APPROXIMATING LÉVY PROCESSES WITH A VIEW TO OPTION PRICING

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Abstract. We examine how to approximate a Lévy process by a hyperexponential jump-diffusion (HEJD) process, composed of Brownian motion and of an arbitrary number of sums of compound Poisson processes with double exponentially distributed jumps. This approximation will facilitate the pricing of exotic options since HEJD processes have a degree of tractability that other Lévy processes do not have. The idea behind this approximation has been applied to option pricing by [2] and [14]. In this paper we introduce a more systematic methodology for constructing this approximation which allow us to compute the intensity rates, the mean jump sizes and the volatility of the approximating HEJD process (almost) analytically. Our methodology is very easy to implement. We compute vanilla option prices and barrier option prices using the approximating HEJD process and we compare our results to those obtained from other methodologies in the literature. We demonstrate that our methodology gives very accurate option prices and that these prices are more accurate than those obtained from existing methodologies for approximating Lévy processes by HEJD processes.

1. Introduction

The purpose of this paper is to examine, with a view to option pricing, how a Lévy process can be approximated by a jump-diffusion process with jumps consisting of sums of compound Poisson processes with double exponentially distributed jumps. Firstly, we need to introduce some concepts and notation.

We define the initial time (today) by t_0 and denote calendar time by t, $t \geq t_0$. Consider a market, which we assume to be free of arbitrage, where the risk-free interest rate is r and in which there is an asset, which pays a dividend yield q, whose price at time t is S_t.

The absence of arbitrage guarantees the existence of a risk-neutral equivalent martingale measure. However, as we will utilise Lévy processes, the market is incomplete and, hence, the risk-neutral equivalent martingale measure is not unique. We will assume that one such measure $\mathbb{Q}$ has been fixed on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{Q})$. We denote by $\mathbb{E}_t^{\mathbb{Q}}[\cdot] := \mathbb{E}^{\mathbb{Q}}[\cdot|\mathcal{F}_t]$ the conditional expectation, under $\mathbb{Q}$, at time t. We assume that, under the risk-neutral measure $\mathbb{Q}$, the asset price evolves as:

$$S_t = S_{t_0}\exp((r-q)(t-t_0) + X_t), \tag{1}$$

where X_t is a Lévy process, mean-corrected such that $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(X_t)] = 1$ for all $t \geq t_0$, with $X_{t_0} = 0$.

Examples of Lévy processes which have been used in finance include Variance Gamma (henceforth VG) [21], CGMY [10] (also known as the KoBol process, [7] [8]), Normal Inverse Gaussian (henceforth NIG) [3] and Generalised Hyperbolic [13]. For more details on these processes, see [26] and [11]. All the aforementioned processes have infinite activity (that is to say there are an infinite number of jumps in a finite time interval, see [26] - actually, to be precise, in the CGMY model, the Y parameter must be non-negative to have infinite activity). We will refer to these aforementioned infinite activity Lévy processes by

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the collective noun General Classes of Lévy processes. The motivation for using General Classes of Lévy processes is two-fold. Firstly, they can capture empirically observed features of time-series data such as high-frequency jumps and excess kurtosis. Secondly, and much more relevantly for us, they can be fitted (especially if time-changed) to implied volatility surfaces (see e.g. [10]).

The characteristic function is known (in essentially closed form) for all the Lévy processes mentioned above. Results from [19], [18] and [27] (see also Section 4), show that given the characteristic function, it is straightforward to price vanilla (standard European) options very rapidly. However, pricing exotic options is much more complicated for General Classes of Lévy processes. Whereas analytical solutions exist for many simple exotic options in the model of [4] and [22], analytical results are rarely available for General Classes of Lévy processes. This means that pricing exotic options generally requires computationally intensive methodologies such as Monte Carlo simulation. Various techniques for simulating Lévy processes have been developed in the literature. However, many of them, broadly-speaking, come down to approximating the Lévy process by a jump-diffusion process.

There is a class of processes which does allow for analytical results (up to Laplace inversion) for a range of exotic options. These are jump-diffusion processes which have a Brownian motion component as well as a jump component formed from sums of compound Poisson processes with double exponentially distributed jumps (henceforth HEJD for hyperexponential jump-diffusion). Analytical results for various types of barrier options, for lookback options, for Russian options and first passage time distributions are developed variously in [16], [17], [28], [20], [2], [14] and [9] under a HEJD process or a special case of it (the double exponential jump-diffusion in [15]) or a generalisation of it (HEJD where both volatility and jump intensity are stochastic, see [9] and [23] for more details). Hence, we can see that a HEJD process has, from the point of view of pricing exotic options, a considerable degree of tractability which General Classes of Lévy processes such as CGMY and NIG do not have. As an aside, we also mention that Monte Carlo simulation of HEJD processes, without discretization error, is very straightforward (see, for example,[26]).

Let us assume that the Lévy measure of the process X_t has a Lévy density given by $\nu : \mathbb{R}\backslash\{0\} \to [0, \infty)$ which is completely monotone (see [24], page 388, for definition and properties). It is easily verified that HEJD, VG, NIG, Generalised Hyperbolic and CGMY (the latter provided the parameter $Y \geq -1$) processes all satisfy this condition. Then, by Bernstein's theorem we can express the function ν in the form:

$$\nu(x) = \mathbf{1}_{(0,\infty)}(x) \int_0^{+\infty} e^{-ux} \mu_+(du) + \mathbf{1}_{(-\infty,0)}(x) \int_{-\infty}^0 e^{-ux} \mu_-(du) \tag{2}$$

where $\mathbf{1}_A$ denotes the indicator function of the set A and where $\mu_+(du)$, $\mu_-(du)$ are measures on the intervals $(0, \infty)$, $(-\infty, 0)$ respectively. Observing the forms of the integrands and noting that an integral can be approximated as a discrete sum, equation (2) immediately suggests how to approximate the Lévy process by a HEJD process (where jumps whose magnitudes are smaller than some certain level being approximated by Brownian motion). After constructing this approximation, one can then use the analytical results available to price exotic options. This idea was introduced into the option pricing literature by [2] and then developed further by [14].

The methodologies employed by [2] and by [14] to approximate the Lévy process in question by a HEJD process are described in those papers in more depth. However, a brief precis is as follows. They fitted a total of fourteen compound Poisson processes with exponentially distributed jumps, seven producing up jumps and seven producing down jumps. They chose (based on intuition) some points x at which to approximate the Lévy density $\nu(x)$. They chose (again, based on intuition) some mean jump sizes for the individual exponentially-distributed jumps which constituted the HEJD process. In [2], they did a non-linear least-squares fit, over the choice of the mean jump sizes and the jump intensity rates, between the Lévy density $\nu(x)$ and the Lévy density of the HEJD process evaluated at the chosen points x. In [14], they kept the mean jump sizes the same as the initial guesses and only fitted the jump intensity rates in the least-squares fit. The diffusion component was determined by approximating as Brownian motion all the jumps whose magnitude were less than the smallest mean jump sizes (for both up and down jumps).

One might describe these fitting procedures as a little ad-hoc. This is not a criticism. Indeed, firstly, the approximation of the Lévy density was by no means the central point of either of those papers, secondly, the fitting procedures are intuitive and easy to implement and, thirdly, based on results reported there and also based on results contained within this paper (see Section 5), the resulting barrier option prices are reasonably accurate. However, the question might still be asked as to whether there might be an alternative methodology. By contrast with [2] and [14], approximating Lévy processes by HEJD processes is the central point of this paper. Our starting point is the desire to find a more systematic methodology of approximating Lévy processes by HEJD processes which has the following six features:

1. No non-linear least-squares fitting is required.
2. No guessing of the mean jump sizes is required.
3. The methodology is equally as intuitive and easy to implement as the procedures described above.
4. The methodology has a robust way of approximating the very small jumps by Brownian motion.
5. The methodology yields more accurate vanilla option prices than the procedures described above.
6. The methodology yields more accurate barrier option prices than the procedures described above.

Three comments are in order about the last six points. Firstly, our desire to avoid non-linear least-squares fitting is based on the fact that such methods can be unstable or ill-posed because the algorithm may find a local minimum rather than the global minimum. Secondly, although we wish to have more accurate vanilla option prices, this will primarily be in order that we can benchmark the accuracy of our methodology (since we know we can accurately price vanilla options for both General Classes of Lévy processes and for the HEJD process). Thirdly, although we will focus mostly on pricing barrier options, we believe our methodology should also be of interest for pricing other exotic options, either by Laplace transform methods (for example, for lookback options and Russian options, see [17] and [29]) or by Monte Carlo simulation.

We describe our methodology, which does in fact have our six desirable features, in this paper. The rest of this paper is structured as follows. In Section 2 we very briefly review the key properties of Lévy processes that we will use later. In Section 3 we explain our algorithm to calculate the parameters of the approximating hyperexponential jump-diffusion process. In sections 4 and 5 we present numerical comparisons and results for vanilla option prices and for barrier option prices. Section 6 is a short conclusion. In the appendices, we present our results in graphical form.

2. Lévy processes – basic facts

2.1. **Key features of Lévy Processes.** We consider a Lévy process $(X_t)_{t\geq t_0}$, with $X_{t_0} = 0$ a.s. Let $\Phi_t(z) = \mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(izX_t)]$ be the characteristic function of X_t, where $z \in \mathbb{R}$. The characteristic exponent $\phi(z)$, defined by $\Phi_t(z) = \exp((t-t_0)\phi(z))$, for each $t \geq t_0$, satisfies the Lévy-Khintchine formula:

$$\phi(z) = i\gamma z - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{+\infty} (\exp(izx) - 1 - izx\mathbf{1}_{[0,1]}(|x|))\nu(dx), \tag{3}$$

for some $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and a measure ν on $\mathbb{R}\backslash\{0\}$ with $\int_{-\infty}^{+\infty}(1\wedge x^2)\nu(dx) < \infty$. Here, ν is the *Lévy measure* of the process X_t. If ν is absolutely continuous with respect to the Lebesgue measure, i.e. $\nu(dx) = \nu(x)\, dx$, then the function $\nu : \mathbb{R}\backslash\{0\} \to [0,\infty)$ is the *Lévy density*. See [24], Section 2.8, Theorem 8.1 for more details.

Remark 2.1. The term $izx\mathbf{1}_{[0,1]}(|x|)$ in equation (3) ensures that the integral converges. However, if the Lévy measure decays fast enough as $|x| \to \infty$ so that the integral $\int_{\mathbb{R}\backslash[-1,1]} x\nu(dx)$ exists (this is the case for the Lévy processes we will consider here, such as CGMY and NIG, but not the case in general, for the α-stable process for example), we can replace $izx\mathbf{1}_{[0,1]}(|x|)$ by izx. If $\int_{[-1,1]}|x|\nu(dx) < \infty$ (which corresponds to the process $(X_t)_{t\geq t_0}$ with finite variation), then we do not need this term at all. Note that changing the form of this term is, loosely speaking, the same as changing the drift of $(X_t)_{t\geq t_0}$.

Remark 2.2. Note also that if the Lévy density ν decays at least as fast as the function $x \mapsto e^{-A|x|}$, for some $A > 0$, when $|x| \to \infty$, then we can extend the characteristic exponent ϕ to the strip $\{z \in \mathbb{C} : \Im(z) \in (-A, A)\}$ in the complex plane. This is clear from formula (3). In the rest of the paper we assume that such an extension exists for $A \geq 1$, which clearly holds for the main models of interest (see Appendix A). The extension of the characteristic exponent ϕ will be used in our approximation algorithm (see Subsection 4.1). Note also that the assumption in this remark is not unreasonable if we are modelling the asset price process by (1) because it is equivalent to requiring that S_t and S_t^{-1} have finite first moments for all $t \geq t_0$.

We define the mean-corrected characteristic exponent $\phi_{MC}(z)$, via $\phi_{MC}(z) \equiv \phi(z) - iz\phi(-i)$. It is straightforward to see that $\phi_{MC}(z)$ is the characteristic exponent of a Lévy process X_t which satisfies $\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_t}\right] = 1$. Since in applications to pricing theory, the drift of the Lévy process is determined by the no arbitrage condition (see equation (1)), it is sufficient to consider Lévy processes which satisfy $\mathbb{E}^{\mathbb{Q}}_{t_0}\left[e^{X_t}\right] = 1$. Since we shall only consider Lévy processes whose Lévy density decays sufficiently quickly and without a Gaussian component, we can always write the characteristic exponent in the form:

$$\phi(z) \quad = \quad \int_{-\infty}^{+\infty}(e^{izx} - 1 - i\beta zx)\nu(x)dx \ - \ i\alpha z\int_{-\infty}^{+\infty}(e^{x} - 1 - \beta x)\nu(x)dx, \tag{4}$$

where $\alpha = 1$ gives the mean-corrected characteristic exponent and $\beta = 0$ gives a process of finite variation. Since we would also like the Lévy density ν to take the form of equation (2), we must, by Bernstein's theorem, assume that it is completely monotonic. To summarise, from now on we assume the following.

Assumption 2.3. The Lévy density $\nu : \mathbb{R}\backslash\{0\} \to [0, \infty)$ in (3) is completely monotonic, it decays at least as fast as the function $x \mapsto e^{-A|x|}$, for some $A \geq 1$, when $|x| \to \infty$, and the measures $\mu_\pm$ in (2) are absolutely continuous with respect to the Lebesgue measure.

2.2. **The Hyperexponential Jump-Diffusion Process.** The hyperexponential jump-diffusion (henceforth HEJD) process, with an arbitrary number N (we assume N is even for notational simplicity) of compound Poisson processes has the form

$$X_t = \sigma W_t + \sum_{i=1}^{N/2}\sum_{k=1}^{N_{t,i}} J^+_{ik} - \sum_{i=N/2+1}^{N}\sum_{k=1}^{N_{t,i}} J^-_{ik},$$

where W_t denotes a standard Brownian motion with $W_{t_0} = 0$ and where $N_{t,i}$, for each $i = 1, 2, ..., N$, denotes a Poisson (counting) process with $N_{t_0,i} = 0$ and random variables $J^+_{ik}, J^-_{(i+N/2)k}$, for $i = 1, ..., N/2$, $k \in \mathbb{N}$, are independent exponentially distributed. Furthermore, we denote by a_i and c_i respectively the intensity rates of the Poisson processes corresponding to up jumps and down jumps, and we denote by b_i and d_i the reciprocals of the mean jump sizes for the up and down jumps respectively, i.e.

$$\mathbb{E}^{\mathbb{Q}}_{t_0}[N_{t,i}] = a_i(t - t_0), \qquad \mathbb{E}^{\mathbb{Q}}[J^+_{ik}] = \frac{1}{b_i}, \qquad \text{for } 1 \leq i \leq N/2,$$

$$\mathbb{E}^{\mathbb{Q}}_{t_0}[N_{t,i}] = c_i(t - t_0), \qquad \mathbb{E}^{\mathbb{Q}}[J^-_{ik}] = \frac{1}{d_i}, \qquad \text{for } N/2 + 1 \leq i \leq N.$$

The characteristic exponent $\phi_N(z)$ of this process has the form (choose $\alpha = 1$ for the mean-corrected form):

$$\begin{aligned}\phi_N(z) &= -\frac{1}{2}\sigma^2(z^2+i\alpha z) + \sum_{i=1}^{N/2}\left[a_i b_i\left(\frac{1}{b_i - iz} - \frac{1}{b_i}\right) - i\alpha z a_i b_i\left(\frac{1}{b_i-1} - \frac{1}{b_i}\right)\right] \\ &+ \sum_{i=N/2+1}^{N}\left[c_i d_i\left(\frac{1}{d_i + iz} - \frac{1}{d_i}\right) - i\alpha z c_i d_i\left(\frac{1}{d_i+1} - \frac{1}{d_i}\right)\right]. \end{aligned} \tag{5}$$

3. Approximation methodology

3.1. **Initial presentation.** Our aim in this section is to approximate, in distribution, a given Lévy process by a HEJD process. We know that the convergence in distribution follows from the convergence of corresponding characteristic exponents (or characteristic functions). Hence, we seek to approximate the characteristic exponent ϕ of the Lévy process by the characteristic exponent of a HEJD process. In addition to Assumption 2.3 without loss of generality we can suppose that the Lévy process has no Brownian component and can hence express ϕ on the set $\{z \in \mathbb{C} : \Im(z) \in (-A, A)\}$ using formula (4).

Given N, we seek to approximate ϕ by the characteristic exponent ϕ_N of a HEJD process (see (5)). In other words we must choose $a_i,\ b_i,\ c_i,\ d_i$ and σ^2 so that the approximation by the HEJD process is as accurate as possible. If we substitute equation (2) into equation (4), change variables $u \to -u$ and $x \to -x$ in the integrals over $(-\infty, 0)$ and switch the order of integration, we get:

$$\begin{aligned}\phi(z) &= \int_0^{+\infty} \mu_+(u) g^+_{\alpha,\beta}(u,z)\, du + \int_0^{+\infty} \mu_-(-u) g^-_{\alpha,\beta}(u,z)\, du, \quad \text{where} \\ g^{\pm}_{\alpha,\beta}(u,z) &\equiv \left(\frac{1}{u \mp iz} - \frac{1}{u} \mp \frac{i\beta z}{u^2}\right) - i\alpha z\left(\frac{1}{u \mp 1} - \frac{1}{u} \mp \frac{\beta}{u^2}\right)\end{aligned}$$

and $\mp$ is the opposite sign of $\pm$. In the mean-corrected case ($\alpha = 1$) we recognize that the terms $g^+_{\alpha,\beta}(u,z)$ and $g^-_{\alpha,\beta}(u,z)$ are of the same form as the summands appearing in the characteristic exponent (5), where the exponentially distributed jumps have mean sizes $1/u$ and the compound Poisson processes have intensity rates equal to $1/u$.

We define $h^{\pm}_{\alpha,\beta}(u,z) \equiv -g^{\pm}_{\alpha,\beta}(u,z)/(z^2+i\alpha z)$, introduce $\overline{\theta}_+,\ \overline{\theta}_- \in \mathbb{R}_+$ and express the exponent ϕ as

$$\begin{aligned}\phi(z) &= \int_0^{\overline{\theta}_+} \mu_+(u) g^+_{\alpha,\beta}(u,z)\, du + \int_0^{\overline{\theta}_-} \mu_-(-u) g^-_{\alpha,\beta}(u,z)\, du \\ &- (z^2+i\alpha z)\left(\int_{\overline{\theta}_+}^{+\infty} \mu_+(u) h^+_{\alpha,\beta}(u,z)\, du + \int_{\overline{\theta}_-}^{+\infty} \mu_-(-u) h^-_{\alpha,\beta}(u,z)\, du\right).\end{aligned} \tag{6}$$

The identity in (6) suggests an approximation scheme where the first two integrals are replaced by sums:

$$\phi(z) \simeq \sum_{i=1}^{N/2} \omega_i^+ \mu_+(u_i^+) g^+_{\alpha,\beta}(u_i^+, z) + \sum_{i=1+\frac{N}{2}}^{N} \omega_i^- \mu_-(-u_i^-) g^-_{\alpha,\beta}(u_i^-, z) - \frac{1}{2}(\Sigma^+ + \Sigma^-)(z^2 + i\alpha z). \tag{7}$$

In (7) ω_i^+ and u_i^+, for $i = 1, ..., N/2$, are respectively weights and abscissas coming from a $N/2$-point Gauss-Legendre quadrature rule (see [1]) on the interval $(0, \overline{\theta}_+)$, ω_i^- and u_i^-, for $i = 1 + N/2, ..., N$, are respectively weights and abscissas coming from a $N/2$-point Gauss-Legendre quadrature rule on the interval $(0, \overline{\theta}_-)$, and:

$$\Sigma^+ \equiv 2\int_{\overline{\theta}_+}^{+\infty} \mu_+(u) h^+_{\alpha,\beta}(u,z) du \quad \text{and} \quad \Sigma^- \equiv 2\int_{\overline{\theta}_-}^{+\infty} \mu_-(-u) h^-_{\alpha,\beta}(u,z) du. \tag{8}$$

Observing the form of equation (7) we see that we have written the characteristic exponent of the Lévy process in a form that resembles the characteristic exponent of a HEJD process. In terms of the parameters

$a_i,\ b_i,\ c_i$ and d_i, we have:

$$\begin{aligned} a_i b_i &= \omega_i^+ \mu_+(u_i^+) \quad \text{and} \quad b_i = u_i^+,\ 1 \le i \le N/2 \\ c_i d_i &= \omega_i^- \mu_-(-u_i^-) \quad \text{and} \quad d_i = u_i^-,\ 1 + N/2 \le i \le N. \end{aligned} \tag{9}$$

However, there are two sources of error in our proposed approximation.

1. The discretization error: in equation (6), we replace the integrals by finite sums.

2. The truncation error: in the third and fourth terms of equation (6) respectively, we would like the integrands $\mu_+(u)h^+_{\alpha,\beta}(u,z)$ and $\mu_-(-u)h^-_{\alpha,\beta}(u,z)$ and hence Σ^+ and Σ^- to be independent of z. However, clearly, neither Σ^+ nor Σ^- is independent of z and this prevents us from identifying the parameter σ^2 as being equal to $\Sigma^+ + \Sigma^-$. Note that for a Lévy density ν it can be shown easily that $\mu_+(u)$ and $\mu_-(-u)$ must grow slower than a quadratic in u, as $u \to \infty$, and therefore, observing the forms of $h^+_{\alpha,\beta}(u,z)$ and $h^-_{\alpha,\beta}(u,z)$, we have $\lim_{\overline{\theta}_\pm \to \infty} \Sigma^\pm = 0$. Hence the error in the third and fourth terms could be viewed as a truncation error in the upper limit of the integrals.

Note how these two errors work in opposite directions. Indeed, for a fixed N, if $\overline{\theta}_+$ and $\overline{\theta}_-$ increase, we would intuitively expect that the discretization error gets larger while the truncation error gets smaller. If $\overline{\theta}_+$ and $\overline{\theta}_-$ decrease, we would intuitively expect that the discretization error gets smaller while the truncation error gets larger. We will analyse the discretization error and the truncation error in the next two sections.

3.2. **Truncation error.** Let us now have a closer look at the term $(\Sigma^+ + \Sigma^-)(z^2 + i\alpha z)/2$. We momentarily view this term as the characteristic exponent of a random variable. We can then calculate the second and third central moments μ_2 and μ_3 of this random variable by differentiating its characteristic function twice and three times at $z = 0$ respectively. We obtain:

$$\mu_2 \equiv 2\int_{\overline{\theta}_+}^{+\infty} \frac{\mu_+(u)}{u^3} du + 2\int_{\overline{\theta}_-}^{+\infty} \frac{\mu_-(-u)}{u^3} du, \quad \mu_3 \equiv 6\int_{\overline{\theta}_+}^{+\infty} \frac{\mu_+(u)}{u^4} du - 6\int_{\overline{\theta}_-}^{+\infty} \frac{\mu_-(-u)}{u^4} du.$$

In practice, the third central moment μ_3 should be small in magnitude relative to μ_2. In fact, for a symmetrical Lévy process (i.e. with $\mu_+(u) = \mu_-(-u)$ for all $u \in \mathbb{R}_+$ and if we were to choose $\overline{\theta}_+ = \overline{\theta}_-$) μ_3 would be identically equal to zero. In any event, it is certainly true that if $\overline{\theta}_+$ and $\overline{\theta}_-$ are large enough, then $|\mu_3| \ll \mu_2$. Furthermore, in the mean-corrected case $\alpha = 1$, a simple calculation shows that the term $(\Sigma^+ + \Sigma^-)(z^2 + i\alpha z)/2$ behaves asymptotically like

$$(z^2 + iz)\left[\int_{\overline{\theta}_+}^{\infty} \mu_+(u)\left(\frac{1}{u^3}\right) du + \int_{\overline{\theta}_-}^{\infty} \mu_-(-u)\left(\frac{1}{u^3}\right) du\right] = \frac{1}{2}\mu_2(z^2 + iz)$$

if $\overline{\theta}_+$ and $\overline{\theta}_-$ are both much, much larger than $|z|$. We recognize the right-hand side as the mean-corrected characteristic exponent of Brownian motion with variance μ_2.

Our approximation, therefore, is to replace the term $(\Sigma^+ + \Sigma^-)(z^2 + i\alpha z)/2$ in equation (7) by the term $\mu_2(z^2 + i\alpha z)/2$. This is intuitively equivalent to approximating small jumps (both up and down) by Brownian motion with variance μ_2.

In order for us to justify this approximation, we would want the term $\mu_2(z^2 + i\alpha z)/2$ to be as close as possible to $(\Sigma^+ + \Sigma^-)(z^2 + i\alpha z)/2$. When $z = 0$, both terms equal zero and hence there is no approximation. This suggests that, in order to get a handle on the truncation error, we need to compare $(\Sigma^+ + \Sigma^-)(z^2 + i\alpha z)/2$ and $\mu_2(z^2 + i\alpha z)/2$ when evaluated at some value of z, z_{large} say, such that $|z|$ is large. Once we have chosen z_{large} (see Section 4), this suggest a measure of the truncation error TE:

$$\begin{aligned} TE &\equiv \frac{1}{2}|\mu_2(z_{\text{large}}^2 + i\alpha z_{\text{large}}) - (\Sigma^+ \ + \ \Sigma^-)\,(z_{\text{large}}^2 + i\alpha z_{\text{large}})| \\ &= |z_{\text{large}}^2 + i\alpha z_{\text{large}}| \left| \int_0^{\frac{1}{\overline{\theta}_+}} \mu_+\left(\frac{1}{t}\right) t\ dt + \int_0^{\frac{1}{\overline{\theta}_-}} \mu_-\left(-\frac{1}{t}\right) t\ dt \right. \\ &- \left. \int_0^{\frac{1}{\overline{\theta}_+}} \mu_+\left(\frac{1}{t}\right) h^+_{\alpha,\beta}\left(\frac{1}{t}, z_{\text{large}}\right) \frac{1}{t^2} dt \ - \ \int_0^{\frac{1}{\overline{\theta}_-}} \mu_-\left(-\frac{1}{t}\right) h^-_{\alpha,\beta}\left(\frac{1}{t}, z_{\text{large}}\right) \frac{1}{t^2}\ dt \right|, \end{aligned}$$

where we have performed the substitution $t = 1/u$, in order to help evaluate these integrals (since we will have to compute them numerically and we want to avoid infinite limits). We will indicate how to choose z_{large} in Section 4.

3.3. **Discretization error.** We estimate the integrals $\int_0^{\overline{\theta}_+} \mu_+(u) g^+_{\alpha,\beta}(u,z) du$ and $\int_0^{\overline{\theta}_-} \mu_-(-u) g^-_{\alpha,\beta}(u,z)\ du$ by a numerical method such as a Gauss-Legendre quadrature rule, using some number of points N_{large} where $N_{large} \gg N$ (or alternatively, by an adaptive Gauss-Lobatto quadrature which will compute the integrals accurate to some very small pre-specified tolerance). We then take the difference between these (very accurate) estimates and those obtained by a $N/2$-point Gauss-Legendre quadrature rule on the interval $(0, \overline{\theta}_+)$ and by a $N/2$-point Gauss-Legendre quadrature rule on the interval $(0, \overline{\theta}_-)$ (see equation (7)).

We have to choose a value of z at which the integrals are computed. From the definitions of $g^+_{\alpha,\beta}(u,z)$ and $g^-_{\alpha,\beta}(u,z)$, we know that when $z = 0$, $g^+_{\alpha,\beta}(u,0)$ and $g^-_{\alpha,\beta}(u,0)$ are both identically equal to zero for all u. Hence, in order to get a meaningful estimate of the discretization error, we need to evaluate the integrands at some value of z such that $|z|$ is large. We elect to evaluate them at the same z_{large} that we use in estimating the truncation error. Hence, we get an estimate for the discretization error:

$$\begin{aligned} DE &\equiv \left| \sum_{i=1}^{N/2} \omega_i^+ \mu_+(u_i^+) g^+_{\alpha,\beta}(u_i^+, z_{\text{large}}) - \int_0^{\overline{\theta}_+} \mu_+(u) g^+_{\alpha,\beta}(u, z_{\text{large}})\ du \right. \\ &+ \left. \sum_{i=1+N/2}^{N} \omega_i^- \mu_-(-u_i^-) g^-_{\alpha,\beta}(u_i^-, z_{\text{large}}) - \int_0^{\overline{\theta}_-} \mu_-(-u) g^-_{\alpha,\beta}(u, z_{\text{large}})\ du \right|. \end{aligned} \tag{10}$$

3.4. **Initial estimates.** We have examined the forms of the discretization error and the truncation error. Now we need a way to choose the limits $\overline{\theta}_+$ and $\overline{\theta}_-$ of the integrals. Since, intutively speaking, the errors act in opposite directions, a possible criterion is to find $\overline{\theta}_+$ and $\overline{\theta}_-$ such that the discretization error DE and the truncation error TE are equal, using, for example, a solver-type methodology. In other words, we search for $\overline{\theta}_+$ and $\overline{\theta}_-$ such that $|TE - DE|^2$ is minimised and we do so in the hope that this minimum is zero. Using a solver-type methodology does not usually guarantee that the algorithm finds the global minimum rather than a local minimum, which would contradict the hypothesis that our model satisfies the feature 1 (no non-linear least-squares fitting is required) in the introduction. In this generality we cannot guarantee existence of a global minimum of $|TE - DE|^2$ on a two-dimensional domain $\mathbb{R}^2_+$ for a general pair of functions μ_+ and μ_-. However, numerical experimentation (when the imaginary part of z_{large} is less than one in magnitude) with models of interest (see Appendix A) appears to support the claim that there exists a unique point $(\overline{\theta}_+, \overline{\theta}_-) \in \mathbb{R}^2_+$ at which the difference of errors equals zero.

Once we find $\overline{\theta}_+$ and $\overline{\theta}_-$ such that the discretization and truncation errors are equal, we can get ω_i^+ and u_i^+, for $i = 1, ..., N/2$, from a $N/2$-point Gauss-Legendre quadrature rule on the interval $(0, \overline{\theta}_+)$ and likewise we can get ω_i^- and u_i^-, for $i = 1 + N/2, ..., N$, from a $N/2$-point Gauss-Legendre quadrature rule on the interval $(0, \overline{\theta}_-)$. We can then immediately get estimates for a_i, b_i, for $1 \le i \le N/2$, c_i, d_i, for $1 + N/2 \le i \le N$ as indicated in equation (9) and for σ^2, via $\sigma^2 = \mu_2$.

3.5. **Initial estimates via a simplified algorithm.** The strategy described in the last section is certainly feasible and it would be applicable to any Lévy process with a completely monotonic Lévy density but it does rely on being able to solve uniquely for the limits $\overline{\theta}_+$ and $\overline{\theta}_-$ via a solver-type methodology. It would be preferable to simplify the algorithm in order to reduce the problem to solving for a single parameter using a simple one-dimensional root finder method such as bisection. We will describe such a simplified algorithm in this subsection. In order to do this, we must make a further assumption about the Lévy density.

Assumption 3.1. In this subsection only, we further assume that the Lévy density can be expressed in the form:

$$\begin{array}{rcll} L(x)\exp(-r_+x) & = & \int_0^\infty e^{-ux}\mu_+(u)du & \text{for } x>0, \\ L(-x)\exp(r_-x) & = & \int_{-\infty}^0 e^{-ux}\mu_-(u)du & \text{for } x<0, \end{array} \tag{11}$$

where $L:(0,\infty)\to\mathbb{R}$ is completely monotonic, constants $r_+,r_-\in\mathbb{R}$ and the functions $\mu_+:(0,\infty)\to\mathbb{R}$ and $\mu_-:(-\infty,0)\to\mathbb{R}$ are the densities of the corresponding measures in expression (2).

Note that Assumption 3.1 gives a relationship between the Laplace transforms of the functions $u\mapsto\mu_+(u)$ and $u\mapsto\mu_-(-u)$, where $u\in(0,\infty)$. By inverting Laplace transforms we find the following identity

$$\mu_+(u+r_+)H(u+r_+)=\mu_-(-(u+r_-))H(u+r_-)=\mu(u) \quad \text{for all } u>0, \tag{12}$$

where H is the Heaviside function and μ is the Laplace inverse of L. Since $\mu(u)=0$ for all $u<0$, equality (12) implies $\mu_+(u)=0$, for all $u<\max(r_+,0)$, and $\mu_-(-u)=0$, for all $u<\max(r_-,0)$.

We now use the asymmetry created by the representation (11) of the Lévy density. If we make the change of variables $u\to u-r_+$ in the first and third terms and $u\to u-r_-$ in the second and fourth terms of equation (6), then, using identity (12), we can express the characteristic function as:

$$\begin{array}{rcl} \phi(z) & = & \displaystyle\int_0^{\overline{\theta}_+-r_+}\mu(u)g^+_{\alpha,\beta}(u+r_+,z)\,du \;+\; \int_0^{\overline{\theta}_--r_-}\mu(u)g^-_{\alpha,\beta}(u+r_-,z)\,du \\ & - & \displaystyle(z^2+i\alpha z)\left(\int_{\overline{\theta}_+-r_+}^{+\infty}\mu(u)h^+_{\alpha,\beta}(u+r_+,z)\,du+\int_{\overline{\theta}_--r_-}^{+\infty}\mu(u)h^-_{\alpha,\beta}(u+r_-,z)\,du\right). \end{array} \tag{13}$$

The parameters r_+ and r_- provide natural scalings. Hence, we assume that the upper limits in the the first two integrals in equation (13) are equal, i.e. $\overline{\theta}_+-r_+=\overline{\theta}_--r_-\equiv\overline{\theta}$, say. With the change of variable $t=1/u$ in the last two integrals of (13) the characteristic function becomes:

$$\begin{array}{rcl} \phi(z) & = & \displaystyle\int_0^{\overline{\theta}}\mu(u)(g^+_{\alpha,\beta}(u+r_+,z)+g^-_{\alpha,\beta}(u+r_-,z))\,du \\ & - & \displaystyle(z^2+i\alpha z)\int_0^{\frac{1}{\overline{\theta}}}\frac{\mu(1/t)}{t^2}(h^+_{\alpha,\beta}(1/t+r_+,z)+h^-_{\alpha,\beta}(1/t+r_-,z))\,dt. \end{array}$$

As before, we approximate the first two integrals by sums. Specifically, we use weights ω_i and abscissas u_i, for $1\le i\le N/2$, obtained from a $N/2$-point Gauss-Legendre quadrature rule on the interval $(0,\overline{\theta})$. Furthermore, we define ω_i and u_i, for $1+N/2\le i\le N$, by $\omega_i=\omega_{i-N/2}$ and $u_i=u_{i-N/2}$. We approximate the third and fourth terms by a Gaussian term as in Subsection 3.2. Then we can approximate the characteristic function $\phi(z)$ in the form:

$$\begin{array}{rcl} \phi(z) & \simeq & \displaystyle\sum_{i=1}^{N/2}\omega_i\mu(u_i)g^+_{\alpha,\beta}(u_i+r_+,z)+\sum_{i=1+N/2}^{N}\omega_i\mu(u_i)g^-_{\alpha,\beta}(u_i+r_-,z)-\frac{1}{2}\sigma^2(z^2+i\alpha z), \quad \text{where} \\ \sigma^2 & \equiv & \displaystyle 2\int_0^{\frac{1}{\overline{\theta}}}\frac{\mu(1/t)}{t^2}\left(\frac{1}{(1/t+r_+)^3}+\frac{1}{(1/t+r_-)^3}\right)dt. \end{array} \tag{14}$$

The discretization error $DE_{\overline{\theta}}$ is given by (10), where μ_+ and μ_- are substituted by μ and, as mentioned above, $\overline{\theta}_+ = \overline{\theta} + r_+$, $\overline{\theta}_- = \overline{\theta} + r_-$. Exactly as in Subsection 3.2, the truncation error is:

$$TE_{\overline{\theta}} \quad = \quad |z_{\text{large}}^2 + i\alpha z_{\text{large}}||\sigma^2 - \int_0^{\frac{1}{\overline{\theta}}} \frac{\mu(1/t)}{t^2}(h^+_{\alpha,\beta}(1/t + r_+, z_{\text{large}}) + (h^-_{\alpha,\beta}(1/t + r_-, z_{\text{large}}))dt|.$$

We now determine $\overline{\theta}$ by computing the root of the equation $DE_{\overline{\theta}} = TE_{\overline{\theta}}$. We no longer need to use a solver-type methodology - a simple bisection method will suffice. We then obtain the reciprocals of the mean jump sizes b_i, for $1 \leq i \leq N/2$, and d_i, for $1 + N/2 \leq i \leq N$, and (initial estimates for) the intensities rates a_i, for $1 \leq i \leq N/2$, and c_i, for $1 + N/2 \leq i \leq N$, of the approximating HEJD process:

$$(15) \qquad b_i = u_i + r_+, \quad d_i = u_i + r_- \quad \text{and} \quad a_i = \frac{\omega_i}{b_i}\mu(u_i), \quad c_i = \frac{\omega_i}{d_i}\mu(u_i).$$

The diffusion variance σ^2 of the approximating HEJD process is given by equation (14).

Remark 3.2. 1. Notice that the reciprocals of the mean jump sizes are scaled by r_+ for the up jumps, and by r_- for the down jumps, which seems a very intuitive result.
2. In the special case of an NIG process (appendix A), the form of $\mu(u)$ implies that we can actually use a $N/2$-point Gauss-Legendre quadrature rule on the interval $(\alpha, \overline{\theta})$ (rather than on the interval $(0, \overline{\theta})$).
3. In the case that $\alpha = 0$ and the imaginary part of z_{large} is equal to zero, it is possible to prove that $DE_{\overline{\theta}}$ is an increasing function of $\overline{\theta}$ and $TE_{\overline{\theta}}$ is a decreasing function of $\overline{\theta}$. Hence, we can deduce that, in this case, the equation $DE_{\overline{\theta}} = TE_{\overline{\theta}}$ certainly has a unique root in $(0, \infty)$. We will use this observation to motivate our choice of z_{large} (see Section 4) (as an aside, the uniqueness of the root does, in fact, hold in greater generality).

3.6. **Refining the results.** Let us summarize our proposed algorithm up to this point. Under Assumption 3.1, using standard results from Gauss-Legendre quadrature we have estimates for a_i, b_i, $c_{i+N/2}$, $d_{i+N/2}$, $i = 1, \dots, N/2$ (see (15)) and σ^2 (see (14)) which are essentially analytic. It is shown in [25] that these estimates would allow us to compute prices of vanilla options under a HEJD process which are quite close to the prices of vanilla options under the Lévy process in equation. However, the prices are not as close as we would like. Therefore, we now seek to refine our parameter estimates.

The key idea is to refine only the estimates for the parameters that enter linearly into the characteristic exponent. Therefore from now on we regard the mean jump sizes $1/b_i$ and $1/d_{i+N/2}$ as fixed. For each $i = 1, \dots, N/2$ we denote by $a_i^{(0)}$, $c_{i+N/2}^{(0)}$ the initial estimates of a_i, $c_{i+N/2}$ from (15) and by $\sigma^{(0)2}$ the initial value σ^2 obtained in Subsections 3.4 (or 3.5). We now seek to refine our initial estimates by finding the values of a_i, $c_{i+N/2}$, for $i = 1, \dots, N/2$, and σ^2 which most closely match (in both real and imaginary parts) the characteristic exponent of a HEJD process (multiplied by a carefully chosen weighting function $z \mapsto \Omega(z)$) with the characteristic exponent of the Lévy process in question (multiplied by the same weighting function $z \mapsto \Omega(z)$) at some judiciously chosen points z_k, $k = 1, ..., m$, in $\mathbb{C}$ (see Subsection 4.1).

As mentioned above, an important point is that the characteristic exponent of the HEJD process is linear in a_i, c_i and σ^2. Indeed, this is the very reason why we choose to fit the characteristic exponents (multiplied by a weighting function) rather than the characteristic functions.

Essentially, we now have to solve a linear system of the form $Ax = b$, where $A \in \mathbb{R}^{2m\times(N+1)}$, $x \in \mathbb{R}^{N+1}$ and $b \in \mathbb{R}^{2m}$, where $x = [a_1 \dots a_{N/2}\ c_{1+N/2} \dots c_N\ \kappa\sigma^2]^T$, and where, for $1 \leq k \leq m$, b_{2k-1} and b_{2k} are given by the real and imaginary parts of $\Omega(z_k)\left[\phi_\alpha(z_k) + (\kappa - 1)\frac{1}{2}\sigma^2(z_k^2 + i\alpha z_k)\right]$ respectively, and where, for $1 \le k \le m$, $1 \le j \le N+1$, $A_{2k-1,j}$ and $A_{2k,j}$ are respectively given by the real and imaginary parts of:

$$\begin{cases} \Omega(z_k)\left(\left(\dfrac{b_j}{b_j - iz_k} - 1\right) - i\alpha z_k\left(\dfrac{b_j}{b_j - 1} - 1\right)\right) & \text{if } 1 \le j \le \frac{N}{2}, \\ \Omega(z_k)\left(\left(\dfrac{d_j}{d_j + iz_k} - 1\right) - i\alpha z_k\left(\dfrac{d_j}{d_j + 1} - 1\right)\right) & \text{if } \frac{N}{2} + 1 \le j \le N, \\ -\Omega(z_k)\left(\dfrac{z_k^2 + i\alpha z_k}{2}\right) & \text{if } j = N+1, \end{cases} \tag{16}$$

and where $\phi_\alpha(z)$ is the characteristic exponent of the Lévy process we are trying to approximate. The parameters α and κ take values in the set $\{0,1\}$ and their roles are explained below.

If we try to solve this linear system directly, we will have two problems. Firstly, since we fit both the real and imaginary part of the characteristic exponents, we will have an even number of equations $2m$. If we decide to fit $a_i, c_{i+N/2}$, $i = 1, \ldots, N/2$, and σ^2 the number of parameters will be odd, and the linear system will not be square. In any event, we may wish to have the flexibility to choose m such that $2m > N+1$. Secondly, solving the linear system directly does not guarantee that $a_i, c_{i+N/2}$ and σ^2 are all positive.

To allow us to fit the characteristic exponents at a number of points m, possibly such that $2m > N+1$, and to try to ensure a_i, $c_{i+N/2}$ and σ^2 are all positive, we will use Tikhonov regularization. Specifically, instead of solving $Ax = b$, we seek x that minimizes $|Ax - b|^2 + \varepsilon^2 |x - x_0|^2$, where $\varepsilon \in \mathbb{R}^+$ and where $x_0 \in \mathbb{R}^{N+1}$ is the vector of our initial estimates $x_0 = [a_1^{(0)} \ldots a_{N/2}^{(0)}\, c_{1+N/2}^{(0)} \cdots c_N^{(0)}\ \kappa\sigma^{(0)2}]^T$. The solution is given by:

$$x = x_0 + (A^T A + \varepsilon^2 I)^{-1} A^T (b - Ax_0). \tag{17}$$

We choose some ε (see Section 4) that will allow us to reach a compromise between the necessity that the coefficients a_i, $c_{i+N/2}$, $i = 1, \ldots, N/2$, and σ^2 be non-negative and the precision of our solution. We allow two possibilities in solving the linear system given by (16) via equation (17).

- We can keep our initial estimate for $\sigma^{(0)2}$ or refine it, by choosing $\kappa = 0$ or $\kappa = 1$ respectively.
- We can decide to fit the characteristic exponents, or the mean-corrected characteristic exponents, by choosing $\alpha = 0$ or $\alpha = 1$ respectively.

In the next section, we will make explicit our choices of the points z_k, $k = 1, ..., m$ (where we fit the characteristic exponents), of z_{large} (which we use in calculating the truncation and discretization errors) and of the weighting function $\Omega(z)$.

4. Vanilla options

In [27], the value of an option, at time t_0, when the asset price is S_{t_0}, whose payoff function is $\min(S_T, K)$, where S_T is the asset price at maturity $T > t_0$ and K is the strike, is shown to be given by

$$f(S_{t_0}, K, T) = \frac{Ke^{-r(T-t_0)}}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{izk} \frac{\Phi_{T-t_0}(-z)}{z^2 - iz}\, dz, \tag{18}$$

where r is the interest rate, q is the dividend yield, $k = \ln\left(\dfrac{K}{S_{t_0}}\right) - (r-q)(T-t_0)$, Φ_{T-t_0} is the mean-corrected characteristic function of the Lévy process, ν is the imaginary part of z and $\nu \in (0,1)$. In both [18] and [27], the integral is evaluated at $\nu = 1/2$. This gives $z = u + i/2$, where u is real, that we use in equation (18). Hence, after some simplification,

$$\begin{aligned} f(S_{t_0}, K, T) &= \frac{1}{\pi}\sqrt{S_{t_0} K}\, e^{-\frac{T-t_0}{2}(r+q)} \int_0^{+\infty} \Re\left(e^{iuk} \frac{\Phi_{T-t_0}(-u - i/2)}{u^2 + 1/4}\right) du \\ &= \frac{1}{\pi}\sqrt{S_{t_0} K}\, e^{-\frac{T-t_0}{2}(r+q)} \int_0^{\pi} \Re\left(e^{i\tan(y/2)k/2}\, \Phi_{T-t_0}\left(-\frac{1}{2}\tan\left(-\frac{y}{2}\right) - \frac{i}{2}\right)\right) dy, \end{aligned} \tag{19}$$

where we have made the substitution $y = 2\arctan(2u)$. Call and put vanilla option prices, at time t_0, are given by $S_{t_0}e^{-q(T-t_0)} - f(S_{t_0}, K, T)$ and $Ke^{-r(T-t_0)} - f(S_{t_0}, K, T)$ respectively.

Our task now is to find an algorithm for choosing the points z_k, $k = 1, \ldots, m$, where we fit the characteristic exponent and the parameter z_{large} which is used to calculate the discretization and the truncation errors. We also need to specify the form of the weighting function Ω which appears in (16). We will use the form of the integrals in equations (18) and (19) to motivate our choices, because matching vanilla prices in the original Lévy process and in the (approximating) HEJD process is the same as matching the integrals in equations (18) and (19) with the corresponding characteristic functions.

4.1. **Determination of the points z_k.** The form of the integrals in equations (18) and (19) suggests that the points z_k, for $k = 1, ..., m$, where we try to match the characteristic exponents, should lie on one of the lines $\pm(u + i/2)$, $u \in \mathbb{R}$, in the complex plane. If we choose K to be the at-the-money strike (i.e. $k = 0$), the integrand in equation (19) has the form $\Re\left(\Phi_{T-t_0}(-\tan(-y/2)/2 - i/2)\right)$. To price the option we need to evaluate this integral, so a natural way of choosing the points z_k is to choose them so that if we were to have chosen these points z_k as abscissas in the numerical evaluation of this integral, the integral would be as precise as possible. We can again use Gauss-Legendre quadrature, to find the abscissas at which we would evaluate the integral. Discretizing the integral as a discrete sum, we obtain

$$\int_0^{\pi} \Re\left(\Phi_{T-t_0}\left(-\frac{1}{2}\tan\left(-\frac{y}{2}\right) - \frac{i}{2}\right)\right)\, dy \quad \approx \quad \sum_{k=1}^{m} \omega_k \Re\left(\Phi_{T-t_0}\left(-\frac{1}{2}\tan\left(-\frac{y_k}{2}\right) - \frac{i}{2}\right)\right),$$

where the weights ω_k and the abscissas y_k come from a m-point Gauss-Legendre quadrature rule on the interval $(0, \pi)$. This gives us the points z_k, namely $z_k = -1/2\tan(y_k/2) - i/2$, where we will choose to do the fitting of the characteristic exponents described in Section 3.6. We still have to determine the number of points m we will use. We would like m to be greater than $N/2$ but, at the same time, we would like m not to be too large. We found $m \approx 3N/4$ gives very good results, in practice.

4.1.1. *Determination of* z_{large}. In Subsection 3.5, we evaluated the discretization error $DE_{\overline{\theta}}$ and the truncation error $TE_{\overline{\theta}}$ at some point z_{large} such that $|z_{\text{large}}|$ is large. To choose the precise value of z_{large}, we can once again draw intuition from the form of the integrals in equations (18) and (19). A possible choice would be to choose z_{large} of the form $z_{\text{large}} = u_{\text{large}} + i/2$, where u_{large} is real. The disadvantage of this choice is that we cannot prove that the equation $DE_{\overline{\theta}} = TE_{\overline{\theta}}$ has a unique root. Hence, in view of the remark at the end of Section 3.4, we actually choose z_{large} of the form $z_{\text{large}} = u_{\text{large}}$, where u_{large} is positive. We now have to find u_{large}. Since the integrand $\Re(\Phi_{T-t_0}(-u - i/2)/(u^2 + 1/4))$ in equation (19) converges to 0 as $u \to \infty$, we choose u_{large} such that the integrand is smaller than some specified threshold. Typical values of this threshold are between 10^{-4} and 10^{-10}, depending on how fast the integrand converges to 0. Note that, as an aside, since we expect that $u_{\text{large}} \gg 1/2$, we expect little difference between the two possible choices.

4.1.2. *Determination of the weighting function* $z \mapsto \Omega(z)$. We now have to choose a form for the weighting function Ω that we used in Subsection 3.6. Noting that we made the choice $z = u + i/2$, where u is real, in going from equation (18) to (19), we choose $\Omega(z)$ in the form $\Omega(u + i/2)$. If we observe the form of the integral in equation (19), we see that $\Re\left(\Phi_{T-t_0}(-u - i/2)/(u^2 + 1/4)\right)$ tends to zero very rapidly as u tends to infinity and it does so because both $\Re\left(\Phi_{T-t_0}(-u - i/2)\right) \to 0$ and $1/(u^2 + 1/4) \to 0$ as $u \to \infty$. This means that the contribution to the integral when evaluating the integrand at large u is negligible compared to the contribution to the integral when evaluating the integrand when u is close to zero. This suggests the use of a weighting function $\Omega(z) \equiv \Omega(u + i/2)$ that also tends to zero as $u \to \infty$. If we do not do this (for example, if we were to choose $\Omega(z) = 1$, and noting that the characteristic exponent of the HEJD process contains a term proportional to z^2), then when trying to match the characteristic exponents in equation (16) of Subsection 3.6 we, intuitively speaking, give too much weight to values of the characteristic exponent when z is large in modulus. The observation that the characteristic exponent of the HEJD process contains a term proportional to z^2 leads us to propose a weighting function $\Omega(z) \equiv \frac{1}{z^2 - iz}$, which is equivalent to $\Omega(u + i/2) = \frac{1}{u^2+1/4}$.

4.2. **Vanilla option prices.** In this section we will present our results for vanilla option prices obtained from the approximating HEJD process and compare them to those obtained from the Lévy process we want to approximate. In this paper, we will only consider vanilla option prices for the case where the Lévy process is a NIG process, whose parameters we take as given (for examples of VG and CGMY processes, we refer the reader to Le Saux (2008)). Using the algorithm described in Section 3 and choosing z_{large}, $\Omega(z)$ and z_k, $k = 1, ..., m$ as explained above, we obtain the parameters a_i, b_i, $c_{i+N/2}$, $d_{i+N/2}$, $i = 1, \ldots, N/2$, and σ of the approximating HEJD process. When solving the linear system in equation (16) via equation (17), we found by numerical experimentation that we obtained the best results when we kept our initial estimate for $\sigma^{(0)2}$ (i.e. $\kappa = 0$) and when we fitted the mean-corrected characteristic exponents (i.e. $\alpha = 1$). Therefore, all the results we report in Sections 4 and 5 will use these choices. We used $\varepsilon = 10^{-5}$ in equation (17). For the results presented in this section, we set the number of points z_k, m, equal to 10.

The NIG parameters were those obtained by a calibration to the market prices of vanilla options on the Eurostoxx 50 equity index in [14]: $\alpha = 8.858$, $\beta = -5.808$ and $\delta = 0.174$ (see Appendix A.1 for the definition of the model). We analyse vanilla option prices obtained through equation (19) under the NIG process using five different approaches:

(a) Using equation (19) with the characteristic function for the NIG process. Clearly this approach will give us the "true" values against which we can benchmark the accuracy of approaches (b), (c), (d) and (e). For the remaining four approaches, we used equation (19) with the characteristic function for the HEJD process where we have fitted $N = 14$ Poisson processes (seven up and seven down).

(b) We used a solver-type methodology to find the roots $\overline{\theta}_+$ and $\overline{\theta}_-$ of the equation $TE = DE$ as in Section 3.4. Numerical experimentation appeared to confirm that we had found unique roots. We then used equation (17). We used a precision of 10^{-5} for the threshold that determines z_{large}.

(c) We proceeded as in approach (b). We then further revised our estimate for σ^2 by exactly matching the variance of the NIG process and of the HEJD process. The revised estimate was positive. The estimates for a_i, b_i, $c_{i+N/2}$, $d_{i+N/2}$, $i = 1, \ldots, N/2$, are exactly as in approach (b).

(d) We used the intensity rates, mean jump sizes and diffusion volatility from [14]. This data was supplied by Marc Jeannin and Martijn Pistorius to whom we again express our thanks.

(e) We used the simplified algorithm of Subsection 3.5 where we only do a one-dimensional search for $\overline{\theta}$ by computing the root of the equation $DE_{\overline{\theta}} = TE_{\overline{\theta}}$. We then used equation (17). We used a precision of 10^{-8} for the threshold that determines z_{large}.

We valued vanilla options with an initial asset price $S_{t_0} = 100$, risk-free rate $r = 0$, dividend yield $q = 0$ and time to maturity equal to one year. We priced options with 41 different strikes where the strikes were of the form $100 \exp(y)$ where the value of y ranged from -0.8 to 0.8 in intervals of 0.04. Hence, the strikes varied from approximately 44.93 to approximately 222.55. For the options with strikes greater than or equal to 100, we valued call options, else we valued put options. We then converted these prices to implied volatilities (expressed as percentages). We verified (by increasing the number of points used in the numerical integration) that all the implied volatilities reported were accurate to at least five decimal places. The results are in Figure 1 (and in tabular format in a spreadsheet available online [12]).

We can take approach (a) as giving the "true" values, then we can take the differences in implied volatilities between approach (a) and the other four approaches as being a measure of the error of the methodology for approximating a NIG process by a HEJD process consisting of fourteen Poisson processes. For approaches (b), (c), (d) and (e), the root-mean-square errors were 0.2044, 0.1989, 4.2558 and 0.1595 (expressed as implied volatility percentage points) respectively and the maximum absolute errors across all 41 strikes were 0.6537, 0.6385, 10.3291 and 0.5329 for approaches (b), (c), (d) and (e) respectively.

We can see that approaches (b), (c), (d) and (e) all fit quite well for low strikes but the fit is visibly less good for high strikes. This is particularly the case for approach (d) where the quality of the fit is much worse than for approaches (b), (c) and (e). Because the errors are much larger for approach (d), we have not included approach (d) in the graph of the errors in Figure 1. Based on the root-mean-square errors and maximum absolute errors across all 41 strikes, approach (c) works a little better than approach (b). However, inspection of Figure 1 shows that, in fact, the slightly better performance of approach (c) is explained by the fact that it performs better at very high strikes. Across the range of strikes from, say, 65.0 to, say, 155.0 (which might be more relevant in practice), approach (b) works better than approach (c). We see that while exactly matching the variance seems appealing, it is not, in fact, particularly successful. [25] shows that moment matching (she also considers matching the third moment i.e. the skew) does not perform well for the case of the VG process either. The fact that moment matching does not perform well can be explained by the fact that moment matching is essentially equivalent to matching the characteristic function near the origin. However, moment matching may actually make worse the fit between the characteristic functions of the Lévy process and the approximating HEJD process away from the origin and hence produce less accurate vanilla option prices. Approach (e) clearly works the best. It has the smallest root-mean-square error and the smallest maximum absolute error.

5. Barrier option price comparisons

We will now proceed to examine and compare barrier option prices obtained from the approximation of the Lévy process by a HEJD process which we described in Section 3.

Our approach is as follows. We take as given the parameters of the Lévy process in question. Using the results of Section 3, we approximate the Lévy process by a HEJD process. We then price barrier options using the methodology of [9] which relies on the fact that the Laplace Transform of the barrier option price can be computed essentially in closed form. The Gaver-Stehfest algorithm is then used to invert the Laplace Transform and obtain the barrier option price. We will use the terminology nTerms to denote the number of terms used in the Gaver-Stehfest algorithm. We will refer to this methodology as the HEJDCC methodology.

In order to give us a benchmark against which to compare our results, we will also utilise another approach. [5], [6] describe (henceforth the BoyarLeven methodology) FFT-based algorithms for pricing barrier options under General Classes of Lévy processes which are not based on approximating the Lévy process by a HEJD process. It should be said that the BoyarLeven methodology uses numerical methods and hence can't be and won't be literally exact. However, it does not approximate the Lévy process in question by a HEJD process at the outset as our approach does and it does appear to give accurate barrier option prices. Hence, we take as a working hypothesis that the option prices obtained from the BoyarLeven methodology are the most accurate and, therefore, we can and will use prices obtained from the BoyarLeven methodology to benchmark the accuracy of our HEJDCC methodology. These prices were provided to us by Mitya Boyarchenko. We again express our thanks to him and to Sergei Levendorskii.

In Subsection 5.1, we will also compare the prices from the HEJDCC methodology against prices reported in [14]. They also price barrier options by approximating the Lévy process in question by a HEJD process but their methodology to do this approximation is very different. We will refer to their methodology as the JPHEJD methodology.

The rest of this section is structured as follows: In Subsection 5.1, we compare single barrier option prices under the NIG process using the HEJDCC, BoyarLeven and JPHEJD methodologies. In Subsection 5.2, we compare double barrier option prices under the CGMY process for different values of the Y parameter using the HEJDCC and BoyarLeven methodologies. The above comparisons all use fourteen Poisson processes for the HEJD process.

5.1. **Single barrier option price comparisons under NIG.** In this section, we will price single barrier options under the NIG process using the HEJDCC, JPHEJD and BoyarLeven methodologies. The data we will use is the same as in Table 1 of [5] and in Table 2 of [14].

We price down-and-out put barrier options with the barrier set at 2100. The options have a strike of 3500, a maturity of one year, the risk-free rate $r = 0.03$ and the dividend yield $q = 0$. We price the options with 32 different initial asset prices which are expressed as a percentage of 3500. The percentages are: 64.0, 66.0,...,126.0. Hence, the initial asset prices varied from 2240 to 4410.

The NIG parameters are the same as we used in Section 4: $\alpha = 8.858$, $\beta = -5.808$ and $\delta = 0.174$. We used approaches (b) and (e) of Section 4 and fitted a HEJD process with fourteen Poisson processes (seven up and seven down - this is the same number as [14] used) using the results of Section 3. The values of the parameters a_i, b_i, $c_{i+N/2}$, $d_{i+N/2}$, $i = 1, \ldots, N/2$ and σ that we obtained are in a spreadsheet available online [12].

We then priced the barrier options using the methodology of [9]. Since [9] always works with double barrier options, we priced the options as if they were double barrier knockout put options and set the upper barrier level to 21000 which makes the corresponding knockout probability negligible. We used two different values of nTerms, namely 12 and 14 for approach (b) (labelled HEJDCC (Approach (b)), nTerms = 12 and HEJDCC (Approach (b)), nTerms = 14 respectively) but for approach (e) we only used a value of nTerms of 12 (labelled HEJDCC (Approach (e)) (simplified algorithm)). As a comparison, we also display the prices of the barrier options using the BoyarLeven methodology and those from [14]. The results are displayed in graphical form in Figure 2 (and in tabular format in a spreadsheet available online [12]).

Overall, the agreement between the prices using the different methodologies and approaches is good. However, it is clear that the prices obtained from the HEJDCC methodology are much closer than those of the JPHEJD methodology to the prices obtained from the BoyarLeven methodology. In fact, over the whole range of initial asset prices, the lines in Figure 2 depicting the prices using the BoyarLeven methodology and those using the HEJDCC methodology (for both approaches / values of nTerms) essentially lie on top of each other.

We take as our working hypothesis that the BoyarLeven methodology is the most accurate and we compute the root-mean-square proportional error (i.e. the errors relative to the BoyarLeven prices) for the HEJDCC methodology (we use approach (b) with nTerms set to 12 for this comparison but the other cases give qualitatively the same results) and for the JPHEJD methodology. The root mean-square errors were (for the HEJDCC methodology) 0.00312 and (for the JPHEJD methodology) 0.04401. Hence, the root-mean-square errors in the HEJDCC methodology are about one-fourteenth the root-mean-square errors in the JPHEJD methodology.

We believe that the reason for the better performance of the HEJDCC methodology is that the procedure for fitting a HEJD process to the NIG process is much better.

5.2. **Double barrier option price comparisons under CGMY.** In this section, we will price double barrier options under the CGMY process for different values of the Y parameter using the HEJDCC and BoyarLeven methodologies. The data we will use is closely based on that in Table 1 of [6].

We price two types of double barrier options, namely double barrier knockout put (henceforth DBKP) options and double-no-touch (henceforth DNT) options. For both types, the lower barrier is 2800 and the upper barrier is 4200. If the asset price trades at a level equal to or outside the barriers at any time up to and including maturity, the options are knocked out and expire worthless. If the options are not knocked out, then the DBKP options have the same payoff at maturity as vanilla put options with strike 3500 and the DNT options pay one unit of account at maturity. The risk-free rate $r = 0.03$, the dividend yield $q = 0$ and all options have a maturity of 0.1 years. We price both types of options with 75 different initial asset prices which are expressed as a percentage of 3500. The percentages are: 80.1, 80.2,(i.e. in intervals of 0.1),...,82.0, then 83.0, 84.0, (i.e. in intervals of 1.0),...,118.0, then 118.1, 118.2, (i.e. in intervals of 0.1 again),..., 119.9. We have smaller intervals when the initial asset price is close to the barriers in order to more closely examine the behaviour of the option prices in these regions. The change in intervals at 82.0 and 118.0 is the reason why the graphs (see Figures 3, 4 and 5) appear to exhibit slight kinks at 82.0 and 118.0 - had we used constant intervals throughout the range 80.1 to 119.9, these kinks would not be present.

We priced both types of barrier option for all the different initial asset prices for three different combinations of CGMY parameters. In all three cases, we used $C = 1$, $G = 9$, $M = 8$. We varied Y. The three different values of Y were: (1) $Y = 0.25$, (2) $Y = 0.5$ and (3) $Y = 1.25$.

For each combination of CGMY parameters, we fitted a HEJD process with fourteen Poisson processes (seven up and seven down) using the results of Section 3. In all cases, we used the simplified algorithm where we only do a one-dimensional search for $\overline{\theta}$ by computing the root of the equation $DE_{\overline{\theta}} = TE_{\overline{\theta}}$. The values of the parameters a_i, b_i, $c_{i+N/2}$, $d_{i+N/2}$, $i = 1, \ldots, N/2$ and σ are available in [25].

We then priced the barrier options using the methodology of [9]. We priced these options with two different values of nTerms (the number of terms used in the Laplace inversion), namely 12 and 14. As a comparison, we also display the prices of the barrier options using the BoyarLeven methodology. The results are displayed in graphical form in Figures 3, 4 and 5 (and in tabular format in [25]).

Overall, the agreement between the prices using the HEJDCC methodology and the BoyarLeven methodology is very good - especially when the initial asset price is not too close to either barrier. However, the agreement does deteriorate a little when the initial asset price is very close to either barrier.

We compute the root-mean-square proportional differences (by which we mean proportional differences to the [6] prices) (for nTerms set equal to 14), over all 75 different initial asset prices. The values obtained were:

(1) $Y = 0.25$, for DBKP options 0.0836, for DNT options 0.1130.

(2) $Y = 0.5$, for DBKP options 0.1090, for DNT options 0.1158.

(3) $Y = 1.25$, for DBKP options 0.0605, for DNT options 0.0561.

We see that the infinite variation case ($Y = 1.25$) performs very well.

5.3. **Summary of barrier option price comparisons.** We have shown that, by approximating the Lévy process in question by a HEJD process, we can very accurately price barrier options as long as the initial asset price is not too close to the barrier (or barriers). We have shown that this is true whether the Lévy process has finite or infinite variation. We have illustrated that our methodology for approximating the Lévy process by a HEJD process yields more accurate barrier option prices than the methodology of [14]. However, pricing barrier options by approximating the Lévy process by a HEJD process works somewhat less well when the initial asset price is very close to the barrier (or barriers).

6. Conclusions

General Classes of Lévy processes provide a model that can capture the effect of smiles and skews in implied volatilities, which the standard model of [4] and [22] cannot. Whilst pricing vanilla options remains straightforward, it is much more difficult to price exotic options, since few, if any, analytical results exist. Hyperexponential jump-diffusion (HEJD) processes are attractive because they are considerably more tractable than General Classes of Lévy processes and this facilitates the pricing of exotic options.

In this paper, our aim was to focus on the methodology of approximating Lévy processes by HEJD processes. We thus developed an algorithm that determines the parameters of the approximating process in a more systematic way than the procedures in the extant literature and that satisfies the six desirable features presented in the introduction. Indeed, our methodology is intuitive and easy to implement and does not require non-linear least-squares fitting in the determination of the parameters of the HEJD process. Our methodology calculates intensity rates and the mean jump sizes of the HEJD process, thus there is no need to guess the mean jump sizes. It also computes the magnitude of the very small jumps of the Lévy process in question, below which the very small jumps are approximated by Brownian motion.

Moreover, comparing our results to those obtained from existing procedures for approximating Lévy processes by a HEJD process, we demonstrate that our methodology computes more accurate option prices, both for vanilla and barrier options.

Appendix A. The NIG and CGMY processes

A.1. **The Normal Inverse Gaussian Process.** The Normal Inverse Gaussian (NIG) process was introduced by [3]. It has parameters $\alpha > 0$,$-\alpha < \beta < \alpha$ and $\delta > 0$. Its unit time characteristic function is given by

$$\Phi_{NIG}(u; \alpha, \beta, \delta) = \exp(-\delta(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2})).$$

From the form of the Lévy density ([14]), one can show that, in terms of equation (2), the functions $\mu_+(u)$ and $\mu_-(u)$ can be written in the form:

$$\mu_+(u) = \frac{\delta\alpha}{\pi}\sqrt{\left(\frac{u+\beta}{\alpha}\right)^2 - 1}\ \mathbf{1}_{(\alpha-\beta,\infty)}(u), \qquad \mu_-(u) = \frac{\delta\alpha}{\pi}\sqrt{\left(\frac{u+\beta}{\alpha}\right)^2 - 1}\ \mathbf{1}_{(\alpha+\beta,\infty)}(-u).$$

In terms of Section 3.5 and equation (11),

$$\mu(u) = \frac{\delta\alpha}{\pi}\sqrt{\left(\frac{u}{\alpha}\right)^2 - 1}\ \mathbf{1}_{(\alpha,\infty)}(u), \quad \text{and} \quad r_+ = -\beta,\ r_- = \beta.$$

A.2. **The CGMY Process.** The CGMY process was introduced by [10], and is also called the KoBoL process. For $Y \neq 0, 1$, its unit time characteristic function is given by

$$\Phi_{CGMY}(u; C, G, M, Y) = \exp(C\Gamma(-Y)\left[(M - iu)^Y + (G + iu)^Y - M^Y - G^Y\right]),$$

where $C, G, M > 0$ and $Y < 2$.

The Lévy density is only completely monotonic if $Y \geq -1$, in which case [14] show that, in terms of equation (2), the functions $\mu_+(u)$ and $\mu_-(u)$ are:

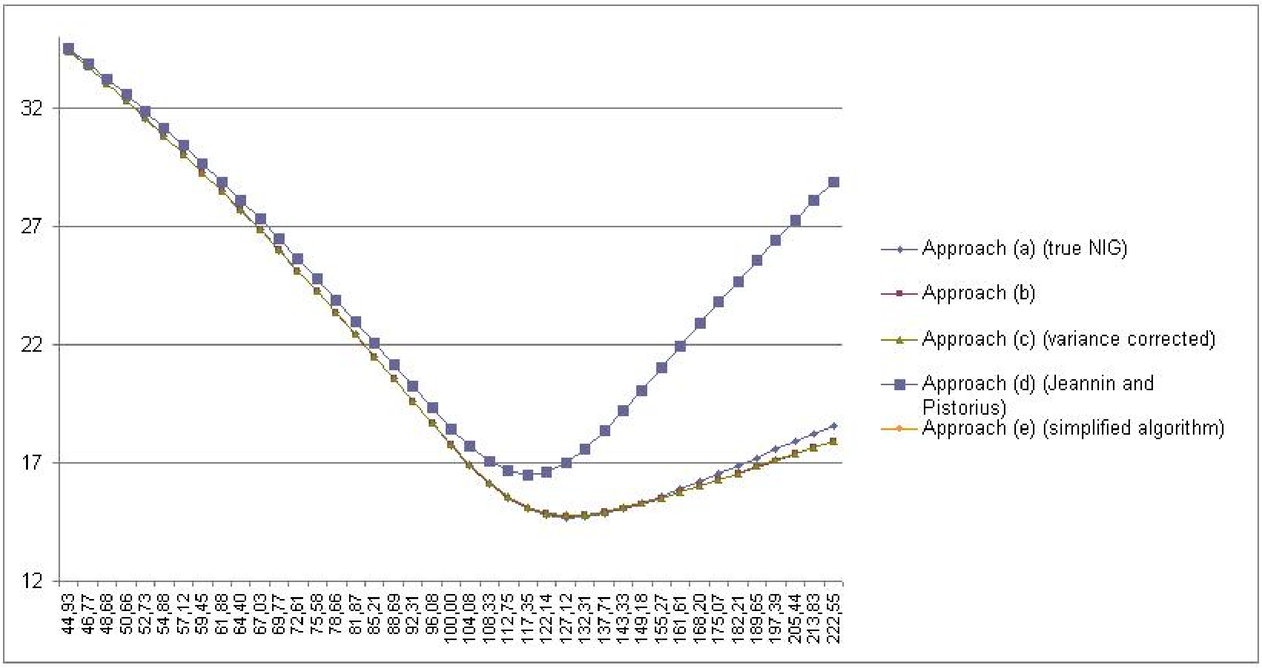
$$\mu_+(u) = C\frac{(u - M)^Y}{\Gamma(1+Y)}\ \mathbf{1}_{(M,\infty)}(u), \qquad \mu_-(u) = C\frac{(-u - G)^Y}{\Gamma(1+Y)}\ \mathbf{1}_{(G,\infty)}(-u).$$

In terms of Section 3.5 and equation (11),

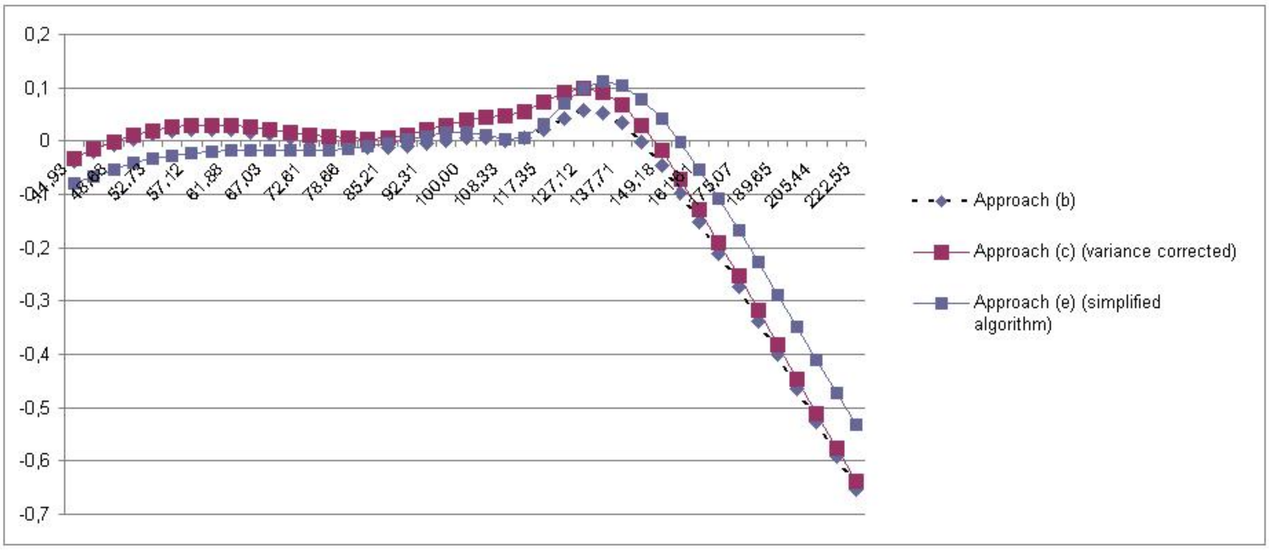
$$\mu(u) = C\frac{u^Y}{\Gamma(1+Y)}\ \mathbf{1}_{(0,\infty)}(u), \quad \text{and} \quad r_+ = M,\ r_- = G.$$

Appendix B. Vanilla option price results

- Results for $S_{t_0} = 100$, $r = 0$, $q = 0$ and $T = 1$.



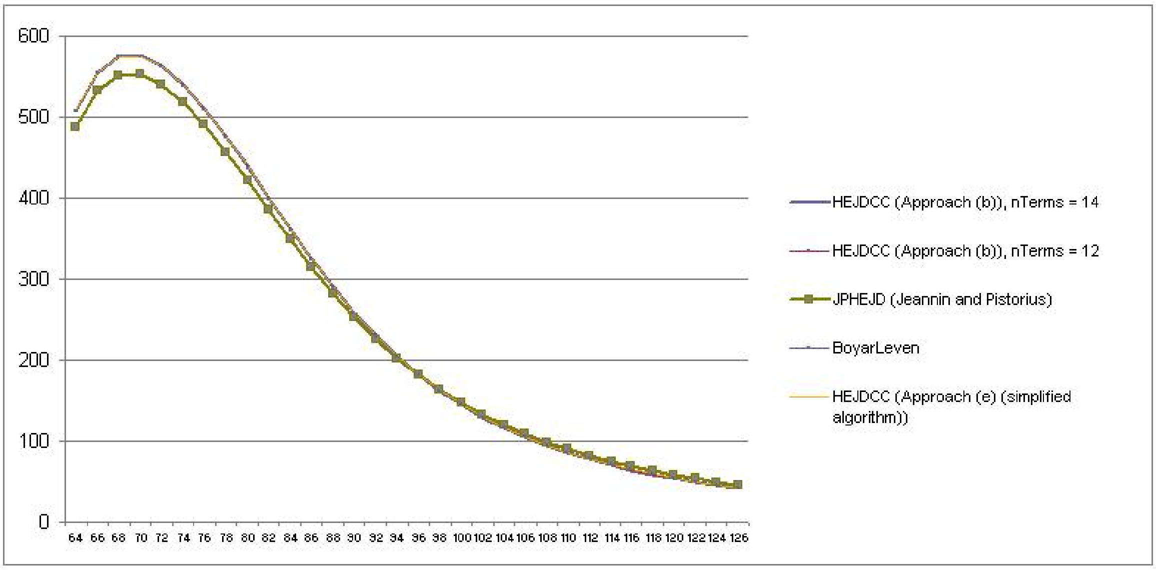
(a) Implied volatilities (expressed as a percentage)



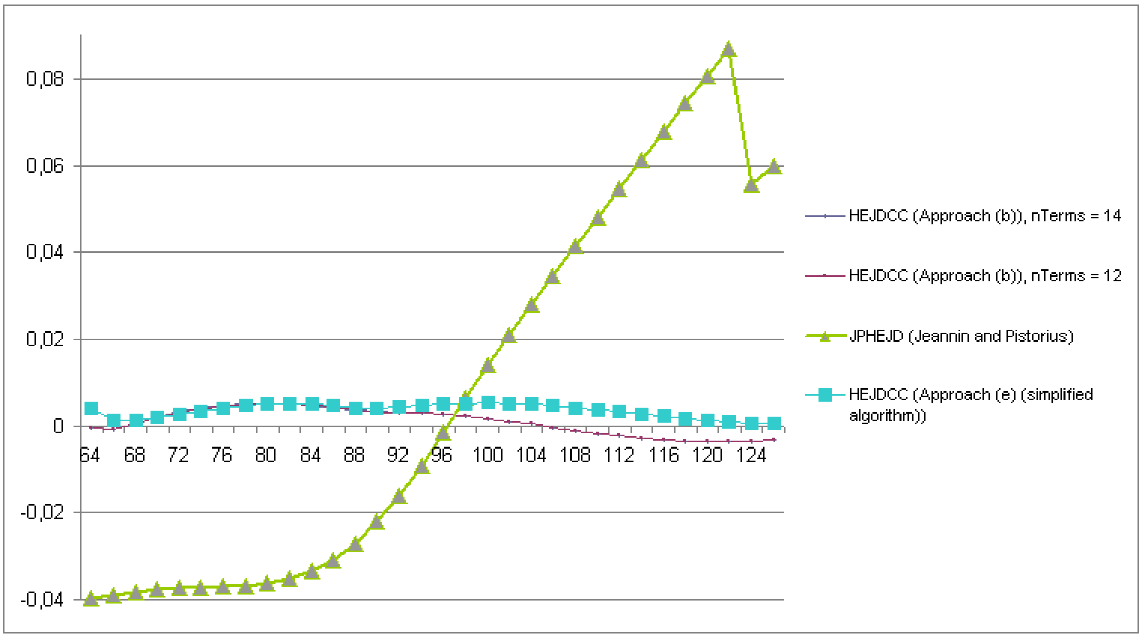
(b) Errors in implied volatilities (expressed as percentage points) compared to the original NIG process

Figure 1. NIG results with $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$ and $N = 14$.

• Single barrier option price comparisons under NIG



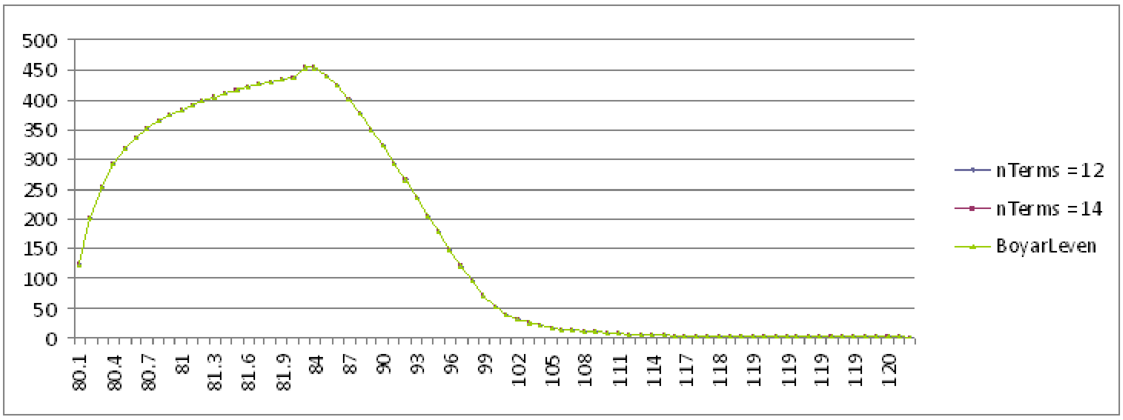
(a) Graph of option price against spot, where the spot is expressed as a percentage of 3500, for percentages from 64 to 126.



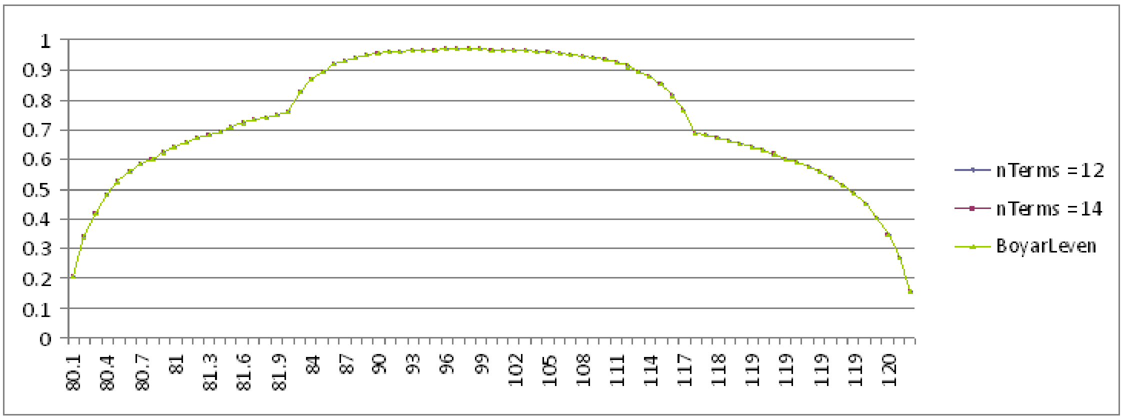
(b) Graph of proportional errors against spot, where the spot is expressed as a percentage of 3500, for percentages from 64 to 126.

Figure 2. NIG results with $\alpha = 8.858$, $\beta = -5.808$, $\delta = 0.174$ and $N = 14$.

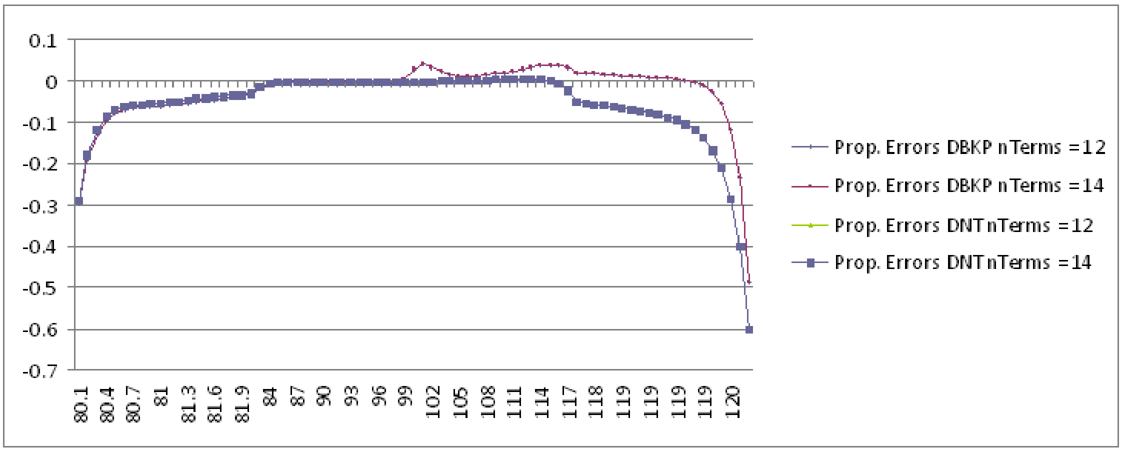
- Double barrier option price comparisons under CGMY



(a) Double barrier knockout put option prices

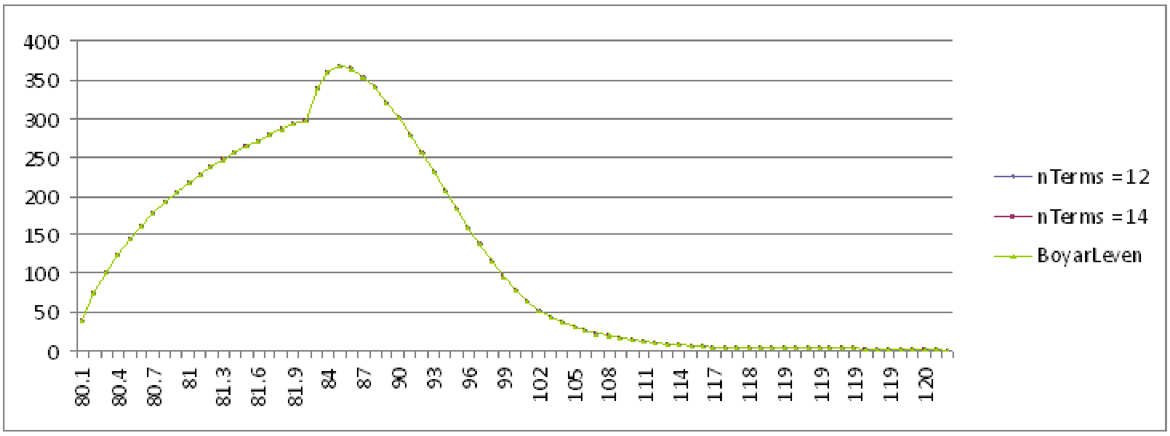


(b) DNT option prices

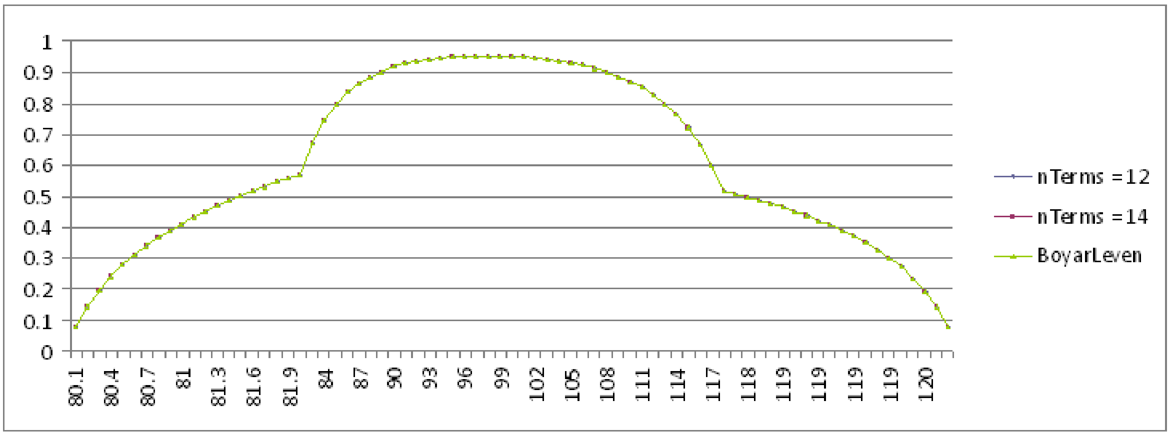


(c) Proportional errors for both double barrier knockout put (DBKP) option prices and double-no-touch (DNT) option prices

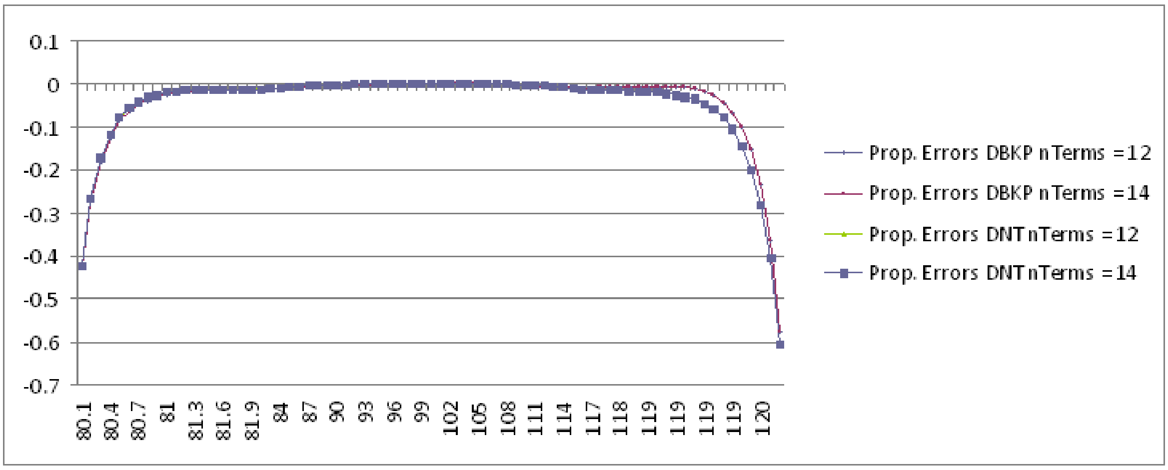
FIGURE 3. CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 0.25$ and $N = 14$.



(a) Double barrier knockout put option prices

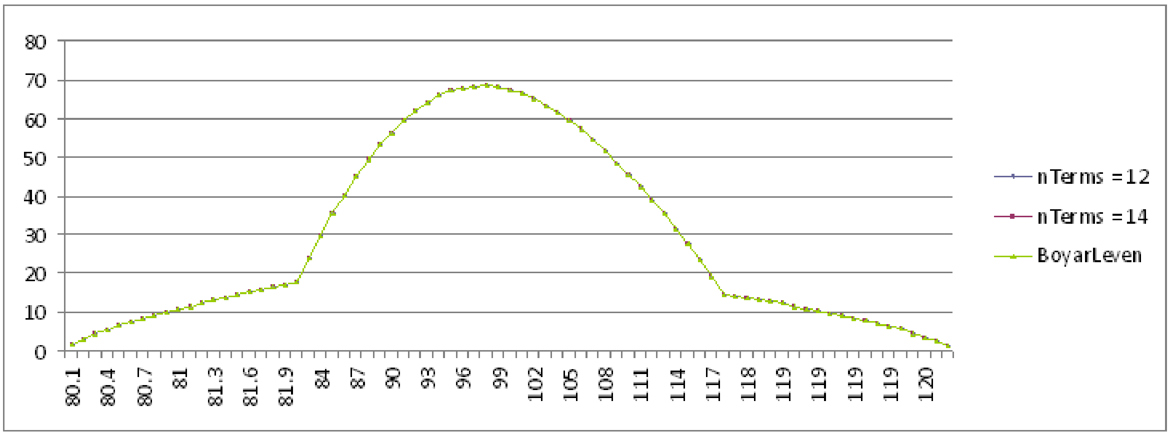


(b) DNT option prices

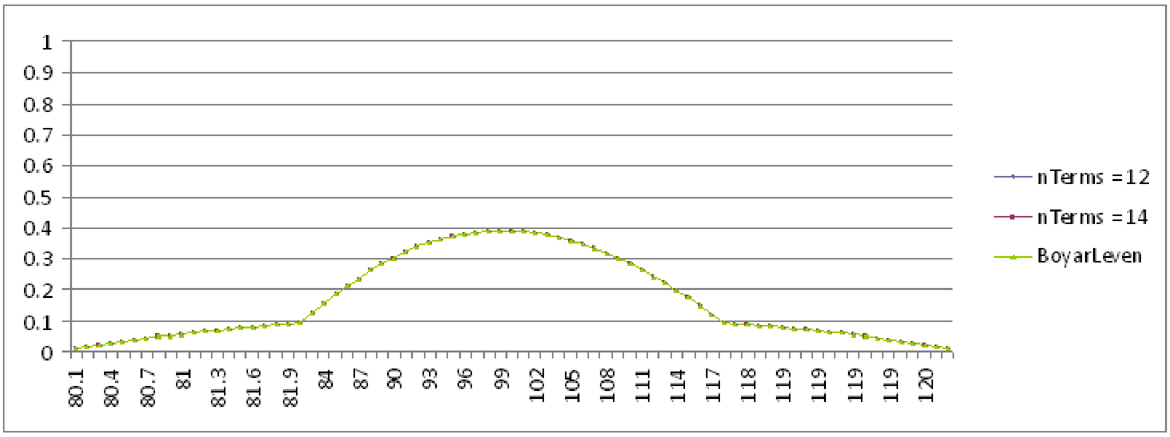


(c) Proportional errors for both double barrier knockout put (DBKP) option prices and double-no-touch (DNT) option prices

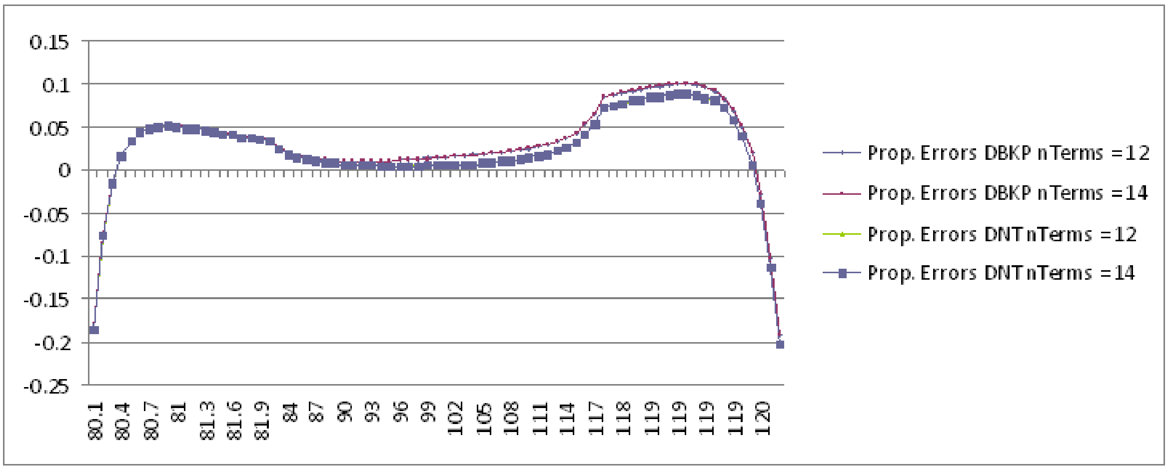
FIGURE 4. CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 0.5$ and $N = 14$.



(a) Double barrier knockout put option prices



(b) DNT option prices



(c) Proportional errors for both double barrier knockout put (DBKP) option prices and double-no-touch (DNT) option prices

FIGURE 5. CGMY results with $C = 1$, $G = 9$, $M = 8$, $Y = 1.25$ and $N = 14$.

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