Asymptotic formulae for implied volatility in the Heston model[*]

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Abstract

In this paper we prove an approximate formula expressed in terms of elementary functions for the implied volatility in the Heston model. The formula consists of the constant and first order terms in the large maturity expansion of the implied volatility function. The proof is based on saddlepoint methods and classical properties of holomorphic functions.

1 Introduction

The Heston model introduced in 1993, [19], has become one of the most widely used stochastic volatility models in the derivatives market (see [14], [29], [2], [3], [28]). In this paper, we provide a closed-form formula for the implied volatility in this model for a maturity-dependent strike $K = S_0 \exp(xt)$:

$$\hat{\sigma}_t^2(x) = \hat{\sigma}_\infty^2(x) + t^{-1} \frac{8 \hat{\sigma}_\infty^4(x)}{4x^2 - \hat{\sigma}_\infty^4(x)} \log \left(\frac{A(x)}{A_{\mathrm{BS}}(x, \hat{\sigma}_\infty(x), 0)} \right) + o\left(t^{-1} \right) \tag{1}$$

as the maturity t tends to infinity, where $\hat{\sigma}_\infty^2$ is defined in (15), A in (9) and A_{BS} in (14). For a constant strike $K = S_0 \exp(x)$, we obtain the following formula:

$$\sigma_t^2(x) = 8V^*(0) + t^{-1} 4 \left(x \left(2 p^*(0) - 1 \right) - 2 \log \left(-A(0) \sqrt{2V^*(0)} \right) \right) + o\left(t^{-1} \right) \tag{2}$$

as the maturity t tends to infinity, where V^* is given by (8) and p^* by (30).

In practice, stochastic volatility models are first calibrated on market data, then used for pricing. Concerning the pricing step, accurate algorithms rely either on PDE (see [20], [23]), or accurate quadrature [2] methods. As the calibration step is based on optimisation algorithms, the lack of a closed-form formula for the implied volatility makes it very time consuming. For instance, the SABR stochastic volatility model has become very popular because a closed-form approximation formula for the implied volatility was derived in [18] and hence made the model easily tractable. Likewise, perturbation methods as developed in [13] have proved to be very useful for obtaining a closed-form approximation formula of option prices. Although these methods only hold under some constraints on the parameters, they provide

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useful initial reference points for calibration. In Section 4, we show that our formula is very accurate for some parameters calibrated on market data. It is less accurate for maturities shorter than five years. However, it is still useful as a first guess for calibration purposes.

It is a well-known fact that for a fixed strike, the implied volatility flattens as the maturity increases (see [34], [32]); this is confirmed by formula (2) above, the zeroth order term of which was already known (see [29], [12]). However, the maturity-dependent strike formulation in formula (1) above reveals that the implied volatility smile does not flatten but rather spreads out in a very specific way as the maturity increases.

In the fixed strike case, Lewis pioneered the research on large-time asymptotics of implied volatility in stochastic volatility models in [29] by studying the first eigenvalue and eigenfunction of the differential operator associated with the two-dimensional stochastic volatility SDE. More recently, Tehranchi [35] studied the large-time behaviour of the implied volatility when the stock price is a non-negative local martingale and obtained an analog of formula (2) in that setting. Comparatively, there has been a profusion of work on small-time asymptotics, based on differential geometry techniques ([24], [25]), PDE methods [6] or large deviations techniques ([11] and [10]). Likewise, many papers have studied the behaviour of the implied volatility smile in the wings (see [4], [5], [16], [17], [26]).

The proof of our main result in this paper, Theorem 2.1, is based on two methods: first, we use saddlepoint approximation methods to study the behaviour of the call price function as an inverse Fourier transform. This idea has already been applied by several authors, including [9], [15], [1] and [31] in order to speed up the computation of option pricing algorithms based on inverse Fourier transforms. We are also able to obtain the saddlepoint in closed form, thus avoiding any numerical approximations in determining it. The second step in our proof relies on Cauchy's integral theorem and contour integration for holomorphic functions in order to obtain precise estimates of call option prices in the large maturity limit.

The paper is organised as follows. Section 2 contains the large-time asymptotic formula for call options under the Heston and the Black-Scholes models, both in the maturity-dependent and in the fixed strike case. The proof of the main theorem, Theorem 2.1, is given in Section 5. In section 3, we translate these results into implied volatility asymptotics and prove formulae (1) and (2) above. In Section 4, we calibrate the Heston model and provide numerical examples based on formulae (1) and (2).

2   Large-time behaviour of call options

Throughout this article, we work on a model $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t \geq 0}$ supporting two Brownian motions, and satisfying the usual conditions. Let $(S_t)_{t \geq 0}$ denote a stock price process and we let $X_t := \log(S_t)$. Interest rates and dividends are considered null. We assume the following Heston dynamics for the log-stock price:

$$
\begin{aligned}
&dX_t = -\tfrac{1}{2}Y_t dt + \sqrt{Y_t}\,\mathrm{d}W_t^1, \ X_0 = x_0 \in \mathbb{R}, \\
&dY_t = \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t}\,\mathrm{d}W_t^2, \ Y_0 = y_0 > 0, \\
&\mathrm{d}\langle W^1, W^2 \rangle_t = \rho\mathrm{d}t
\end{aligned}
\tag{3}
$$

with $\kappa, \theta, \sigma, y_0 > 0, |\rho| < 1$ and $2\kappa\theta > \sigma^2$, which ensures that 0 is an unattainable boundary for the process Y (so the SDE admits a unique strong solution, see Proposition 2.13 in [21]). Let us define $\bar{\kappa} := \kappa - \rho\sigma$, $\bar{\rho} := \sqrt{1 - \rho^2}$, and $\bar{\theta} := \kappa\theta/\bar{\kappa}$. We also assume $\bar{\kappa} > 0$. This assumption ensures (see [12] for instance) that moments of S_t greater than 1 exist for all times t. Let V be the limiting log moment generating function of X_t defined as

$$V(p) := \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}\left(\exp\left(p(X_t - x_0)\right)\right), \tag{4}$$

for all p such that the limit exists and is finite. It follows from [3] that V is a well defined and strictly convex function (p_-, p_+) and is infinite outside, where

$$p_\pm := \left(-2\kappa\rho + \sigma \pm \sqrt{\sigma^2 + 4\kappa^2 - 4\kappa\rho\sigma}\right)/(2\sigma\bar{\rho}), \tag{5}$$

with $p_- < 0$ and $p_+ > 0$. Furthermore the function V takes the following form

$$V(p) = \frac{\kappa\theta}{\sigma^2}\left(\kappa - \rho\sigma p - d(-\mathrm{i}p)\right), \quad \text{for } p \in (p_-, p_+), \tag{6}$$

where

$$d(k) := \sqrt{(\kappa - \mathrm{i}\rho\sigma k)^2 + \sigma^2(\mathrm{i}k + k^2)}, \quad \text{for } k \in \mathbb{C}, \tag{7}$$

and we take the principal branch for the complex square root function in (7). Let us now define the following function

$$V^*(x) := p^*(x)x - V(p^*(x)), \quad \text{for all } x \in \mathbb{R}, \tag{8}$$

where the function $p^* : \mathbb{R} \to (p_-, p_+)$ is defined by the explicit formula in (30). Since the image of p^* is (p_-, p_+), then the function V^* is well defined on $\mathbb{R}$ and has an explicit form from (6) and (30). The following properties of V^* are easy to prove

(a) $V^{*'}(x) = p^*(x)$ for all $x \in \mathbb{R}$;

(b) $V^{*''}(x) > 0$ for all $x \in \mathbb{R}$;

(c) $x \mapsto V^*(x)$ is non-negative, has a unique minimum at $-\theta/2$ and $V^*(-\theta/2) = 0$;

(d) $x \mapsto V^*(x) - x$ is non-negative, has a unique minimum at $\bar{\theta}/2$ and $V^*\left(\bar{\theta}/2\right) = \bar{\theta}/2$.

From the definition (8) of V^* and relation (31), the equality in (a) follows. The inequality in (b) is a consequence of (a) and Proposition 5.4. Now, (a), (b) and Proposition 5.4 imply that $-\theta/2$ is the only local minimum of the function V^* and is therefore a global minimum. The definition of V^* given in (8) implies $V^*(-\theta/2) = -V(0) = 0$. Since the stock price S is a true martingale, we have $V(1) = 0$ and Proposition 5.4 implies that $V^*\left(\bar{\theta}/2\right) = \bar{\theta}/2 > 0$. This proves (c). From (a) and Proposition 5.4, we know that the function $x \mapsto V^*(x) - x$ has a unique minimum attained at $\bar{\theta}/2$ and $V^*\left(\bar{\theta}/2\right) - \bar{\theta}/2 = 0$. Therefore (b) implies (d). Note that V^* can also be understood as the Fenchel-Legendre transform of V.

2.1 Large-time behaviour of call options under the Heston model

In this section, we derive the asymptotic behaviour of call option prices under the Heston dynamics (3) as the maturity t tends to infinity, both for maturity-dependent and for fixed strikes. The next theorem is essentially the main result of the paper and its proof is quite involved, so we postpone it to Section 5.

Theorem 2.1. *For the Heston model and the assumptions above, we have the following asymptotic behaviour for the price of a call option with strike $S_0 \exp(xt)$ for all $x \in \mathbb{R}$,*

$$
\frac{1}{S_0} \mathbb{E} \left(S_t - S_0 \mathrm{e}^{xt} \right)^+ = \left(1 - \mathrm{e}^{xt} \right) \mathbf{1}_{\{x < -\theta/2\}} + \mathbf{1}_{\{-\theta/2 < x < \bar{\theta}/2\}} + \frac{1}{2} \mathbf{1}_{\{x = \bar{\theta}/2\}} + \left(1 - \frac{1}{2} \mathrm{e}^{-\theta t/2} \right) \mathbf{1}_{\{x = -\theta/2\}}
$$
$$
+ \, (2\pi t)^{-1/2} \exp\left(-(V^*(x) - x)\, t \right) A(x) \left(1 + O\left(1/t \right) \right), \quad \text{as } t \to \infty,
$$

where

$$
A(x) := \frac{1}{\sqrt{V''(p^*(x))}}
\begin{cases}
\dfrac{U(p^*(x))}{p^*(x)(p^*(x) - 1)}, & \text{if } x \in \mathbb{R} \setminus \left\{ -\theta/2, \bar{\theta}/2 \right\}, \\[3mm]
-1 - \mathrm{sgn}(x) \left(\dfrac{1}{6} \dfrac{V'''(p^*(x))}{V''(p^*(x))} - U'(p^*(x)) \right), & \text{if } x \in \left\{ -\theta/2, \bar{\theta}/2 \right\},
\end{cases}
\tag{9}
$$

where

$$
U(p) := \left(\frac{2d(-\mathtt{i}p)}{\kappa - \rho \sigma p + d(-\mathtt{i}p)} \right)^{2\kappa\theta/\sigma^2} \exp\left(\frac{y_0}{\kappa\theta} V(p) \right),
\tag{10}
$$

V is defined in (6), p^* in (30), V^* in (8) and d in (7) and $\mathrm{sgn}(x)$ equals 1 if x is positive and -1 otherwise.

Remark 2.2. Note that, from property (b) on page 3, the square root in the function A is a strictly positive real number.

Remark 2.3. It is proved in Proposition 5.4 that $p^*(-\theta/2) = 0$ and $p^*(\bar{\theta}/2) = 1$. Note further that $U'(0) = (\theta - y_0)/(2\kappa)$ and $U'(1) = (y_0 - \bar{\theta})/(2\bar{\kappa})$.

Remark 2.4. The condition $\bar{\kappa} > 0$ is usually assumed in the literature (see [22] and [3] for instance). $\bar{\kappa}$ is the mean reversion level of the Y process under the so-called Share measure (see [12] for details). If $\bar{\kappa} \leq 0$, the Y process will not be mean reverting and thus will not have the required ergodic behaviour under the Share measure, and grows exponentially as $t \to \infty$. In the equities market, this is not a huge issue as the correlation ρ is almost always negative.

Remark 2.5. Theorem 2.1 is similar in spirit to the saddlepoint approximation for a density of random variable X given in Butler [8]

$$
f_X(x) \approx \left(2\pi K''(p^*(x)) \right)^{-1/2} \exp\left(K(p^*(x)) - xp^*(x) \right),
$$

where $K(p) := \log \mathbb{E}(\exp(pX))$, and $p^*(x)$ is the unique solution to $K'(p^*(x)) = x$. Here $X := X_t - x_0$, $K(p) = tV(p) + O(1)$, and we substitute x to xt so that

$$
f_{X_t - x_0}(xt) \approx \left(2\pi V''(p^*(x))\, t \right)^{-1/2} \exp\left(V(p^*(x))\, t - xp^*(x)t \right) = \left(2\pi V''(p^*(x))\, t \right)^{-1/2} \exp\left(-V^*(x)\, t \right).
$$

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In order to precisely compare our result to the existing literature, we prove the following lemma, which gives the asymptotic behaviour of vanilla call options when the strike K is fixed, independent of the maturity. This theorem was derived in [29], Chapter 6; a rigorous proof is detailed here in Appendix A.

Lemma 2.6. *Under the same assumptions as Theorem 2.1, for any $x \in \mathbb{R}$, we have the following behaviour for a call option with fixed strike $K = S_0 \exp(x)$,*

$$\frac{1}{S_0}\mathbb{E}\left(S_t - K\right)^+ = 1 + \frac{A(0)}{\sqrt{2\pi t}} \exp\left(x(1 - p^*(0)) - V^*(0)t\right)\left(1 + O(1/t)\right), \quad as \ t \to \infty.$$

2.2 Large-time behaviour of the the Black-Scholes call option formula

By a similar analysis, we can deduce the large-time asymptotic call price for the Black-Scholes model. This result is of fundamental importance for us as it will allow us to compute the implied volatility by comparing the Black-Scholes and the Heston call option prices. Throughout the rest of the paper, we let $C_{\mathrm{BS}}(S_0, K, t, \Sigma)$ denote the Black-Scholes price of a European call option written on a reference stock price S, with strike $K > 0$, initial stock price $S_0 > 0$, time-to-maturity $t \geq 0$, and volatility $\Sigma > 0$ (with zero interest rate and zero dividend). In this section, we further assume that the variance Σ^2 is of the form $\sigma^2 + a_1/t$, for each maturity t, where $\sigma > 0$ and $a_1 > -\sigma^2 t$. Similar to Section 2, let us define the function $V_{\mathrm{BS}} : \mathbb{R} \to \mathbb{R}$ as in (4), where $X_t := \log(S_t)$. In the Black-Scholes case, it reads

$$V_{\mathrm{BS}}(p) = p(p-1)\Sigma^2/2, \quad \text{for all } p \in \mathbb{R}. \tag{11}$$

Similarly to (8), we can define the functions $V_{\mathrm{BS}}^* : \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}$ and $p_{\mathrm{BS}}^* : \mathbb{R} \to \mathbb{R}$, by

$$V_{\mathrm{BS}}^*\left(x, \Sigma\right) := \left(x + \Sigma^2/2\right)^2 / \left(2\Sigma^2\right), \quad \text{for all } x \in \mathbb{R}, \ \Sigma \in \mathbb{R}_+^*, \tag{12}$$

and

$$p_{\mathrm{BS}}^*(x) := \left(x + \Sigma^2/2\right)/\Sigma^2, \quad \text{for all } x \in \mathbb{R}. \tag{13}$$

The following proposition, proved in Appendix B, gives the behaviour of the Black-Scholes price as the maturity tends to infinity.

Proposition 2.7. *With the assumptions above, for all $x \in \mathbb{R}$, we have the following asymptotic behaviour for the Black-Scholes call option formula in the large-strike, large-time case*

$$
\begin{aligned}
\frac{1}{S_0}C_{\mathrm{BS}}\left(S_0, S_0 e^{xt}, t, \sqrt{\sigma^2 + a_1/t}\right) &= \left(1 - e^{xt}\right)\mathbb{1}_{\{x < -\sigma^2/2\}} + \mathbb{1}_{\{-\sigma^2/2 < x < \sigma^2/2\}} + \frac{1}{2}\mathbb{1}_{\{x = \sigma^2/2\}} \\
&+ \left(1 - \frac{1}{2}e^{-\sigma^2 t/2}\right)\mathbb{1}_{\{x = -\sigma^2/2\}} \\
&+ \left(2\pi t\right)^{-1/2} A_{\mathrm{BS}}(x, \sigma, a_1)\exp\left(-\left(V_{\mathrm{BS}}^*(x, \sigma) - x\right)t\right)\left(1 + O(1/t)\right),
\end{aligned}
$$

where

$$A_{\mathrm{BS}}(x, \sigma, a_1) := \exp\left(\frac{1}{8}a_1\left(\frac{4x^2}{\sigma^4} - 1\right)\right)\frac{\sigma^3}{x^2 - \sigma^4/4}\mathbb{1}_{\{x \neq \pm\sigma^2/2\}} + \frac{a_1/2 - 1}{\sigma}\mathbb{1}_{\{x = \pm\sigma^2/2\}}. \tag{14}$$

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Remark 2.8. If we set $a_1 = 0$ in Proposition 2.7, we obtain the large-time expansion for a call option under the standard Black-Scholes model with volatility σ and log-moneyness equal to xt.

As in the Heston model above, we derive here the equivalent of Proposition 2.7 when the strike does not depend on the maturity anymore.

Lemma 2.9. *With the assumptions above, we have the following behaviour for the Black-Scholes call option formula in the fixed-strike, large-time case*

$$\frac{1}{S_0} C_{\mathrm{BS}}\left(S_0, S_0 e^x, t, \sqrt{\sigma^2 + a_1/t}\right) = 1 - \frac{2}{\sigma\sqrt{2\pi t}} \exp\left(-\sigma^2 t/8 + x/2 - a_1/8\right)(1 + O(1/t)).$$

This lemma is immediate from the Black-Scholes formula given in Appendix B and the approximation (A-2) for the Gaussian cumulative distribution function.

3 Large-time behaviour of implied volatility

The previous section dealt with large-time asymptotics for call option prices. In this section, we translate these results into asymptotics for the implied volatility. Recall that [12], [29], and [22] have already derived the leading order term for the implied volatility in the large-time, fixed-strike case. Our goal here is to obtain the leading order and the correction term in the large-time, large-strike case. Theorem 3.1 provides the main result, i.e. the large-time behaviour of the implied volatility in the large strike case. In the following, $\hat{\sigma}_t(x)$ will denote the implied volatility corresponding to a vanilla call option with maturity t and (maturity-dependent) strike $S_0 \exp(xt)$ in the Heston model (3). We now define the functions $\hat{\sigma}_\infty^2 : \mathbb{R} \to \mathbb{R}_+$ and $\hat{a}_1 : \mathbb{R} \to \mathbb{R}$ by

$$\hat{\sigma}_\infty^2(x) := 2\left(2V^*(x) - x + 2\left(\mathbf{1}_{x \in (-\theta/2, \bar{\theta}/2)} - \mathbf{1}_{x \in \mathbb{R} \setminus (-\theta/2, \bar{\theta}/2)}\right)\sqrt{V^*(x)^2 - V^*(x)x}\right), \quad \text{for all } x \in \mathbb{R} \quad (15)$$

and

$$\hat{a}_1(x) := 2 \begin{cases} \left(x^2/\hat{\sigma}_\infty^4(x) - 1/4\right)^{-1} \log\left(A(x)/A_{\mathrm{BS}}(x, \hat{\sigma}_\infty(x), 0)\right), & x \in \mathbb{R} \setminus \left\{-\frac{\theta}{2}, \frac{\bar{\theta}}{2}\right\}, \\ 1 - \frac{\hat{\sigma}_\infty(x)}{\sqrt{V''(p^*(x))}}\left(1 + \mathrm{sgn}(x)\left(\frac{V'''(p^*(x))}{6V''(p^*(x))} - U'(p^*(x))\right)\right), & x \in \left\{-\frac{\theta}{2}, \frac{\bar{\theta}}{2}\right\}, \end{cases} \quad (16)$$

where A is defined in (9), A_{BS} in (14), U in (10), V in (6), p^* in (30) and V^* in (8). They are all completely explicit, so that the functions $\hat{\sigma}_\infty^2$ and $\hat{a}_1$ are also explicit. From the properties of V^* proved on page 3, $V^*(x)$ and $V^*(x) - x$ are non-negative, so that $\hat{\sigma}_\infty^2(x)$ is a well defined real number for all $x \in \mathbb{R}$. Then the following theorem holds:

Theorem 3.1. *The functions $\hat{\sigma}_\infty$ and $\hat{a}_1$ are continuous on $\mathbb{R}$ and*

$$\forall x \in \mathbb{R}, \ \hat{\sigma}_t^2(x) = \hat{\sigma}_\infty^2(x) + \hat{a}_1(x)/t + o(1/t) \ \text{ as } t \to \infty,$$

Remark 3.2. The functions A and A_{BS} are not continuous at $-\theta/2$ and $\bar{\theta}/2$, (see Figure 1) however $\hat{a}_1$ is continuous by this theorem.

Proof. We first prove that the functions $\hat{\sigma}_\infty$ and $\hat{a}_1$ are continuous. In fact, the continuity of the function $\hat{\sigma}_\infty$ follows from properties (c) and (d) on page 3. We observe that the two functions $x \mapsto A(x)$ and $x \mapsto A_{\mathrm{BS}}(x, \hat{\sigma}_\infty(x), 0)$ have poles at $\bar{\theta}/2$ and $-\theta/2$ and their quotient is strictly positive for $x \in \mathbb{R} \backslash \{-\theta/2, \bar{\theta}/2\}$. Therefore the function $\hat{a}_1$ is continuous on this complement. Elementary calculations show that in the neighbourhood of $\bar{\theta}/2$, we have the following expansion (where $\Theta := (\bar{\theta}/V''(1))^{1/2}$)

$$\hat{\sigma}_\infty^2(x) = \bar{\theta} + 2(1 - \Theta)(x - \bar{\theta}/2) + \frac{2}{V''(1)}\left(1 - \frac{1}{\Theta} + \frac{V'''(1)}{6V''(1)^2}\Theta\right)(x - \bar{\theta}/2)^2 + O\left(|x - \bar{\theta}/2|^3\right).$$

This expansion can be used to obtain

$$\frac{A(x)}{A_{\mathrm{BS}}(x, \hat{\sigma}_\infty(x), 0)} = 1 + \frac{1}{V''(1)}\left(U'(1) - 1 - \frac{V'''(1)}{6V''(1)} + \frac{1}{\Theta}\right)(x - \bar{\theta}/2) + O\left(|x - \bar{\theta}/2|^2\right),$$

which implies the equality $\lim_{x \to \bar{\theta}/2} \hat{a}_1(x) = \hat{a}_1(\bar{\theta}/2)$. A similar argument shows continuity of $\hat{a}_1$ at $-\theta/2$.

We now prove the formula in the theorem in the case $x > \bar{\theta}/2$. Note that $\hat{\sigma}_\infty(x)$ as defined in (15) satisfies the following quadratic equation

$$V^*(x) - x = V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x)) - x, \quad \text{for all } x \in \mathbb{R}, \tag{17}$$

where V^* is given by (8) and V_{BS}^* by (12). We now prove the theorem in two steps: first, we prove the convergence of the implied variance to $\hat{\sigma}_\infty^2(x)$, then we prove the first order correction term. As a first step, we have to prove that, for all $\delta > 0$, there exists $t^*(\delta) > 0$ such that for all $t > t^*(\delta)$, $|\hat{\sigma}_t(x) - \hat{\sigma}_\infty(x)| \le \delta$. By Theorem 2.1 and (17) we know that for all $\epsilon > 0$, there exists $t^*(\epsilon)$ such that for all $t > t^*(\epsilon)$ we have the lower bound

$$\exp\left(-\left(V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x)) - x + \epsilon\right)t\right) = \exp\left(-\left(V^*(x) - x + \epsilon\right)t\right) \le \frac{1}{S_0}\mathbb{E}\left(S_t - S_0 e^{xt}\right)^+, \tag{18}$$

and the upper bound

$$\frac{1}{S_0}\mathbb{E}\left(S_t - S_0 e^{xt}\right)^+ \le \exp\left(-\left(V^*(x) - x - \epsilon\right)t\right) = \exp\left(-\left(V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x)) - x - \epsilon\right)t\right). \tag{19}$$

Note that

$$\hat{\sigma}_\infty^2(x) - 2x = 4\left((V^*(x) - x) - \sqrt{(V^*(x) - x)^2 + (V^*(x) - x)x}\right) < 0,$$

since $V^*(x) - x > 0$ by property (d) on page 3. For x fixed, the function $\Sigma \mapsto V_{\mathrm{BS}}^*(x, \Sigma) - x$ defined on $(0, \sqrt{2x})$ is continuous and strictly monotonically decreasing, where V_{BS}^* is given in (12). Thus, for any $\delta > 0$ such that $\hat{\sigma}_\infty(x) \pm \delta \in (0, \sqrt{2x})$, define

$$\epsilon_1(\delta) := \left(V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x)) - V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x) + \delta)\right)/2 > 0,$$

$$\epsilon_2(\delta) := \left(V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x) - \delta) - V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x))\right)/2 > 0,$$

where $\lim_{\delta \to 0} \epsilon_1(\delta) = \lim_{\delta \to 0} \epsilon_2(\delta) = 0$. Combining (18), (19) and Proposition 2.7, there exists $t^*(\delta)$ such that for all $t > t^*(\delta)$,

$$\frac{1}{S_0}C_{\mathrm{BS}}\left(S_0, S_0 e^{xt}, t, \hat{\sigma}_\infty(x) - \delta\right) \le \exp\left(-\left(V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x) - \delta) - x - \epsilon_2(\delta)\right)t\right) \le \frac{1}{S_0}\mathbb{E}\left(S_t - S_0 e^{xt}\right)^+,$$

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and

$$\frac{1}{S_0}\mathbb{E}\left(S_t - S_0\mathrm{e}^{xt}\right)^+ \leq \exp\left(-(V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x) + \delta) - x + \epsilon_1(\delta))t\right) \leq \frac{1}{S_0}C_{\mathrm{BS}}\left(S_0, S_0\mathrm{e}^{xt}, t, \hat{\sigma}_\infty(x) + \delta\right).$$

Thus, by the monotonicity of the Black-Scholes call option formula as a function of the volatility, we have the following bounds for the implied volatility $\hat{\sigma}_t(x)$ at maturity t

$$\hat{\sigma}_\infty(x) - \delta \leq \hat{\sigma}_t(x) \leq \hat{\sigma}_\infty(x) + \delta.$$

For the second step of the proof, we show that for all $\delta > 0$, there exists $t^*(\delta)$ such that, for all $t > t^*(\delta)$,

$$|\hat{\sigma}_t^2(x) - \hat{\sigma}_\infty^2(x) - \hat{a}_1(x)/t| \leq \delta/t. \tag{20}$$

Note that definition in (16) implies

$$A(x) = A_{\mathrm{BS}}(x, \hat{\sigma}_\infty(x), \hat{a}_1(x)), \quad \text{for all } x \in \mathbb{R},$$

so that, by Theorem 2.1, we know that for all $\epsilon > 0$, there exists $t^*(\epsilon)$ such that for all $t > t^*(\epsilon)$ we have

$$\frac{1}{S_0}\mathbb{E}\left(S_t - S_0\mathrm{e}^{xt}\right)^+ \leq \frac{A_{\mathrm{BS}}(x, \hat{\sigma}_\infty(x), \hat{a}_1(x))}{\sqrt{2\pi t}}\mathrm{e}^{-(V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x)) - x)t}\mathrm{e}^{\epsilon}, \tag{21}$$

and

$$\frac{A_{\mathrm{BS}}(x, \hat{\sigma}_\infty(x), \hat{a}_1(x))}{\sqrt{2\pi t}}\mathrm{e}^{-(V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x)) - x)t}\mathrm{e}^{-\epsilon} \leq \frac{1}{S_0}\mathbb{E}\left(S_t - S_0\mathrm{e}^{xt}\right)^+.$$

Let $\delta > 0$ and $\epsilon(\delta) := \left(4x^2/\hat{\sigma}_\infty^4(x) - 1\right)\delta/16 > 0$. From (21), there exists $t^*(\delta)$, such that for all $t > t^*(\delta)$,

$$\frac{1}{S_0}\mathbb{E}\left(S_t - S_0\mathrm{e}^{xt}\right)^+ \leq \frac{A_{\mathrm{BS}}(x, \hat{\sigma}_\infty(x), \hat{a}_1(x) + \delta)}{\sqrt{2\pi t}}\mathrm{e}^{-(V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x)) - x)t}\mathrm{e}^{-\epsilon(\delta)}$$
$$\leq \frac{1}{S_0}C_{\mathrm{BS}}\left(S_0, S_0\mathrm{e}^{xt}, t, \sqrt{\hat{\sigma}_\infty^2(x) + (\hat{a}_1(x) + \delta)/t}\right),$$

and

$$\frac{1}{S_0}\mathbb{E}\left(S_t - S_0\mathrm{e}^{xt}\right)^+ \geq \frac{A_{\mathrm{BS}}(x, \hat{\sigma}_\infty(x), \hat{a}_1(x) - \delta)}{\sqrt{2\pi t}}\mathrm{e}^{-(V_{\mathrm{BS}}^*(x, \hat{\sigma}_\infty(x)) - x)t}\mathrm{e}^{\epsilon(\delta)}$$
$$\geq \frac{1}{S_0}C_{\mathrm{BS}}\left(S_0, S_0\mathrm{e}^{xt}, t, \sqrt{\hat{\sigma}_\infty^2(x) + (\hat{a}_1(x) - \delta)/t}\right).$$

By monotonicity of the Black-Scholes price as a function of the volatility, we obtain

$$\hat{\sigma}_\infty^2(x) + (\hat{a}_1(x) - \delta)/t \leq \hat{\sigma}_t^2(x) \leq \hat{\sigma}_\infty^2(x) + (\hat{a}_1(x) + \delta)/t,$$

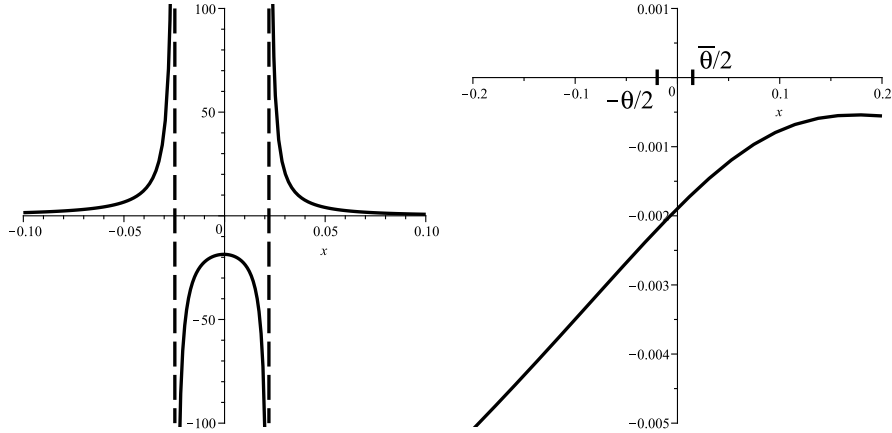which proves (20). The proof of the theorem in the other case $x \leq -\theta/2$ is analogous. $\qquad\square$

Figure 1: We plot the two functions A (left) and $\hat{a}_1$ (right) with the calibrated parameters on page 10. The function A_{BS} in Proposition 2.7 has the same shape as A. Here, the two poles of A and A_{BS} are $-\theta/2 \approx -0.025$ and $\bar{\theta}/2 \approx 0.022$.

3.1 The large-time, fixed-strike case

This section is the translation of Lemmas 2.6 and 2.9 in terms of implied volatility asymptotics, and improves the understanding of the behaviour of the Heston implied volatility in the long term. Let $\sigma_t(x)$ denote the implied volatility corresponding to a vanilla call option with maturity t and fixed strike $K = S_0 \exp(x)$ in the Heston model (3). Let us define the function $a_1 : \mathbb{R} \to \mathbb{R}$ by

$$a_1(x) := -8 \log\left(-A(0)\sqrt{2V^*(0)}\right) + 4\left(2\,p^*(0) - 1\right)x, \quad \text{for all } x \in \mathbb{R}, \tag{22}$$

where A is defined in (9), V^* in (8) and p^* in (30). Elementary calculations show that $A(0) < 0$. From the properties of V^* on page 3, $a_1(x)$ is then well defined as a real number for all $x \in \mathbb{R}$.

Theorem 3.3. *With the assumptions above, we have the following behaviour for the implied volatility in the fixed-strike case*

$$\sigma_t^2(x) = 8V^*(0) + a_1(x)/t + o\left(1/t\right), \quad \text{for all } x \in \mathbb{R}, \text{ as } t \to \infty,$$

where V^* is given by (8) and a_1 by (22).

Proof. The proof of this theorem is very similar to the proof of Theorem 3.1, so we do not detail it as much as we did, in particular we only prove the upper bound, the lower bound being analogous; also we deal with both the zeroth order term and the first order term at the same time. By Lemma 2.6 and (22), we know that for all $\epsilon > 0$, there exists a $t^*(\epsilon)$ such that for all $t > t^*(\epsilon)$, we have

$$\frac{1}{S_0}\mathbb{E}\left(S_t - S_0 \mathrm{e}^x\right)^+ \leq \left(1 + (2\pi t)^{-1/2} A(0) \exp\left(x(1 - p^*(0)) - V^*(0)t\right)\right) \mathrm{e}^\epsilon$$

$$= \left(1 - \frac{1}{2\sqrt{\pi V^*(0)t}} \exp\left(x/2 - a_1(x)/8 - V^*(0)t\right)\right) \mathrm{e}^\epsilon.$$

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Now, for any $\delta > 0$, by a continuity argument, we can then find $\epsilon(\delta) > 0$ such that

$$\left(1 - \frac{1}{2\sqrt{\pi V^*(0)t}} \mathrm{e}^{x/2 - a_1(x)/8 - V^*(0)t} \right) \mathrm{e}^{\epsilon(\delta)} = \left(1 - \frac{1}{2\sqrt{\pi V^*(0)t}} \mathrm{e}^{x/2 - (a_1(x)+\delta)/8 - V^*(0)t} \right) \mathrm{e}^{-\epsilon(\delta)}.$$

Combining these equations, there exists $t^*(\delta) > 0$ such that, for all $t > t^*(\delta)$,

$$\frac{1}{S_0} \mathbb{E} \left(S_t - \mathrm{e}^x \right)^+ \le \frac{1}{S_0} C_{BS} \left(S_0, S_0 \mathrm{e}^x, t, \sqrt{8V^*(0) + (a_1(x) + \delta)/t} \right)$$

Thus, by the monotonicity of the Black-Scholes formula as a function of the volatility, we obtain

$$\sigma_t^2(x) \le V^*(0) + (a_1(x) + \delta)/t.$$

Because δ can be chosen as small as we wish, the theorem follows. $\qquad\square$

4 Numerical results

We present here some numerical evidence of the validity of the volatility asymptotic formula in Theorem 3.1. We calibrate the Heston model on the European vanilla options on the Eurostoxx 50 on February, 15th, 2006. The option maturities range from one year to nine years and the initial spot S_0 equals 3729.79. The calibration, performed using the Zeliade Quant Framework, by Zeliade Systems, on the whole implied volatility surface gives the following parameters: $\kappa = 1.7609$, $\theta = 0.0494$, $\sigma = 0.4086$, $y_0 = 0.0464$ and $\rho = -0.5195$. We plot the implied volatility smile of the calibrated Heston model for maturities $t = 5$ and $t = 9$ years (see Figure 2).

5 Proof of Theorem 2.1

The proof of the theorem is divided into a series of steps: we first write the Heston call price in terms of an inverse Fourier transform of the characteristic function of the stock price (23). Then we prove a large-time estimate for the characteristic function (Lemma 5.1). The next step is to deform the contour of integration of the inverse Fourier transform through the saddlepoint of the integrand (Equation (30) and Proposition 5.4). Finally, studying the behaviour of the integral around this saddlepoint (Proposition 5.9) and bounding the remaining terms (Lemma 5.7) completes the proof. The special cases $x = -\theta/2$ and $x = \bar{\theta}/2$ in formula (9) are proved in Sections 5.3 and 5.4.

5.1 The Lee-Fourier inversion formula for call options

Using similar notation to Lee [27], set

$$A_{t,X} := \left\{ \nu \in \mathbb{R} : \mathbb{E} \left(\exp \left(\nu \left(X_t - x_0 \right) \right) \right) < \infty \right\}, \quad \text{for all } t \ge 0,$$

and define the characteristic function $\phi_t : \mathbb{R} \to \mathbb{C}$ of $X_t - x_0$ by

$$\phi_t(z) := \mathbb{E} \left(\exp \left(\mathrm{i}z(X_t - x_0) \right) \right), \quad \text{for all } t \ge 0.$$
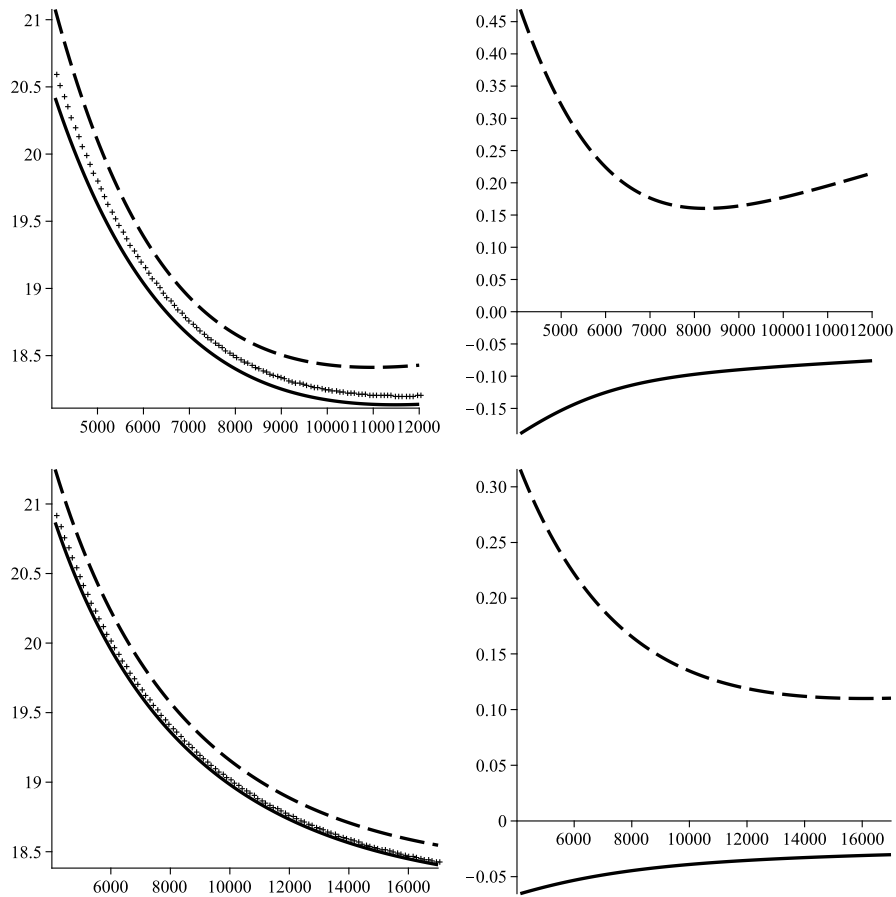
Figure 2: The left plots represent the leading order term $\hat{\sigma}_\infty$ (dashed) defined in (15), the asymptotic formula in Theorem 3.1 (solid line) and the true implied volatility (crosses) for the calibrated parameters (see page 10) as functions of the strike K. The right plots represent the errors between the true implied volatility and $\hat{\sigma}_\infty$ (dashed) and between the true implied volatility and our formula. From top to bottom, these plots correspond to the maturities $t = 5$ and 9 years. All the values are given as percentage.

From Theorem 1 in [28] and Proposition 3.1 in [3], we know that $\phi_t(z)$ can be analytically extended for any $z \in \mathbb{C}$ such that $-\Im(z) \in A_{t,X}$, and from our assumptions on the parameters in Section 2, $(p_-, p_+) \subseteq A_{t,X}$ for all $t \geq 0$. By Theorem 5.1 in [27], for any $\alpha \in (p_-, p_+)$, we have the following Fourier inversion formula for the price of a call option on S_t

$$
\begin{aligned}
\frac{1}{S_0}\mathbb{E}\left(S_t - K\right)^+ = {}& \phi_t(-\mathtt{i})\mathbb{1}_{\{0<\alpha<1\}} + \left(\phi_t(-\mathtt{i}) - \mathrm{e}^x\phi_t(0)\right)\mathbb{1}_{\{\alpha<0\}} + \frac{1}{2}\phi_t(-\mathtt{i})\mathbb{1}_{\{\alpha=1\}} \\
& + \left(\phi_t(-\mathtt{i}) - \frac{\phi_t(0)}{2}\mathrm{e}^{xt}\right)1_{\{\alpha=0\}} + \frac{1}{\pi}\int_{\gamma_+}\Re\left(\mathrm{e}^{-\mathtt{i}zx}\frac{\phi_t(z-\mathtt{i})}{\mathtt{i}z - z^2}\right)\mathrm{d}z,
\end{aligned}
$$

where $x := \log(K/S_0)$ and $\gamma_+ : \mathbb{R} \to \mathbb{C}$ is a contour such that $\gamma_+(u) := u - \mathtt{i}(\alpha-1)$. The first four terms on the right hand side are complex residues that arise when we cross the pole of $\left(\mathtt{i}z - z^2\right)^{-1}$ at $z = 0$. We now set $k = \mathtt{i} - z$, substitute x to xt, and use the fact that S_t is a true martingale for all $t \geq 0$ (see Proposition 2.5 in [3]). From now on, as k will always denote a complex number, we use the notation $k = k_r + \mathtt{i}k_i$ for $k_r, k_i \in \mathbb{R}$. Note that

$$
\Re\left(\mathrm{e}^{\mathtt{i}kxt}\frac{\phi_t(-k)}{\mathtt{i}k - k^2}\right) = \mathbb{E}\left(\Re\left(\mathrm{e}^{\mathtt{i}kxt}\frac{\mathrm{e}^{-\mathtt{i}k(X_t - x_0)}}{\mathtt{i}k - k^2}\right)\right),
$$

that $k_r \mapsto \Re\left(\mathrm{e}^{\mathtt{i}kxt}\frac{\mathrm{e}^{-\mathtt{i}k(X_t-x_0)}}{\mathtt{i}k-k^2}\right)$ is an even function and $k_r \mapsto \Im\left(\mathrm{e}^{\mathtt{i}kxt}\frac{\mathrm{e}^{-\mathtt{i}k(X_t-x_0)}}{\mathtt{i}k-k^2}\right)$ an odd function. Clearly the normalised call price $\frac{1}{S_0}\mathbb{E}(S_t - S_0\exp(xt))^+$ is real, so if we take the real part of both sides and break up the integral, we obtain

$$
\begin{aligned}
\frac{1}{S_0}\mathbb{E}\left(S_t - S_0\mathrm{e}^{xt}\right)^+ = {}& \mathbb{1}_{\{0<\alpha<1\}} + \left(1 - \mathrm{e}^{xt}\right)\mathbb{1}_{\{\alpha<0\}} + \frac{1}{2}\mathbb{1}_{\{\alpha=1\}} + \left(1 - \frac{1}{2}\mathrm{e}^{xt}\right)\mathbb{1}_{\{\alpha=0\}} \\
& + \frac{\exp(xt)}{2\pi}\Re\left(\left(\int_{\gamma_\alpha} + \int_{\zeta_\alpha}\right)\mathrm{e}^{\mathtt{i}kxt}\frac{\phi_t(-k)}{\mathtt{i}k - k^2}\mathrm{d}k\right),
\end{aligned}
\tag{23}
$$

for any $R > 0$, where, for any $\alpha \in \mathbb{R}$, we define the contours

$$
\gamma_\alpha : (-\infty, -R] \cup [R, +\infty) \to \mathbb{C} \text{ such that } \gamma_\alpha(u) := u + \mathtt{i}\alpha,
\tag{24}
$$

and

$$
\zeta_\alpha : (-R, R) \to \mathbb{C} \text{ such that } \zeta_\alpha(u) := u + \mathtt{i}\alpha.
\tag{25}
$$

For ease of notation, we do not write explicitly the dependence of these contours on R. We will see later how to choose R. In the following lemma, we characterise the large-time asymptotic behaviour of the characteristic function ϕ_t.

Lemma 5.1. *For all $k \in \mathbb{C}$ such that $-k_i \in (p_-, p_+)$, we have*

$$
\phi_t(k) = \exp\left(V(\mathtt{i}k)t\right)U(\mathtt{i}k)\left(1 + \epsilon(k,t)\right), \quad \text{as } t \to \infty,
$$

$\Re(d(k)) > 0$, and $\epsilon(k,t) = O\left(\mathrm{e}^{-t\Re(d(k))}\right)$, where U is defined in (10), p_-, p_+ in (5) and V is the analytic continuation of formula (6).

If $-k_i$ is not in (p_-, p_+), this large time behaviour of ϕ_t still holds, but $\Re(d(k))$ might be null (for instance if $k_r = 0$) so that $\epsilon(k,t)$ does not tend to 0 as $t \to \infty$.

Proof. From [2], we have, for all $k \in \mathbb{C}$ such that $-k_i \in (p_-, p_+)$,

$$\phi_t(k) = \exp\left(V(\mathtt{i}k)t - \frac{2\kappa\theta}{\sigma^2}\log\left(\frac{1 - g(k)\mathrm{e}^{-d(k)t}}{1 - g(k)}\right)\right)\exp\left(\frac{y_0}{\kappa\theta}V(\mathtt{i}k)\frac{1 - \mathrm{e}^{-d(k)t}}{1 - g(k)\mathrm{e}^{-d(k)t}}\right), \tag{26}$$

where d is defined in (7), V is the analytic extension of formula (6) and the correct branch for the complex logarithm and the complex square root function is the principal branch (see also [2] and [28]) and $g : \mathbb{C} \to \mathbb{C}$ is defined by

$$g(k) := \frac{\kappa - \mathtt{i}\rho\sigma k - d(k)}{\kappa - \mathtt{i}\rho\sigma k + d(k)}, \quad \text{for all } k \in \mathbb{C}. \tag{27}$$

For all $k \in \mathbb{C}$ such that $-k_i \in (p_-, p_+)$, we have $\Re(d(k)) > 0$. Let

$$\epsilon_1(k, t) := \left(1 - g(k)\mathrm{e}^{-d(k)t}\right)^{-2\kappa\theta/\sigma^2} \quad \text{and} \quad \epsilon_2(k, t) := \exp\left\{-\frac{2d(k)V(\mathtt{i}k)y_0}{\kappa\theta\left(\kappa - \rho\sigma\mathtt{i}k + d(k)\right)}\left(\mathrm{e}^{d(k)t} - g(k)\right)^{-1}\right\}.$$

Then we have

$$\phi_t(k) = \exp\left(V(\mathtt{i}k)t\right)U(\mathtt{i}k)\epsilon_1(k, t)\epsilon_2(k, t), \quad \text{for all } t \geq 0.$$

Then, as t tends to infinity we have

$$\epsilon_1(k, t) = 1 + \frac{2\kappa\theta g}{\sigma^2}\mathrm{e}^{-d(k)t} + O\left(\mathrm{e}^{-2d(k)t}\right) \quad \text{and} \quad \epsilon_2(k, t) = 1 + \frac{c}{\mathrm{e}^{d(k)t} - 1} + O\left(\left(\mathrm{e}^{d(k)t} - 1\right)^{-2}\right)$$

for some constant c. Set $\epsilon(k, t) := \epsilon_1(k, t)\epsilon_2(k, t) - 1$ and the lemma follows. $\qquad\square$

5.2 The saddlepoint and its properties

We first recall the definition of a saddlepoint in the complex plane (see [7]):

Definition 5.2. Let $F : \mathcal{Z} \to \mathbb{C}$ be an analytic complex function on an open set $\mathcal{Z}$. A point $z_0 \in \mathcal{Z}$ such that the complex derivative $\frac{\mathrm{d}F}{\mathrm{d}z}$ vanishes is called a saddlepoint.

Note that the function $V : (p_-, p_+) \to \mathbb{R}$ defined in (4) can be analytically extended and we define the function $F : \mathcal{Z} \to \mathbb{C}$ by

$$F(k) := -\mathtt{i}kx - V(-\mathtt{i}k), \quad \text{where } \mathcal{Z} := \{k \in \mathbb{C} : k_i \in (p_-, p_+)\}. \tag{28}$$

Note that the exponent of the integrand in (23) has the form $-F(k)t$ by Lemma 5.1, therefore the saddlepoint properties of F given in the following elementary lemma are fundamental.

Lemma 5.3. *The saddlepoints of the complex function $F : \mathcal{Z} \to \mathbb{C}$ are given by*

$$z_0^{\pm} = \mathtt{i}\frac{\sigma - 2\kappa\rho \pm (\kappa\theta\rho + x\sigma)\eta(x^2\sigma^2 + 2x\kappa\theta\rho\sigma + \kappa^2\theta^2)^{-1/2}}{2\sigma\bar{\rho}^2} \in \mathcal{Z},$$

where $\eta := \sqrt{\sigma^2 + 4\kappa^2 - 4\kappa\rho\sigma}$.

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Proof. Since we are looking for the saddlepoint in $\mathcal{Z}$, we can use the representation in (6) for the function V. Therefore the equation $F'(z) = 0$ is quadratic and hence has the two purely imaginary solutions $z_0^\pm$ since the expression $(x^2\sigma^2 + 2x\kappa\theta\rho\sigma + \kappa^2\theta^2) = (x\sigma + \kappa\theta\rho)^2 + \kappa^2\theta^2(1-\rho^2)$ is strictly positive for any $x \in \mathbb{R}$. It is also clear from the definition of p_- and p_+ given in (5) and the assumptions on the coefficients that $\Im\left(z_0^\pm\right) \in (p_-, p_+)$, and therefore $z_0^\pm \in \mathcal{Z}$ are saddlepoints of F. $\square$

The next task is to choose the saddlepoint of the function F in such a way that it converges to the saddlepoint of the function $F_{\mathrm{BS}}(k) := -\mathtt{i}kx - V_{BS}(-\mathtt{i}k)$ for all $k \in \mathcal{Z}$, where V_{BS} is given by (11), in the Black-Scholes model as both the volatility of volatility and the correlation in model (3) tend to zero. It is easy to see that the saddlepoint of F_{BS} equals $\mathtt{i}p_{\mathrm{BS}}^*(x)$ for any $x \in \mathbb{R}$, were p_{BS}^* is given by (13). We can rewrite $\Im\left(z_0^\pm\right)$ defined in Lemma 5.3 as

$$\Im\left(z_0^\pm\right) = \frac{1}{2\bar{\rho}^2} - \frac{\kappa\rho}{\sigma\bar{\rho}^2} \pm \frac{\kappa\theta\rho\eta(x^2\sigma^2 + 2x\kappa\theta\rho\sigma + \kappa^2\theta^2)^{-1/2}}{2\sigma\bar{\rho}^2} \pm \frac{x\sigma\eta(x^2\sigma^2 + 2x\kappa\theta\rho\sigma + \kappa^2\theta^2)^{-1/2}}{2\sigma\bar{\rho}^2}, \quad (29)$$

where $\bar{\rho}$ is defined page 3. The first term converges to $1/2$ and the last one to $\pm x/\theta$ as (ρ, σ) tends to 0. When both ρ and σ tend to 0, a Taylor expansion at first order of the third term gives

$$\frac{\kappa\theta\rho\eta(x^2\sigma^2 + 2x\kappa\theta\rho\sigma + \kappa^2\theta^2)^{-1/2}}{2\sigma\bar{\rho}^2} = \frac{\rho\eta}{2\sigma\bar{\rho}^2}.$$

Take now the positive sign in (29), then the second and third terms cancel out in the limit because η converges to 2κ as (ρ, σ) tends to 0. In that case, we have $\lim_{(\rho,\sigma)\to 0} p^*(x) = 1/2 + x/\theta = p_{\mathrm{BS}}^*(x)$ for all $x \in \mathbb{R}$, where the Black-Scholes variance is equal to θ. Otherwise, if we take the negative sign in (29), we do not recover p_{BS}^*, because the function $(\rho, \sigma) \mapsto \rho/\sigma$ has no limit as the pair (ρ, σ) tends to 0. So that, we define

$$p^*(x) := \Im(z_0^+(x)) = \frac{\sigma - 2\kappa\rho + (\kappa\theta\rho + x\sigma)\eta(x^2\sigma^2 + 2x\kappa\theta\rho\sigma + \kappa^2\theta^2)^{-1/2}}{2\sigma\bar{\rho}^2}, \quad \text{for all } x \in \mathbb{R}. \quad (30)$$

Proposition 5.4. *The function $p^* : \mathbb{R} \to (p_-, p_+)$, where p_- and p_+ are defined in (5), is strictly increasing, infinitely differentiable and satisfies the following properties*

$$p^*\left(-\theta/2\right) = 0, \quad p^*\left(\bar{\theta}/2\right) = 1, \quad \lim_{x\to-\infty} p^*(x) = p_- \quad \text{and} \lim_{x\to+\infty} p^*(x) = p_+,$$

as well as the equation

$$V'\left(p^*(x)\right) = x, \quad \text{for all } x \in \mathbb{R}. \quad (31)$$

Proof. The equation (31) follows from the definition of the saddlepoint. The other properties in the proposition are a direct consequence of the explicit formula for p^* given in (30). $\square$

Remark 5.5. In the fixed strike case, i.e. when $x = 0$, we obtain

$$p^*(0) = \frac{-2\kappa\rho + \sigma + \rho\eta}{2\sigma\bar{\rho}^2}.$$

The corresponding saddlepoint $\mathtt{i}p^*(0)$ is the same as the one in Chapter 6 in [29].

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The following lemma is of fundamental importance and will be the key tool for Proposition 5.9.

Lemma 5.6. *Let $k \in \mathcal{Z}$. Then, for any $k_i \in (p_-, p_+)$, the function $k_r \mapsto \Re(-\mathrm{i}kx - V(-\mathrm{i}k))$ has a unique minimum at 0 and is strictly decreasing (resp. increasing) for $k_r \in (-\infty, 0)$ (resp. $k_r \in (0, \infty)$).*

Proof. Note that the statement in the lemma is equivalent to the map $k_r \mapsto -\Re(V(-\mathrm{i}(k_r + \mathrm{i}k_i)))$ having a unique minimum at $k_r = 0$ for any $k_i \in (p_-, p_+)$ and being increasing (resp. decreasing) on the positive (resp. negative) halfline. Let $k_i \in (p_-, p_+)$, then

$$\Re\left(V(-\mathrm{i}(k_r + \mathrm{i}k_i))\right) = \frac{\kappa\theta}{\sigma^2}\left(\kappa - \rho\sigma k_i - \Re\left(\sqrt{u(k_r) + \mathrm{i}v(k_r)}\right)\right),$$

where

$$u(k_r) := \sigma^2\bar{\rho}^2 k_r^2 - \sigma^2\bar{\rho}^2 k_i^2 - \sigma(2\kappa\rho - \sigma)k_i + \kappa^2 \quad \text{and} \quad v(k_r) := \left(2\kappa\rho - \sigma + 2\sigma\bar{\rho}^2 k_i\right)\sigma k_r.$$

From the identity and the fact that the principal value of the square-root is used, we get

$$\Re\left(\sqrt{u(k_r) + \mathrm{i}v(k_r)}\right) = \frac{1}{2}\sqrt{2u(k_r) + 2\sqrt{u^2(k_r) + v^2(k_r)}} \tag{32}$$

is monotonically increasing in u, u^2 and v^2. First, note that $u'(k_r) = 2\sigma^2\bar{\rho}^2 k_r$, hence u is a parabola with a unique minimum at $k_r = 0$, so that, from (32), it suffices to prove the following claim:

Claim: For every $k_i \in (p_-, p_+)$, the function $g := u^2 + v^2$ has a unique (strictly positive) minimum attained at $k_r = 0$ and is strictly increasing (resp. decreasing) for $k_r > 0$ (resp. $k_r < 0$).

Let us write $u(k_r) = \sigma^2\bar{\rho}^2 k_r^2 + \psi(k_i)$, for all $k_r \in \mathbb{R}$, where $\psi(k_i) := \kappa^2 - \sigma^2\bar{\rho}^2 k_i^2 - \sigma(2\kappa\rho - \sigma)k_i$. We have

$$g(k_r) = \sigma^4\bar{\rho}^4 k_r^4 + \left(2\bar{\rho}^2\psi(k_i) + \left(2\kappa\rho - \sigma + 2\sigma\bar{\rho}^2 k_i\right)^2\right)\sigma^2 k_r^2 + \psi(k_i)^2 \quad \text{for all } k_r \in \mathbb{R}. \tag{33}$$

The coefficient $\sigma^2\bar{\rho}^4$ and the constant κ^2 are strictly positive, so the claim follows if $\chi(k_i) > 0$ for all $k_i \in (p_-, p_+)$, where

$$\chi(k_i) := \sigma^2\left(2\bar{\rho}^2\psi(k_i) + \left(2\kappa\rho - \sigma + 2\sigma\bar{\rho}^2 k_i\right)^2\right) = 2\sigma^4\bar{\rho}^4 k_i^2 + 2\sigma^3\bar{\rho}^2\left(2\kappa\rho - \sigma\right)k_i + \kappa^2\sigma^2 + \sigma^2(2\kappa\rho - \sigma)^2.$$

The discriminant is $\Delta_\chi = -4\sigma^6\bar{\rho}^4\left[2\kappa^2 + (2\kappa\rho - \sigma)^2\right] < 0$, so that χ has no real root and is hence always strictly positive. This proves the claim and concludes the proof of the lemma. $\square$

The following two results complete the proof of Theorem 2.1, by studying the behaviour of the two integrals in (23) as the time to maturity tends to infinity. The following lemma proves that the integral along $\gamma_{p^*(x)}$ is negligible and Proposition 5.9 hereafter provides the asymptotic behaviour of the integral along the contour $\zeta_{p^*(x)}$.

Lemma 5.7. *For any $m > V^*(x)$, there exists $R(m) > 0$ such that for every $k \in \mathcal{Z}$ with $|k_r| > R(m)$, we have*

$$|\exp(\mathrm{i}kxt)\phi_t(-k)| \leq \exp(-mt), \quad \text{for all } t \geq 1. \tag{34}$$

Therefore

$$\left|\mathrm{e}^{xt}\int_{\gamma_{p^*(x)}} \mathrm{e}^{\mathrm{i}kxt}\frac{\phi_t(-k)}{\mathrm{i}k - k^2}\mathrm{d}k\right| = O\left(\mathrm{e}^{-(m-x)t}\right), \tag{35}$$

where the contour $\gamma_{p^*(x)}$ is defined in (24).

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Remark 5.8. (i) For every $x \in \mathbb{R}$, we have $V^*(x) > x$ by (d) on page 3 and hence $m - x > 0$. Therefore the modulus of the integral (35) tends to zero exponentially in time and in m.

(ii) Recall that $p^*(x) \in (p_-, p_+)$ by Proposition 5.4 and hence inequality (34) can be applied when estimating integral (35).

Proof. We only need to prove (35). Recall from Lemma 5.1, after some rearrangements, that

$$\phi_t(-k) = e^{(t + y_0/(\kappa\theta))V(-\mathrm{i}k)} (1 - g(-k))^{2\kappa\theta/\sigma^2} \left(1 + O\left(e^{-t\Re(d(-k))}\right)\right).$$

It follows from equations (7), (6) and (27) that

$$\Re\left(d\left(-(k_r + \mathrm{i}k_i)\right)\right) \sim \sigma\bar{\rho}|k_r|, \quad \text{as } |k_r| \to \infty,$$

$$V\left(-\mathrm{i}(k_r + \mathrm{i}k_i)\right) \sim -\kappa\theta\bar{\rho}|k_r|/\sigma, \quad \text{as } |k_r| \to \infty,$$

$$\lim_{|k_r| \to \infty} g\left(-(k_r + \mathrm{i}k_i)\right) = (\rho - \mathrm{i}\bar{\rho})^2 \neq 1 \text{ since } |\rho| < 1.$$

Hence there exists a constant $C > 0$, independent of k and t, such that the following inequality holds

$$\left|e^{\mathrm{i}kxt}\phi_t(-k)\right| \leq C \exp\left[-t\left(k_i x - 1 + \frac{\kappa\theta\bar{\rho}}{\sigma}|k_r|\right)\right].$$

Define $R(m) := \max\{\sigma(m + 1 - k_i x + \log(C))/(\kappa\theta\bar{\rho}), 1\}$. Then if $|k_r| > \max\{R(m), R\}$, where the positive constant R is given in definition (24), the equality (35) follows. $\square$

Proposition 5.9. *For any $R > 0$ and $x \in \mathbb{R} \setminus \{-\theta/2, \bar{\theta}/2\}$, we have as $t \to \infty$,*

$$\frac{\exp(xt)}{2\pi}\Re\left(\int_{\zeta_{p^*(x)}} e^{\mathrm{i}kxt} \frac{\phi_t(-k)}{\mathrm{i}k - k^2}\mathrm{d}k\right) = \frac{\exp\left(-(V^*(x) - x)t\right)}{\sqrt{2\pi t}} \left(A(x) + O\left(1/t\right)\right), \tag{36}$$

where A is given in (9), V^* in (8) and $\zeta_{p^*(x)}$ in (25).

Proof. Let $x \in \mathbb{R} \setminus \{-\theta/2, \bar{\theta}/2\}$. Applying Lemma 5.1 on the compact interval $[-R, R]$, we have

$$\int_{\zeta_{p^*(x)}} e^{\mathrm{i}kxt} \frac{\phi_t(-k)}{\mathrm{i}k - k^2}\mathrm{d}k = \int_{\zeta_{p^*(x)}} \frac{U(-\mathrm{i}k)}{\mathrm{i}k - k^2} e^{(\mathrm{i}kx + V(-\mathrm{i}k))t} (1 + \epsilon(k, t))\, \mathrm{d}k,$$

for t large enough. By Lemma 5.6, we know that $k_r \mapsto -\Re(\mathrm{i}(k_r + \mathrm{i}p^*(x))x + V(-\mathrm{i}(k_r + \mathrm{i}p^*(x))))$ has a unique minimum at $k_r = 0$ and the value of the function at this minimum equals $V^*(x)$ by the definition of V^*. The functions V and U are analytic along the contour of integration and thus, by Theorem 7.1, section 7, chapter 4 in [30], we have

$$\Re\left(\int_{\zeta_{p^*(x)}} \frac{U(-\mathrm{i}k)}{\mathrm{i}k - k^2} e^{(\mathrm{i}kx + V(-\mathrm{i}k))t}\mathrm{d}k\right) = \frac{e^{xt}}{\sqrt{\pi t}} e^{-V^*(x)t} \left(\frac{U(p^*(x))}{\sqrt{2V''(p^*(x))}} + O(1/t)\right)$$

$$= \frac{\exp(-(V^*(x) - x)t)}{\sqrt{2\pi t}} \left(A(x) + O(1/t)\right)$$

as t tends to infinity. The $\epsilon(k, t)$ term is a higher order term which we can ignore at the level we are interested in. $\square$

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Lemma 5.7 and Proposition 5.9 complete the proof of Theorem 2.1 for the general case. Concerning the two special cases, we first introduce a new contour, the path of steepest descent, which represents the optimal (in a sense made precise below) path of integration. Note that, the general case can also be proved using this path, but Lemma 5.6 simplifies the proof.

5.3 Construction of the path of steepest descent

We first recall the definition of the path of steepest descent before computing it explicitly for the Heston model in the large time case.

Definition 5.10. (see Stein&Shakarchi [33]) Let $z := x + \mathtt{i}y$, $x, y \in \mathbb{R}$ and $F : \mathbb{C} \to \mathbb{C}$ be an analytic complex function. The steepest descent contour $\gamma : \mathbb{R} \to \mathbb{C}$ is a map such that

- $\Re(F)$ has a minimum at some point $z_0 \in \gamma$ and $\Re(F''(z_0)) > 0$ along γ.

- $\Im(F)$ is constant along γ.

These two conditions imply that $F'(z_0) = 0$.

The following lemma computes the path of steepest descent explicitly in the Heston case for the function F given in (28) passing through $\mathtt{i}p^*(x)$.

Lemma 5.11. *The path of steepest descent γ in the Heston model is the map $\gamma : \mathbb{R} \to \mathbb{C}$ defined by*

$$\gamma(s) := s + \mathtt{i}k_i(s), \quad \text{for all } s \in \mathbb{R},$$

where

$$k_i(s) := -\frac{\beta - (\kappa\theta\rho + x\sigma)\sqrt{\psi(s)}}{2\kappa\theta\sigma\xi\bar{\rho}^2}, \tag{37}$$

$\beta := \kappa\theta\xi(2\kappa\rho - \sigma)$, $\psi(s) := 4\sigma^2\bar{\rho}^2\xi^2 s^2 + \kappa^2\theta^2\left((2\kappa\rho - \sigma)^2 + 4\kappa^2\bar{\rho}^2\right)\xi$ *and* $\xi := (\kappa\theta\rho + x\sigma)^2 + \kappa^2\theta^2\bar{\rho}^2$.

Note that ξ is strictly positive, so that the function k_i is well defined.

Proof. By definition, the contour of steepest descent is such that the function $\Im(F \circ \gamma)$ remains constant. So we look for the map γ such that $\Im(F(\gamma(s))) = 0$, for all $s \in \mathbb{R}$ because $F(\gamma(0))$ is already real. Using the identity $\Im\left(\sqrt{x + \mathtt{i}y}\right) = 4\left(2x + 2\sqrt{x^2 + y^2}\right)^{-1/2}$, for all $x, y \in \mathbb{R}$, we find that the function $F \circ \gamma$ is real along the contour $\gamma : s \mapsto s + \mathtt{i}k_i(s)$. Note also that this contour is orthogonal to the imaginary axis at $k_i(0)$ (see Exercise 2, Chapter 8 in [33]). $\qquad\square$

Remark 5.12.

- The contour γ depends on x, but for clarity we do not write this dependence explicitly.

- The construction of γ is such that the saddlepoint defined in (30) satisfies $\mathtt{i}p^*(x) = \gamma(0)$.

- We have $k_i(s) = \Im(\gamma(s))$ is an even function of s, i.e. γ is symmetric around the imaginary axis.

We now prove Theorem 2.1 in the two special cases $x \in \{-\bar{\theta}/2, \bar{\theta}/2\}$. In these cases, we need a result similar to Proposition 5.9, as Lemma 5.7 still holds, i.e. we need the asymptotic behaviour of the integral in (36) for the two special cases. The problem with these special cases is that $(\mathtt{i}k - k^2)^{-1}$ in the integrand in (36) has a pole at the saddlepoint, so we need to deform the contour using Cauchy's integral theorem and take the real part to remove the singularity, before we can use a saddlepoint expansion.

5.4 Proof of the call price expansion for the special cases

We here prove Theorem 2.1 in the case $x = \bar{\theta}/2$ for which $p^* (\bar{\theta}/2) = 1$, $V^* (\bar{\theta}/2) = \bar{\theta}/2$, and for simplicity we also assume that $\kappa < (\sigma - 2\rho^2\sigma)/(2\rho)$ (the other cases follows similarly). From (37), we see that in this case, γ lies below the horizontal contour $\gamma_H : \mathbb{R} \to \mathbb{C}$ such that $\gamma_H(s) := s + \mathtt{i}$ (in the other case, γ lies above γ_H). We want to construct a new contour leaving the pole outside. Let $\epsilon > 0$ and $\gamma_\epsilon : (-\pi, \pi] \to \mathbb{C}$ denote the clockwise oriented circular keyhole contour parametrised as $\gamma_\epsilon(\theta) := \mathtt{i} + \epsilon\,\mathrm{e}^{\mathtt{i}\theta}$ around the pole. To leave the pole outside the new contour of integration, we need to follow γ on $\mathbb{R}_-$, switch to the keyhole contour as soon as we touch it, follow it clockwise (above the pole), and get back to γ on $\mathbb{R}_+$. As γ is below γ_H, it intersects γ_ϵ on its lower half, which can be analytically represented as $\gamma_\epsilon^- : [-\epsilon, \epsilon] \to \mathbb{C}$ such that $\gamma_\epsilon^-(s) := \mathtt{i} + s - \mathtt{i}\sqrt{\epsilon^2 - s^2}$. From (37), the two contours intersect at $s^* = \pm\epsilon$. Choose now $0 < \epsilon < \delta < R$ (Lemma 5.13 makes the choice of δ precise and ϵ must be such that $1 + 2\epsilon < p_+$) and define the following contours (they are all considered anticlockwise, see Figure 3)

- $\gamma_{\delta,R} : [-R, -\delta] \cup [\delta, R] \to \mathbb{C}$ given by $\gamma_{\delta,R}(u) = u + \mathtt{i}k_i(\delta)$, with $k_i(\delta)$ defined in (37);

- $\gamma_{\epsilon,\delta}$ is the restriction of γ to the union of the intervals $[-\delta, -\epsilon] \cup [\epsilon, \delta]$;

- γ_ϵ^U is the portion of the circular keyhole contour γ_ϵ which lies above γ, i.e. the upper half keyhole contour as well as the two sections of γ_ϵ between γ and γ_H;

- $\Gamma_{R,\epsilon,\delta}^\pm$ are the two vertical strips joining $\pm R + \mathtt{i}(1 + 2\epsilon)$ to $\pm R - \mathtt{i}k_i(\delta)$.

By Cauchy's integral theorem, we now have

$$\left(\int_{\gamma_{\delta,R}} + \int_{\gamma_{\epsilon,\delta}} + \int_{\gamma_\epsilon^U} + \int_{\Gamma_{R,\epsilon,\delta}^\pm} - \int_{\zeta_{1+2\epsilon}} \right) \frac{\phi_t(-k)}{\mathtt{i}k - k^2} \exp\left(\mathtt{i}k\bar{\theta}t/2\right) \mathrm{d}k = 0, \tag{38}$$

Recall that the curves $\zeta_{1+2\epsilon}$ and $\gamma_{1+2\epsilon}$ are defined in (25) and (24) respectively and rewrite (38) as

$$\left(\int_{\gamma_{1+2\epsilon}} + \int_{\zeta_{1+2\epsilon}} \right) \frac{\phi_t(-k)}{\mathtt{i}k - k^2}\mathrm{e}^{\mathtt{i}k\bar{\theta}t/2}\mathrm{d}k = \left(\int_{\gamma_{\delta,R}} + \int_{\Gamma_{R,\epsilon,\delta}^\pm} + \int_{\gamma_{1+2\epsilon}} + \int_{\gamma_\epsilon^U} \right) \frac{\phi_t(-k)}{\mathtt{i}k - k^2}\mathrm{e}^{\mathtt{i}k\bar{\theta}t/2}\mathrm{d}k$$
$$+ \int_{\gamma_{\epsilon,\delta}} \frac{\phi_t(-k)}{\mathtt{i}k - k^2}\mathrm{e}^{\mathtt{i}k\bar{\theta}t/2}\mathrm{d}k. \tag{39}$$

The integral on the lhs is equal to the normalised call price $\frac{2\pi}{S_0}\mathrm{e}^{-\bar{\theta}t/2}\mathbb{E}\left(S_t - S_0\mathrm{e}^{\bar{\theta}t/2}\right)^+$ by Theorem 5.1 in Lee [27], which is independent of ϵ (this holds because $1 + 2\epsilon < p_+$). For k close to $\mathtt{i}$, we have

$$\frac{\phi_t(-k)}{\mathtt{i}k - k^2} \exp\left(\mathtt{i}k\bar{\theta}t/2\right) = \left(\frac{\mathtt{i}}{k - \mathtt{i}} + O(1) \right)\mathrm{e}^{-\bar{\theta}t/2},$$
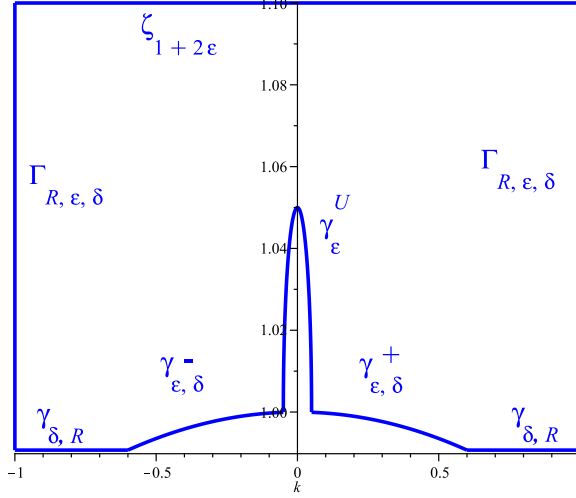
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Figure 3: Here we have plotted the closed contour of integration in (38) for $\epsilon = 0.05$, $\delta = 0.6$, $R = 1$.

so that

$$\int_{\gamma_\epsilon^U} \frac{\phi_t(-k)}{\mathrm{i}k - k^2} \exp\left(\mathrm{i}k\bar\theta t/2\right) \mathrm{d}k = (\pi + O\left(\epsilon\right)) \, \mathrm{e}^{-\bar\theta t/2}. \tag{40}$$

Lemma 5.13 gives the behaviour of the last integral on the rhs of (39) as ϵ tends to 0 for δ small enough.

The other integrals can be bounded as follows. By Lemma 5.7, the integral along $\gamma_{1+2\epsilon}$ is $O\left(\mathrm{e}^{-\bar\theta t/2}\right)$, for $t > t^*(m)$, $R > R(m)$, as ϵ tends to 0.

The curves $\Gamma_{R,0,\delta}^\pm$ are both vertical strips of length δ and therefore their images are compact sets. Applying the tail estimate of Lemma 5.7 along $\Gamma_{R,0,\delta}^\pm$, we know that for any $m > \bar\theta/2$, there exist $t(m)$ and $R(m)$ such that

$$\int_{\Gamma_{R,0,\delta}^\pm} \frac{\phi_t(-k)}{\mathrm{i}k - k^2} \mathrm{e}^{\mathrm{i}k\bar\theta t/2} \mathrm{d}k = O\left(\mathrm{e}^{-mt}\right), \quad \text{for all } t > t(m), |k| > R(m).$$

Lemma 5.6 implies that the real function $k_r \mapsto \Re\left(-\mathrm{i}(k_r + \mathrm{i}k_i)\bar\theta/2 - V(-\mathrm{i}(k_r + \mathrm{i}k_i))\right)$ attains its global minimum at 0 for any fixed $k_i \in (p_-, p_+)$ and is strictly decreasing (resp. increasing) for $k_r < 0$ (resp. $k_r > 0$). It therefore follows that the function $u \mapsto \Re\left(-\mathrm{i}\gamma_{\delta,R}(u)\bar\theta/2 - V(-\mathrm{i}\gamma_{\delta,R}(u))\right)$, where $u \in [-R, -\delta] \cup [\delta, R]$, attains its minimum value $g(\delta) := \Re\left((k_i(\delta) - \mathrm{i}\delta)\bar\theta/2 - V(k_i(\delta) - \mathrm{i}\delta)\right)$, where $k_i(\delta)$ is defined in (37), at the points $u = \pm\delta$. It can be checked directly that $g(0) = \bar\theta/2$, $g'(0) = 0$ and $g''(0) > 0$ and hence for every $\delta > 0$ there exists $\epsilon_0 > 0$ such that the following inequality holds

$$\Re\left(-\mathrm{i}\gamma_{\delta,R}(u)\bar\theta/2 - V(-\mathrm{i}\gamma_{\delta,R}(u))\right) > \bar\theta/2 + \epsilon_0 \quad \text{for all} \quad u \in [-R, -\delta] \cup [\delta, R].$$

Therefore Lemma 5.1 yields the following inequality

$$\left| \int_{\gamma_{\delta,R}} \frac{\phi_t(-k)}{\mathrm{i}k - k^2} \mathrm{e}^{\mathrm{i}k\bar\theta t/2} \mathrm{d}k \right| \leq \mathrm{e}^{-(\bar\theta/2 + \epsilon_0)t} \int_{\gamma_{\delta,R}} \left| \frac{U(-\mathrm{i}k)(1 + \epsilon(k,t))}{\mathrm{i}k - k^2} \right| \mathrm{d}k = O\left(\exp\left(-\left(\bar\theta/2 + \epsilon_0\right)t\right)\right).$$

We now prove the following lemma about the integral along $\gamma_{\epsilon,\delta}$ as ϵ tends to 0.

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Lemma 5.13. *For $\delta > 0$ and sufficiently small we have*

$$\lim_{\epsilon \searrow 0} \int_{\gamma_{\epsilon,\delta}} \frac{\phi_t(-k)}{\mathtt{i}k - k^2} \exp\left(\mathtt{i}k\bar{\theta}t/2\right) \mathrm{d}k = \sqrt{\frac{2\pi}{V''(1)t}} \mathrm{e}^{-\bar{\theta}t/2} \left(-1 - \frac{1}{6}\frac{V'''(1)}{V''(1)} + U'(1)\right) (1 + O(1/t)),$$

where U is given by (10) and V by (6).

Proof. Recall that γ is the contour of steepest descent defined in Lemma 5.11 and that the curve $\gamma_{\epsilon,\delta}$ is its restriction to the intervals $[-\delta, -\epsilon] \cup [\epsilon, \delta]$. Note that $s \mapsto \Re\left(\frac{\phi_t(-\gamma(s))\gamma'(s)}{\mathtt{i}\gamma(s) - \gamma(s)^2}\mathrm{e}^{\mathtt{i}\gamma(s)\bar{\theta}t/2}\right)$ is an even function and $s \mapsto \Im\left(\frac{\phi_t(-\gamma(s))\gamma'(s)}{\mathtt{i}\gamma(s) - \gamma(s)^2}\mathrm{e}^{\mathtt{i}\gamma(s)\bar{\theta}t/2}\right)$ is an odd function. We therefore obtain

$$\int_{\gamma_{\epsilon,\delta}} \frac{\phi_t(-k)}{\mathtt{i}k - k^2} \exp\left(\mathtt{i}k\bar{\theta}/2t\right) \mathrm{d}k = \int_{[-\delta,-\epsilon]\cup[\epsilon,\delta]} \Re\left(\frac{\phi_t(-\gamma(s))\gamma'(s)}{\mathtt{i}\gamma(s) - \gamma(s)^2} \exp\left(\mathtt{i}\gamma(s)\bar{\theta}t/2\right)\right) \mathrm{d}s. \qquad (41)$$

From (37), for s around 0, we have $\left(\mathtt{i}\gamma(s) - \gamma(s)^2\right)^{-1} = \mathtt{i}/s - 1 + O(s)$, so $\Re\left(\left(\mathtt{i}\gamma(s) - \gamma(s)^2\right)^{-1}\right) = -1 + O(s)$, i.e. taking the real part removes the singularity at $k = \mathtt{i}$. Using Lemma 5.1, we then have

$$\int_{-\delta}^{\delta} \Re\left(\frac{\phi_t(-\gamma(s))\gamma'(s)}{\mathtt{i}\gamma(s) - \gamma(s)^2}\mathrm{e}^{\mathtt{i}\gamma(s)\bar{\theta}t/2}\right) \mathrm{d}s = \int_{-\delta}^{\delta} \Re\left(q(s)\right) \mathrm{e}^{\mathtt{i}\gamma(s)\bar{\theta}t/2 + V(-\mathtt{i}\gamma(s))t} \mathrm{d}s + O\left(\mathrm{e}^{-mt}\right),$$

for some $m > 0$ large enough, where we define the function $q : \mathbb{R}\backslash\{0\} \to \mathbb{C}$ by

$$q(s) := \frac{U(-\mathtt{i}\gamma(s))\gamma'(s)}{\mathtt{i}\gamma(s) - \gamma(s)^2}, \quad \text{for all } s \in \mathbb{R}.$$

Then, from (37), we have the following expansion

$$q(s) = \left(\frac{\mathtt{i}}{s} - \left(\frac{V'''(1)}{6V''(1)} + 1\right)\right)(1 - \mathtt{i}U'(1)s) + O\left(s^3\right). \qquad (42)$$

We can therefore extend the function q to the map $q : B_\delta(0)\backslash\{0\} \to \mathbb{C}$ for some $\delta > 0$, where $B_\delta(0) := \{z \in \mathbb{C} : |z| < \delta\}$ is an open disc of radius δ. Note that for $s \in \mathbb{R}$ we have $\Re\left(q(s)\right) = -1 - V'''(1)/\left(6V''(1)\right) + U'(1) + O(s)$ and hence the function $\Re(q) : [-\delta, \delta] \to \mathbb{R}$ does not have a singularity at $s = 0$.

Recall that if a function $G : B_\delta(0)\backslash\{0\} \to \mathbb{C}$ has a Laurent series expansion

$$G(z) = \frac{\mathtt{i}a_{-1}}{z} + \sum_{n=0}^{\infty} a_n z^n, \quad \text{for } z \in B_\delta(0) \setminus \{0\},$$

with $a_{-1} \in \mathbb{R}$, then the function $\Re(G) : \mathbb{R} \cap B_\delta(0) \to \mathbb{R}$ has an analytic continuation on the whole disc $B_\delta(0)$. It follows from (42) that there exists a holomorphic function $Q : B_\delta(0) \to \mathbb{C}$ such that $Q(s) = \Re(q(s))$ for any $s \in (-\delta, \delta)$. Thus by Theorem 7.1, Chapter 4 of [30], we have

$$\int_{-\delta}^{\delta} \Re\left(q(s)\right)\mathrm{e}^{\mathtt{i}\gamma(s)\bar{\theta}t/2 + V(-\mathtt{i}\gamma(s))t}\mathrm{d}s = \int_{-\delta}^{\delta} Q(s)\mathrm{e}^{\mathtt{i}\gamma(s)\bar{\theta}t/2 + V(-\mathtt{i}\gamma(s))t}\mathrm{d}s$$

$$= \sqrt{\frac{2\pi\,\mathrm{e}^{-\bar{\theta}t}}{V''(1)t}} \left(-1 - \frac{1}{6}\frac{V'''(1)}{V''(1)} + U'(1)\right)(1 + O(1/t)).$$

$\square$

Letting ϵ go to 0 in equation (39), applying Lemma 5.13 and the bounds developed above for the other integrals in (39) the theorem follows in the case $x = \bar{\theta}/2$. The case $x = -\theta/2$ is analogous.

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APPENDIX

A Proof of Theorem 2.6

The proof is analogous to the proof of Theorem 2.1. The residue in Theorem 2.1 is equal to 1 (arising from the $\mathbf{1}_{\{-\theta/2<x<\bar{\theta}/2\}}$ term), and from (23), the integral part is equal, for R large enough, reads

$$\frac{\exp(xt)}{2\pi}\Re\left(\left(\int_{\zeta_{p^*(0)}}+\int_{\gamma_{p^*(0)}}\right)\frac{\phi_t(-k)}{\mathrm{i}k-k^2}\mathrm{e}^{\mathrm{i}kxt}\mathrm{d}k\right),$$

and the behaviour of these integrals follows exactly the lines of the proof of Theorem 2.1.

B Proof of Proposition 2.7

Let us now consider a squared volatility of the form $\hat{\sigma}_t^2=\sigma^2+a_1/t>0$, then the Black-Scholes call option reads

$$\frac{1}{S_0}C_{\mathrm{BS}}\left(S_0,S_0\mathrm{e}^{xt},t,\hat{\sigma}_t\right)=\Phi\left(\frac{-x+\left(\sigma^2+a_1/t\right)/2}{\sqrt{\sigma^2+a_1/t}}\sqrt{t}\right)-\mathrm{e}^{xt}\Phi\left(\frac{-x-\left(\sigma^2+a_1/t\right)/2}{\sqrt{\sigma^2+a_1/t}}\sqrt{t}\right), \quad \text{(A-1)}$$

and let $z_{\pm}=\left(-x\pm\frac{1}{2}\left(\sigma^2+a_1/t\right)\right)\sqrt{t}/\sqrt{\sigma^2+a_1/t}$. Recall that (see [30])

$$\Phi(-z)=1-\Phi(z)=\frac{\exp\left(-z^2/2\right)}{z\sqrt{2\pi}}\left(1+O\left(1/z^2\right)\right), \quad \text{as } z\to+\infty \quad \text{(A-2)}$$

The case $x>\sigma^2/2$. As $\sigma^2/2=\lim\limits_{t\to\infty}\hat{\sigma}_t^2/2$, there exists t^* such that for all $t>t^*$, $x>\hat{\sigma}_t^2/2=(\sigma+a_1/t)^2/2$. From (A-1), we have, using a Taylor expansion for $z_{\pm}$,

$$\frac{1}{S_0}C_{\mathrm{BS}}(S_0,S_0\mathrm{e}^{xt},t,\hat{\sigma}_t) = \mathrm{e}^{\frac{1}{8}a_1\left(4x^2/\hat{\sigma}^4-1\right)}\left(\frac{\sigma}{x-\sigma^2/2}-\frac{\sigma}{x+\sigma^2/2}\right)\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(-x+\sigma^2/2)^2t}{2\sigma^2}\right)(1+O(1/t))$$

$$= (2\pi t)^{-1/2}\exp\left(-(V_{\mathrm{BS}}^*(x,\sigma)-x)t\right)A_{\mathrm{BS}}(x,\sigma,a_1)\left(1+O(1/t)\right).$$

The cases $x<-\sigma^2/2$ and $-\sigma^2/2<x<\sigma^2/2$ follow likewise.

The case $x=\sigma^2/2$. From (A-1), we have

$$\frac{1}{S_0}C_{\mathrm{BS}}\left(S_0,S_0\mathrm{e}^{\sigma^2t/2},t,\hat{\sigma}_t\right) = \Phi\left(\frac{a_1/2}{\sqrt{\sigma^2t+a_1}}\right)-\frac{\mathrm{e}^{\sigma^2t/2}\sqrt{\sigma^2t+a_1}}{xt+(\sigma^2t+a_1)/2}\mathrm{e}^{-\frac{(-xt-(\sigma^2t+a_1)/2)^2}{2(\sigma^2t+a_1)}}(1+O(1/t))$$

$$= \frac{1}{2}+\frac{a_1/2}{\sigma\sqrt{2\pi t}}-\frac{1}{\sigma\sqrt{2\pi t}}(1+O(1/t))=\frac{1}{2}+\frac{A_{\mathrm{BS}}\left(\sigma^2/2,\sigma,a_1\right)}{\sqrt{2\pi t}}(1+O(1/t)).$$

The case $x=-\sigma^2/2$ is analogous.

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