JOINT ASYMPTOTIC DISTRIBUTION OF CERTAIN PATH FUNCTIONALS OF THE REFLECTED PROCESS

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Abstract. Let $\tau(x)$ be the first time that the reflected process Y of a Lévy process X crosses $x > 0$. The main aim of this paper is to investigate the joint asymptotic distribution of the path functionals $Y(t) = X(t) - \inf_{0\leq s\leq t} X(s)$, $Z(x) = Y(\tau(x)) - x$, and $m(t) = \sup_{0\leq s\leq t} Y(s) - y^*(t)$ for a certain non-linear curve $y^*(t)$. We prove that under Cramér's condition on $X(1)$ the functionals $Y(t)$ and $Z(x)$ are asymptotically independent as $\min\{t, x\} \to \infty$ and characterise the law of the limit $(Y(\infty), Z(\infty))$. Moreover, if $y^*(t) = \gamma^{-1}\log(t)$ and $\min\{t, x\} \to \infty$ in such a way that $t\exp\{-\gamma x\} \to 0$ (γ denotes the Cramér coefficient), then we show that $Y(t)$, $Z(x)$ and $m(t)$ are asymptotically independent and derive the explicit form of the joint weak limit $(Y(\infty), Z(\infty), m(\infty))$. The proof is based on the theorem of Doney & Maller [7] together with our characterisation of the law $(Y(\infty), Z(\infty))$.

1. Introduction and main results

The reflected process Y of a Lévy process X is a strong Markov process on $\mathbb{R}_+ \doteq [0,\infty)$ equal to X reflected at its running infimum. The reflected process is of great importance in many areas of probability, ranging from the fluctuation theory for Lévy processes (e.g. [2, Ch. VI] and the references therein) to mathematical statistics (e.g. [16, 18], CUSUM method of cumulative sum), queueing theory (e.g. [1, 17]), mathematical finance (e.g. [10, 14], drawdown as risk measure), mathematical genetics (e.g. [13] and references therein) and many more. The aim of this paper is to study the weak limiting behaviour of the following functionals of the reflected process Y:

$$(1.1)\qquad Y(t) \doteq X(t) - \inf_{0\leq s\leq t} X(s), \qquad Z(x) \doteq Y(\tau(x)) - x, \qquad m(t) \doteq Y^*(t) - y^*(t),$$

where $t, x \in \mathbb{R}_+$ and y^* is a specific non-linear curve to be specified shortly. Here $\tau(x)$ and $Y^*(t)$ denote the first entry time of Y into the interval (x,∞) and the supremum up to time t of the reflected process respectively,

$$\tau(x) \doteq \inf\{t \geq 0 : Y(t) > x\} \quad (\inf\emptyset \doteq \infty), \qquad Y^*(t) \doteq \sup_{0\leq s\leq t} Y(s).$$

In this paper we restrict to Lévy processes satisfying the Cramér assumption and a standard non-lattice condition:

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Assumption 1. The mean of $X(1)$ is finite, *Cramér's condition*, $E[\mathrm{e}^{\gamma X(1)}] = 1$ for $\gamma > 0$, holds, $E[\mathrm{e}^{\gamma X(1)}|X(1)|] < \infty$ and either the Lévy measure of X is non-lattice or 0 is regular for $(0, \infty)$.

Cramér's condition implies that X tends to $-\infty$ almost surely, and hence, by a classical time reversal argument, the reflected process Y has a weak limit $Y(\infty)$ equal in distribution to the ultimate supremum $\sup_{t\geq 0} X(t)$. The second functional, the overshoot $Z(x)$, admits a weak limit $Z(\infty)$ described in Section 3 (see Proposition 3). The question of interest is that of the weak asymptotics of the vector $(Y(t), Z(x))$.

Theorem 1. *$Y(t)$ and $Z(x)$ are asymptotically independent, as $\min\{t, x\} \to \infty$, in the sense that*

$$\begin{aligned}\lim_{\min\{x,t\}\to\infty} E[\exp(-uY(t) - vZ(x))] &= E[\exp(-uY(\infty))]E[\exp(-vZ(\infty))] \\ &= \lim_{\min\{x,t\}\to\infty} E[\exp(-uY(t))]E[\exp(-vZ(x))],\end{aligned}$$

for $u, v \in \mathbb{R}_+$. $(Y(t), Z(x))$ converges weakly to the law $(Y(\infty), Z(\infty))$ given by the Laplace transform

$$E[\exp(-uY(\infty) - vZ(\infty))] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(u)}, \qquad \text{for all} \quad u, v \in \mathbb{R}_+, \tag{1.2}$$

where ϕ is the Laplace exponent of the ascending ladder-height subordinator of X.[1] *In particular, the law of the sum $Y(\infty) + Z(\infty)$ is exponential with mean $1/\gamma$.*

We now turn to the weak asymptotics of the triplet $(Y(t),\, Z(x),\, m(t))$. To avoid degeneracies we specify the centering curve to be given by

$$y^*(t) = \gamma^{-1} \log(t), \qquad t \in \mathbb{R}_+\backslash\{0\}. \tag{1.3}$$

This choice is informed by Iglehart [11], where in the analogous random walk setting $x(n) = \gamma^{-1} \log n$ was chosen as centering sequence, and by the main result in Doney & Maller [7], which implies that running maximum $m(t)$ of Y after centering by the curve $y^*(t)$ given in (1.3) converges weakly to a Gumbel distribution (see [8, Ch. 3] for the form of the Gumbel distribution and Sect. 5 below for a simple derivation of the distribution of $m(\infty)$ deploying [7, Thm. 1]). A question of interest is if and when the asymptotic independence of $Y(t)$ and $Z(x)$ extends to that of the triplet $Y(t)$, $Z(x)$ and $m(t)$. A priori, it appears unlikely that $Z(x)$ and $m(t)$ are asymptotically independent in general, for x and t tending to infinity in an arbitrary way. In the next result we give a sufficient condition for such asymptotic independence to hold, namely that $\min\{x, t\} \to \infty$ such that

$$x - y^*(t) \to \infty, \qquad \text{or equivalently } t \exp\{-\gamma x\} \to 0.$$

Since, by [7], the process Y^* has weakly convergent random fluctuations around the deterministic curve y^*, the assumption $x - y^*(t) \to \infty$ in effect forces the process Y to reach the level x for the first time after time t. The result is as follows.

[1] Note that the Cramér condition implies $E[X(1)] < 0$ and hence $\phi(0) > 0$ (see (2.1) for definition of ϕ and Section 2 for more details on ladder processes), making the formula in (1.2) well defined.

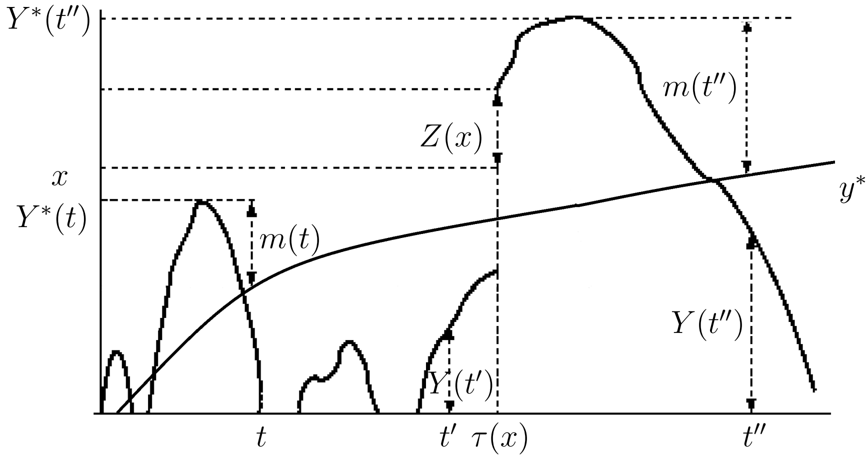

FIGURE 1. This schematic figure of a path of Y depicts the values of the three functionals in (1.1) at times t, t', and t'' before and after the reflected process crosses the level x, and the curve $y^*(t) = \gamma^{-1}\log(t)$.

Theorem 2. *Let $\min\{t,x\} \to \infty$ such that $t\exp\{-\gamma x\} \to 0$. Then the triplet $(Y(t), Z(x), m(t))$ converges weakly and the law of the weak limit $(Y(\infty), Z(\infty), m(\infty))$ is determined by the Fourier-Laplace transform*

$$E\left[\exp\left(-uY(\infty) - vZ(\infty) + \mathrm{i}\beta m(\infty)\right)\right] = \frac{\gamma}{\gamma+v}\cdot\frac{\phi(v)}{\phi(u)}\cdot\Gamma\left(1-\frac{\mathrm{i}\beta}{\gamma}\right)\cdot\exp\left[\mathrm{i}\beta\gamma^{-1}\log\left(\ell C_\gamma\widehat{\phi}(\gamma)\right)\right] \tag{1.4}$$

for all $u, v \in \mathbb{R}_+$, $\beta \in \mathbb{R}$, where $\widehat{\phi}$ is the Laplace exponent of the decreasing ladder-height process, $\widehat{L}^{-1}$ is the decreasing ladder-time processes with $\ell \doteq 1/E[\widehat{L}^{-1}(1)]$ (see Section 2 for the definitions of $\widehat{\phi}$ and $\widehat{L}^{-1}$), $\Gamma(\cdot)$ denotes the gamma function and the constant C_γ is given by

$$C_\gamma \doteq \frac{\phi(0)}{\gamma\phi'(-\gamma)}. \tag{1.5}$$

In particular, $Y(t)$, $Z(x)$ and $m(t)$ are asymptotically independent: for any $a, b \in \mathbb{R}_+$ and $c \in \mathbb{R}$

$$P(Y(t) \le a, Z(x) \le b, m(t) \le c) = P(Y(t) \le a)P(Z(x) \le b)P(m(t) \le c) + o(1).$$[2]

The remainder of the paper is devoted to the proofs of Prop. 3 and Thms. 1 and 2. After notation and setting are fixed in Sect. 2, the law of the asymptotic overshoot $Z(\infty)$ is established in Sect. 3 and the proof of the asymptotic independence is given in Sect. 4. Drawing on these results we present the proofs of Prop. 3 and Thms. 1 and 2 in Sect. 5 by deploying a number of classical facts from the fluctuation theory of Lévy processes.

2. Setting and notation

The formula for the law of the asymptotic overshoot follows from Lemma 5 and Prop. 7 established in Sect. 3. The asymptotic independence in Thms. 1 and 2 is a consequence of Lemma 10 proved in Sect. 4. We next briefly define the setting and notation to be used throughout.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\ge 0}, P)$ be a filtered probability space that carries a Lévy process X satisfying As. 1. Here $\Omega \doteq D(\mathbb{R})$ is the Skorokhod space of real-valued functions that are right-continuous

[2]Here we use the definiton $f(t,x) = o(1)$ if $\lim_{\min\{t, x-y*(t)\}\to\infty} f(t,x) = 0$.

on $\mathbb{R}_+$ and have left-limits on $(0,\infty)$, X is the coordinate process, $\{\mathcal{F}(t)\}_{t\geq 0}$ denotes the completed filtration generated by X, which is right-continuous, and $\mathcal{F}$ is the completed σ-algebra generated by $\{X(t)\}_{t\geq 0}$. For any $x\in\mathbb{R}$ denote by P_x the probability measure on $(\Omega,\mathcal{F})$ under which $X-x$ is a Lévy process. We refer to [2, Ch. I] for further background on Lévy processes.

Let L be a local time[3] at zero of the reflected process $\widehat{Y}=\{\widehat{Y}(t)\}_{t\geq 0}$ of the dual $\widehat{X}\doteq -X$, i.e. $\widehat{Y}(t)\doteq X^*(t)-X(t)$, where $X^*(t)\doteq\sup_{0\leq s\leq t}X(s)$. The ladder-time process $L^{-1} = \{L^{-1}(t)\}_{t\geq 0}$ is equal to the right-continuous inverse of L. The ladder-height process $H=\{H(t)\}_{t\geq 0}$ is given by $H(t)\doteq X(L^{-1}(t))$ for all $t\geq 0$ with $L^{-1}(t)$ finite and by $H(t)\doteq +\infty$ otherwise. Let ϕ be the Laplace exponent of H,

$$\phi(\theta)\doteq -\log E[\mathrm{e}^{-\theta H(1)}\mathbf{I}_{\{H(1)<\infty\}}],\qquad \text{for any}\quad \theta\in\mathbb{R}_+, \tag{2.1}$$

where $\mathbf{I}_A$ denotes the indicator of a set A. Analogously, define the local time $\widehat{L}$ of Y at zero, the decreasing ladder-time and ladder-height subordinators $\widehat{L}^{-1}$ and $\widehat{H}$ with $\widehat{\phi}$ the Laplace exponent of $\widehat{H}$. See [2, Sec. VI.1] for more details on ladder subordinators. Note that the Cramér assumption implies $E[X(1)]<0$, making Y (resp. $\widehat{Y}$) a recurrent (resp. transient) Markov process on $\mathbb{R}_+$. Hence $\phi(0)>0$ and the stopping time $\tau(x)$ is a.s. finite for any $x\in\mathbb{R}_+$, so that H is a killed subordinator under P and the overshoot $Z(x)$ a P-almost surely defined random variable.

We now briefly review elements of Itô's excursion theory that will be used in the proof. We refer to [9], [6] and [2, Ch. IV] for a general treatment and further references. Consider the Poisson point process of excursions away from zero associated to the strong Markov process Y. For each moment $t\in\mathbb{R}_+$ of local time, let $\epsilon(t)\in\mathcal{E}=\{\varepsilon\in\Omega:\varepsilon\geq 0\}$ denote the excursion at t:

$$\epsilon(t)\doteq\begin{cases}\left\{\begin{array}{ll} Y\left(s+\widehat{L}^{-1}(t-)\right), & s\in[0,\widehat{L}^{-1}(t)-\widehat{L}^{-1}(t-)) \\ 0, & s\geq \widehat{L}^{-1}(t)-\widehat{L}^{-1}(t-)\end{array}\right\}, & \text{if } \widehat{L}^{-1}(t-)<\widehat{L}^{-1}(t), \\ \Upsilon, & \text{otherwise,}\end{cases} \tag{2.2}$$

where $\Upsilon\equiv 0$ is the null function, $\widehat{L}^{-1}(t-)\doteq\lim_{s\uparrow t}\widehat{L}^{-1}(s)$ if $t>0$ and $\widehat{L}^{-1}(0-)=0$ otherwise. Definition (2.2) uses the fact $\widehat{L}(\infty)\doteq\lim_{s\to\infty}\widehat{L}(s)=\infty$ P-a.s., which holds by the recurrence of Y. Itô [12] proved that ϵ is a Poisson point process under P. Let n be the intensity (or excursion) measure on $(\mathcal{E},\mathcal{G})$ of ϵ, where $\mathcal{G}$ is the Borel σ-algebra on the Polish space $\mathcal{E}$. In Sections 4 and 3, for any Borel-measurable non-negative (or integrable) functional $F:\mathcal{E}\to\mathbb{R}$ we denote $n(F)=n(F(\varepsilon))\doteq\int_{\mathcal{E}}F\,\mathrm{d}n$. In this notation the equality $n(A)=n(\mathbf{I}_A)$ holds for any $A\in\mathcal{G}$ and, if $n(A)\in(0,\infty)$, we denote $n(B|A)\doteq n(B\cap A)/n(A)$ for any $B\in\mathcal{G}$.

For an excursion $\varepsilon\in\mathcal{E}$, let $\rho(x,\varepsilon)$ (for any $x>0$) and $\zeta(\varepsilon)$ be the first time that ε enters the interval (x,∞) and the lifetime of ε respectively:

$$\rho(x,\varepsilon)\doteq\inf\{s\geq 0:\varepsilon(s)>x\}\qquad \zeta(\varepsilon)\doteq\inf\{t>0:\varepsilon(t)=0\}, \tag{2.3}$$

where here and troughout we set $\inf\emptyset\doteq\infty$. For brevity we sometimes write $\rho(x)$ (resp. ζ) instead of $\rho(x,\varepsilon)$ (resp. $\zeta(\varepsilon)$). Note that $\zeta(\epsilon(t))$ is given in terms of $\widehat{L}^{-1}$ by $\zeta(\epsilon(t))=\widehat{L}^{-1}(t)-\widehat{L}^{-1}(t-)$ for

[3] In the case 0 is not regular for $[0,\infty)$, only a finite number of maxima of X are attained in any compact time interval. In this case we work with the right-continuous version of local time L.

any $t \in \mathbb{R}_+$. We refer to [2, Ch. O.5] for a treatment of Poisson point processes, the compensation formula and the properties of its characteristic measure.

3. Limiting overshoot of the reflected process

In this section we prove the following result, which plays a role in Theorems 1 and 2.

Proposition 3. (i) *The weak limit $Z(\infty)$ of $Z(x)$ as $x \to \infty$ has Laplace transform*

$$E[\mathrm{e}^{-vZ(\infty)}] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(0)} \qquad \text{for all} \quad v \in \mathbb{R}_+, \tag{3.1}$$

where ϕ is the Laplace exponent of the ascending ladder-height subordinator of X.[4]
(ii) *Let $m \doteq \lim_{u\to\infty} \phi(u)/u$ and ν_H denote the Lévy measure of the Laplace exponent ϕ with the tail function $\overline{\nu}_H(x) \doteq \nu_H((x,\infty))$, $x > 0$. Then the law of the asymptotic overshoot $Z(\infty)$ is given by*

$$P(Z(\infty) > x) = \frac{\gamma}{\phi(0)}\mathrm{e}^{-\gamma x} \int_x^\infty \mathrm{e}^{\gamma y}\, \overline{\nu}_H(y)\, \mathrm{d}y, \quad x \in [0,\infty), \quad \text{and} \quad P(Z(\infty) = 0) = \frac{\gamma}{\phi(0)} m. \tag{3.2}$$

In particular, $Z(\infty)$ is a continuous random variable except possibly at the origin.

The formula in (3.1) of Prop. 3, which characterises the law of the limiting overshoot $Z(\infty)$ is implied by the main result in [15]. As this formula constitutes a key step in the proofs of Theorems 1 and 2, we give in this section an independent proof of Prop. 3 based on excursion theory alone. This approach is in the spirit of the present paper and should be contrasted with the result in [15], which crucially relies on the renewal theorem.

The proof of Prop. 3 is as follows: we first establish Cramér's asymptotics for the exit probabilities of X from a finite interval. Then we describe the distribution of the overshoot $Z(x)$, defined in (1.1), in terms of the excursion measure n and apply the result from the first step to find the relevant asymptotics under the excursion measure, which in turn yield the Laplace transform of the limiting law $Z(\infty)$. Finally, formula (3.2) is established in Section 3.1.

Let $T(x)$ and $\widehat{T}(x)$ denote the first-passage times of X into the intervals (x,∞) and $(-\infty,-x)$ respectively for any $x \in \mathbb{R}_+$,

$$T(x) \doteq \inf\{t \geq 0 : X(t) \in (x,\infty)\}, \qquad \widehat{T}(x) \doteq \inf\{t \geq 0 : X(t) \in (-\infty,-x)\}. \tag{3.3}$$

and define the overshoot

$$K(x) \doteq X(T(x)) - x \qquad \text{on the event } \{T(x) < \infty\}.$$

Denote by $f(x) \sim g(x)$ as $x \uparrow \infty$ the functions $f, g : \mathbb{R}_+ \to (0,\infty)$ satisfying $\lim_{x\uparrow\infty} \frac{f(x)}{g(x)} = 1$.

Proposition 4. (i) (Asymptotic two-sided exit probability) *For any $z > 0$ we have*

$$P(T(x) < \widehat{T}(z)) \sim C_\gamma \mathrm{e}^{-\gamma x}\left(1 - E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\right]\right) \qquad \text{as} \quad x \to \infty, \tag{3.4}$$

where the constant C_γ is given in (1.5).

[4]Note that the Cramér condition implies $E[X(1)] < 0$ and hence $\phi(0) > 0$ (see (2.1) for definition of ϕ and Section 2 for more details on ladder processes), making the formula in (3.1) well defined.

(ii) (Asymptotic overshoot) *Let $u \in \mathbb{R}_+$ and fix $z > 0$. Then we have as $x \to \infty$:*

$$(3.5) \quad E\left[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\widehat{T}(z)\}}\right] \sim C(u)\mathrm{e}^{-\gamma x}\left(1 - E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\right]\right), \quad \text{with} \quad C(u) \doteq \frac{\gamma}{\gamma+u}\cdot\frac{\phi(u)}{\phi(0)}\cdot C_\gamma$$

and C_γ in (1.5).

Remarks. **(i)** Let $P_x^{(\gamma)}$ be the Cramér measure on $(\Omega, \mathcal{F})$. Its restriction to $\mathcal{F}(t)$ is given by

$$P_x^{(\gamma)}(A) \doteq E_x[\mathrm{e}^{\gamma(X(t)-x)}\mathbf{I}_A], \qquad A \in \mathcal{F}(t), \qquad t \in \mathbb{R}_+.$$

Here E_x is the expectation under P_x and $\mathbf{I}_A$ is the indicator of A. Under As. 1 it follows that $P_x^{(\gamma)}$ is a probability measure and $X - x$ is a Lévy process under $P_x^{(\gamma)}$ with $E_x^{(\gamma)}[X(1) - x] \in (0, \infty)$.
(ii) Since the overshoot of X is the same as that of its ladder process, the weak limit under $P^{(\gamma)}$ of $K(x)$ as $x \to \infty$, needed in the proof of Proposition 4, follows from [4, Thm. 1]. The second ingredient of the proof of Proposition 4 is the Cramér estimate for Lévy processes [3].
(iii) Note that the random variable $X(\widehat{T}(z))$ under the expectation in (3.4) is well-defined P-a.s., since As. 1 implies that the Lévy process X drifts to $-\infty$ P-a.s.

Proof. **(i)** Recall that, under As. 1, [3] shows that Cramér's estimate remains valid for the Lévy process X (with C_γ defined in (1.5)):

$$(3.6) \qquad P(T(y) < \infty) \sim C_\gamma \mathrm{e}^{-\gamma y} \quad \text{as } y \to \infty.$$

By the strong Markov property and spatial homogeneity of X it follows that

$$(3.7) \quad P(T(x) < \widehat{T}(z)) = P(T(x) < \infty) - \int_{(-\infty,-z]} P_y(T(x) < \infty)P(X(\widehat{T}(z)) \in \mathrm{d}y, \widehat{T}(z) < T(x)).$$

The translation invariance of X and Cramér's estimate (3.6) imply the following equality

$$(3.8) \qquad P_y(T(x) < \infty) \;=\; C_\gamma \mathrm{e}^{-\gamma x}\mathrm{e}^{\gamma y}(1 + r(x-y)) \quad \text{for all} \quad x > y,$$

where $\lim_{x'\to\infty} r(x') = 0$. Equality (3.8) applied to the identity in (3.7) yields

$$(3.9) \qquad \begin{aligned} C_\gamma^{-1}\mathrm{e}^{\gamma x}P(T(x) < \widehat{T}(z)) \;&=\; 1 - E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}\right] \\ &+\; r(x) - E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}r(x - X(\widehat{T}(z)))\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}\right]. \end{aligned}$$

Since $X(\widehat{T}(z)) \le -z < 0$ on the event $\{\widehat{T}(z) < \infty\}$, which satisfies $P(\widehat{T}(z) < \infty) = 1$ by As. 1, the dominated convergence theorem implies

$$E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\right] = E\left[\mathrm{e}^{\gamma X(\widehat{T}(z))}\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}\right] + o(1) \qquad \text{as } x \to \infty.$$

An application of the dominated convergence theorem to the second expectation on the right-hand side of equality (3.9), together with the fact that r vanishes in the limit as $x \to \infty$, proves the first statement in the proposition.

(ii) Recall that the Laplace exponent ϕ of the increasing ladder-height process H is a strictly concave function that satisfies $\phi(-\gamma) = 0$ so that the right-derivative $\phi'(-\gamma)$ is strictly positive. Under the measure $P^{(\gamma)}$ the identity $\phi^{(\gamma)}(\gamma + u) = \phi(u)$ holds for any $u \in \mathbb{R}_+$ and hence, since $\phi'(-\gamma) = E^{(\gamma)}[X_1] > 0$, X drifts to $+\infty$ as $t \to \infty$, i.e. $P^{(\gamma)}(T(x) < \infty) = 1$ for any $x > 0$. Therefore, under As. 1, under $P^{(\gamma)}$ the ladder-height process H is a non-lattice subordinator with $E^{(\gamma)}[H(1)] \in (0, \infty)$. Since the overshoot $K(x)$ is equal to that of H over x, [4, Thm. 1] implies that the weak limit $K(x) \xrightarrow{\mathcal{D}} K(\infty)$, as $x \to \infty$, exists. Since $x \mapsto \mathrm{e}^{-ux}$ is uniformly continuous on $\mathbb{R}_+$, [5, p. 16, Thm. 2.1] implies $\lim_{x\uparrow\infty} E^{(\gamma)}[\mathrm{e}^{-uK(x)}] = E^{(\gamma)}[\mathrm{e}^{-uK(\infty)}]$ for any fixed $u \geq 0$. A version of the Wiener-Hopf factorisation of X (see e.g. [2, p.183]) under the measure $P^{(\gamma)}$ yields

$$\int_0^\infty q\mathrm{e}^{-qx} E^{(\gamma)}\left[\mathrm{e}^{-uK(x)}\right] \mathrm{d}x = \frac{q}{\phi(q-\gamma)} \cdot \frac{\phi(q-\gamma)-\phi(u-\gamma)}{q-u} \qquad \text{for any } q, u > 0. \tag{3.10}$$

Since the function $x \mapsto E^{(\gamma)}\left[\mathrm{e}^{-uK(x)}\right]$ is bounded, the dominated convergence theorem implies that in the limit as $q \downarrow 0$ we get $E^{(\gamma)}[\mathrm{e}^{-uK(\infty)}] = \phi(u-\gamma)/(u\phi'(-\gamma))$. The Esscher change of measure formula implies the following for any $u \geq 0$ ($C(u)$ is defined in (3.5)):

$$E[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\infty\}}] = \mathrm{e}^{-\gamma x} \cdot E^{(\gamma)}[\mathrm{e}^{-(\gamma+u)K(x)}] \sim C(u)\mathrm{e}^{-\gamma x} \quad \text{as } x \to \infty. \tag{3.11}$$

Furthermore, since the expectation in (3.11) is bounded as $x \to \infty$, there exists a bounded function $R : \mathbb{R}_+ \to \mathbb{R}$, such that $E[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\infty\}}] = C(u)\mathrm{e}^{-\gamma x}(1 + R(x))$ for $x > 0$, and $\lim_{x\to\infty} R(x) = 0$. The strong Markov property at $\widehat{T}(z)$ and an argument analogous to the one used in the proof of Proposition 4(i) (cf. (3.9)) yields

$$\begin{aligned} & C(u)^{-1}\mathrm{e}^{\gamma x}E[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\widehat{T}(z)\}}] \\ = \; & 1 - E[\mathrm{e}^{\gamma X(\widehat{T}(z))}\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}] + R(x) - E[\mathrm{e}^{\gamma X(\widehat{T}(z))}R(x - X(\widehat{T}(z)))\mathbf{I}_{\{\widehat{T}(z)<T(x)\}}], \end{aligned}$$

which implies equivalence (3.5). □

Since the expectation $E^{(\gamma)}[X_1]$ is strictly positive, the reflected process Y, under $P^{(\gamma)}$, is transient and $\widehat{L}(\infty)$ is an exponentially distributed random variable, independent of the killed subordinator $\{(\widehat{L}^{-1}(t), \widehat{H}(t))\}_{t\in[0,\widehat{L}(\infty))}$. As a consequence, the excursion process $\epsilon' = \{\epsilon'(t)\}_{t\geq 0}$, given by the formula in (2.2) for $t < \widehat{L}(\infty)$ and the cemetery state ∂ after $\widehat{L}(\infty)$, is, under $P^{(\gamma)}$, a Poisson point process killed at an independent exponential time with mean $E^{(\gamma)}[\widehat{L}(\infty)]$. Put differently, ϵ' is equal to (2.2) up to the first entry into $\{\varepsilon \in \mathcal{E} : \zeta(\varepsilon) = \infty\}$ and killed after that. In the rest of the paper we will denote by $n^{(\gamma)}$ the excursion measure under $P^{(\gamma)}$ of the killed Poisson point process ϵ'.

Lemma 5. *For any $x > 0$ the random variable $\widehat{L}(\tau(x))$ is exponentially distributed under P (resp. $P^{(\gamma)}$) with parameter $n(\rho(x) < \zeta)$ (resp. $n^{(\gamma)}(\rho(x) < \zeta)$) and the following equality holds:*

$$P(Z(x) > y) \;=\; n(\varepsilon(\rho(x,\varepsilon)) - x > y|\rho(x) < \zeta) \qquad \textit{for any} \quad y \in \mathbb{R}_+.$$

Proof. The definitions of the Poisson point process ϵ in (2.2) and the first-passage time $\rho(x, \varepsilon)$ in (2.3) imply the equality $\widehat{L}(\tau(x)) = T_A \doteq \inf\{t \geq 0 : \epsilon(t) \in A\}$ where $A \doteq \{\varepsilon \in \mathcal{E} : \rho(x,\varepsilon) < \zeta(\varepsilon)\}$. The first statement in the lemma follows since T_A is exponentially distributed with parameter $n(A)$ (e.g. [2, Sec. O.5, Prop. O.2]). The second statement is a consequence of the fact that $\epsilon(T_A)$ follows an n-uniform distribution (i.e. $P(\epsilon(T_A) \in B) = n(B|A)$ for any $B \in \mathcal{G}$, see e.g. [2, Sec. O.5, Prop. O.2]) and taking B to be equal to $\{\varepsilon \in \mathcal{E} : \rho(x,\varepsilon) < \zeta(\varepsilon), \varepsilon(\rho(x,\varepsilon)) - x > y\}$. □

Conversely, one may also express n as a ratio of expectations under the measure P. To derive such a representation, for any $x > 0$, define the random variable $K_F(x)$ by

$$K_F(x) \doteq \sum_g F(\epsilon_g)\mathbf{I}_{\{g<\tau(x)\}}, \tag{3.12}$$

where the sum runs over all left-end points g of excursion intervals, $\epsilon_g \doteq \epsilon(\widehat{L}(g))$, and $F : \mathcal{E} \to \mathbb{R}$ is Borel-measurable and non-negative (note that $F \equiv 1$ implies $K_F(x) \equiv 1$ P- and $P^{(\gamma)}$-almost surely).

Lemma 6. **(i)** *Define* $\widehat{\mathcal{V}}(x) \doteq E\left[\widehat{L}(\tau(x))\right]$ *and* $\widehat{\mathcal{V}}^{(\gamma)}(x) \doteq E^{(\gamma)}\left[\widehat{L}(\tau(x))\right]$. *Then the following hold:*

$$n(F) \quad = \quad \widehat{\mathcal{V}}(x)^{-1} E\left[K_F(x)\right], \qquad n^{(\gamma)}(F) = \widehat{\mathcal{V}}^{(\gamma)}(x)^{-1} E^{(\gamma)}\left[K_F(x)\right]. \tag{3.13}$$

In particular we have $\widehat{\mathcal{V}}(x) \cdot n(\rho(x) < \zeta) = 1$ *and* $\widehat{\mathcal{V}}^{(\gamma)}(x) \cdot n^{(\gamma)}(\rho(x) < \zeta) = 1$.

(ii) *The following holds* $n^{(\gamma)}(F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}) = n(\mathrm{e}^{\gamma\varepsilon(\rho(x,\varepsilon))}F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}})$. *Hence we have*

$$n^{(\gamma)}(\rho(x,\varepsilon) < \zeta(\varepsilon)) = n(\mathrm{e}^{\gamma\varepsilon(\rho(x,\varepsilon))}\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}). \tag{3.14}$$

(iii) *For any* $z \in (0, \infty)$ *the following holds as* $x \to \infty$:

$$n^{(\gamma)}(\rho(x,\varepsilon) < \zeta(\varepsilon)) \sim \widehat{\phi}(\gamma) \qquad \text{and} \qquad \mathrm{e}^{\gamma x} n(\varepsilon(\rho(z,\varepsilon)) > x, \rho(z,\varepsilon) < \zeta(\varepsilon)) = o(1). \tag{3.15}$$

Proof of Lemma 6. **(i)** The proof of (3.13) is identical under both measures. Hence we give the argument only under P. Note that for any left-end point g of an excursion interval the following equality holds: $F(\epsilon_g)\mathbf{I}_{\{g<\tau(x)\}} = F(\epsilon_g)\mathbf{I}_{\{g\le\tau(x)\}}$. Since for every $\varepsilon \in \mathcal{E}$ the process $t \to F(\varepsilon)\mathbf{I}_{\{t\le\tau(x)\}}$ is left-continuous and adapted, an application of the compensation formula of excursion theory for the Poisson point process ϵ defined in (2.2) to $K_F(x)$ (see e.g. [2, Cor. IV.11]) yields representation (3.13). The second statement follows by taking $F = \mathbf{I}_{\{\rho(x)<\zeta\}}$ in (3.13), since in that case $K_F(x) = \mathbf{I}_{\{\tau(x)<\infty\}}$.
(ii) Define $G(\varepsilon) \doteq F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}$ and let $K_G(x)$ as in (3.12). The Esscher change of measure formula and the compensation formula in [2, Cor. IV.11] yield

$$E^{(\gamma)}\left[K_G(x)\right] \quad = \quad E\left[\int_0^\infty \mathrm{e}^{\gamma X(t-)}\mathbf{I}_{\{t\le\tau(x)\}}\mathrm{d}\widehat{L}(t)\right] n\left(\mathrm{e}^{\gamma\varepsilon(\rho(x,\varepsilon))}F(\varepsilon)\mathbf{I}_{\{\rho(x,\varepsilon)<\zeta(\varepsilon)\}}\right). \tag{3.16}$$

A change of variable $t = \widehat{L}^{-1}(u)$ under the expectation on the right-hand side of (3.16), Fubini's theorem and $P^{(\gamma)}$-a.s. equality $\{\widehat{L}^{-1}(u-) \le \tau(x)\} = \{\widehat{L}^{-1}(u) \le \tau(x)\}$ yield

$$E\left[\int_0^\infty \mathrm{e}^{\gamma X(t-)}\mathbf{I}_{\{t\le\tau(x)\}}\mathrm{d}\widehat{L}(t)\right] = E^{(\gamma)}\left[\int_0^{\widehat{L}(\infty)} \mathbf{I}_{\{\widehat{L}^{-1}(u-)\le\tau(x)\}}\mathrm{d}u\right] = \widehat{\mathcal{V}}^{(\gamma)}(x).$$

The final equality follows from $\{\widehat{L}^{-1}(u-) \le \tau(x)\} = \{u \le \widehat{L}(\tau(x))\}$. Equality in (3.13) under $P^{(\gamma)}$ applied to $K_G(x)$ and (3.16) now imply the formula in part (ii) of the lemma.
(iii) By Lemma 5 the random variable $\widehat{L}(\tau(x))$ is exponentially distributed under $P^{(\gamma)}$ with parameter $n^{(\gamma)}(\rho(x) < \zeta)$. Hence $n^{(\gamma)}(\rho(x) < \zeta) = -\log P^{(\gamma)}(\widehat{L}(\tau(x)) > 1)$ and the dominated convergence theorem implies $\lim_{x\uparrow\infty} n^{(\gamma)}(\rho(x) < \zeta) = -\log P^{(\gamma)}(\widehat{L}(\infty) > 1) = -\log P^{(\gamma)}(\widehat{L}^{-1}(1) < \infty)$, which is equal to $\widehat{\phi}^{(\gamma)}(0) = \widehat{\phi}(\gamma)$ by the elementary equality $\widehat{\phi}^{(\gamma)}(u) = \widehat{\phi}(\gamma + u)$, $u \ge 0$. Chebyshev's inequality and part (ii) of the lemma imply $\mathrm{e}^{\gamma x} n(\varepsilon(\rho(z,\varepsilon)) > x, \rho(z,\varepsilon) < \zeta(\varepsilon)) \le n(\mathrm{e}^{\gamma\varepsilon(\rho(z,\varepsilon))}\mathbf{I}_{\{\varepsilon(\rho(z,\varepsilon))>x,\rho(z,\varepsilon)<\zeta(\varepsilon)\}}) = n^{(\gamma)}(\varepsilon(\rho(z,\varepsilon)) > x, \rho(z,\varepsilon) < \zeta(\varepsilon))$. The final expression tends to zero as $x \uparrow \infty$ by the dominated convergence theorem and the lemma follows. □

We now apply Lemma 6 to establish the asymptotic behaviour of certain integrals against the excursion measure as $x \to \infty$.

Proposition 7. *Let $u \geq 0$. Then, as $x \to \infty$, we have*

$$n(\mathrm{e}^{-u(\varepsilon(\rho(x))-x)}|\rho(x)<\zeta) \longrightarrow C(u)\cdot C_\gamma^{-1} = \frac{\gamma}{\gamma+u}\cdot\frac{\phi(u)}{\phi(0)}. \tag{3.17}$$

In particular, as $x \to \infty$, $Z(x)$ converges weakly to a random variable $Z(\infty)$ with Laplace transform $E[\exp(-uZ(\infty))] = C(u)\cdot C_\gamma^{-1}$.

Remark. Recall the result of Doney & Maller [7, Thm. 1] (C_γ is defined in (1.5)):

$$n(\rho(x)<\zeta) \sim C_\gamma\, \widehat{\phi}(\gamma)\, \mathrm{e}^{-\gamma x} \qquad \text{as} \quad x\to\infty. \tag{3.18}$$

Proof of Proposition 7. Fix $M > 0$ and recall that, under the probability measure $n(\,\cdot\,|\rho(M)<\zeta)$, the coordinate process has the same law as the first excursion of Y away from zero with height larger than M. For any $x > M$, the following identity holds:

$$n(\mathrm{e}^{-u(\varepsilon(\rho(x))-x)}|\rho(x)<\zeta) \quad = \quad n(\mathrm{e}^{-u(\varepsilon(\rho(x))-x)}\mathbf{I}_{\{\rho(x)<\zeta\}}|\rho(M)<\zeta)\frac{n(\rho(M)<\zeta)}{n(\rho(x)<\zeta)}. \tag{3.19}$$

The definitions of the point process ϵ in (2.2) and of the compensator measure n, together with the strong Markov property under the probability measure $n(\,\cdot\,|\rho(M)<\zeta)$, imply that $\varepsilon\circ\theta_{\rho(M)}$ has the same law as the process X with entrance law $n(\varepsilon(\rho(M,\varepsilon))\in \mathrm{d}z|\rho(M)<\zeta)$ and killed at the epoch of the first passage into the interval $(-\infty,0]$. We therefore find

$$\begin{aligned} n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}\mathbf{I}_{\{\rho(x)<\zeta\}}|\rho(M)<\zeta) &= n\left(\mathrm{e}^{-u(\varepsilon(\rho(M,\varepsilon))-x)}\mathbf{I}_{\{\varepsilon(\rho(M,\varepsilon))>x\}}|\rho(M)<\zeta\right) \\ &+ \int_{[M,x]} E_z\left[\mathrm{e}^{-uK(x)}\mathbf{I}_{\{T(x)<\widehat{T}(0)\}}\right] n(\varepsilon(\rho(M,\varepsilon))\in\mathrm{d}z|\rho(M)<\zeta), \end{aligned} \tag{3.20}$$

where $K(x) = X(T(x)) - x$. By the second equality in (3.15) of Lemma 6, we have as $x \uparrow \infty$:

$$\mathrm{e}^{\gamma x} n\left(\mathrm{e}^{-u(\varepsilon(\rho(M,\varepsilon))-x)}\mathbf{I}_{\{\varepsilon(\rho(M,\varepsilon))>x\}}|\rho(M)<\zeta\right) \leq \mathrm{e}^{\gamma x}\frac{n\left(\varepsilon(\rho(M,\varepsilon))>x, \rho(M,\varepsilon)<\zeta(\varepsilon)\right)}{n(\rho(M)<\zeta)} = o(1).$$

This estimate, spatial homogeneity of X and equations (3.19) and (3.20) yield as $x\to\infty$:

$$\begin{aligned} &n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}|\rho(x)<\zeta) \\ &\quad = \quad o(1) + \int_{[M,x]} E\left[\mathrm{e}^{-uK(x-z)}\mathbf{I}_{\{T(x-z)<\widehat{T}(z)\}}\right] \frac{n(\varepsilon(\rho(M,\varepsilon))\in\mathrm{d}z, \rho(M)<\zeta)}{n(\rho(x)<\zeta)}. \end{aligned} \tag{3.21}$$

Formula (3.5) of Proposition 4 implies the following equality:

$$E\left[\mathrm{e}^{-uK(x-z)}\mathbf{I}_{\{T(x-z)<\widehat{T}(z)\}}\right] = C(u)\mathrm{e}^{-\gamma x}\left(1-G(z)+R(x-z)\right)\mathrm{e}^{\gamma z}, \tag{3.22}$$

where $G, R : \mathbb{R}_+ \to \mathbb{R}$ are bounded functions such that $G(z) = E[\mathrm{e}^{\gamma X(\widehat{T}(z))}]$ and $\lim_{x'\to\infty} R(x') = 0$. Therefore the equality in (3.21), the asymptotic behaviour of $n(\rho(x)<\zeta)$ given in (3.18) and Lemma 6 (ii) imply the following identity as $x\to\infty$:

$$\begin{aligned} n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}|\rho(x)<\zeta) &= A_\gamma(u) n^{(\gamma)}(\varepsilon(\rho(M,\varepsilon))\in[M,x], \rho(M,\varepsilon)<\zeta(\varepsilon)) + o(1) \\ &+ \quad A_\gamma(u) n^{(\gamma)}\left([R(x-\varepsilon(\rho(M,\varepsilon))) - G(\varepsilon(\rho(M,\varepsilon)))]\, I_{\{\varepsilon(\rho(M,\varepsilon))\in[M,x], \rho(M,\varepsilon)<\zeta(\varepsilon)\}}\right), \end{aligned} \tag{3.23}$$

where $A_\gamma(u) \doteq C(u)/(C_\gamma\widehat{\phi}(\gamma))$. By (3.23) the limit $\lim_{x\to\infty} n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}|\rho(x)<\zeta)$ exists and the dominated convergence theorem yields

$$\lim_{x\to\infty} n(\mathrm{e}^{-u(\varepsilon(\rho(x,\varepsilon))-x)}|\rho(x)<\zeta) = A_\gamma(u)\left(n^{(\gamma)}(\rho(M)<\zeta) - n^{(\gamma)}\left(G(\varepsilon(\rho(M,\varepsilon)))I_{\{\rho(M,\varepsilon)<\zeta(\varepsilon)\}}\right)\right).$$

Since this equality holds for any $M > 0$ and the left-hand side does not depend on M, if the right-hand side has a limit as $M \to \infty$, then the equality also holds in this limit. Note that (3.15) of Lemma 6 (iii) implies $\lim_{M\to\infty} n^{(\gamma)}(\rho(M) < \zeta) = \widehat{\phi}(\gamma)$. Since $G(z) = E[\mathrm{e}^{\gamma X(\widehat{T}(z))}]$ it holds $G(\varepsilon(\rho(M,\varepsilon))) \le \mathrm{e}^{-\gamma M}$ and an application of the dominated convergence theorem yields (3.17). By combining with Lemma 5 we find the stated form of Laplace transform of $Z(\infty)$. □

3.1. **Proof of Prop. 3.** **(i)** Equation (3.1) is established in Proposition 7.
(ii) The Wiener-Hopf factorisation of X [2, p. 166] implies the following identity for some $k \in (0,\infty)$:

$$-\log E[\mathrm{e}^{\theta X(1)}] = k\phi(-\theta)\widehat{\phi}(\theta), \qquad \theta \in \mathbb{C},\ \Re(\theta) = 0. \tag{3.24}$$

By analytic continuation and As. 1, identity (3.24) holds for all $\theta \in \mathbb{C}$ with $\Re(\theta) \in [0,\gamma)$. Furthermore, continuity implies that (3.24) remains valid for $\theta = \gamma$. As $\widehat{H}$ is a non-zero subordinator (recall $E[X(1)] < 0$), we have $\widehat{\phi}(\gamma) > 0$ and hence $\phi(-\gamma) = 0$.

By (1.2) the Laplace transform of $x \mapsto P(Z(\infty) > x)$ is $(1 - \frac{\gamma}{\phi(0)}\phi(v)/(v+\gamma))/v$. The Lévy-Khintchine formula for ϕ and integration by parts imply $\phi(v) = \phi(0) + v(m + \int_0^\infty \mathrm{e}^{-vx}\overline{\nu}_H(x)\,\mathrm{d}x)$ for any $v \ge -\gamma$. Since $\phi(-\gamma) = 0$, we have $\int_0^\infty \mathrm{e}^{\gamma y}\overline{\nu}_H(y)\mathrm{d}y = \phi(0)/\gamma - m$. A direct Laplace inversion, based on this representation of ϕ, yields the left-hand side of formula (3.2). The atom at zero is obtained by taking the limit in (3.1) of part (i) as $v \to \infty$.

4. Asymptotic independence

In this section we establish the asymptotic independence of the triplet $(Y(t), Z(x+y), M(t,x))$ as $\min\{t,x,y\} \to \infty$, i.e. for any $a, b \in \mathbb{R}_+$ and $c \in \mathbb{R}$

$$P(Y(t) \le a, Z(x+y) \le b, M(t,x) \le c) = P(Y(t) \le a)P(Z(x+y) \le b)P(M(t,x) \le c) + o(1).$$ [5]

where

$$M(t,x) \doteq Y^*(t) - x, \qquad t, x \in \mathbb{R}_+. \tag{4.1}$$

From this we deduce (see Lemma 11 below) he asymptotic independence of $(Y(t), X(x), m(t))$ as $\min\{t,x\} \to \infty$ and $x - y^*(t) \to \infty$, described in Theorem 2. We start with the following observations concerning the large time behaviour of the local time $\widehat{L}$:

Lemma 8. *The following statements hold true:* **(i)** *We have $E[\widehat{L}^{-1}(1)] \in (0,\infty)$.*
(ii) *As in Thm. 2 denote $\ell = 1/E[\widehat{L}^{-1}(1)]$. For any $\delta \in (0,\ell/2)$ we have*

$$\limsup_{\min\{x,t\}\to\infty} P(\widehat{L}(\tau(x)) \in t[\ell-\delta,\ell+\delta]) \le \frac{4}{\mathrm{e}\ell}\delta.$$

(iii) *The following limit holds: $P(\widehat{L}(t) = \widehat{L}(\tau(x))) \longrightarrow 0$ as $\min\{x,t\} \to \infty$;*
(iv) *For any $\delta_1, \delta_2 \in [0,1/4)$ we have*

$$\limsup_{\min\{x,t\}\to\infty} P(\widehat{L}(t(1-\delta_1)) \le \widehat{L}(\tau(x)) \le \widehat{L}(t(1+\delta_2))) \le \frac{8}{\mathrm{e}}\max\{\delta_1,\delta_2\}. \tag{4.2}$$

For any fixed $s \in (0,\infty)$ it holds $P(\widehat{L}((t-s)\vee 0) \le \widehat{L}(\tau(x)) < \widehat{L}(t)) \longrightarrow 0$ as $\min\{x,t\} \to \infty$.

[5] $f(t,x,y) = o(1)$ $(\min\{x,y,t\} \to \infty)$ if $\lim_{\min\{t,x,y\}\to\infty} f(t,x,y) = 0$

Remarks. **(i)** Part (iii) in Lemma 8 implies that, as x and t tend to infinity, the probability that the excursion straddling t is the first excursion with height larger than x tends to zero. This fact is in line with the asymptotic independence of $Z(\infty)$ and $Y(\infty)$. Part (iv) of Lemma 8 has analogous interpretation.
(ii) The important role played by Lemma 8 in the proof of the asymptotic independence in Thms. 1 and 2 lies in the fact that, the limits in parts (iii) and (vi) do not require the point (t, x) in $(0,\infty)^2$ to tend to infinity along a specific trajectory but only for its norm $\min\{x, t\}$ to increase beyond all bounds.
(iii) In contrast to Lemma 8(iv) the inequality $\limsup_{\min\{x,t\}\to\infty} P(\widehat{L}(\tau(x)) < \widehat{L}(t) \le \widehat{L}(\tau(x+z))) > 0$ holds for any fixed $z > 0$. To show this, recall $\widehat{L}(t)/t \to \ell$ a.s. as $t \uparrow \infty$ (see e.g. proof of Lemma 8(iii) below) and note that for any small $\delta > 0$ we have $P(\widehat{L}(\tau(x)) < \widehat{L}(t) \le \widehat{L}(\tau(x+z))) \ge P(\widehat{L}(\tau(x)) < t(\ell-\delta), \widehat{L}(\tau(x+z)) \ge t(\ell+\delta)) + o(1)$. Hence by Lemma 5 and equality (3.18) we find

$$\begin{aligned} &P(\widehat{L}(\tau(x)) < t(\ell-\delta), \widehat{L}(\tau(x+z)) \ge t(\ell+\delta)) \\ &\quad\ge\; P(\widehat{L}(\tau(x+z)) \ge t(\ell+\delta)) - P(\widehat{L}(\tau(x)) \ge t(\ell-\delta)) \\ &\quad=\; \mathrm{e}^{-t(\ell+\delta)\, n(\rho(x+z)<\zeta)} - \mathrm{e}^{-t(\ell-\delta) n(\rho(x)<\zeta)} \to \mathrm{e}^{-(\ell+\delta)C_\gamma\widehat{\phi}(\gamma)\mathrm{e}^{-\gamma z}} - \mathrm{e}^{-(\ell-\delta)C_\gamma\widehat{\phi}(\gamma)} > 0, \end{aligned}$$

where $\min\{x, t\} \to \infty$ in such a way that $t\mathrm{e}^{-x\gamma} \to 1$ and ρ is given in (2.3). Since $z > 0$, the final inequality clearly holds for $\delta = 0$ and hence by continuity for all $\delta > 0$ sufficiently small.

Proof of Lemma 8. **(i)** This part of the lemma is known. For completeness a short proof, based on the Wiener-Hopf factorisation, is given in the Appendix.
(ii) For any $x, t \in (0,\infty)$, Lemma 5 implies $P(\widehat{L}(\tau(x)) > t) = \mathrm{e}^{-t\, n(B(x))}$ for all $t \ge 0$, where $B(x) \doteq \{\rho(x) < \zeta\}$ with ρ defined in (2.3). Therefore for any $\delta \in (0, \ell/2)$ the following holds:

$$P(\widehat{L}(\tau(x)) \in t[\ell-\delta, \ell+\delta]) = \mathrm{e}^{-t\,\ell\, n(B(x))}\left(\mathrm{e}^{\delta\, t\, n(B(x))} - \mathrm{e}^{-\delta\, t\, n(B(x))}\right).$$

Lagrange's theorem implies that there exists $\xi_{t,x} \in (-\delta, \delta)$ such that

$$\begin{aligned} P(\widehat{L}(\tau(x)) \in t[\ell-\delta, \ell+\delta]) &= 2\delta t n(B(x))\mathrm{e}^{(\xi_{t,x}-\ell)tn(B(x))} \\ &\le 2\delta t n(B(x))\mathrm{e}^{-tn(B(x))\ell/2} \le \delta 4/(\mathrm{e}\ell), \end{aligned}$$

where the inequality follows from $|\xi_{t,x}| < \ell/2$. Since $t, x \in (0,\infty)$ were arbitrary, this concludes the proof of part (ii).
(iii) Since $\widehat{L}^{-1}$ is a subordinator under P, the strong law of large numbers (see e.g. [2, p.92]) implies that, as $t \to \infty$, the ratio $t/\widehat{L}^{-1}(t)$ tends to ℓ almost surely. Hence, for any $\delta \in (0, \ell/2)$,

$$P\left(\widehat{L}(t)/t \in [\ell-\delta, \ell+\delta]\right) = 1 + o(1), \qquad \text{as } t \to \infty. \tag{4.3}$$

Equation (4.3) yields the following as $\min\{x, t\} \to \infty$:

$$\begin{aligned} P(\widehat{L}(t) = \widehat{L}(\tau(x))) &= P(\widehat{L}(t) = \widehat{L}(\tau(x)), \widehat{L}(t) \in t[\ell-\delta, \ell+\delta]) + o(1) \\ &\le P(\widehat{L}(\tau(x)) \in t[\ell-\delta, \ell+\delta]) + o(1). \end{aligned}$$

Hence part (ii) yields $\limsup_{\min\{x,t\}\to\infty} P(\widehat{L}(t) = \widehat{L}(\tau(x))) \le \delta 4/(\mathrm{e}\ell)$. Since $\delta \in (0, \ell/2)$ was arbitrary and probabilities are non-negative quantities, the limit in part (iii) follows.

(iv) Note that for any $\alpha \geq 0$ the quotient $\widehat{L}(t\alpha)/t$ tends to $\ell\alpha$ P-a.s. as $t \to \infty$. For any $\delta_1, \delta_2 \in [0, 1/4)$ we therefore find that the probability of the event

$$A_{\delta_1,\delta_2}(t,x) = \{\widehat{L}(t(1-\delta_1)) \leq \widehat{L}(\tau(x)) \leq \widehat{L}(t(1+\delta_2))\}$$

satisfies the following as $\min\{x,t\} \to \infty$:

$$\begin{aligned} P(A_{\delta_1,\delta_2}(t,x)) &= P(A_{\delta_1,\delta_2}(t,x), \widehat{L}(t(1-\delta_1)), \widehat{L}(t(1+\delta_2)) \in t[\ell(1-\delta), \ell(1+\delta)]) + o(1) \\ &\leq P(\widehat{L}(\tau(x)) \in t[\ell(1-\delta), \ell(1+\delta)]) + o(1), \end{aligned} \tag{4.4}$$

for any $\delta \in (2\max\{\delta_1, \delta_2\}, 1/2)$. Since $0 < \delta\ell < \ell/2$, part (ii) of the lemma and inequality (4.4) imply that $\limsup_{\min\{x,t\}\to\infty} P(A_{\delta_1,\delta_2}(t,x)) \leq \delta 4/\mathrm{e}$. Therefore the first inequality in part (iv) is satisfied. The second limit in part (iv) follows by noting that, for any $s \in \mathbb{R}_+$ and $\delta_1 \in (0, 1/4)$, the inclusion $\{\widehat{L}((t-s) \vee 0) \leq \widehat{L}(\tau(x)) < \widehat{L}(t)\} \subset A_{\delta_1,0}(t,x)$ holds for all (t,x) with large $\min\{x,t\}$. Hence by (4.2) we have

$$\limsup_{\min\{x,t\}\to\infty} P(\widehat{L}((t-s) \vee 0) \leq \widehat{L}(\tau(x)) < \widehat{L}(t)) \leq \delta_1 8/\mathrm{e}.$$

Since δ_1 can be chosen arbitrarily small, this proves part (iv) and hence the lemma. □

Before moving to the proof of the asymptotic independence of $Y(t), Z(x+y)$ and $M(x,t)$, we establish the asymptotic behaviour of certain convolutions that will arise in the proof. For any $x \in \mathbb{R}_+$, recall that $T(x)$ is given in (3.3).

Lemma 9. *For $a \in [0, \infty)$ and any family of sets $F(t) \in \mathcal{F}$, $t \in \mathbb{R}_+$, we have*

$$\begin{aligned} \int_{[0,t]} P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t-s))\, P(T(a) \in \mathrm{d}s) \\ = P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t))\, P(Y(t) > a) + o(1), \qquad \textit{as } \min\{y,t\} \to \infty. \end{aligned} \tag{4.5}$$

Proof of Lemma 9. The proof of this lemma is based on Lemma 8. Since $Y(t)$ and $\sup_{0\leq s\leq t} X(s)$ are equal in law (by time reversal) and $P(T(a) = t) \to 0$ as $t \to \infty$ (as $X_t \to -\infty$ by As. 1), it holds

$$P(T(a) \leq t) = P(Y(t) > a) + o(1) \qquad \text{as } t \to \infty.$$

Hence, to prove equality (4.5), it is sufficient to establish

$$\int_{[0,t]} \Big(P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t)) - P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t-s))\Big)\, P(T(a) \in \mathrm{d}s) = o(1) \tag{4.6}$$

as $\min\{y,t\} \to \infty$. Since the local time $\widehat{L}$ is non-decreasing, the integrand in (4.6) satisfies

$$|P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t)) - P(F(t), \widehat{L}(\tau(y)) < \widehat{L}(t-s))| \leq P(\widehat{L}(t-s) \leq \widehat{L}(\tau(y)) < \widehat{L}(t)).$$

Hence Lemma 8(iv) and the dominated convergence theorem imply that (4.6) holds. This completes the proof of the lemma. □

We move next to the asymptotic independence of $Y(t), Z(x+y)$ and $M(t,x)$.

Lemma 10. *For any* $t, x \in (0,\infty)$, $a, b \in \mathbb{R}_+$, $c \in \mathbb{R}$, $y \in [0,x]$ *and Borel sets* $A, B, C \in \mathcal{B}(\mathbb{R})$ *with* $A = (-\infty, a]$, $B = (-\infty, b]$ *and* $C = (-\infty, c]$ *denote*

$$\pi_1(t,A) = P(Y(t) \in A), \quad \pi_2(x,B) = P(Z(x) \in B), \quad \pi_3(t,y) = P(\widehat{L}(\tau(y)) < \widehat{L}(t)).$$

Recall the definition of $M(t,x)$ *in* (4.1). *We have as* $\min\{t, y, x-y\} \to \infty$

$$P(Y(t) \in A, Z(x) \in B) = \pi_1(t,A)\pi_2(x,B) + o(1), \tag{4.7}$$

$$P(Y(t) \in A, Z(x) \in B, \widehat{L}(\tau(y)) < \widehat{L}(t)) = \pi_1(t,A)\pi_2(x,B)\pi_3(t,y) + o(1), \tag{4.8}$$

$$P(Y(t) \in A, Z(x) \in B, M(t,y) \in C) = \pi_1(t,A)\pi_2(x,B)P(M(t,y) \in C) + o(1). \tag{4.9}$$

Proof of Lemma 10. Fix $t, x \in \mathbb{R}_+ \backslash \{0\}$, $y \in [0,x]$, $a, b \in \mathbb{R}_+$ arbitrary, with $A = (-\infty, a]$, $B = (-\infty, b]$. As first step we note that by a classical application of excursion theory[6] involving $G(\tau(x)) = \sup\{s < \tau(x) : Y(s) = 0\} = \widehat{L}^{-1}(\widehat{L}(\tau(x))-)$ the random elements $\mathcal{A} := \{Y(s) : 0 \le s \le G(\tau(x))\}$ and $\mathcal{A}' := \epsilon(\widehat{L}(\tau(x)))$ are independent. Hence the sets $\{Z(x) \in B\}$ and $\{\widehat{L}(\tau(y)) > \widehat{L}(t), Y(t) \in A\}$, which are measurable with respect to $\sigma(\mathcal{A}')$ and $\sigma(\mathcal{A})$ respectively, are independent, that is,

$$\begin{aligned} &P(\widehat{L}(\tau(y)) > \widehat{L}(t), Y(t) \in A, Z(x) \in B) \\ &\quad = P(\widehat{L}(\tau(y)) > \widehat{L}(t), Y(t) \in A) P(Z(x) \in B). \end{aligned} \tag{4.10}$$

As second step we establish another asymptotic factorisation. Since $s \mapsto \mathbf{I}_{\{\tau(x)<s\le t, Z(x)\in B\}}$ is left-continuous and adapted an application of the compensation formula of excursion theory (see e.g. [2, Cor. IV.11]) yields

$$\begin{aligned} &P\left(\widehat{L}(\tau(x)) < \widehat{L}(t), Y(t) \in A, Z(x) \in B\right) \\ &\quad = E\left[\sum_g \mathbf{I}_{\{\tau(x)<g\le t, Z(x)\in B\}} \mathbf{I}_{\{\epsilon_g(t-g)\in A, t-g<\zeta(\epsilon_g)\}}\right] \\ &\quad = E\left[\int_{[0,t]} \mathbf{I}_{\{\tau(x)<s\le t, Z(x)\in B\}} n(\varepsilon(t-s) \in A, t-s < \zeta(\varepsilon)) \mathrm{d}\widehat{L}(s)\right], \end{aligned} \tag{4.11}$$

where the sum is over all left-end points of excursion intervals. Let $e(q)$ be an exponential random time with mean $1/q$ defined by extending the probability space to $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \times P')$. Replacing t by $e(q)$ in (4.11) and denoting $\mathbb{P} := P \times P'$ we have by the lack of memory property of $e(q)$

$$\begin{aligned} &\mathbb{P}\left(\widehat{L}(\tau(x)) < \widehat{L}(e(q)), Y(e(q)) \in A, Z(x) \in B\right) \\ &= \mathbb{E}\left[\int_{[0,e(q)]} \mathbf{I}_{\{\tau(x)<s\le e(q), Z(x)\in B\}} \mathrm{d}\widehat{L}(s)\right] \mathbb{E}[n(\varepsilon(e(q)) \in A, e(q) < \zeta(\varepsilon))] \\ &= \mathbb{E}\left[\int_{[0,e(q)]} \mathbf{I}_{\{\tau(x)<s\le e(q), Z(x)\in B\}} \mathrm{d}\widehat{L}(s)\right] \mathbb{E}[n(e(q) < \zeta(\varepsilon))] \frac{\mathbb{E}[n(\varepsilon(e(q)) \in A, e(q) < \zeta(\varepsilon))]}{\mathbb{E}[n(e(q) < \zeta(\varepsilon))]} \\ &= \mathbb{P}\left(\widehat{L}(\tau(x)) < \widehat{L}(e(q)), Z(x) \in B\right) \mathbb{P}(Y(e(q)) \in A), \end{aligned} \tag{4.12}$$

[6] This can be seen to follow directly as a consequence of the splitting property [2, Sec O.5, Prop. O.2] of the Poisson point process ϵ at the first entrance time $\mathbb{H}_{B'} = \inf\{s \le 0 : \epsilon(s) \in B'\}$ of ϵ into the set $B' = \{\varepsilon \in \mathcal{E} : \rho(x,\varepsilon) < \zeta(\varepsilon)\}$.

where the equality in the final line follows by similar applications of the compensation formula. Dividing the LHS and RHS of (4.12) by q and inverting the Laplace transform in q, and deploying (4.5) in Lemma 9 we have

$$P(\widehat{L}(\tau(x)) < \widehat{L}(t), Y(t) \in A, Z(x) \in B) \\ = P(\widehat{L}(\tau(x)) < \widehat{L}(t), Z(x) \in B)\pi_1(t, A) + o(1), \qquad \text{as } \min\{x, t\} \to \infty. \tag{4.13}$$

Taking note of the following equality for any $y, t \in (0, \infty)$ and set $E \in \mathcal{F}$:

$$\mathbb{P}(E, \widehat{L}(\tau(y)) > \widehat{L}(t)) + P(E, \widehat{L}(\tau(y)) = \widehat{L}(t)) \\ = P(E) - P(E, \widehat{L}(\tau(y)) < \widehat{L}(t)), \tag{4.14}$$

and applying (4.10) and (4.13) (with $B = \mathbb{R}_+$) yields as $\min\{x, t\} \to \infty$

$$\begin{aligned} & P(Y(t) \in A, Z(x) \in B) \\ &= \pi_1(t, A)P(\widehat{L}(\tau(x)) < \widehat{L}(t), Z(x) \in B) + P(\widehat{L}(\tau(x)) > \widehat{L}(t), Y(t) \in A)\pi_2(x, B) \\ &\quad + P(\widehat{L}(\tau(x)) = \widehat{L}(t), Y(t) \in A, Z(x) \in B) + o(1) \\ &= \pi_1(t, A)\pi_2(x, B) + R(t, x) + o(1), \end{aligned}$$

where $R(t, x) = P(\widehat{L}(\tau(x)) = \widehat{L}(t), Y(t) \in A, Z(x) \in B) - P(\widehat{L}(\tau(x)) = \widehat{L}(t), Y(t) \in A)\pi_2(x, B) - P(\widehat{L}(\tau(x)) = \widehat{L}(t), Z(x) \in B)\pi_1(t, A) + \pi_1(t, A)\pi_2(x, B)P(\widehat{L}(\tau(x)) = \widehat{L}(t))$. Observing that $R(t, x) = o(1)$ when $\min\{x, t\} \to \infty$ by Lemma 8(iii) the proof of (4.7) is complete.

Equation (4.8) follows similarly, by combining the equality (4.14) (with $E = \{Y(t) \in A, Z(x) \in B\}$) with Lemma 8(iii) and the identities (4.7), (4.10), and (4.13) (with $B = \mathbb{R}_+$).

Finally, take $C = (-\infty, c]$ for an arbitrary fixed $c \in \mathbb{R}$. In order to prove equality (4.9) note that the following inclusions hold for any $y \in \mathbb{R}_+$:

$$\begin{aligned} \{M(t, y) \in C\} = \{Y^*(t) \le y + c\} &\subset \{\widehat{L}(t) \le \widehat{L}(\tau((y + c)^+))\} \quad \text{and} \\ \{\widehat{L}(t) \le \widehat{L}(\tau((y + c)^+))\} \cap \{M(t, y) \notin C\} &\subset \{\widehat{L}(\tau((y + c)^+)) = \widehat{L}(t)\} \end{aligned}$$

(recall that $\tau(x)$ is defined for $x \in \mathbb{R}_+$). These inclusions, together with Lemma 8(iii), imply that the following equality holds for any family of events $E(t, x) \in \mathcal{F}$, $t, x \in \mathbb{R}_+$, as $\min\{t, y, x - y\} \to \infty$:

$$P\left(E(t, x), \widehat{L}(t) \le \widehat{L}(\tau((y + c)^+))\right) = P\left(E(t, x), M(t, y) \in C\right) + o(1). \tag{4.15}$$

Since $\min\{t, y, x - y\} \to \infty$, for the fixed $c \in \mathbb{R}$ the inequalities $0 \le y + c \le x$ hold for all large y and x. In particular (4.8), applied to the complement $\{\widehat{L}(\tau(y + c)) < \widehat{L}(t)\}^c = \{\widehat{L}(\tau(y + c)) \ge \widehat{L}(t)\}$, Lemma 8(iii) and (4.15) yield the following equalities:

$$\begin{aligned} P(Y(t) \in A, Z(x) \in B, M(t, y) \in C) &= P(Y(t) \in A, Z(x) \in B, \widehat{L}(t) \le \widehat{L}(\tau(y + c))) + o(1) \\ &= P(Y(t) \in A)P(Z(x) \in B)P(\widehat{L}(t) \le \widehat{L}(\tau(y + c))) + o(1) \\ &= P(Y(t) \in A)P(Z(x) \in B)P(M(t, y) \in C) + o(1) \end{aligned}$$

as $\min\{t, y, x - y\} \to \infty$, which establishes (4.9). This concludes the proof of the lemma. □

Lemma 11. **(i)** *As* $\min\{x,t\}\to\infty$, $Y(t)$ *and* $Z(x)$ *satisfy*

$$E[\exp(-uY(t)-vZ(x))] = E[\exp(-uY(t))]E[\exp(-vZ(x))]+o(1), \quad \textit{for any } u,v\in\mathbb{R}_+\backslash\{0\}. \tag{4.16}$$

(ii) *As* $\min\{x,t\}\to\infty$ *such that* $t\exp(-\gamma x)\to 0$, $Y(t)$, $Z(x)$ *and* $m(t)$ *satisfy*

$$\begin{aligned} E[\exp(-uY(t)-vZ(x)\pm\beta m(t))\mathbf{I}_{\{\pm m(t)<0\}}] &= E[\exp(-uY(t))]\,E[\exp(-vZ(x))]\times \\ &\times E[\exp(\pm\beta m(t))\mathbf{I}_{\{\pm m(t)<0\}}]+o(1), \qquad \textit{for any } u,v,\beta\in\mathbb{R}_+\backslash\{0\}. \end{aligned} \tag{4.17}$$

In particular, we have

$$\begin{aligned} E[\exp(-uY(t)-vZ(x)-\beta|m(t)|-b\, s(m(t)))] &= E[\exp(-uY(t))]\,E[\exp(-vZ(x))]\times \\ &\times E[\exp(-\beta|m(t)|-b\, s(m(t)))]+o(1), \qquad \textit{for any } u,v,\beta,b\in\mathbb{R}_+\backslash\{0\}, \end{aligned} \tag{4.18}$$

where $s:\mathbb{R}\to(-\infty,\infty]$ *is given by* $s(x)=\pm1$ *for* $\pm x\in\mathbb{R}_+\backslash\{0\}$, *and* $s(0)=+\infty$.

Proof. **(i)** Fix $u,v\in\mathbb{R}_+\backslash\{0\}$ arbitrary. By integrating both sides of the identity in (4.7) in Lemma 10 over $\mathbb{R}^2$ against the measure $\mathbf{I}_{\mathbb{R}_+\times\mathbb{R}_+}(a,b)a\,b\,\exp(-ua-vb)\mathrm{d}a\mathrm{d}b$ we have (4.16) by noting that the integral of the $o(1)$ term in (4.7) tends to zero by the dominated convergence theorem (as it is bounded by one).
(ii) The proof is a modification of the argument in part (i). Let now $u,v,w\in\mathbb{R}_+\backslash\{0\}$ be arbitrary. Integrating both sides of the identity in (4.9) in Lemma 10 over $\mathbb{R}^3$ against the measures $\mathbf{I}_{\mathbb{R}^2_+\times\mathbb{R}_+\backslash\{0\}}(a,b,c)\,\exp(-ua-vb-wc)\mathrm{d}a\mathrm{d}b\mathrm{d}c$ (with $\mathbb{R}^2_+=(\mathbb{R}_+)^2$) and applying the dominated convergence theorem shows that also in this case the integral of the $o(1)$ tends to zero, which yields the "$-$"-version of (4.17). The "$+$"-version of follows similarly by integrating both sides of the identity in (4.9) against the measure $\mathbf{I}_{\mathbb{R}^2_+\times\mathbb{R}_-\backslash\{0\}}(a,b,c)\exp(-ua-vb+wc)\mathrm{d}a\mathrm{d}b\mathrm{d}c$ (with $\mathbb{R}_-=\mathbb{R}\backslash\mathbb{R}_+$). As (4.18) follows as direct consequence of (4.17), the proof is complete. □

5. Proofs of Theorems 1 and 2

5.1. **Proof of Thm. 1.** We first observe that $Y(t)$ and $Z(x)$ each admit a weak limit $Y(\infty)$, $Z(\infty)$ as $t,x\to\infty$. Existence and the form of the Laplace transform of $Z(\infty)$ are given in Proposition 7. As far as the weak limit of $Y(t)$ is concerned we note that the duality lemma for Lévy processes implies that the supremum $X^*(t)=\sup_{0\le s\le t}X(s)$ and $Y(t)$ have the same law for any fixed $t\ge0$. Since, by As. 1, $E[X(1)]<0$, and the process $\{X^*(t)\}_{t\ge0}$ is non-decreasing, it converges a.s. as $t\uparrow\infty$ to $X^*(\infty)\doteq\sup_{s\ge0}X(s)$. Therefore $Y(t)$ converges weakly to a limit $Y(\infty)$ that has the same law as $X^*(\infty)$ and that is characterised by its Laplace transform $E[\mathrm{e}^{-uY(\infty)}]=\phi(0)/\phi(u)$, $u\in\mathbb{R}_+$ (see [2, p. 163]). The joint Laplace transform of $(Y(\infty),Z(\infty))$ now follows from (4.16) in Lemma 11(i).

Finally, the factorisation of the exponential distribution is obtained by setting $u=v$ in (1.2).

5.2. **Proof of Thm. 2.** We first establish that the elements ℓ and C_γ in the last factor in (1.4) are both strictly positive and finite. Since ϕ is strictly concave with $\phi(-\gamma)=0$, the right-derivative of ϕ at $-\gamma$ satisfies $\phi'(-\gamma)>0$ and the constant C_γ in (1.5) is well-defined. By Lemma 8(i) we have $\ell\in(0,\infty)$.

It follows from [7, Thm. 1] that if t tends to infinity then $m(t)$ converges in distribution to $m(\infty)$, which follows a Gumbel distribution,

$$P(m(\infty) < z) = \exp\left(-\ell\, C_\gamma\, \widehat{\phi}(\gamma)\, \mathrm{e}^{-\gamma z}\right), \qquad \text{for all} \quad z \in \mathbb{R}. \tag{5.1}$$

We give below a short proof of (5.1) based on [7, Thm. 1]. The joint Fourier-Laplace transform and asymptotic independence now follow from a direct calculation using (5.1) and (4.18) in Lemma 11(ii) (which implies that $Y(\infty)$, $Z(\infty)$ and $(|m(\infty)|, \mathrm{sgn}(m(\infty)))$ are independent, and hence $Y(\infty)$, $Z(\infty)$ and $m(\infty)$ are).

To establish (5.1) we show that, as $\min\{x,t\} \to \infty$ and $t\mathrm{e}^{-\gamma x} \to 1$, the following holds:

$$P(Y^*(t) - x < z) \;=\; \exp(-t\,\ell\, n(\rho(x+z) < \zeta)) + o(1) \quad \text{for any } z \in \mathbb{R}. \tag{5.2}$$

Since (3.18) implies $tn(\rho(x+z) < \zeta) \to C_\gamma \widehat{\phi}(\gamma)\mathrm{e}^{-\gamma z}$ as $\min\{x,t\} \to \infty$ and $t\mathrm{e}^{-\gamma x} \to 1$, the limit in (5.1) follows from (5.2).

To complete the proof we now verify the claim in (5.2). Note that $\tau(x+z) \to \infty$ P-a.s. as $x \to \infty$ and, as shown in the proof of Lemma 8, the law of large numbers implies that $\widehat{L}(t)/t \to \ell$ P-a.s. as $t \to \infty$, where $\ell = 1/E\left[\widehat{L}^{-1}(1)\right]$ (recall from Lemma 8(i) that $0 < \ell < \infty$). Therefore $\widehat{L}(\tau(x+z))/\tau(x+z)$ tends to ℓ P-a.s as $x \to \infty$. In particular, for any $\delta > 0$, we have

$$P(\widehat{L}(\tau(x+z))/\tau(x+z) \in (\ell-\delta, \ell+\delta)) = 1 + o(1) \qquad \text{as} \quad x \to \infty.$$

Hence as $\min\{x,t\} \to \infty$ the following holds

$$\begin{aligned} P(Y^*(t) < x+z) \;&=\; P(\tau(x+z) > t, \widehat{L}(\tau(x+z))/\tau(x+z) \geq \ell-\delta) + o(1) \\ &\leq\; P(\widehat{L}(\tau(x+z)) > t(\ell-\delta)) + o(1). \end{aligned}$$

Similarly, it follows that as $\min\{x,t\} \to \infty$ we have

$$\begin{aligned} P(Y^*(t) < x+z) \;&\geq\; P(\widehat{L}(\tau(x+z)) > \widehat{L}(t), \widehat{L}(t) \leq t(\ell+\delta)) \\ &\geq\; P(\widehat{L}(\tau(x+z)) > t(\ell+\delta), \widehat{L}(t) \leq t(\ell+\delta)) = P(\widehat{L}(\tau(x+z)) > t(\ell+\delta)) + o(1). \end{aligned}$$

By Lemma 5, $\widehat{L}(\tau(x+z))$ is exponentially distributed with parameter $n(\rho(x) < \zeta)$ and hence we find

$$\exp(-(\ell+\delta)t\, n(\rho(x+z) < \zeta)) + o(1) \leq P(Y^*(t) < x+z) \leq \exp(-(\ell-\delta)t\, n(\rho(x+z) < \zeta)) + o(1).$$

Since this result holds for any $\delta > 0$, the equality in (5.2) follows. □

Appendix A. Proof of Lemma 8(i)

By analytical continuation and As. 1 it follows that identity (3.24) remains valid for all $\theta \in \mathbb{C}$ with $\Re(\theta) \in [0, \gamma)$. Therefore on the event $\{H(1) < \infty\}$ the random variable $H(1)$ admits finite exponential moments and in particular $E\left[H(1)\mathbf{I}_{\{H(1)<\infty\}}\right] < \infty$. Since $E[X(1)] \in (-\infty, 0)$, the ladder-height process of the dual process $\widehat{X} = -X$ satisfies $P(\widehat{H}(1) < \infty) = 1$. Furthermore, we have $P(H(1) < \infty) < 1$. Definition (2.1) of ϕ, its analogue for $\widehat{\phi}$, the Wiener-Hopf factorisation in (3.24) and the dominated convergence theorem imply that the following identity holds for all $\theta \in (0, \gamma)$:

$$-\frac{E[X(1)\mathrm{e}^{\theta X(1)}]}{kE[\mathrm{e}^{\theta X(1)}]} = \frac{E[H(1)\mathrm{e}^{\theta H(1)}\mathbf{I}_{\{H(1)<\infty\}}]}{E[\mathrm{e}^{\theta H(1)}\mathbf{I}_{\{H(1)<\infty\}}]}\log E\left[\mathrm{e}^{-\theta \widehat{H}(1)}\right] - \frac{E[\widehat{H}(1)\mathrm{e}^{-\theta\widehat{H}(1)}]}{E[\mathrm{e}^{-\theta\widehat{H}(1)}]}\log E\left[\mathrm{e}^{\theta H(1)}\mathbf{I}_{\{H(1)<\infty\}}\right].$$

As. 1 implies that in the limit as $\theta \to 0$ this equality yields $E[\widehat{H}(1)] \in (0, \infty)$.

The inequality $\left|\widehat{X}\left(\min\{t, \widehat{L}^{-1}(1)\}\right)\right| \leq \widehat{H}(1) + X^*(\infty)$ holds for all $t \in \mathbb{R}_+$. Cramér's estimate (3.6) implies that $X^*(\infty)$ is integrable. Since $\left\{\widehat{X}(t) - tE[\widehat{X}(1)]\right\}_{t\geq 0}$ is a martingale we have

$$E\left[\widehat{X}\left(\min\{t, \widehat{L}^{-1}(1)\}\right)\right] = E\left[\widehat{X}(1)\right] E\left[\min\{t, \widehat{L}^{-1}(1)\}\right] \quad \text{for all} \quad t \in \mathbb{R}_+.$$

The dominated and monotone convergence theorems applied to each side of this equality respectively imply Wald's identity for the $\{\mathcal{F}_t\}$-stopping time $\widehat{L}^{-1}(1)$: $E\left[\widehat{H}(1)\right] = -E\left[X(1)\right] E\left[\widehat{L}^{-1}(1)\right]$. In particular we obtain $\ell^{-1} = E\left[\widehat{L}^{-1}(1)\right] \in (0, \infty)$, proving Lemma 8(i). □

References

[1] S. Asmussen. *Applied probability and queues*, volume 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2003. Stochastic Modelling and Applied Probability.

[2] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[3] J. Bertoin and R. A. Doney. Cramér's estimate for Lévy processes. *Statist. Probab. Lett.*, 21(5):363–365, 1994.

[4] J. Bertoin, K. Van Harn, and F.W. Steutel. Renewal theory and level passage by subordinators. *Stat. Prob. Lett.*, 45(1):65–69, 1999.

[5] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[6] R. M. Blumenthal. *Excursions of Markov processes*. Birkhäuser.

[7] R. A. Doney and R. A. Maller. Cramér's estimate for a reflected Lévy process. *Ann. Appl. Probab.*, 15(2):1445–1450, 2005.

[8] P. Embrechts, C. Klüppelberg, and T. Mikosch. *Modelling Extremal Events for Insurance and Finance*. Springer, Berlin, 1997.

[9] P. Greenwood and J. Pitman. Fluctuation identities for Lévy processes and splitting at the maximum. *Adv. in Appl. Probab.*, 12(4):893–902, 1980.

[10] O. Hadjiliadis and J. Večeř. Drawdowns preceding rallies in the Brownian motion model. *Quant. Finance*, 6(5):403–409, 2006.

[11] D.L. Iglehart. Extreme values in the $GI/G/1$ queue. *Ann. Math. Statist.*, 43:627–635, 1972.

[12] K. Itô. Poisson point processes attached to Markov processes. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. III: Probability theory*, pages 225–239, Berkeley, Calif., 1972. Univ. California Press.

[13] S. Karlin and A. Dembo. Limit distributions of maximal segmental score among Markov-dependent partial sums. *Adv. in Appl. Probab.*, 24(1):113–140, 1992.

[14] A. Mijatović and M. Pistorius. On the drawdown of completely asymmetric Lévy processes. *Stochastic Processes and Applications*, 122(11):3812–3836, 2012.

[15] A. Mijatović and M. Pistorius. Buffer-overflows: joint limit laws of undershoots and overshoots of reflected processes. *to appear in Stochastic Processes and Applications*, 2015.

[16] G.V. Moustakides. Optimality of the CUSUM procedure in continuous time. *Ann. Statist.*, 32(1):302–315, 2004.

[17] N. U. Prabhu. *Stochastic storage processes*, volume 15 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 1998. Queues, insurance risk, dams, and data communication.

[18] A. N. Shiryaev. Minimax optimality of the method of cumulative sums (CUSUM) in the continuous time case. *Uspekhi Mat. Nauk*, 51(4(310)):173–174, 1996.

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