ASYMPTOTIC INDEPENDENCE OF THREE STATISTICS OF MAXIMAL SEGMENTAL SCORES

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Abstract. Let $\xi_1, \xi_2, \ldots$ be an iid sequence with negative mean. The (m,n)-segment is the subsequence $\xi_{m+1}, \ldots, \xi_n$ and its *score* is given by $\max\{\sum_{m+1}^n \xi_i, 0\}$. Let R_n be the largest score of any segment ending at time n, R_n^* the largest score of any segment in the sequence $\xi_1, \ldots, \xi_n$, and O_x the overshoot of the score over a level x at the first epoch the score of such a size arises. We show that, under the Cramér assumption on ξ_1, asymptotic independence of the statistics R_n, $R_n^* - y$ and O_{x+y} holds as $\min\{n, y, x\} \to \infty$. Furthermore, we establish a novel Spitzer-type identity characterising the limit law O_∞ in terms of the laws of $(1,n)$-scores. As corollary we obtain: (1) a novel factorization of the exponential distribution as a convolution of O_∞ and the stationary distribution of R; (2) if $y = \gamma^{-1} \log n$ (where γ is the Cramér coefficient), our results, together with the classical theorem of Iglehart [6], yield the existence and explicit form of the joint weak limit of $(R_n, R_n^* - y, O_{x+y})$.

1. Introduction and the main result

Consider a sequence of iid random variables $\{\xi_i\}_{i\in\mathbb{N}}$ with negative mean and denote by $S = \{S_n\}_{n\in\mathbb{N}^*}$ the random walk corresponding to $\{\xi_i\}$: $S_0 \doteq 0$ and $S_n \doteq \sum_{i=1}^n \xi_i$. For any $m < n$ with $n \in \mathbb{N}$, $m \in \mathbb{N}^* \doteq \mathbb{N} \cup \{0\}$, the *segmental score* of the (m,n)-segment $\{\xi_i\}_{i=m+1}^n$ of $\{\xi_i\}$ is given by the maximum of the sum of the elements in the segment and zero (as usual we denote $x^+ \doteq \max\{x, 0\}$, $x \in \mathbb{R}$):

$$\left(\sum_{i=1+m}^{n} \xi_i\right)^+ = (S_n - S_m)^+.$$

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The notion of segmental scores arises naturally in several areas of applied probability and statistics. For their application in the study of DNA sequences see *e.g.* [3] and [7]. Segmental scores also play an important role in sequential change point detection problems of mathematical statistics (*e.g.* CUSUM test), see [14] and [12, 13]. Moreover, in sequential analysis in the context of abortion epidemiology, the maximal segmental score is proposed in [8] as a test statistic for the detection of a *one-sided epidemic alternative* for the increase in the mean of a sequence of independent random variables (see also [5] for related applications of the epidemic alternative in experimental neurophysiology). For the role of the segmental scores in queueing theory see [1, 2, 6]. It is of interest in all of these applications to quantify the fluctuations of the segmental scores. In the recent paper [11] (see also [10]) this problem is studied under heavy-tailed step-size distributions, where an appropriate scaling of segmental scores is necessary for the analysis. In the case of an exponentialy thin positive tail, *i.e.* under the Cramér assumption, no scaling is required and the asymptotics of the fluctuations of the segmental scores can be analysed directly, which is the aim of the present paper.

Two natural statistics measuring the fluctuations of the segmental scores are R_n, the largest score of any (m,n)-segment (*i.e.* of any segment ending at time n), and R_n^*, the largest score of any of segment in $\{\xi_i\}_{i=1}^n$ (i.e. the largest score that has arisen up to time n). More precisely, for $n \in \mathbb{N}$, we have

$$R_n \doteq \max_{m\in\{0,\ldots,n-1\}} (S_n - S_m)^+ \qquad \text{and} \qquad R_n^* \doteq \max_{m,k\in\{0,\ldots,n\},m<k} (S_k - S_m)^+,$$

A third statistic quantifying the fluctuations of segmental scores is the first segmental score larger than $x > 0$, which is given by $R_{H(x)}$ with $H(x)$ the first time an increment of the random walk larger than x occurs:

$$H(x) \doteq \min\{k \in \mathbb{N} : \exists\, m < k \text{ such that } S_k - S_m > x\}.$$

The main contribution of this paper is to give sufficient conditions for the three statistics

$$R_n, \qquad Q_{n,x} \doteq R_n^* - x, \qquad O_x \doteq R_{H(x)} - x,$$

of the maximal increments of the walk S to be asymptotically independent in the sense that the joint CDF is asymptotically equal to the product of the marginal CDFs of the statistics:

Definition. A family of random vectors $\{(U_z^1, \dots, U_z^d)\}_{z \in \mathcal{Z}}$ on a given probability space, indexed by $z \in \mathcal{Z} \subset [0, \infty)^l$, $d, l \in \mathbb{N}$, is *asymptotically independent* if the joint CDF is asymptotically equal to a product of the CDFs of the components: i.e. for any $a_i \in (-\infty, \infty]$, $i = 1, \dots, d$, it holds

$$P(U_z^1 \le a_1, \dots, U_z^d \le a_d) = \prod_{i=1}^{d} P(U_z^i \le a_i) + o(1) \qquad \text{as } \min\{z_1, \dots, z_l\} \to \infty.$$

Our result states that the asymptotic independence of the three statistics above essentially holds under the Cramér assumption on the step-size distribution (which in particular implies $E[\xi_1] < 0$):

Assumption 1. The distribution of ξ_1 has finite mean, is non-lattice and satisfies *Cramér's condition*, i.e. $E[\mathrm{e}^{\gamma \xi_1}] = 1$ for some $\gamma \in (0, \infty)$, and $E[|\xi_1|\mathrm{e}^{\gamma \xi_1}]$ is finite.

Theorem 1. *Under As. 1, the triplet $\{(R_n, Q_{n,y}, O_{y+x})\}_{n \in \mathbb{N}, x, y \in \mathbb{R}_+}$ is asymptotically independent, where $\mathbb{R}_+ \doteq [0, \infty)$. Furthermore, the following limit in distribution holds: $O_x \xrightarrow{\mathcal{D}} O_\infty$ as $x \to \infty$, where O_∞ is a non-negative distribution with the characteristic function*

$$E[\mathrm{e}^{\mathrm{i}\theta\, O_\infty}] = \frac{\gamma}{\gamma - \mathrm{i}\theta} \cdot \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - E\left[\mathrm{e}^{\mathrm{i}\theta\, S_n^+} \right] \right) \right\}, \qquad \textit{for all } \theta \in \mathbb{R}. \tag{1.1}$$

Remarks. (i) A classical time-reversal argument implies that R_n and $\max_{m \in \{0, \dots, n\}} S_m$ have the same law for every $n \in \mathbb{N}$. Hence R_n converges in distribution, as $n \uparrow \infty$, to $S_\infty^* \doteq \sup_{n \in \mathbb{N}} S_n$, which is finite (as $E[\xi_1] < 0$ by As. 1) and follows a distribution characterised by Spitzer's identity (see [2, p.230]).

(ii) Note that Spitzer's identity [2, p.230] and a time-reversal argument imply that the second factor in (1.1) is equal to $1/E[\mathrm{e}^{\mathrm{i}\theta R_\infty}]$. The asymptotic independence of Thm. 1 therefore yields the joint law of the weak limit (R_∞, O_∞). In particular, the limit law $R_\infty + O_\infty$ of the sum $R_n + O_x$, as $\min\{x, n\} \to \infty$, is characterised by the identity

$$E\left[\mathrm{e}^{\mathrm{i}\theta(R_\infty + O_\infty)}\right] = \frac{\gamma}{\gamma - \mathrm{i}\theta}, \quad \forall \theta \in \mathbb{R}, \qquad \text{and hence} \qquad \gamma \cdot (R_\infty + O_\infty) \sim \mathrm{Exp}(1).$$

This establishes a novel factorization of the exponential distribution Exp(1) as the convolution of the distribution of the asymptotic overshoot and the stationary distribution of a reflected random walk with step-size distribution satisfying As. 1. Note further that, unlike in the Wiener-Hopf factorisation, here the supports of the factorising random variables are in general not disjoint.

(iii) Note that $Q_{n,x}$ does not admit a non-degenerate weak limit along any sequence (n, x) with $\min\{n, x\} \to \infty$. A sufficient condition for the weak convergence of the statistic $Q_{n,x}$ is given in Iglehart [6, Thm. 2]: if $x(n) = \gamma^{-1}\log(Kn)$, for a certain positive constant K, then $\gamma\, Q_{n,x(n)}$ converges weakly to a Gumbel distribution as $n \uparrow \infty$.

(iv) The main technical fact established in this paper is that asymptotically, as $\min\{x, y, n\} \to \infty$, the probability that R crosses the level $x + y$ for the first time during the excursion of R away from 0 straddling time n vanishes (see Section 2.2). This fact, in conjunction with the independence of distinct excursions, essentially implies the asymptotic independence in Thm. 1.

2. Proof of Thm. 1

We start with the observation that R is a reflected random walk and that R^* and $H(x)$ may be represented as path-functionals of R:

$$R_n = S_n - \min_{m\in\{0,\dots,n\}} S_m, \qquad R_n^* = \max_{k\in\{1,\dots,n\}} R_k \quad \text{and} \quad H(x) = \min\{n \in \mathbb{N} : R_n > x\}.$$

The proof of the asymptotic independence is based on (i) an application of asymptotic results for reflected Lévy processes (reviewed in Section 2.1) to the compound Poisson process obtained by subordinating S by a Poisson process and (ii) asymptotic de-Poissonization results, which are established by splitting arguments (given in Section 2.2). Finally, the proof of Eqn. (1.1) is presented in Section 2.3.

2.1. **Maximal increments of Lévy processes.** Let $X = \{X(t)\}_{t\geq 0}$ be a Lévy process under a probability measure P, *i.e.* a P-a.s. càdlàg process started at $X(0) = 0$ with stationary independent increments (refer to [4] for background on the theory of Lévy processes). Denote by $Y = \{Y(t)\}_{t\geq 0}$

the Lévy process reflected at its running infimum and by $\tau(x)$ the first time an increment of X of size larger than $x \in \mathbb{R}_+$ occurs:

$$Y(t) \doteq X(t) - \inf_{0\le s\le t} X(s) = \sup_{0\le s\le t}\{X(t)-X(s)\}, \quad \tau(x) \doteq \inf\{t\ge 0 : Y(t) > x\}\ (\inf\emptyset = \infty).$$

Let $Y^*(t) \doteq \sup_{0\le s\le t} Y(s)$ be the supremum of Y up to time t. The three statistics of interest take the following form for any $t, x \in \mathbb{R}_+$:

$$Y(t), \qquad M(t,x) \doteq Y^*(t) - x, \qquad Z(x) \doteq Y(\tau(x)) - x. \tag{2.1}$$

Denote by $\widehat{L}$ a local time at zero of Y and let L be a local time at zero of the reflected process $\widehat{Y} = \{\widehat{Y}(t)\}_{t\ge 0}$ of the dual $\widehat{X} \doteq -X$, $\widehat{Y}(t) \doteq \sup_{0\le s\le t} X(s) - X(t)$. The ladder-time process $L^{-1} = \{L^{-1}(t)\}_{t\ge 0}$ is equal to the right-continuous inverse of L (see [4, Ch. IV] for a definition of local time and its inverse). The ladder-height process $H = \{H(t)\}_{t\ge 0}$ is given by $H(t) \doteq X(L^{-1}(t))$ for all $t \ge 0$ with $L^{-1}(t)$ finite and by $H(t) \doteq +\infty$ otherwise. Let ϕ be the Laplace exponent of H,

$$\phi(\theta) \doteq -\log E[\mathrm{e}^{-\theta H(1)} \mathbf{I}_{\{H(1)<\infty\}}], \qquad \text{for any} \quad \theta \in \mathbb{R}_+, \tag{2.2}$$

where $\mathbf{I}_A$ denotes the indicator of a set A and $E[\cdot]$ is the expectation under P.

Assumption 2. The mean of $X(1)$ is finite, the Cramér condition for X is satisfied, i.e.

$$\text{there exists a } \gamma \in (0,\infty) \text{ such that } E[\mathrm{e}^{\gamma X(1)}] = 1, \tag{2.3}$$

$E[\mathrm{e}^{\gamma X(1)}|X(1)|] < \infty$, and either the Lévy measure of X is non-lattice or 0 is regular for $(0,\infty)$.

Remark. Display (2.3) implies $E[X(1)] < 0$, making Y (resp. $\widehat{Y}$) a recurrent (resp. transient) Markov process on $\mathbb{R}_+$. Hence $\phi(0) > 0$ and the stopping time $\tau(x)$ is a.s. finite for any $x \in \mathbb{R}_+$, so that H is a killed subordinator and the overshoot $Z(x)$ is a well-defined random variable.

Thm. 2, which provides a key step in the proof of Thm. 1, was established in [9, Thm. 1, Lem. 3].

Theorem 2. *Under As. 2 the following statements hold:*

(i) *The triplet* $\{(Y(t), Z(x+y), M(t,x))\}_{t,x,y\in\mathbb{R}_+}$ *is asymptotically independent.*

(ii) *The limit in distribution holds:* $Z(x) \xrightarrow{\mathcal{D}} Z(\infty)$ *as* $x \to \infty$, *where*

$$E[\mathrm{e}^{-vZ(\infty)}] = \frac{\gamma}{\gamma+v}\cdot\frac{\phi(v)}{\phi(0)} \qquad \textit{for all} \quad v \in \mathbb{R}_+. \tag{2.4}$$

(iii) *It holds that* $\lim_{x\wedge t\to\infty} P(\widehat{L}(t) = \widehat{L}(\tau(x))) = 0$ *and for any* $\delta_1, \delta_2 \in [0, 1/4)$ *we have*

$$\limsup_{x\wedge t\to\infty} P\left(\widehat{L}(t(1-\delta_1)) \le \widehat{L}(\tau(x)) \le \widehat{L}(t(1+\delta_2))\right) \le \frac{8}{\mathrm{e}}\max\{\delta_1, \delta_2\}. \tag{2.5}$$

2.2. **Proof of the asymptotic independence.** Let the random walk $\{S_n\}_{n\in\mathbb{N}^*}$ and an independent Poisson process $\{N(t)\}_{t\ge0}$ with unit intensity rate be defined on the same probability space, and define a compound Poisson process $X = \{X(t)\}_{t\ge0}$ by

$$X(t) \doteq S_{N(t)}, \quad t \ge 0. \tag{2.6}$$

For any $t > 0$, let $[t] \doteq \max\{n \in \mathbb{N}^* : n < t\}$ denote the largest integer which is smaller than t, and set $[0] \doteq 0$. For any $t, x, y > 0$ let A, B, C and A', B', C' be the sets

$$\begin{aligned} &A \doteq \{Y(t) \le w\}, \qquad B \doteq \{Z(x+y) \le v\}, \qquad C \doteq \{M(t,y) > z\},\\ &A' \doteq \{R_{[t]} \le w\}, \qquad B' \doteq \{O_{x+y} \le v\}, \qquad C' \doteq \{Q_{[t],y} > z\}, \end{aligned} \tag{2.7}$$

where $w, v \in [0,\infty]$ and $z \in [-\infty, \infty)$ are arbitrary (the statistics $Y(t), Z(x+y), M(t,y)$ correspond to the Lévy process X in (2.6), see (2.1)). Since X satisfies As. 2 (as S satisfies As. 1), the asymptotic independence in Thm. 1 will follow from Thm. 2 once we show that if t, x, and y tend to infinity, in such a way that $t \wedge x \wedge y$ tends to infinity,[1] we have:

$$|P(A') - P(A)| \vee |P(B') - P(B)| \vee |P(C') - P(C)| = o(1), \tag{2.8}$$

$$|P(A\cap B\cap C) - P(A'\cap B'\cap C')| = o(1). \tag{2.9}$$

[1]Here and throughout we use the notation: $a \vee b \doteq \max\{a,b\}$, $a \wedge b \doteq \min\{a,b\}$ for any $a, b \in \mathbb{R}$.

Indeed, the triangle inequality implies

$$\begin{aligned}|P(A'\cap B'\cap C')-P(A')P(B')P(C')| &\leq |P(A'\cap B'\cap C')-P(A\cap B\cap C)| \\ &+ |P(A\cap B\cap C)-P(A)P(B)P(C)| \\ &+ |P(A)P(B)P(C)-P(A')P(B')P(C')|,\end{aligned}$$

which tends to zero if $t\wedge x\wedge y\to\infty$ in view of Eqns. (2.8) and (2.9) and Thm. 2(i) (recall that $P(A\cap B\cap C)-P(A)P(B)P(C)=o(1)$ as $t\wedge x\wedge y\to\infty$).

2.2.1. *Proof of convergence in Eqn.* (2.8). We proceed by showing the convergence of the three differences of probabilities by treating the three cases separately. Convergence of the first difference to zero follows by a duality argument. Since $S^*_n \doteq \max_{k\in\{0,\ldots,n\}} S_k$ increases to the random variable S^*_∞ as $n\to\infty$ and $N(t)$ tends to infinity as $t\to\infty$ (both P-a.s.), we have that the running supremum $X^*(t)$ of X also increases to S^*_∞ as $t\to\infty$ P-a.s. As a consequence, the indicators $\mathbf{I}_{\{X^*(t)\leq w\}}$ and $\mathbf{I}_{\{S^*_{[t]}\leq w\}}$ decrease to $\mathbf{I}_{\{S^*_\infty\leq w\}}$ as $t\to\infty$ P-a.s., and we have that both $P(X^*(t)\leq w)$ and $P(S^*_n\leq w)$ converge to $P(S^*_\infty\leq w)$. By duality, the random variables $R_{[t]}$ and $Y(t)$ have the same laws as $S^*_{[t]}$ and $X^*(t)$, respectively, so that we find

$$\Delta^A(t)\doteq P(A')-P(A)=o(1)\qquad \text{as } t\to\infty. \tag{2.10}$$

The second difference is zero since definition (2.6) yields $O_{x+y}=Z(x+y)$ and hence the events B and B' coincide, implying the equality $P(B)=P(B')$ for all $x,y>0$ and $v\in[0,\infty]$.

In order to establish that the third difference in Eqn. (2.8), $\Delta^C(t,y)\doteq|P(C)-P(C')|$, tends to zero (for any $z\in[-\infty,\infty)$) as $t\wedge y$ tends to infinity, we need to control the deviation of the Poisson random variable $N(t)$ away from its mean t as $t\uparrow\infty$. Pick $\delta\in(0,1/4)$, define the events

$$A_\delta(t)\doteq\{N(t(1-\delta))<[t]<N(t(1+\delta))\}, \tag{2.11}$$

and note that the law of large numbers for subordinators [4, p. 92] implies $P(A_\delta(t)) \to 1$ as $t \to \infty$. Recalling that $Q_{[t],y} = R^*_{[t]} - y$ and $M(t,y) = Y^*(t) - y$, we have

$$\Delta^C(t,y) = |P(R^*_{[t]} > y+z, A_\delta(t)) - P(Y^*(t) > y+z, A_\delta(t))| + o(1) \quad \text{as } t \wedge y \to \infty.$$

Since $\{R^*_n\}_{n\in\mathbb{N}}$ is a non-decreasing process and $R^*_{N(s)} = Y^*(s)$, $s \in \mathbb{R}_+$, the following holds:

$$P(Y^*(t(1-\delta)) > y+z, A_\delta(t)) \le P(R^*_{[t]} > y+z, A_\delta(t)) \le P(Y^*(t(1+\delta)) > y+z, A_\delta(t)). \tag{2.12}$$

Hence, as $t \wedge y \to \infty$, we find

$$\begin{aligned} 0 \le \Delta^C(t,y) &\le P(Y^*(t(1-\delta)) \le y+z < Y^*(t(1+\delta)), A_\delta(t)) + o(1) \\ &\le P(t(1-\delta) \le \tau(y+z) < t(1+\delta)) + o(1) \\ &\le P(\widehat{L}(t(1-\delta)) \le \widehat{L}(\tau(y+z)) \le \widehat{L}(t(1+\delta))) + o(1). \end{aligned}$$

Eqn. (2.5) in Theorem 2(iii) implies

$$0 \le \limsup_{t\wedge y\to\infty} \Delta^C(t,y) \le \frac{8}{\mathrm{e}}\delta \qquad \forall \delta \in (0,1/4). \tag{2.13}$$

Therefore we have $\limsup_{t\wedge y\to\infty} \Delta^C(t,y) = \lim_{t\wedge y\to\infty} \Delta^C(t,y) = 0$ and Eqn. (2.8) follows. □

2.2.2. *Proof of convergence in Eqn.* (2.9). By the strong law of large numbers, the definition of $A_\delta(t)$ in (2.11) and the fact $B = B'$, the difference $\Delta^*(t,x,y) \doteq P(A\cap B\cap C) - P(A'\cap B'\cap C')$ satisfies

$$\begin{aligned} \Delta^*(t,x,y) &= P(A\cap B\cap C\cap (A'\cap C')^c, A_\delta(t)) - P(A'\cap B\cap C'\cap (A\cap C)^c, A_\delta(t)) + o(1) \\ &= P(A\cap B\cap C\cap [((A')^c\cap C')\cup (C')^c], A_\delta(t)) \\ &- P(A'\cap B\cap C'\cap [(A^c\cap C)\cup C^c], A_\delta(t)) + o(1) \\ &= P(A\cap (A')^c\cap E_\delta(t)) - P(A'\cap A^c\cap E_\delta(t)) \qquad (2.14) \\ &+ P(C\cap (C')^c\cap F_\delta(t)) - P(C'\cap C^c\cap F'_\delta(t)) + o(1), \qquad \text{as } t\wedge x\wedge y\to\infty, \end{aligned}$$

where $E_\delta(t) \doteq B\cap C\cap C'\cap A_\delta(t)$, $F_\delta(t) \doteq A\cap B\cap A_\delta(t)$, $F'_\delta(t) \doteq A'\cap B\cap A_\delta(t)$ and $\delta\in(0,1/4)$ is arbitrary but fixed. The second difference of the two probabilities on the right-hand side of (2.14) satisfies the following estimates by the monotonicity of R^* and Y^* and the definition of $A_\delta(t)$ in (2.11) (cf. Eqn. (2.12) above):

$$\begin{aligned}\left|P(C\cap(C')^c\cap F_\delta(t)) - P(C'\cap C^c\cap F'_\delta(t))\right| &\leq P(C\cap(C')^c, A_\delta(t)) + P(C'\cap C^c, A_\delta(t))\\ &\leq 2P\left(Y^*(t(1-\delta))\leq y+z < Y^*(t(1+\delta))\right) + o(1).\end{aligned}$$

Hence, by an argument analogous to the one in (2.13), this difference tends to zero as $t\wedge x\wedge y\to\infty$.

Eqn. (2.9), and hence the asymptotic independence in Thm. 1, will follow once we verify the equality

$$|P(A\cap(A')^c\cap E_\delta(t)) - P(A'\cap A^c\cap E_\delta(t))| = o(1) \quad \text{as } t\wedge x\wedge y\to\infty. \tag{2.15}$$

The proof of the limit in (2.15) is more involved than the one above as, unlike Y^* and R^*, the processes Y and R do not have monotone paths. The main tools in this proof are the weak limit in (2.10) and the strong Markov property of Y. Furthermore, to establish Eqn. (2.15) we will need to control the time between the epochs t and $\widehat{L}^{-1}(\widehat{L}(\tau(a))$ as $t\wedge a\to\infty$ (for $a=x+y$ and $a=z+y$), where $\widehat{L}^{-1}$ is the right-continuous inverse of $\widehat{L}$ (see Section 2.1 and [4, Ch. IV] for definition).

Consider the events $\tilde{D}_\delta(t,a)$, $\bar{D}_\delta(t,a)$ and $D_\delta(t,a)\doteq \tilde{D}_\delta(t,a)\cup\bar{D}_\delta(t,a)$ for any $a>0$:

$$\tilde{D}_\delta(t,a) \doteq \left\{\widehat{L}^{-1}(\widehat{L}(\tau(a))) < (1-\delta)t\right\}, \qquad \bar{D}_\delta(t,a) \doteq \left\{\widehat{L}^{-1}(\widehat{L}(\tau(a))) > (1+\delta)t\right\}. \tag{2.16}$$

Since $D^c_\delta(t,a)\subseteq \left\{\widehat{L}(t(1-\delta))\leq \widehat{L}(\tau(a))\leq \widehat{L}(t(1+\delta))\right\}$, Theorem 2(iii) implies

$$0\leq \limsup_{t\wedge a\to\infty} P(D^c_\delta(t,a)) \leq \frac{8}{\mathrm{e}}\delta \qquad \forall\delta\in(0,1/4). \tag{2.17}$$

Hence, by an argument analogous to (2.13), equality (2.15) will follow if we prove

$$|P(A\cap(A')^c\cap E_\delta(t)\cap\tilde{D}_\delta(t,x+y)) - P(A'\cap A^c\cap E_\delta(t)\cap\tilde{D}_\delta(t,x+y))| = o(1), \tag{2.18}$$

$$|P(A\cap(A')^c\cap E_\delta(t)\cap\bar{D}_\delta(t,x+y)) - P(A'\cap A^c\cap E_\delta(t)\cap\bar{D}_\delta(t,x+y))| = o(1), \tag{2.19}$$

as $t \wedge x \wedge y \to \infty$. With this in mind, we denote by $(\mathcal{F}_t)_{t\geq 0}$ the completed right-continuous filtration generated by the process X in (2.6). Note that N is $(\mathcal{F}_t)$-adapted and define an $(\mathcal{F}_t)$-stopping time $T'_{[t]} \doteq \inf\{s \geq 0 : N(s) = [t]\}$ (i.e. $N(T'_{[t]}) = [t]$). Note further that by (2.6) we have $Y(T'_{[t]}) = R_{[t]}$, $Y^*(T'_{[t]}) = R^*_{[t]}$ and $Z(x+y) = O_{x+y}$ and hence the events A', B', C' in (2.7) can be expressed as:

$$A' = \{Y(T'_{[t]}) \leq w\}, \qquad B' = \{Z(x+y) \leq v\}, \qquad C' = \{Y^*(T'_{[t]}) > y+z\} = \{\tau(y+z) < T'_{[t]}\}.$$

For any time $t \geq 0$ and event K, which may depend on t, define

$$p_K(t) \doteq P(K \cap A_\delta(t)) \tag{2.20}$$

and note that by (2.10) we have $\Delta^A(t) = p_{A\cap(A')^c}(t) - p_{A^c\cap A'}(t) + o(1)$ as $t \uparrow \infty$. The key identity in our proof is (for any $a > 0$) given by

$$\mathbf{I}_{\tilde{D}_\delta(t,a)} P\left(\bar{A} \cap A_\delta(t) \,\middle|\, \mathcal{F}_{\widehat{L}^{-1}\left(\widehat{L}(\tau(a))\right)}\right) = \mathbf{I}_{\tilde{D}_\delta(t,a)} p_{\bar{A}}\left(t - \widehat{L}^{-1}\left(\widehat{L}(\tau(a))\right)\right) \qquad P\text{-a.s.} \tag{2.21}$$

where $\bar{A}$ denotes either $A \cap (A')^c$ or $A^c \cap A'$. It is important to observe that, since the definition of $\bar{A} \cap A_\delta(t)$ depends on the epoch t (cf. (2.11) and the definitions of A, A' in (2.7)), for any $\omega \in \tilde{D}_\delta(t,a)$, this set on the right- (resp. left-) hand side of (2.21) is defined for the epoch $t - \widehat{L}^{-1}\left(\widehat{L}(\tau(a))\right)(\omega)$ (resp. t). A formal construction of the right-hand side of (2.21), which ensures its measurability, requires the *shift operator* θ that can be defined on the canonical probability space and for any $s \in \mathbb{R}_+$ and path ω satisfies $\theta_s(\omega)(\cdot) = \omega(s+\cdot)$ (for details we refer to [4, Ch. 0] and the references there in). The equality in (2.21) holds since Y is a strong $(\mathcal{F}_t)$-Markov process, $Y\left(\widehat{L}^{-1}\left(\widehat{L}(\tau(a))\right)\right) = 0$ and $A_\delta(t)$ is given by (2.11). Note that the stopping time $\widehat{L}^{-1}\left(\widehat{L}(\tau(a))\right)$ is the time when the excursion of Y, during which Y crosses the level a for the first time, ends.

2.2.3. *Proof of* (2.18). We may assume $x > z$ implying $\tau(z+y) \leq \tau(x+y) \leq \widehat{L}^{-1}\left(\widehat{L}(\tau(y+x))\right)$. Hence $B, \bar{C}, \tilde{D}_\delta(t, x+y) \in \mathcal{F}_{\widehat{L}^{-1}\left(\widehat{L}(\tau(y+x))\right)}$, where $\bar{C} \doteq C \cap C' = \{\tau(y+z) < t \wedge T'_{[t]}\}$, and (2.21) yields

$$\begin{aligned} P(\bar{A} \cap B \cap \bar{C} \cap \tilde{D}_\delta(t, y+x), A_\delta(t)) &= E\left[\mathbf{I}_{B\cap\bar{C}\cap\tilde{D}_\delta(t,y+x)} P\left(\bar{A} \cap A_\delta(t) \,\middle|\, \mathcal{F}_{\widehat{L}^{-1}\left(\widehat{L}(\tau(y+x))\right)}\right)\right] \\ &= E\left[\mathbf{I}_{B\cap\bar{C}\cap\tilde{D}_\delta(t,y+x)} p_{\bar{A}}\left(t - \widehat{L}^{-1}\left(\widehat{L}(\tau(x+y))\right)\right)\right]. \end{aligned} \tag{2.22}$$

The left-hand side of (2.18) is by (2.22) bounded above by

$$E\left[\mathbf{I}_{B\cap\bar{C}\cap\tilde{D}_\delta(t,y+x)}\left|p_{A\cap(A')^c}\left(t-\widehat{L}^{-1}\left(\widehat{L}(\tau(x+y))\right)\right)-p_{A^c\cap A'}\left(t-\widehat{L}^{-1}\left(\widehat{L}(\tau(x+y))\right)\right)\right|\right]. \tag{2.23}$$

The expression in (2.23) is $o(1)$ as $t\wedge x\wedge y\to\infty$ by (2.10) (cf. (2.20)) and the dominated convergence theorem: for any sequence (t_n,x_n,y_n), such that $t_n\wedge x_n\wedge y_n\uparrow\infty$, pick an arbitrary subsequence $(t_{n_k},x_{n_k},y_{n_k})$ and note that, on the event $\cap_k\tilde{D}_\delta(t_{n_k},y_{n_k}+x_{n_k})$, the diferrence $t_{n_k}-\widehat{L}^{-1}(\widehat{L}(\tau(y_{n_k}+x_{n_k}))$ is bounded below by $\delta\cdot t_{n_k}$ and hence tends to infinity. On the complement of this event, the random variable under the expectation in (2.23) is clearly zero for all large k. This establishes (2.18).

2.2.4. *Proof of* (2.19). The proof in this case is slightly more complex than the one in Section 2.2.3 as, intuitively speaking, it requires splitting the events in (2.19) in such a way so that the excursions of Y corresponding to $\widehat{L}(\tau(x+y))$ and $\widehat{L}(\tau(z+y))$ are distinct (put differently, Y crosses levels $z+y$ and $x+y$ for the first time during distinct excursions excursions away from 0; recall that $x>z$).

In order to analyse (2.19), note $t(1+\delta)\le\widehat{L}^{-1}\left(\widehat{L}(t(1+\delta))\right)$ and hence $\mathcal{F}_{t(1+\delta)}\subset\mathcal{F}_{\widehat{L}^{-1}\left(\widehat{L}(t(1+\delta))\right)}$. Therefore $\bar{C}\in\mathcal{F}_t$ and $\bar{A}\cap A_\delta(t),\bar{D}_\delta(t,x+y)\in\mathcal{F}_{t(1+\delta)}$ imply $\bar{A}\cap A_\delta(t),\bar{C},\bar{D}_\delta(t,x+y)\in\mathcal{F}_{\widehat{L}^{-1}\left(\widehat{L}(t(1+\delta))\right)}$, where $\bar{C}=C\cap C'$ and $\bar{A}$ as in (2.21). The strong Markov property of Y at the stopping time $\widehat{L}^{-1}\left(\widehat{L}(t(1+\delta))\right)$, the equality $Y\left(\widehat{L}^{-1}\left(\widehat{L}(t(1+\delta))\right)\right)=0$ and the definition of B (cf. (2.7)) yield

$$P\left(B\cap H\,\middle|\,\mathcal{F}_{\widehat{L}^{-1}\left(\widehat{L}(t(1+\delta))\right)}\right)=\mathbf{I}_HP(B),\quad\text{where } H\doteq\{\widehat{L}^{-1}(\widehat{L}(\tau(y+x)))>\widehat{L}^{-1}(\widehat{L}(t(1+\delta)))\}.$$

It holds $\bar{D}_\delta(t,x+y)\cap H^c\subseteq\{\widehat{L}^{-1}(\widehat{L}(\tau(y+x)))=\widehat{L}^{-1}(\widehat{L}(t(1+\delta)))\}$ and hence by Theorem 2(iii) we have $P(\bar{D}_\delta(t,x+y)\cap H^c)=o(1)$ as $t\wedge x\wedge y\to\infty$. Therefore conditioning on $\mathcal{F}_{\widehat{L}^{-1}\left(\widehat{L}(t(1+\delta))\right)}$ yields

$$\begin{aligned}P(\bar{A}\cap A_\delta(t)\cap B\cap\bar{C}\cap\bar{D}_\delta(t,y+x)) &= P(\bar{A}\cap A_\delta(t)\cap\bar{C}\cap\bar{D}_\delta(t,y+x),B\cap H)+o(1)\\ &= P(\bar{A}\cap A_\delta(t)\cap\bar{C}\cap\bar{D}_\delta(t,y+x))P(B)+o(1)\end{aligned} \tag{2.24}$$

as $t\wedge x\wedge y\to\infty$. Note that (2.16), Theorem 2(iii) and the inclusion $\bar{D}_\delta(t,y+z)\cap\bar{C}\subset\{\widehat{L}(t)=\widehat{L}(\tau(y+z))\}$ imply $P(\bar{D}_\delta(t,y+z)\cap\bar{C})\le P(\widehat{L}(t)=\widehat{L}(\tau(y+z)))=o(1)$ as $t\wedge y\to\infty$. Hence, as

$t\wedge x\wedge y\to\infty$, we can decompose the right-hand side of (2.24) further using (2.21) as follows:

$$
\begin{aligned}
&P(\bar{A}\cap A_\delta(t)\cap\bar{C}\cap\bar{D}_\delta(t,y+x))\\
&= P(\bar{A}\cap A_\delta(t)\cap\bar{C}) - P(\bar{A}\cap A_\delta(t)\cap\bar{C}\cap\tilde{D}_\delta(t,y+x)) - P(\bar{A}\cap A_\delta(t)\cap\bar{C}\cap D^c_\delta(t,y+x))\\
&= P(\bar{A}\cap A_\delta(t)\cap\bar{C}\cap\tilde{D}_\delta(t,y+z)) - P(\bar{A}\cap A_\delta(t)\cap\bar{C}\cap\tilde{D}_\delta(t,y+x))\\
&- P(\bar{A}\cap A_\delta(t)\cap\bar{C}\cap D^c_\delta(t,y+z)) - P(\bar{A}\cap A_\delta(t)\cap\bar{C}\cap D^c_\delta(t,y+x)) + o(1)\\
\text{(2.25)}&= E\left[\mathbf{I}_{\bar{C}\cap\tilde{D}_\delta(t,y+z)}p_{\bar{A}}\left(t-\widehat{L}^{-1}\left(\widehat{L}(\tau(z+y))\right)\right) - \mathbf{I}_{\bar{C}\cap\tilde{D}_\delta(t,y+x)}p_{\bar{A}}\left(t-\widehat{L}^{-1}\left(\widehat{L}(\tau(x+y))\right)\right)\right]\\
&- P(\bar{A}\cap A_\delta(t)\cap\bar{C}\cap D^c_\delta(t,y+z)) - P(\bar{A}\cap A_\delta(t)\cap\bar{C}\cap D^c_\delta(t,y+x)) + o(1).
\end{aligned}
$$

Since in (2.25) $\bar{A}$ denotes either $A\cap(A')^c$ or $A^c\cap A'$, the left-hand side in (2.19) is bounded above by the sum (recall $\Delta^A(t)=P(A\cap(A')^c)-P(A'\cap A^c)$, definition (2.20) and equalities (2.24) and (2.25)):

$$
\begin{aligned}
&E\left[\mathbf{I}_{\bar{C}\cap\tilde{D}_\delta(t,y+z)}\left|\Delta^A\left(t-\widehat{L}^{-1}\left(\widehat{L}(\tau(z+y))\right)\right)\right| + \mathbf{I}_{\bar{C}\cap\tilde{D}_\delta(t,y+x)}\left|\Delta^A\left(t-\widehat{L}^{-1}\left(\widehat{L}(\tau(x+y))\right)\right)\right|\right]\\
&\qquad +2P(D^c_\delta(t,y+z)) + 2P(D^c_\delta(t,y+x)) + o(1) \qquad \text{as } t\wedge x\wedge y\to\infty.
\end{aligned}
$$

The expectation converges to zero as $t\wedge x\wedge y\to\infty$ by the same argument as in (2.23) and, since $\delta\in(0,1/4)$ was arbitrary, the bound in (2.17) implies the equality in (2.19). □

2.3. **Proof of the Spitzer-type formula in** (1.1)**.** Let the compound Poisson process X be as in (2.6). It is clear that the overshoots O_x of $\{S_n\}_{n\in\mathbb{N}^*}$ and $Z(x)$ of X coincide (see (2.1) and Section 1 for definitions of $Z(x)$ and O_x respectively): $Z(x)=O_x$ for any level $x\in\mathbb{R}_+$. The ladder-height process of X is a compound Poisson process with Laplace exponent ϕ given by

$$
\frac{\phi(\theta)}{\phi(0)} = \exp\left(\int_0^\infty \mathrm{d}t\int_{[0,\infty)}(1-\mathrm{e}^{-\theta x})t^{-1}P(S_{N(t)}\in\mathrm{d}x)\right) = \exp\left\{\sum_{n=1}^\infty\frac{1}{n}(1-E[\mathrm{e}^{-\theta S_n^+}])\right\}
$$

for any $\theta\in\mathbb{R}_+$ (see e.g. [4, p.166]). The second equality is a consequence of Fubini's theorem and the fact $\int_0^\infty t^{-1}P(N(t)=k)\mathrm{d}t=\Gamma(k)/k!=1/k$, for all $k\in\mathbb{N}$, where Γ denotes the Gamma function. This concludes the proof of (1.1). □

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