\epsilon(x) \leq x \qquad \forall x \in (-1 + h(\epsilon), 1). \tag{5.2}$$

The idea is to use g_ϵ in order to define a continuous-time random walk W^ϵ with increments given by $g_\epsilon\left(B_{T'_n} - B_{T'_{n-1}}\right)$, approximating those of the Brownian motion B. However, since these increments can be zero if during a holding-time interval the Brownian motion B has either moved in the positive direction by less than $h(\epsilon)$ or in either direction by more than 1, we first need to "prune" the Poisson process N' in the following way.[6] We mark N' at each time T'_n, $n \in \mathbb{N}$, with 1 (resp. 0) if the event $\left\{g_\epsilon\left(B_{T'_n} - B_{T'_{n-1}}\right) > 0\right\}$ has (resp. has not) occurred. By the Marking Theorem in [6, Sec. 5.2], this yields a Poisson process on $(0, \infty) \times \{0, 1\}$. The Restriction Theorem (see [6, Sec. 2.2]), applied to the subset $(0, \infty) \times \{1\}$, defines a Poisson process N on $(0, \infty)$ with jump times T_n, $n \in \mathbb{N}$, satisfying

$$T_n - T_{n-1},\ n \in \mathbb{N},\ \text{are exponential IID random variables with mean larger than } \epsilon, \tag{5.3}$$

$$N_t = \max\{n \in \mathbb{N}\cup\{0\} \,:\, T_n \leq t\}, \quad T_{N_t} \leq t < T_{N_t+1} \quad \text{and} \quad \lim_{\epsilon\downarrow 0} T_{N_t} = t \ \mathsf{P}\text{-a.s.}, \tag{5.4}$$

where as before we have $T_{-1} := T_0 := 0$. Note that in the construction of the marked Poisson process it is key that the marks at distinct Poisson points of the original process N' are independent of each other, which is the case in our setting since the Brownian increments over disjoint holding-time intervals of N' are independent.

We can now define the process $W^\epsilon = (W^\epsilon_t)_{t\geq 0}$ by

$$W^\epsilon_t := \sum_{n=0}^{N_t} g_\epsilon\left(B_{T_n} - B_{T_{n-1}}\right).$$

Note that W^ϵ has a countable discrete state space $\epsilon\mathbb{Z}$ and cádlág trajectories. It jumps only finitely many times on any compact time interval, has the same holding times as the Poisson process N, the jumps $W^\epsilon_t - W^\epsilon_{t-} = g_\epsilon\left(B_{T_{N_t}} - B_{T_{N_{t-}}}\right)$ are distributed as $g_\epsilon\left(B_{T_1}\right)$ for all $t > 0$ and *do not* depend on the position of W^ϵ when the jump occurs. Hence W^ϵ is a continuous-time time-homogeneous random

[6]In the construction that follows, for brevity we do not distinguish between the notions of the Poisson point process, corresponding to a Poisson process, and the Poisson process itself. We refer to both objects simply as the Poisson process (see [6] for more details).

walk, making its stochastic exponential $Z^\epsilon := z_0\mathcal{E}(W^\epsilon)$ (see [10, Sec II.7, Thm. 37] for definition) a time-homogeneous Markov chain with a countable state space and cádlág paths (footnote 5 also applies here). If for some $T>0$ it holds

$$\lim_{\epsilon\to 0}\int_0^T \mathsf{E}\left[(Z_t^\epsilon - Z_t)^2\right]\mathrm{d}t = 0, \tag{5.5}$$

where Z is defined in (1.6), then, since the stochastic exponentials Z and Z^ϵ are square integrable on compact intervals (see Lemma 5.2), the Burkholder-Davis-Gundy inequality [10, Sec IV.4, Thm. 48] implies the following almost sure convergence

$$\phi\left(X_T^\epsilon - Y_T^\epsilon(V)\right) \to \phi\left(X_T - Y_T(V)\right), \qquad I_{\{\tau_0(X^\epsilon - Y^\epsilon(V))>T\}} \to I_{\{\tau_0(X-Y(V))>T\}},$$

as $\epsilon\to 0$, for any Brownian motion $V\in\mathcal{V}$, cost function ϕ and volatility functions σ_1,σ_2 given in Section 1.2.1 (the processes $X^\epsilon, Y^\epsilon(V)$ are defined in (1.1) with Z replaced by Z^ϵ and the stopping time $\tau_0(X^\epsilon - Y^\epsilon(V))$ is equal to $\inf\{t\geq 0: X_t^\epsilon = Y_t^\epsilon(V)\}$). The counterexamples from Section 1.2.1 (with a bounded $\phi\in\mathcal{C}^2(\mathbb{R})$) show that the conjecture in Section 1.2 fails (for both Problems **(T)** and **(C)**) in the case of the process Z^ϵ if $\epsilon>0$ is small enough.

In order to complete our counterexample, we need to prove that the limit in (5.5) holds. To this end we establish the following lemma.

Lemma 5.2. *Let Z^ϵ be as defined above and let Z be given by (1.6). Then Z^ϵ and Z are square integrable on compact intervals and there exists a constant $C_0>0$ such that the following holds:*

$$\mathsf{E}\left[|Z_t - Z_t^\epsilon|^2\right] \leq C_0\int_0^t \mathsf{E}\left[|Z_s - Z_s^\epsilon|^2\right]\mathrm{d}s + \alpha(t,\epsilon), \qquad \textit{for all } t\geq 0 \textit{ and small } \epsilon>0,$$

where $\alpha(t,\epsilon)\in[0,\infty)$ satisfies $\lim_{\epsilon\downarrow 0}\alpha(t,\epsilon)=0$ for any $t\geq 0$. Furthermore, on any interval $[0,T]$, $T<\infty$, the function $\alpha(\cdot,\epsilon):[0,T]\to\mathbb{R}$ can be chosen to be bounded uniformly in all small $\epsilon>0$.

Proof. Recall that the definition of the stochastic exponential (see e.g. [10, Sec II.7, Thm. 37]) implies the following representations for Z^ϵ:

$$Z_t^\epsilon = z_0 + \int_0^t Z_{s-}^\epsilon\,\mathrm{d}W_s^\epsilon = z_0 + \sum_{n=1}^{N_t} Z_{T_{n-1}}^\epsilon g_\epsilon\left(B_{T_n} - B_{T_{n-1}}\right) \quad\text{and}\quad Z_t^\epsilon = z_0\prod_{n=1}^{N_t}\left(1 + g_\epsilon\left(B_{T_n} - B_{T_{n-1}}\right)\right),$$

where the sum (resp. product) is taken to be zero (resp. one) if $N_t=0$. The second equality and the definition of the function g_ϵ imply that $0<Z_t^\epsilon\leq z_0 2^{N_t}$ (recall that $z_0>0$), which yields the stated square integrability of Z^ϵ. The first equality and the fact $Z_t = z_0 + \int_0^t Z_s\,\mathrm{d}B_s$ imply the following:

$$Z_t - Z_t^\epsilon = \int_0^{T_{N_t}}(Z_s - Z_s^\epsilon)\,\mathrm{d}B_s + \int_{T_{N_t}}^t Z_s\,\mathrm{d}B_s + \sum_{n=1}^{N_t} Z_{T_{n-1}}^\epsilon\left(B_{T_n} - B_{T_{n-1}} - g_\epsilon\left(B_{T_n} - B_{T_{n-1}}\right)\right).$$

The definition of the Poisson process N, the definition of g_ϵ with the estimate in (5.2), the elementary fact $(a+b+c)^2\leq 9(a^2+b^2+c^2)$ for any positive a,b,c, and the triangle inequality yield

$$A_0\,|Z_t - Z_t^\epsilon|^2 \leq \left|\int_0^{T_{N_t}}(Z_s - Z_s^\epsilon)\,\mathrm{d}B_s\right|^2 + \left|\int_{T_{N_t}}^t Z_s\,\mathrm{d}B_s\right|^2 + B_1\exp(B_0 N_t)h(\epsilon)^2, \tag{5.6}$$

for some constants $A_0, B_0, B_1 > 0$. Since, by (5.3), the random variable N_t is Poisson distributed with the rate less or equal to t/ϵ and $h(\epsilon) = \exp(-1/\epsilon^2)$, we have

$$\lim_{\epsilon\downarrow 0} \mathsf{E}\left[\exp(B_0 N_t) h(\epsilon)^2\right] = 0 \qquad \text{for any } t \geq 0. \tag{5.7}$$

By (5.4) we have $T_{N_t} \leq t$ and hence

$$\left|\int_0^{T_{N_t}} (Z_s - Z_s^\epsilon)\, \mathrm{d}B_s\right|^2 \leq \sup_{u\in[0,T_{N_t}]} \left|\int_0^u (Z_s - Z_s^\epsilon)\, \mathrm{d}B_s\right|^2 \leq \sup_{u\in[0,t]} \left|\int_0^u (Z_s - Z_s^\epsilon)\, \mathrm{d}B_s\right|^2,$$

which by the Burkholder-Davis-Gundy inequality [11, Thm IV.4.1] implies the following bound

$$\mathsf{E}\left[\left|\int_0^{T_{N_t}} (Z_s - Z_s^\epsilon)\, \mathrm{d}B_s\right|^2\right] \leq B_2 \int_0^t \mathsf{E}\left[|Z_s - Z_s^\epsilon|^2\right] \mathrm{d}s \tag{5.8}$$

for some positive constant B_2. An analogous argument yields

$$\left|\int_{T_{N_t}}^t Z_s\, \mathrm{d}B_s\right|^2 \leq \sup_{u\in[0,t]} \left|\int_0^u Z_s\, \mathrm{d}B_s\right|^2 \leq 2 \sup_{u\in[0,t]} Z_u^2. \tag{5.9}$$

The limit in (5.4), estimate (5.9), Doob's L^2-martingale inequality and the Dominated Convergence Theorem imply

$$\lim_{\epsilon\downarrow 0} \mathsf{E}\left[\left|\int_{T_{N_t}}^t Z_s\, \mathrm{d}B_s\right|^2\right] = 0. \tag{5.10}$$

Define $C_0 := B_2/A_0$ and

$$\alpha(t,\epsilon) := \frac{1}{A_0}\mathsf{E}\left[\left|\int_{T_{N_t}}^t Z_s\, \mathrm{d}B_s\right|^2\right] + \frac{B_1}{A_1}\mathsf{E}\left[\exp(B_0 N_t) h(\epsilon)^2\right].$$

The inequality in the lemma now follows from (5.6), (5.7), (5.8) and (5.10). The last statement in the lemma follows from the fact that $N_t \leq N_T$ for any $t \in [0,T]$, the bound in (5.9), the right-hand side of which is independent of ϵ, and Doob's L^2-martingale inequality. □

Coming back to the proof of (5.5), note first that Gronwall's inequality and Lemma 5.2 yield

$$\mathsf{E}\left[|Z_t - Z_t^\epsilon|^2\right] \leq \alpha(t,\epsilon) + \int_0^t \exp(C_0(t-s))\alpha(s,\epsilon)\, \mathrm{d}s \qquad \text{for any } t \geq 0.$$

Hence, for any $T \in (0,\infty)$, we have

$$\int_0^T \mathsf{E}\left[|Z_t - Z_t^\epsilon|^2\right] \mathrm{d}t \leq \int_0^T \alpha(t,\epsilon)\mathrm{d}t + \int_0^T \mathrm{d}t \int_0^t \exp(C_0(t-s))\alpha(s,\epsilon)\, \mathrm{d}s.$$

Since T is fixed and $\alpha(\cdot,\epsilon)$ is bounded uniformly in ϵ on $[0,T]$, the Dominated Convergence Theorem and Lemma 5.2 imply that the right-hand side of this inequality tends to zero and (5.5) follows.

5.3. $(\mathcal{F}_t)$**-Feller process** Z **independent of** B**.** The final counterexample shows that the "tracking" part of the conjecture in Section 1.2 fails for general Feller processes even if Z and B are independent.

Assume that there exist an $(\mathcal{F}_t)$-Brownian motion $B^\perp \in \mathcal{V}$, independent of B, and define the $(\mathcal{F}_t)$-Feller process $Z := z_0 + B^\perp$ with state space $\mathbb{E} := \mathbb{R}$ for any starting point $z_0 \in \mathbb{R}$. Let $\sigma_1(z) := 2z$ and $\sigma_2(z) := z$, for any $z \in \mathbb{R}$, and note that by (1.5) we have $V^I = B$. We will now show that, for the cost function $\phi(x) := x^4$, the first inequality in Problem **(T)** fails, i.e. there exists a Brownian motion $V \in \mathcal{V}$ such that for any $T > 0$ it holds

$$\mathsf{E}_{r,z_0}\left[(R_T(V))^4\right] < \mathsf{E}_{r,z_0}\left[(R_T(V^I))^4\right], \tag{5.11}$$

where $R(V) = X - Y(V)$ (and X, $Y(V)$ given in (1.1) for any $V \in \mathcal{V}$) and $R_0(V) = r$, $Z_0 = z_0$.

To construct such a process V, define the family $V^c = (V^c_t)_{t\geq 0}$, $c \in [-1,1]$, of $(\mathcal{F}_t)$-Brownian motions by

$$V^c_t := \sqrt{1-c^2}B_t + cB^\perp_t,$$

and note that $V^0 = B = V^I$. Therefore the difference process $R(V^c)$ takes the form

$$R_t(V^c) = r + \int_0^t (2Z_s \mathrm{d}B_s - Z_s \mathrm{d}V^c_s) = r + \left(2 - \sqrt{1-c^2}\right)\int_0^t Z_s\,\mathrm{d}B_s - c\int_0^t Z_s\,\mathrm{d}B^\perp_s,$$

and hence we find $\mathrm{d}[R(V^c), R(V^c)]_t = (5 - 4\sqrt{1-c^2})Z_t^2\mathrm{d}t$ and $\mathrm{d}[R(V^c), Z]_t = -cZ_t\mathrm{d}t$.

Lemma 5.3. *Define* $\psi^c(r,z,t) := \mathsf{E}_{r,z}[(R_t(V^c))^4]$ *for any* $r, z \in \mathbb{R}$ *and* $t \geq 0$*. Then we have*

$$\begin{aligned}\psi^c(r,z,t) &= r^4 + 6k(c)r^2z^2t + 3k(c)(r^2 + k(c)z^4 - 4crz^2)t^2 \\ &\quad + k(c)((7k(c) + 8c^2)z^2 - 4cr)t^3 + (7k^2(c)/4 + 2c^2k(c))t^4,\end{aligned}$$

where $k(c) := 5 - 4\sqrt{1-c^2}$ *for any* $c \in [-1,1]$*.*

Proof. The representation in the lemma for the expectation $\psi^c(r,z,t)$ follows from martingale arguments and stochastic calculus. Alternatively to verify the lemma, one can easily check that the function φ, given by the formula above, satisfies the PDE

$$\frac{1}{2}k(c)z^2\frac{\partial^2\varphi}{\partial r^2} - cz\frac{\partial^2\varphi}{\partial r\partial z} + \frac{1}{2}\frac{\partial^2\varphi}{\partial z^2} = \frac{\partial\varphi}{\partial t},$$

with boundary condition $\varphi(r,z,0) = r^4$ and polynomial growth in r and z. An application of the Feynman-Kac formula then yields $\psi^c = \varphi$. $\square$

Note that $k'(0) = 0$ and hence the derivative in c at $c = 0$ of the value function $\psi^c(r, z_0, T)$ equals

$$\frac{\partial\psi^c}{\partial c}(r, z_0, T)\Big|_{c=0} = -r(12z_0^2T + 4T^3).$$

Since this quantity is non-zero for any $r \neq 0$, inequality (5.11) is satisfied (by Lemma 5.3) for some $V = V^c$ with $c \neq 0$ (recall that $V^0 = B = V^I$). An analogous argument can be used to show that the second inequality in Problem **(T)** also fails in this setting.

Appendix A. Proofs of Lemmas 2.1 and 2.2

A.1. **Proof of Lemma 2.1.** It is clear that Lemma 2.1 follows from (1.1) and the basic properties of stochastic integrals if, for any $V \in \mathcal{V}$, we can find a progressively measurable process C and $W \in \mathcal{V}$, such that $-1 \le C_t \le 1$ for all $t \ge 0$ P-a.s., W and B independent and

$$V_t = \int_0^t C_s \,\mathrm{d}B_s + \int_0^t (1 - C_s^2)^{1/2} \,\mathrm{d}W_s. \tag{A.1}$$

By the Kunita-Watanabe inequality [10, Sec II.6, Thm. 25], the signed random measure $\mathrm{d}[V, B]_t$ on the predictable σ-field is absolutely continuous with respect to Lebesgue measure $\mathrm{d}[B, B]_t = \mathrm{d}t$. Hence, there exists a predictable process $C = (C_t)_{t\ge0}$, such that $\mathrm{d}[V, B]_t = C_t\mathrm{d}t$, and for any $s < t$ we have $|[V, B]_t - [V, B]_s| \le t - s$. Therefore, we may assume that $|C_t| \le 1$ and define the processes $D_t := (1 - C_t^2)^{1/2}$ and $M_t := V_t - \int_0^t C_s\mathrm{d}B_s$. Note that the equalities $[M, B]_t = 0$, $[M, M]_t = \int_0^t D_s^2\mathrm{d}s$ and $\int_0^t I_{\{D_s>0\}}D_s^{-2}\mathrm{d}[M, M]_s \le t$ hold. Therefore the continuous local martingale W, given by

$$W_t := \int_0^t I_{\{D_s>0\}}D_s^{-1}\mathrm{d}M_s + \int_0^t I_{\{D_s=0\}}\mathrm{d}B_s^\perp,$$

is well-defined, where $B^\perp \in \mathcal{V}$ is a Brownian motion independent of B. Lévy's characterisation theorem applied to W now yields the representation in (A.1) and hence implies Lemma 2.1. □

A.2. **Proof of Lemma 2.2.** The assumptions on Q and F imply that $\mathsf{E}[|M_t^U|] < \infty$ for all times $t \ge 0$. The additive structure of the process M^U implies that it is sufficient to prove the following almost sure equality:

$$\mathsf{E}_z\left[\sum_{t<s\le t'}[F(s, U_s, Z_s) - F(s, U_s, Z_{s-})]\middle|\mathcal{F}_t\right] \;=\; \mathsf{E}_z\left[\int_t^{t'}(QF(s, U_s, \cdot))(Z_{s-})\,\mathrm{d}s\middle|\mathcal{F}_t\right], \tag{A.2}$$

for any $0 < t < t'$ and $z \in \mathbb{E}$. The jump-chain holding-time description of the continuous-time chain Z, the continuity of the process U and the continuity and boundedness of the function F imply

$$\mathsf{E}_z\left[\sum_{u<s\le u+\Delta u}[F(s, U_s, Z_s) - F(s, U_s, Z_{s-})]\middle|\mathcal{F}_u\right] \;=\; \Delta u(QF(u, U_u, \cdot))(Z_u) + o(\Delta u), \tag{A.3}$$

for any $u > 0$ and small $\Delta u > 0$. In this expression, for each Δu, $o(\Delta u)$ represents an $\mathcal{F}_u$-measurable random variable which is bounded in modulus by $C\Delta u$, for some constant $C > 0$ independent of Δu (here we use assumption (2.4) and the boundedness of F), and $\lim_{\Delta u\downarrow0} \frac{o(\Delta u)}{\Delta u} = 0$ almost surely.

We now decompose the left-hand side of (A.2) into a sum over the time intervals of length $\Delta t > 0$, where $\frac{t'-t}{\Delta t} \in \mathbb{N}$, and apply (A.3) to each summand:

$$\mathsf{E}_z\left[\sum_{t<s\le t'}[F(s, U_s, Z_s) - F(s, U_s, Z_{s-})]\middle|\mathcal{F}_t\right] = \sum_{i=0}^{\frac{t'-t}{\Delta t}-1}\mathsf{E}_z\left[\sum_{i<\frac{s-t}{\Delta t}\le i+1}[F(s, U_s, Z_s) - F(s, U_s, Z_{s-})]\middle|\mathcal{F}_t\right]$$

$$= \frac{o(\Delta t)}{\Delta t} + \Delta t\sum_{i=0}^{\frac{t'-t}{\Delta t}-1}\mathsf{E}_z\left[(QF(t + i\Delta t, U_{t+i\Delta t}, \cdot))(Z_{t+i\Delta t})\middle|\mathcal{F}_t\right]. \tag{A.4}$$

The properties of the random variables $o(\Delta t)$ listed in the paragraph above, the Dominated Convergence Theorem applied to the right-hand side of (A.4) as $\Delta t \downarrow 0$, the definition of the Lebesgue integral and the fact that Z jumps only finitely many times during the time interval $[t, t']$ together imply the equality in (A.2). This concludes the proof of the lemma. □

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