LARGE DEVIATIONS FOR THE EXTENDED HESTON MODEL: THE LARGE-TIME CASE

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Abstract. We study here the large-time behaviour of all continuous affine stochastic volatility models (in the sense of [13]) and deduce a closed-form formula for the large-maturity implied volatility smile. We concentrate on (rescaled) strikes around the money, which are the most common in practice, and extend the results in [4] and [8].

1. Introduction

We are interested here in the large-time behaviour of the process $\left(t^{-1}X_t\right)_{t>0}$, where X is defined via the system of stochastic differential equations

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}\left(a+V_t\right)\mathrm{d}t + \rho\sqrt{V_t}\,\mathrm{d}W_t^1 + \sqrt{a+\left(1-\rho^2\right)V_t}\,\mathrm{d}W_t^2, \qquad && X_0 = x \in \mathbb{R}, \\ \mathrm{d}V_t &= \left(b+\beta V_t\right)\mathrm{d}t + \sqrt{\alpha V_t}\,\mathrm{d}W_t^1, && V_0 = v \in (0,\infty), \end{aligned}$$

with $a, b \geq 0$, $\alpha > 0$, $\beta \in \mathbb{R}$, $\rho \in [-1,1]$ and $\left(W_t^1, W_t^2\right)_{t\geq 0}$ is a two-dimensional standard Brownian motion. The couple $(X_t, V_t)_{t\geq 0}$ represents the restriction to continuous paths of the whole class of affine stochastic volatility models with jumps (ASVM), introduced by Keller-Ressel [13]. In particular it encompasses the popular Heston stochastic volatility model [9], in which $b > 0$ and $\beta < 0$. The weak convergence of the process $\left(t^{-1}X_t\right)_{t>0}$ has been studied in [4, 5] for the Heston model and in [10] for ASVM, via the Gärtner-Ellis theorem from large deviations theory. This convergence is the main ingredient needed to obtain the large-maturity behaviour of the implied volatility in these models. However the authors have imposed technical conditions on the parameters, which ensures that the assumptions of the Gärtner-Ellis theorem are met: (i) the limiting cumulant generating function Λ is essentially smooth inside a domain $\mathcal{D}$ and (ii) the interior $\mathcal{D}$ contains the origin.

Even though these conditions are usually satisfied in practice, they can actually be broken when calibrating the model for volatile markets. In terms of the parameters these two conditions—assumed in [4, 5]—read $\beta < 0$ and $\beta + \rho\sqrt{\alpha} < 0$. The second assumption makes sense on equity markets where the correlation is usually negative. However, on FX markets, the correlation between the asset and its volatility is not necessarily so (see [11] for instance), and a large value of the variance of volatility parameter α can violate this assumption. In [1], Andersen and Piterbarg studied the moment explosions of the Heston model (and other stochastic volatility models). They assume $\beta < 0$, but it appears that the restriction $\beta + \rho\sqrt{\alpha} < 0$ may also be needed. In [18] the authors highlighted the importance of this latter condition by proving that the Heston model remains of Heston form under the Share measure (i.e. taking the share price as the numeraire) with new mean-reversion speed $-(\beta + \rho\sqrt{\alpha})$. This in particular implies that the left wing of the smile could be deduced from the right wing automatically by symmetry.

Date: July 12, 2013.

The authors would like to thank Jim Gatheral and Claude Martini for useful discussions.

This may not be true however when this condition fails. Reversing the symmetry, the case where the mean-reversion $-\beta$ (in the original measure) is positive becomes interesting to study as well.

We show here that—at least in a neighbourhood of the origin—a large deviations principle still holds (as t tends to infinity) for the process $\left(t^{-1}X_t\right)_{t>0}$ when the two conditions (i) and (ii) above fail, i.e. without the technical assumptions of [4, 5, 10]. As an application, we translate this asymptotic behaviour into asymptotics of the implied volatility, corresponding to European vanilla options with payoff $\left(\mathrm{e}^{X_t}-\mathrm{e}^{xt}\right)_+$, for any real number x. In [8], the authors proved that the so-called *Stochastic Volatility Inspired* (SVI) parametric form—first proposed in [7]—of the implied volatility was the genuine limit (as the maturity tends to infinity) of the Heston implied volatility under the same technical conditions as in [4, 5, 10]. We extend the scope of this result by proving that it remains partially true—i.e. on some subsets of the real line—without the technical conditions mentioned above.

In Section 2, we study the limiting behaviour of the limiting cumulant generating function of the process $\left(t^{-1}X_t\right)_{t>0}$ and state the main result of the paper (Theorem 2.13), i.e. a large deviations principle for this process. In Section 3, we translate this LDP into option price and implied volatility asymptotics.

2. LDP for continuous affine stochastic volatility models

2.1. **The model and its effective domain.** Throughout this paper we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ supporting two independent Brownian motions W^1 and W^2. We consider affine stochastic volatility models in the sense of [13] with continuous paths. Let $(X_t, V_t)_{t\geq 0}$ be an affine process with state-space $\mathbb{R}\times\mathbb{R}_+$ which satisfies the following SDE

$$\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}\left(a+V_t\right)\mathrm{d}t + \rho\sqrt{V_t}\,\mathrm{d}W_t^1 + \sqrt{a+\left(1-\rho^2\right)V_t}\,\mathrm{d}W_t^2, && X_0 = x\in\mathbb{R},\\
\mathrm{d}V_t &= \left(b+\beta V_t\right)\mathrm{d}t + \sqrt{\alpha V_t}\,\mathrm{d}W_t^1, && V_0 = v\in(0,\infty),
\end{aligned} \tag{2.1}$$

where the admissible parameter values are given by

$$a, b\geq 0,\quad \alpha>0,\quad \beta\in\mathbb{R}\quad\text{and}\quad \rho\in[-1,1]\,. \tag{2.2}$$

The process $(V_t)_{t\geq 0}$ is a square-root diffusion process and the Yamada-Watanabe conditions [12] ensure that a unique non-negative strong solution exists. The share price process $S=(S_t)_{t\geq 0}$, defined by $S_t := \exp\left(X_t\right)$, is a local martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, and [13, Theorem 2.5] implies that S is a true martingale. The Heston model [9] with mean-reversion rate κ, positive long-time variance level θ, volatility of volatility σ and correlation ρ, is in the class of models given by the SDE in (2.1) (take $a=0$, $b=\kappa\theta>0$, $\beta=-\kappa<0$, $\alpha=\sigma^2$; the correlation parameter ρ has the same role as in (2.1)).

Remark 2.1.

(i) The class of models defined by (2.1) coincides with the class of affine stochastic volatility models with continuous sample paths.

(ii) The parameter a adds modelling flexibility.

(iii) The general form of the instantaneous variance of a continuous affine stochastic volatility process X is given by $a+\widetilde{\alpha}V$ for some $\widetilde{\alpha}>0$. A simple scaling of the process V in (2.1) maps the class of (2.1) to the general case. Without loss of generality we therefore assume $\widetilde{\alpha}=1$.

(iv) The process $U=(U_t)_{t\geq 0}$ defined by $U_t := a+V_t$ for all $t\geq 0$ follows the shifted square-root dynamics (see [14] for applications of the shifted square-root process in pricing theory).

Let us define the cumulant generating function [1] Λ_t of the random variable X_t, where $X_0 = 0$, by

$$\Lambda_t(u) := \log \mathbb{E}\left(\exp\left(uX_t\right)\right), \quad \text{for any} \quad u \in \mathbb{R}, \quad t \geq 0, \tag{2.3}$$

as an extended real number in $(-\infty, \infty]$. The effective domain of Λ_t is defined by $\mathcal{D}_t := \{u \in \mathbb{R} : \Lambda_t(u) < \infty\}$. Note that by the Hölder inequality the function Λ_t is convex on $\mathcal{D}_t$. In order to give the structure of $\Lambda_t(u)$ explicitly we need to define

$$\chi(u) := \beta + u\rho\sqrt{\alpha}, \tag{2.4}$$

as well as

$$\gamma(u) := \left(\chi(u)^2 + \alpha u\left(1-u\right)\right)^{1/2} \quad \text{and} \quad f_t(u) := \cosh\left(\frac{\gamma(u)t}{2}\right) - \frac{\chi(u)}{\gamma(u)}\sinh\left(\frac{\gamma(u)t}{2}\right). \tag{2.5}$$

In Proposition 2.2 we show how to express the cumulant generating function of X in terms of the logarithmic moment generating function of model (2.1) with $a = 0$.

Proposition 2.2. *The logarithmic moment generating function Λ_t defined in* (2.3) *reads*

$$\Lambda_t(u) = \Lambda_t^H(u) + \frac{a}{2}u\left(u-1\right)t, \quad \text{for all } t \geq 0 \text{ and } u \in \mathcal{D}_t,$$

where Λ_t^H is given by (2.3) *for the process X in* (2.1) *with $a = 0$. Furthermore we have*

$$\mathcal{D}_t = \{u \in \mathbb{R} \,:\, \Lambda_t^H(u) < \infty\}$$

and the following formula holds

$$\Lambda_t^H(u) = -\frac{2b}{\alpha}\left(\frac{\chi(u)t}{2} + \log f_t(u)\right) + \frac{u\left(u-1\right)}{f_t(u)\gamma(u)}\sinh\left(\frac{\gamma(u)t}{2}\right)v, \qquad \text{for all} \quad u \in \mathcal{D}_t. \tag{2.6}$$

Proof. It is well known that the logarithmic moment generating function of an affine process X given as a solution of SDE (2.1) is of the form

$$\Lambda_t(u) = \phi_t(u) + \psi_t(u)v \quad \text{for all } t \geq 0 \text{ and } u \in \mathcal{D}_t,$$

where the functions $\phi_t, \psi_t : \mathcal{D}_t \to \mathbb{R}$ satisfy the system of Riccati equations (see e.g. [13])

$$\begin{aligned} \partial_t\phi_t(u) &= F(u, \psi_t(u)), \qquad \phi_0(u) = 0, \\ \partial_t\psi_t(u) &= R(u, \psi_t(u)), \qquad \psi_0(u) = 0, \end{aligned} \tag{2.7}$$

with

$$R\left(u, w\right) := \frac{1}{2}u\left(u-1\right) + \frac{\alpha}{2}w^2 + uw\rho\sqrt{\alpha} + \beta w \qquad \text{and} \qquad F\left(u, w\right) := \frac{a}{2}u\left(u-1\right) + bw.$$

The Riccati equation equation for ψ_t can be solved in closed form

$$\psi_t(u) = \sinh\left(\frac{\gamma(u)t}{2}\right)\frac{u\left(u-1\right)}{\gamma(u)f_t(u)},$$

where the functions γ and f_t are defined in (2.5). The function ϕ_t can be determined by noting that equation (2.7) is equivalent to $\phi_t(u) = \int_0^t F\left(u, \psi_s(u)\right)\mathrm{d}s$. Therefore $\phi_t(u)st = au\left(u-1\right)t/2 + b\int_0^t \psi_s(u)\mathrm{d}s$. The function Λ_t^H can be constructed in an analogous way on the set $\{u \in \mathbb{R} \,:\, \Lambda_t^H(u) < \infty\}$ with R and F as above and $a = 0$. This concludes the proof. □

[1] We will use here the terms "logarithmic moment generating function" and "cumulant generating function" as synonyms.

In order to analyse the effective domain $\mathcal{D}_t$ we need to introduce the quantities u_- and u_+ given by

$$u_- := \begin{cases} \dfrac{1}{2\sqrt{\alpha}} \dfrac{2\beta\rho + \sqrt{\alpha} - \sqrt{\left(2\beta\rho+\sqrt{\alpha}\right)^2 + 4\beta^2\left(1-\rho^2\right)}}{1-\rho^2}, & \text{if } |\rho|<1, \\ -\infty, & \text{if } |\rho|=1 \text{ and } 2\beta\rho+\sqrt{\alpha}\leq 0, \\ -\beta^2/\left(2\beta\rho\sqrt{\alpha}+\alpha\right), & \text{if } |\rho|=1 \text{ and } 2\beta\rho+\sqrt{\alpha}>0, \end{cases} \tag{2.8}$$

and

$$u_+ := \begin{cases} \dfrac{1}{2\sqrt{\alpha}} \dfrac{2\beta\rho + \sqrt{\alpha} + \sqrt{\left(2\beta\rho+\sqrt{\alpha}\right)^2 + 4\beta^2\left(1-\rho^2\right)}}{1-\rho^2}, & \text{if } |\rho|<1, \\ \infty, & \text{if } |\rho|=1 \text{ and } 2\beta\rho+\sqrt{\alpha}\geq 0, \\ -\beta^2/\left(2\beta\rho\sqrt{\alpha}+\alpha\right), & \text{if } |\rho|=1 \text{ and } 2\beta\rho+\sqrt{\alpha}<0. \end{cases} \tag{2.9}$$

Note that the inequalities $u_- \leq 0$ and $u_+ \geq 1$ hold for all admissible values of the parameters and that in the case $|\rho|<1$ the parabola $\gamma(u)^2$ is strictly positive on the interior of the interval $[u_-, u_+]$ between its distinct zeros. In the case $|\rho|=1$ the graph of the function $\gamma(u)^2$ is a line and either u_- or u_+ are infinite. For notational convenience we shall understand the interval $[x,y]\subset\mathbb{R}$ as $[x,\infty)$ if $y=\infty$ and as $(-\infty,y]$ if $x=-\infty$. Proposition 2.3 analyses the structure of the effective domain $\mathcal{D}_t$ of the function Λ_t.

Proposition 2.3. *The effective domain $\mathcal{D}_t$ of the cumulant generating function Λ_t (defined in (2.3)) satisfies $[0,1]\subset\mathcal{D}_t$ for all $t\geq 0$ and any set of admissible parameter values from (2.2). Furthermore the following statements hold.*

(i) *If $\chi(0)\leq 0$ we have:*
  (a) *if $\chi(1)\leq 0$ then $[u_-,u_+]\subset\mathcal{D}_t$ for any $t>0$;*
  (b) *if $\chi(1)>0$ then for all t large enough there exists $\overline{u}(t)\in(1,u_+)$ such that*

$$\lim_{t\to\infty}\overline{u}(t)=1 \quad \textit{and} \quad [u_-,\overline{u}(t))\subset\mathcal{D}_t\subset(-\infty,\overline{u}(t)).$$

(ii) *If $\chi(0)>0$ we have:*
  (a) *if $\chi(1)\leq 0$ then for all large t there exists $\underline{u}(t)\in(u_-,0)$ such that*

$$\lim_{t\to\infty}\underline{u}(t)=0 \quad \textit{and} \quad (\underline{u}(t),u_+]\subset\mathcal{D}_t\subset(\underline{u}(t),\infty);$$

  (b) *if $\chi(1)>0$ then for large t there exist $\underline{u}(t)\in(u_-,0)$ and $\overline{u}(t)\in(1,u_+)$ such that*

$$\lim_{t\to\infty}\underline{u}(t)=0, \quad \lim_{t\to\infty}\overline{u}(t)=1 \quad \textit{and} \quad \mathcal{D}_t=(\underline{u}(t),\overline{u}(t)).$$

Remark 2.4. The following elementary facts are useful in the proof of Proposition 2.3.

(I) Note that $u_-=-\infty$ and $u_+=\infty$ if and only if the conditions $|\rho|=1$ and $\sqrt{\alpha}+2\rho\beta=0$ hold.

(II) The condition $\chi(1)\neq 0$ implies that $u_+>1$ since u_+ is the largest root of the quadratic $\gamma(u)^2$ in (2.5). In particular in (i)(b) and (ii)(b) of Proposition 2.3 the interval $(1,u_+)$ is not empty.

(III) The condition $\chi(0)\neq 0$ implies that $u_-<0$. In particular in (ii) we have $\chi(0)=\beta>0$ and hence the interval $(u_-,0)$ is not empty.

(IV) The interval $[0,1]$ is contained in $\mathcal{D}_t$ for all $t\geq 0$ since the stock price process $(S_0\exp(X_t))_{t\geq 0}$ is a true martingale.

(V) If $\chi(0)=0$ then $u_-=0$ and $u_+=1/(1-\rho^2)$ for $|\rho|<1$ and $u_+=\infty$ for $|\rho|=1$.

Remark 2.5. The variance process $(V_t)_{t\geq 0}$ in (2.1) is a time-changed squared Bessel process (see [2]):

$$(V_t)_{t\geq 0} \triangleq \mathrm{e}^{\beta t} R^2_{\delta,\tau_t},$$

where $\tau_t := \alpha^4\left(1-\mathrm{e}^{-\beta t}\right)/(4\beta)$, and $\left(R^2_{\delta,t}\right)_{t\geq 0}$ is a squared Bessel process of dimension $\delta := 4b/\alpha^4$, i.e. $\mathrm{d}R_t^2 = 2R_t\,\mathrm{d}W_t + \delta\,\mathrm{d}t$ and $R^2_{\delta,0}=0$. The sign of $\chi(0)=\beta$ changes the convexity of the time-change τ_t.

Proof. Proposition 2.2 implies that it is enough to study the effective domain of the cumulant generating function Λ_t^H of the Heston model. It is clear that the function f_t, defined in (2.5) by

$$f_t(u) = \cosh\left(\frac{\gamma(u)t}{2}\right) - \frac{\chi(u)}{\gamma(u)}\sinh\left(\frac{\gamma(u)t}{2}\right),$$

will play a key role in in understanding the set $\mathcal{D}_t$.
Case (i): If we can prove that

$$f_t(u) > 0, \quad \text{for all } u \in [u_-, 1], \tag{2.10}$$

then Proposition 2.2 implies that $[u_-,1]\subset\mathcal{D}_t$ since the functions on both sides of (2.6) can be analytically extended to a neighbourhood of $[u_-,1]$ in the complex plane and hence coincide on the interval.

We now prove (2.10). It follows from the definition of γ in (2.5) that $|\chi(u)/\gamma(u)|\leq 1$ for all $u\in[0,1]$ and hence (2.10) holds on $[0,1]$. It is easy to see that $\lim_{u\searrow u_-}\chi(u)\leq 0$. Since $\chi(0)=\beta\leq 0$ we have $\chi(u)\leq 0$ for all $u\in[u_-,0]$ which implies (2.10).

In case (i)(a) assume first that $u_+<\infty$. Then elementary algebra shows that $\chi(u_+)\leq 0$. Therefore $\chi(u)\leq 0$, and hence $f_t(u)>0$, for all $u\in[1,u_+]$. If $u_+=\infty$ the condition $\chi(1)\leq 0$ implies that $\rho=-1$ and therefore $\chi(u)<0$ for all $u\geq 1$. Hence $f_t(u)\in(0,\infty)$ for all $u\in[1,\infty)=[1,u_+]$. Proposition 2.2 and the analytic continuation argument as above imply $[u_-,u_+]\subset\mathcal{D}_t$.

Recall that in case (i)(b) we have $u_+>1$ (see Remark 2.4 (II)). Let $\overline{u}(t)$ be the smallest solution of the equation $f_t(u)=0$ in the interval $(1,u_+)$. Note that, since γ is strictly positive on the interval $(1,u_+)$, for a fixed t the equation $f_t(u)=0$ can be rewritten as

$$t = F(u), \qquad \text{where} \quad F(u) := \frac{2}{\gamma(u)}\operatorname{arctanh}\left(\frac{\gamma(u)}{\chi(u)}\right). \tag{2.11}$$

This equation has a solution in $(1,u_+)$ for large t since the continuous function F tends to infinity as u decreases to 1 (since $\lim_{u\searrow 1}\gamma(u)/\chi(u)=1$). This also implies that the smallest solution $\overline{u}(t)$ decreases to one. The functions on both sides of (2.6) coincide on $[u_-,1]$, are analytic on some neighbourhood of this interval in the complex plane and the right-hand side in (2.6) is real and finite on $[u_-,\overline{u}(t))$. They must therefore also coincide on $[u_-,\overline{u}(t))$, which in particular implies $[u_-,\overline{u}(t))\subset\mathcal{D}_t$. Formula (2.6) implies that $\overline{u}(t)$ is not an element of $\mathcal{D}_t$ and the convexity of Λ_t yields that $\mathcal{D}_t\cap[\overline{u}(t),\infty)=\emptyset$.
Case (ii): In case (ii)(a) the condition $\chi(1)\leq 0$ implies $\rho<0$ and hence $\chi(u)\leq 0$ for all $u\in[1,u_+]$. Therefore $f_t(u)>0$ on $[1,u_+]$ and hence $[0,u_+]\subset\mathcal{D}_t$. Let $\underline{u}(t)$ be the largest solution of the equation $f_t(u)=0$ in the interval $(u_,0)$. Since $\lim_{u\nearrow 0}(\gamma(u)/\chi(u))=1$, an analogous argument as in the proof of (i)(b) shows that $\underline{u}(t)$ is well defined and the limit in the proposition holds. The proof for the inclusions follows the same steps as in the proof of (i)(b).

In case (ii)(b) we have $\chi(0)=\beta>0$ and $\chi(1)>0$. Therefore the definition of γ, given in (2.5), implies

$$\lim_{u\nearrow 0}\frac{\gamma(u)}{\chi(u)} = 1 \qquad \text{and} \qquad \lim_{u\searrow 1}\frac{\gamma(u)}{\chi(u)} = 1$$

and hence, by (2.11), there exist solutions to the equation $f_t(u) = 0$ in both intervals $(u_-, 0)$ and $(1, u_+)$. Let $\underline{u}(t)$ be the largest solution in $(u_-, 0)$ and $\overline{u}(t)$ the smallest solution in $(1, u_+)$. An analogous argument to the one in the proofs of (i)(b) and (ii)(a) gives the form of $\mathcal{D}_t$. □

2.2. **Large deviation principles and the Gärtner-Ellis theorem.** We review here the key concepts of large deviations for a family of real random variables $(Z_t)_{t\geq 1}$ and state the Gärtner-Ellis theorem (Theorem 2.6). A general reference for all the concepts in this section is [3, Section 2.3]. Assume that the cumulant generating function $\Lambda_t^Z(u) := \log \mathbb{E}\left(\mathrm{e}^{uZ_t}\right)$ is finite on some neighbourhood of the origin and that for every $u \in \mathbb{R}$ the following limit exists as an extended real number

$$\Lambda(u) := \lim_{t\to\infty} t^{-1}\Lambda_t^Z(ut). \tag{2.12}$$

Let $\mathcal{D}_\Lambda := \{u \in \mathbb{R} : |\Lambda(u)| < \infty\}$ be the effective domain of Λ and assume that

$$0 \in \mathcal{D}_\Lambda^o, \tag{2.13}$$

where $\mathcal{D}_\Lambda^o$ is the interior of $\mathcal{D}_\Lambda$ (in $\mathbb{R}$). Since Λ_t^Z is convex for every t by Hölder's inequality, the limit Λ is also convex and the set $\mathcal{D}_\Lambda$ is an interval. Since $\Lambda(0) = 0$, convexity implies that for any $u \in \mathbb{R}$ we have $\Lambda(u) > -\infty$. The function $\Lambda : \mathbb{R} \to (-\infty, \infty]$ is said essentially smooth if (a) it is differentiable in $\mathcal{D}_\Lambda^o$ and (b) it satisfies $\lim_{n\to\infty} |\Lambda'(u_n)| = \infty$ for every sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_\Lambda^o$ that converges to a boundary point of $\mathcal{D}_\Lambda^o$. A cumulant generating function Λ which satisfies (b) is called steep. The Fenchel-Legendre transform Λ^* of Λ is defined by the formula

$$\Lambda^*(x) := \sup\{ux - \Lambda(u) : u \in \mathbb{R}\}, \quad \text{for all } x \in \mathbb{R} \tag{2.14}$$

with an effective domain $\mathcal{D}_{\Lambda^*} := \{x \in \mathbb{R} : \Lambda^*(x) < \infty\}$. Under certain assumptions Λ^* is a good rate function, i.e. is lower semicontinuous (since it is a supremum of continuous functions), satisfies $\Lambda^*(\mathbb{R}) \subset [0, \infty]$ (since $\Lambda(0) = 0$) and the level sets $\{x : \Lambda^*(x) \leq y\}$ are compact for all $y \geq 0$ (see [3, Lemma 2.3.9(a)]). In general Λ^* can be discontinuous and $\mathcal{D}_{\Lambda^*}$ can be strictly contained in $\mathbb{R}$ (see [3, Section 2.3] for elementary examples of such rate functions). We say that the family of random variables $(Z_t)_{t\geq 1}$ satisfies the large deviations principle (LDP) with the good rate function Λ^* if for every Borel measurable set B in $\mathbb{R}$ the following inequalities hold

$$-\inf_{x\in B^o} \Lambda^*(x) \leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left[Z_t \in B\right] \leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left[Z_t \in B\right] \leq -\inf_{x\in\overline{B}} \Lambda^*(x), \tag{2.15}$$

where the interior B^o and the closure $\overline{B}$ of the set B are taken in the topology of $\mathbb{R}$ and $\inf\emptyset = \infty$. It is clear from definition (2.15) that if $(Z_t)_{t\geq 1}$ satisfies the LDP and Λ^* is continuous on $\overline{B}$, then $\lim_{t\to\infty} t \log \mathbb{P}\left[Z_t \in B\right] = -\inf\{\Lambda^*(x) : x \in B\}$. An element $y \in \mathbb{R}$ is an *exposed point* of Λ^* if there exists $u_y \in \mathbb{R}$ such that

$$yu_y - \Lambda^*(y) > xu_y - \Lambda^*(x) \qquad \text{for all} \quad x \in \mathbb{R}\backslash\{y\}. \tag{2.16}$$

Intuitively the exposed points are those at which Λ^* is strictly convex (e.g. the second derivative is continuous and strictly positive). The segments over which Λ^* is affine are not exposed. Note that (2.16) can only hold for $y \in \mathcal{D}_\Lambda$ and, if Λ is differentiable in $\mathcal{D}_\Lambda^o$, then u_y is the unique solution of $\Lambda'(u) = y$. We now state the Gärtner-Ellis theorem the proof of which can be found in [3, Section 2.3].

Theorem 2.6. *Let $(Z_t)_{t\geq 1}$ be a family of random variables for which the function $\Lambda : \mathbb{R} \to (-\infty, \infty]$ in* (2.12) *satisfies* (2.13). *Let F be a closed and G an open set in $\mathbb{R}$. Then the following inequalities hold*

$$\limsup_{t\to\infty} t^{-1}\mathbb{P}\left[Z_t \in F\right] \leq -\inf\{\Lambda^*(x) \, : \, x \in F\},$$
$$\liminf_{t\to\infty} t^{-1}\mathbb{P}\left[Z_t \in G\right] \geq -\inf\{\Lambda^*(x) \, : \, x \in G \cap \mathcal{E}\},$$

where $\mathcal{E} := \{y \in \mathbb{R} \, : \, y$ satisfies (2.16) *with $u_y \in \mathcal{D}_\Lambda^o\}$. Furthermore if Λ is essentially smooth and lower semicontinuous, then the LDP holds for $(Z_t)_{t\geq 1}$ with the good rate function Λ^*.*

2.3. **LDP in affine stochastic volatility models.** In this section we analyse the large deviations behaviour of the family of random variables $Z_t := X_t/t$ for $t \geq 1$. Corollary 2.7—which follows from Propositions 2.2 and 2.3—describes the properties of the cumulant generating function Λ defined in (2.12), and its Fenchel-Legendre transform Λ^* is studied in Proposition 2.10. The main result of this section, Theorem 2.13, states that the family $(Z_t)_{t\geq 1}$ satisfies a large deviations principle.

Corollary 2.7. *The limiting cumulant generating function* (2.12) *for the family of random variables $(X_t/t)_{t\geq 1}$, where $(X_t)_{t\geq 0}$ is defined by SDE* (2.1),*is given by*

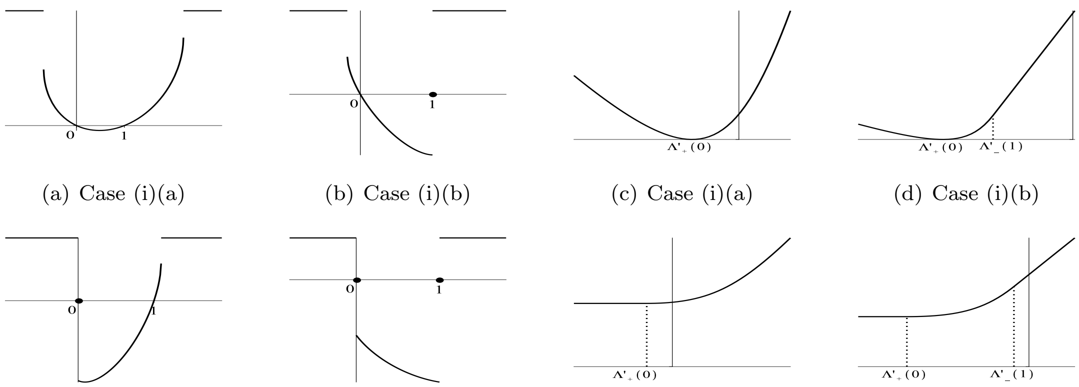
$$\Lambda(u) = \begin{cases} -\dfrac{b}{\alpha}\left(\chi(u) + \gamma(u)\right) + \dfrac{a}{2}u\left(u-1\right), & \text{for all } u \in \mathcal{D}_\Lambda \setminus \{0,1\}, \\ 0, & \text{for } u \in \{0,1\}, \end{cases}$$

with the functions χ and γ given in (2.4) *and* (2.5) *respectively. The function Λ is infinitely differentiable on the interior $\mathcal{D}_\Lambda^o$ of its effective domain. The boundary points u_- and u_+, defined in* (2.8) *and* (2.9), *can be used to describe the effective domain $\mathcal{D}_\Lambda$ as follows.*

- (i) *If $\chi(0) \leq 0$ we have:*
  - (a) *if $\chi(1) \leq 0$ then $\mathcal{D}_\Lambda = [u_-, u_+]$;*
  - (b) *if $\chi(1) > 0$ then $\mathcal{D}_\Lambda = [u_-, 1]$.*
- (ii) *If $\chi(0) > 0$ we have:*
  - (a) *if $\chi(1) \leq 0$ then $\mathcal{D}_\Lambda = [0, u_+]$;*
  - (b) *if $\chi(1) > 0$ then $\mathcal{D}_\Lambda = [0, 1]$.*

Remark 2.8. From Corollary 2.7, the following facts can be deduced immediately for the large deviations behaviour of the family of random variables $(X_t/t)_{t\geq 1}$.

- (I) In case (i)(a) the function Λ is essentially smooth.
- (II) In case (i)(b) (resp. (ii)(a)) the function Λ is steep at the left boundary u_- (resp. right boundary u_+) but not at the right (resp. left) boundary of the effective domain.
- (III) In case (i)(b) (resp. (ii)(a)) the right (resp. left) boundary point of the effective domain is strictly smaller (resp. greater) than u_+ (resp. u_-). This is a consequence of Remark 2.4 (II) and (III).
- (IV) In case (ii)(b) the function Λ is not steep at either of the two boundaries of its effective domain. Furthermore $\mathcal{D}_\Lambda$ is contained in the interior of the interval $[u_-, u_+]$ by Remark 2.4 (II) and (III).
- (V) As a consequence of (I)–(IV) the limiting cumulant generating function Λ is steep at a boundary point of the effective domain if and only if this point is an element of the set $\{u_-, u_+\}$.



(a) Case (i)(a) (b) Case (i)(b) (c) Case (i)(a) (d) Case (i)(b)

(e) Case (ii)(a) (f) Case (ii)(b) (g) Case (ii)(a) (h) Case (ii)(b)

FIGURE 1. The four figures on the left represent the function Λ characterised in Corollary 2.7. The four figures on the right represent the Fenchel-Legendre Λ^* determined in Proposition 2.10. The dotted line on the graphs for Λ^* represent the threshold $\Lambda'_-(1)$ and $\Lambda'_+(0)$ above or below which Λ^* becomes linear.

Note that when u_- (resp. u_+) is not in $\mathcal{D}_\Lambda$ then the function Λ is discontinuous at 0 (resp. at 1). We henceforth define the following extended real numbers

$$\Lambda_-(1) := \lim_{u\nearrow 1} \Lambda(u), \quad \Lambda_+(0) := \lim_{u\searrow 0} \Lambda(u), \quad \Lambda'_-(1) := \lim_{u\nearrow 1} \Lambda'(u), \quad \Lambda'_+(0) := \lim_{u\searrow 0} \Lambda'(u). \tag{2.17}$$

The functions Λ and Λ' are monotone on the intervals $(0,\varepsilon)$ and $(1-\varepsilon,1)$ for small enough ε, hence all the limits exist. Note further that the limit $\Lambda'_+(0)$ (resp. $\Lambda'_-(1)$) is equal to $-\infty$ (resp. ∞) if and only if $\chi(0)=0$ (resp. $\chi(1)=0$).

Remark 2.9. At zero and one the following identities hold

$$\Lambda_+(0) = -\frac{b}{\alpha}\left(\chi(0)+|\chi(0)|\right) \quad \text{and} \quad \Lambda'_+(0) = \begin{cases} \frac{1}{|\chi(0)|}\left(\left(\chi(1)-\chi(0)\right)\Lambda_+(0)-\frac{b}{2}\right)-\frac{a}{2}, & \text{if } \chi(0)\neq 0,\\ -a/2, & \text{if } \chi(0)=0,\ b=0,\\ -\infty, & \text{if } \chi(0)=0,\ b\neq 0,\end{cases}$$

$$\Lambda_-(1) = -\frac{b}{\alpha}\left(\chi(1)+|\chi(1)|\right) \quad \text{and} \quad \Lambda'_-(1) = \begin{cases} \frac{1}{|\chi(1)|}\left(\left(\chi(1)-\chi(0)\right)\Lambda_-(1)+\frac{b}{2}\right)+\frac{a}{2}, & \text{if } \chi(1)\neq 0,\\ a/2, & \text{if } \chi(1)=0,\ b=0,\\ \infty, & \text{if } \chi(1)=0,\ b\neq 0.\end{cases}$$

Note that the inequalities $\Lambda_+(0)\leq 0$ and $\Lambda_-(1)\leq 0$ hold for any admissible set of parameters. The case $\chi(0)=0$ and $b=0$ is rather degenerate, and we refer the reader to Remark 3.5 for further details.

Proposition 2.10. *The Fenchel-Legendre transform Λ^* defined in* (2.14) *for the family of random variables $(X_t/t)_{t\geq 1}$, where $(X_t)_{t\geq 0}$ is given by SDE* (2.1)*, can be represented as follows*

$$\Lambda^*(x) = \begin{cases} xu_x - \Lambda(u_x), & \text{for all } x\in\Lambda'\left(\mathcal{D}^o_\Lambda\right),\\ x-\Lambda_-(1), & \text{for all } x\in\left[\Lambda'_-(1),\infty\right)\cap\left(\mathbb{R}\backslash\Lambda'\left(\mathcal{D}^o_\Lambda\right)\right),\\ -\Lambda_+(0), & \text{for all } x\in\left(-\infty,\Lambda'_+(0)\right]\cap\left(\mathbb{R}\backslash\Lambda'\left(\mathcal{D}^o_\Lambda\right)\right),\end{cases} \tag{2.18}$$

where u_x is the unique solution in $\mathcal{D}_\Lambda^o$ to the equation $\Lambda'(u) = x$ for all $x \in \Lambda'(\mathcal{D}_\Lambda^o)$. Furthermore Λ^* is continuously differentiable on its effective domain $\mathcal{D}_{\Lambda^*}$ and $\mathcal{D}_{\Lambda^*} = \mathbb{R}$.

(i) *The function Λ^* attains its global minimal value $-\Lambda_+(0)$ at $\Lambda'_+(0)$. If $0 \in \mathcal{D}_\Lambda^o$ then the minimum is attained at the unique point $\Lambda'_+(0) = \Lambda'(0)$ and the minimal value is $\Lambda^*(\Lambda'(0)) = \Lambda_+(0) = 0$. If $0 \notin \mathcal{D}_\Lambda^o$ the minimal value is attained at every $x \in \left(-\infty, \Lambda'_+(0)\right] \cap (\mathbb{R}\backslash\Lambda'(\mathcal{D}_\Lambda^o))$*

(ii) *The function $x \mapsto \Lambda^*(x) - x$ attains its global minimal value $-\Lambda_-(1)$ at $\Lambda'_-(1)$. If $1 \in \mathcal{D}_\Lambda^o$ then the minimum value $\Lambda_-(1) = \Lambda(1) = 0$ is attained at the unique point $\Lambda'_-(1) = \Lambda'(1)$ which is therefore the unique solution to the equation $\Lambda^*(x) = x$. If $1 \notin \mathcal{D}_\Lambda^o$ the function $x \mapsto \Lambda^*(x) - x$ attains the minimal value at every $x \in \left[\Lambda'_-(1)\infty\right) \cap (\mathbb{R}\backslash\Lambda'(\mathcal{D}_\Lambda^o))$.*

Remark 2.11.

(i) Since Λ is a strictly convex smooth function on $\mathcal{D}_\Lambda^o$, the first derivative Λ' is invertible on this interval and u_x is a strictly increasing, differentiable function of x on $\Lambda'(\mathcal{D}_\Lambda^o)$. Furthermore the equality $(\Lambda^*)'(x) = u_x$ holds for any $x \in \Lambda'(\mathcal{D}_\Lambda^o)$.

(ii) Corollary 2.7 implies the following form for the interval $\Lambda'(\mathcal{D}_\Lambda^o)$:

$$\Lambda'(\mathcal{D}_\Lambda^o) = \begin{cases} \mathbb{R}, & \text{if } \chi(0) \le 0,\ \chi(1) \le 0, \\ \left(-\infty, \Lambda'_-(1)\right), & \text{if } \chi(0) \le 0,\ \chi(1) > 0, \\ \left(\Lambda'_+(0), \infty\right), & \text{if } \chi(0) > 0,\ \chi(1) \le 0, \\ \left(\Lambda'_+(0), \Lambda'_-(1)\right), & \text{if } \chi(0) > 0,\ \chi(1) > 0. \end{cases} \tag{2.19}$$

Hence the second case in (2.18) corresponds to $\chi(1) > 0$ and the third case occurs when $\chi(0) > 0$.

(iii) When a is null, the unique solution u_x to the equation $\Lambda'(u) = x$, when $x \in \Lambda'(\mathcal{D}_\Lambda^o)$ is given by

$$u_x = \frac{1}{2(1-\rho^2)\sqrt{\alpha}}\left(2\rho\beta + \sqrt{\alpha} + \frac{p(x)\,\xi}{\sqrt{p(x)^2 + b^2(1-\rho^2)}}\right), \tag{2.20}$$

where $p(x) := b\rho + x\sqrt{\alpha}$ and $\xi := \sqrt{\left(2\rho\beta + \sqrt{\alpha}\right)^2 + 4\beta^2(1-\rho^2)}$. This, together with (2.18), yields an explicit formula for the rate function Λ^*. Note that u_x is well defined as a limit when $|\rho|$ tends to 1 and

$$u_x = \frac{1}{4}\frac{b - 2\beta x}{2\beta + \rho\sqrt{\alpha}}\frac{4b\beta + \rho(b + 2\beta x)\sqrt{\alpha}}{\left(b\rho + x\sqrt{\alpha}\right)^2}, \qquad \text{whenever } \rho \in \{-1, 1\}. \tag{2.21}$$

(iv) When the parameter a is not null, we do not have a closed-form representation for u_x, and hence not for the function Λ^* either. However computing Λ^* is a simple root-finding exercise and the smoothness of the function Λ makes it computationally quick.

Proof of Proposition 2.10. Let $u_x \in \mathcal{D}_\Lambda^o$ be the unique solution of $\Lambda'(u) = x$, which exists by Remark 2.11 (i). It is clear from definition (2.14) that, for $x \in \Lambda'(\mathcal{D}_\Lambda^o)$, the Fenchel-Legendre Λ^* takes the form given in the proposition.

Assume now that $\Lambda'_-(1)$ is finite. This is equivalent to $\chi(1) \neq 0$ which implies that for every $u \in \mathcal{D}_\Lambda^o$ we have $u < 1$. Then for any $x \in \left[\Lambda'_-(1), \infty\right) \cap (\mathbb{R}\backslash\Lambda'(\mathcal{D}_\Lambda^o))$ the inequality $\Lambda_-(1) - \Lambda(u) \le x(1-u)$ holds by the Lagrange theorem (and the fact that Λ' is strictly increasing). Hence formula (2.18) follows.

If $\Lambda'_+(0)$ is finite, then for every $u \in \mathcal{D}_\Lambda^o$ we have $u > 0$. For any $x \in \left(-\infty, \Lambda'_+(0)\right] \cap (\mathbb{R}\backslash\Lambda'(\mathcal{D}_\Lambda^o))$ the inequality $ux - \Lambda(u) \le -\Lambda_+(0)$ holds for all $u \in \mathcal{D}_\Lambda^o$. Hence formula (2.18) follows.

The function Λ^* is continuously differentiable on $\mathbb{R}$ by (2.18) and Remark 2.11 (i). Note that, if $0 \in \mathcal{D}_\Lambda^o$, at the minimum we have $u_x = 0$. This implies by definition that the minimum of Λ^* is attained at $\Lambda'(0) = x$. The case $0 \notin \mathcal{D}_\Lambda^o$ follows in a similar way.

If $1 \in \mathcal{D}_\Lambda^o$, then by differentiating the formula in (2.18) we find that the minimum of $x \mapsto \Lambda^*(x) = x$ is attained if and only if $u_x = 1$, which is equivalent to $\Lambda'(1) = x$. If $1 \notin \mathcal{D}_\Lambda^o$, it is easy to see that the minimum is attained for all $x \geq \Lambda'_-(1)$. This concludes the proof. □

Before stating the main theorem of this paper, let us define a probability measure $\widetilde{\mathbb{P}}$, known as the Share measure, via the Radon-Nikodym derivative $\mathrm{d}\widetilde{\mathbb{P}}/\mathrm{d}\mathbb{P}$ which at time t takes the form e^{X_t}. Since $(\mathrm{e}^{X_t})_{t\geq 0}$ is a martingale, $\widetilde{\mathbb{P}}$ is a well-defined probability measure. The cumulant generating functions and consequently the Fenchel-Legendre transforms of X under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are related by

$$(2.22)\quad \widetilde{\Lambda}(u) = \Lambda(u+1), \quad \text{for all } u \text{ such that } (1+u) \in \mathcal{D}_\Lambda, \quad \text{and} \quad \widetilde{\Lambda}^*(x) = \Lambda^*(x) - x, \quad \text{for all } x \in \mathbb{R}.$$

The following proposition gives explicit conditions on the parameters ensuring that zero lies in $\Lambda'(\mathcal{D}_\Lambda^o)$, equivalently that $\Lambda'_+(0) < 0 < \Lambda'_-(1)$. This proposition will be fundamental in the next section in order to determine the large-time behaviour of option prices.

Proposition 2.12. *The origin belongs to $\Lambda'(\mathcal{D}_\Lambda^o)$ if and only if*

(i) $\beta \leq 0$ *and one of the following conditions hold:*
- $\chi(1) \leq 0$;
- $\chi(1) > 0$ *and* $\xi_- \leq 0$;
- $0 < \chi(1) < \alpha b/\xi_-$ *and* $\xi_- > 0$;

(ii) $\beta > 0$ *and the two following conditions hold simultaneously:*
- *either* $\chi(1) \leq 0$ *or* $\chi(1) > 0$ *and* $\xi_- \leq 0$ *or* $0 < \chi(1) < \alpha b/\xi_-$ *and* $\xi_- > 0$;
- *either* $\xi_+ \geq 0$ *or* $\xi_+ < 0$ *and* $\beta \in (0, -\alpha b/\xi_+)$;

where $\xi_\pm := 4b\rho\sqrt{\alpha} \pm a\alpha$.

Proof. The proposition follows by Remark 2.9, i.e. by a careful study of the behaviour of the function Λ' at the boundaries of its effective domain, provided in (2.19). When $\chi(0) = \beta < 0$, then clearly $\Lambda'_+(0) < 0$ and we simply need to ensure that $\Lambda'_-(1) > 0$. This is clearly satisfied when $\chi(1) \leq 0$. Assume that $\chi(1) > 0$ and let $\phi := 4b\rho\sqrt{\alpha} - a\alpha$. A straightforward computation shows that $\Lambda'_-(1) > 0$ if and only if (a) $\chi(1) < \alpha b/\phi$ when $\phi > 0$, (b) $\chi(1) > \alpha b/\phi$ when $\phi < 0$ or (c) always when $\phi = 0$. Since we are already imposing $\chi(1) > 0$, the above shows that $\Lambda'_-(1) > 0$ if and only if (a) $\phi \leq 0$ or (b) $\phi > 0$ and $\chi(1) < \alpha b/\phi$. This proves (i). Let us now consider the case $\beta = \chi(0) > 0$. Then by convexity of the function Λ, we need to ensure that $\Lambda'_-(0) < 0$ and $\Lambda'_-(1) > 0$. The proposition then follows by a careful identification of each case. □

We are now equipped to state the main theorem of this paper.

Theorem 2.13. *The family $(X_t/t)_{t\geq 1}$, with X defined in* (2.1)*, satisfies a large deviations principle under $\mathbb{P}$ (resp. under $\widetilde{\mathbb{P}}$) on $\Lambda'(\mathcal{D}_\Lambda^o)$ with rate function Λ^* described in Proposition 2.10 (resp. $\widetilde{\Lambda}^*$ in* (2.22)*).*

Proof. The proof of this theorem follows from the Gärtner-Ellis theorem. Note that the function Λ is not necessarily steep at the boundary of its effective domain, which is the reason why we can only state a large deviations principle on $\Lambda'(\mathcal{D}_\Lambda^o)$ rather than on the whole real line. □

Remark 2.14. The absence of steepness of the limiting cumulant generating function Λ can be circumvented by applying an extended version of the Gärtner-Ellis theorem, based on a time-dependent change of measure. Likewise, the fact that the origin may not be inside the interior of the effective domain of Λ can be dealt with using the results in [15]. However, the main issue here, which does not seem to have been tackled in the literature, is the discontinuity of Λ at the boundaries of its effective domain. This fact seems (numerically) to break the large deviations principle, and we leave this study for future research.

3. Asymptotics of option prices and implied volatilities

In this section we relate the rate function Λ^* governing the large deviations of the family $(X_t/t)_{t\geq 1}$ to the option prices in the case of model (2.1) and the Black-Scholes model. These asymptotic option prices will then be translated into implied volatility asymptotics.

3.1. **Asymptotics of option prices.**

Theorem 3.1 and Corollary 3.3 below describe the limiting behaviour of European option prices respectively in the model (2.1) and in the Black-Scholes model when the maturity tends to infinity. These results were proved in [10] and we recall them here to highlight the importance of proving a large deviations principle under both probability measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$.

Theorem 3.1. *If the origin lies within the interval $\Lambda'(\mathcal{D}_\Lambda^o)$ and if $(X_t/t)_{t\geq 1}$ satisfies the LDP under both $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ with the respective good rate functions Λ^* and $\widetilde{\Lambda}^*$, the asymptotic behaviour of a covered call option with payoff $\mathrm{e}^{X_t} - \left(\mathrm{e}^{X_t} - \mathrm{e}^{xt}\right)^+$ is given by*

$$\lim_{t\to\infty} t^{-1} \log\left(1 - \mathbb{E}\left[\left(\mathrm{e}^{X_t} - \mathrm{e}^{xt}\right)^+\right]\right) = x - \Lambda^*(x), \quad \text{if } x \in \left[\Lambda'_+(0), \Lambda'_-(1)\right].$$

Remark 3.2. Note that, since we only have a partial LDP (Theorem 2.13), we do not obtain call and put option price asymptotics for all possible strikes. However, take some $x \in (0, \Lambda'_-(1))$. Since the limits are uniform in a neighbourhood of the origin, for any $y > 0$, we can find some t such that $y = xt$, which then gives us option price asymptotics for any fixed (independent of time) positive strike. This is the most relevant case in practice, which is the reason why we only focus on these covered call option prices.

Let us consider the Black-Scholes model where the process $(X_t)_{t\geq 0}$ satisfies the SDE $\mathrm{d}X_t = -\Sigma^2/2\mathrm{d}t + \Sigma\mathrm{d}W_t$, with $\Sigma > 0$. Its limiting cumulant generating function reads $\Lambda_{\mathrm{BS}}(u) = u(u-1)\Sigma^2/2$ for all $u \in \mathbb{R}$, and we define its Fenchel-Legendre transform (2.14) $\Lambda^*_{\mathrm{BS}}(\cdot, \Sigma)$. Since the function $\partial_x \Lambda'_{\mathrm{BS}}(\cdot, \Sigma)$ is strictly increasing on the whole real line, the equation $\Lambda'_{\mathrm{BS}}(u) = x$ has a unique solution $u_x \in \mathbb{R}$ for any real number x. It is straightforward to see that $u_x = x/\Sigma^2 + 1/2$ and hence $\Lambda^*_{\mathrm{BS}}(x, \Sigma) = \left(x + \Sigma^2/2\right)^2 / \left(2\Sigma^2\right)$ for all $x \in \mathbb{R}$. From this characterisation it is immediate to see that $\partial_x \Lambda^*_{\mathrm{BS}}(x, \Sigma) = 0$ if and only if $x = -\Sigma^2/2$ and $\partial_x \Lambda^*_{\mathrm{BS}}(x, \Sigma) = 1$ if and only if $x = \Sigma^2/2$. The following corollary applies Theorem 3.1 to the Black-Scholes model. A more complete version of it can be found in [10].

Corollary 3.3. *Under the Black-Scholes model, we have the following asymptotics.*

$$\lim_{t\to\infty} \frac{1}{t} \log\left(1 - \mathbb{E}\left(\mathrm{e}^{X_t} - \mathrm{e}^{xt}\right)_+\right) = \begin{cases} 2x + \Sigma^2, & \text{if } x \leq -3\Sigma^2/2, \\ x - \Lambda^*_{\mathrm{BS}}(x, \Sigma), & \text{if } x \in \left(-3\Sigma^2/2, \Sigma^2/2\right], \\ 0, & \text{if } x > \Sigma^2/2. \end{cases}$$

3.2. **Implied volatility asymptotics.** We now translate the large-maturity asymptotics for option prices proved above to the study of the implied volatility. Proposition 3.4 provides the limit of the implied volatility for continuous affine stochastic volatility models (2.1). For any real number x, let $\sigma_t(x)$ represent the Black-Scholes implied volatility of a European call option with strike price $S_0 \mathrm{e}^{xt}$ in the model (2.1). Let us further define the function $\sigma_\infty : \left(\Lambda'_+(0), \Lambda'_-(1)\right) \to \mathbb{R}_+$ by

$$\sigma_\infty^2(x) := 2\left(2\Lambda^*(x) - x + 2(x)\left(\Lambda^*(x)\left(\Lambda^*(x) - x\right)\right)^{1/2}\right), \quad \text{for all } x \in \left(\Lambda'_+(0), \Lambda'_-(1)\right). \tag{3.1}$$

The following proposition gives the behaviour of the implied volatility σ_t as t tends to infinity for all affine stochastic volatility models with continuous paths. Note again that we restrict here the range of possible strikes. In view of Remark 3.2, however, this ensures that all observable strikes—for large enough maturities—are encompassed in this result.

Proposition 3.4. *The function σ_∞ defined in (3.1) is continuous. Furthermore, if $b \neq 0$ and the origin lies within the interval $\Lambda'(\mathcal{D}_\Lambda^o)$, then the equality $\lim_{t\to\infty} \sigma_t(x) = \sigma_\infty(x)$ holds for all $x \in \left(\Lambda'_+(0), \Lambda'_-(1)\right)$.*

Proof. From Theorem 3.1 and Corollary 3.3, the implied volatility σ_∞ satisfies the quadratic equation $\Lambda^*(x) = \Lambda^*_{\mathrm{BS}}(x, \sigma_\infty(x))$, for all $x \in \left(\Lambda'_+(0), \Lambda'_-(1)\right)$. The proof of the corollary therefore consists of (a) finding the correct root of this quadratic equation and (b) proving the the function $\sigma_t(x)$ converges to this root for all x in the corresponding subset of the real line. The proof is analogous to the proof of [10, Theorem 14], and we therefore omit it for brevity. We also refer the reader to the recent work [6] for the general methodology to transform option price asymptotics into implied volatility asymptotics. □

Remark 3.5. From Corollary 2.7, the case $b = 0$ can be handled directly since the limiting cumulant generating function reads $\Lambda(u) = \frac{1}{2}au(u-1)$ for all $u \in \mathcal{D}_\Lambda$. Using Proposition 2.10 and [10], we immediately deduce the following limiting smiles:

- (i)(a) it is immediate that σ_∞^2 is everywhere equal to a;
- (i)(b) $\Lambda^*(x) = x - \Lambda_-(1) = x$ for $x > \Lambda'_-(1) = a/2$ and $\Lambda^*(x) = \Lambda^*_{\mathrm{BS}}(x, \sqrt{a})$ otherwise. Therefore $\sigma_\infty^2(x)$ is equal to $2x$ for $x > a/2$ and is equal to a for all $x \le a/2$;
- (ii)(a) $\Lambda^*(x) = 0$ for $x < \Lambda'_+(0) = -a/2$ and $\Lambda^*(x) = \Lambda^*_{\mathrm{BS}}(x, \sqrt{a})$ otherwise. Therefore $\sigma_\infty^2(x)$ is equal to $-2x$ for $x < -a/2$ and is equal to a for all $x \ge -a/2$;
- (ii)(b) $\Lambda^*(x) = 0$ for $x < \Lambda'_+(0) = -a/2$, that $\Lambda^*(x) = x$ for $x < \Lambda'_+(1) = a/2$ and that $\Lambda^*(x) = \Lambda^*_{\mathrm{BS}}(x, \sqrt{a})$ otherwise. Therefore $\sigma_\infty^2(x)$ is equal to $-2x$ for $x < -a/2$, to $2x$ when $x > a/2$ and to a when $x \in [-a/2, a/2]$.

Remark 3.6. The remark above implies that when considering strikes of the form e^z for fixed $z \in \mathbb{R}$, the total variance map $t \mapsto \widetilde{\sigma}_t^2(z)t \equiv \sigma_t^2(x/t)t$ converges to infinity as t tends to infinity.

3.3. **Convergence of the implied volatility of the Heston model to SVI.** In [7], Gatheral proposed the so-called 'Stochastic Volatility Inspired' (SVI) parameterisation of the implied volatility smile. Using the closed-form representation of the rate function Λ^* (Proposition 2.10 and Equation (2.20)) in the Heston model $a = 0$, Gatheral and Jacquier [8] proved that this parameterisation was indeed the true limit of the Heston implied volatility smile as the maturity tends to infinity for strikes of the form $S_0 \mathrm{e}^{xt}$, whenever both conditions $\chi(0) < 0$ and $\chi(1) < 0$ are met. Corollary 3.7 below extends their result without these conditions. Its proof follows from straightforward manipulations of Formula (3.1) and we therefore omit it. Recall that the SVI parameterisation for the implied variance reads

$$\sigma^2_{\text{SVI}}(x) = \frac{\omega_1}{2}\left(1 + \omega_2\rho x + \sqrt{(\omega_2 x + \rho)^2 + 1 - \rho^2}\right), \qquad \text{for all } x \in \mathbb{R}, \tag{3.2}$$

where $(\omega_1, \omega_2) \in \mathbb{R}^2$ and $\rho \in [-1, 1]$. Let us further define the mappings

$$\omega_1 := \frac{4b}{\alpha(1-\rho^2)}\left(\sqrt{\left(2\beta + \rho\sqrt{\alpha}\right)^2 + \alpha(1-\rho^2)} + \left(2\beta + \rho\sqrt{\alpha}\right)\right) \qquad \text{and} \qquad \omega_2 := \frac{\sqrt{\alpha}}{b}. \tag{3.3}$$

Corollary 3.7. *If $a = 0$, $b \neq 0$ and if $0 \in \Lambda'(\mathcal{D}^o_\Lambda)$, the asymptotic implied volatility* (3.1) *satisfies $\sigma^2_\infty(x) = \sigma^2_{\text{SVI}}(x)$ under the mappings* (3.3) *for all $x \in \left(\Lambda'_+(0), \Lambda'_-(1)\right)$.*

Remark 3.8.

(a) The case $b = 0$ was treated in Remark 3.5.

(b) When $a = 0$, the quantities in Remark 2.9 simplify to

$$\Lambda_+(0) = -\frac{2b\beta}{\alpha}, \qquad \Lambda'_+(0) = -\frac{b}{2\sqrt{\alpha}}\left(4\rho + \frac{\sqrt{\alpha}}{\beta}\right), \qquad \text{when } \chi(0) > 0,$$

$$\Lambda_-(1) = -\frac{2b}{\alpha}\left(\beta + \rho\sqrt{\alpha}\right), \qquad \Lambda'_-(1) = -\frac{b}{2\sqrt{\alpha}}\left(4\rho + \frac{\sqrt{\alpha}}{\beta + \rho\sqrt{\alpha}}\right), \qquad \text{when } \chi(1) > 0.$$

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