Optimal trade execution for Gaussian signals with power-law resilience

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We compute an explicit solution for the optimal **signal-adaptive liquidation** strategy for a trader subject to transient price impact with **power-law resilience** and **zero temporary price impact**, with a **Gaussian signal** driven by an underlying Brownian motion $W$.

The optimal selling speed $u^*_t$ turns out to be a **Gaussian Volterra process** of the form

$$u^*_t = u^0_t + \bar{u}(t) + \int_0^t k(u, t)dW_t,$$

where $k(.,.)$ and $\bar{u}$ satisfy a family of **Fredholm integral equations** of the 1st kind which can be solved in closed-form, $u^0(t)$ is the (deterministic) solution for the no-signal case given in Gatheral et al.[GSS12], and the integral equation becomes a Fredholm equation of the 2nd kind if we add temporary price impact. For a certain class of price process $P$ we can re-express $u_t$ in terms of $P$ itself using an **inversion formula**.

This generalizes **Gatheral, Schied & Slynko**[GSS12] for the no-signal case, and complements **Neuman & Voß**[NV20] for the case of exponential resilience, and has the advantage over [NV20] that we impose the full liquidation constraint $X_T = 0$ (as opposed to a quadratic penalty of the form $\alpha X_T^2$) and we do not require having non-zero temporary price impact (adding temporary price impact has a regularizing effect).
We work on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) throughout, with a filtration \((\mathcal{F}_t)_{t \geq 0}\) which satisfies the usual conditions, and \(\mathbb{E}_t(.)\) will denote \(\mathbb{E}(.)|\mathcal{F}_t)\) throughout. We consider a model with transient price impact, where the price we pay for an asset at time \(t\) is

\[
S_t = P_t + \int_0^t G(t-s) dX_s
\]  

(1)

where \(G(t) = ct^{-\gamma}\) for some constant \(c > 0\).

Here \(P\) is some \(\mathcal{F}_t\)-progressively measurable process \(P\) with \(\mathbb{E}(P_t^2) < \infty\) for all \(t \in [0, T]\) (which we refer to as the unaffected price process), where \(X_t = X_0 - \int_0^t u_s ds\) is the number of shares held at time \(t\), which we assume is absolutely continuous in \(t\) so \(u_t\) is the selling speed. Note that we do not assume that \(S\) is a semimartingale (as is usually assumed in the literature). We let \(\mathcal{U}\) denote the space of admissible controls equal to the space of \(\mathcal{F}_t\)-progressively measurable processes \(u\) such that \(|V(u)| < \infty\) where \(V(u)\) is the expected profit/loss (see next slide for formula).
We let $\mathcal{U}_x^0$ denote the set of $u \in \mathcal{U}$ with $X_0 = x$ and $X_T = 0$, and set
\[ \xi_t := \mathbb{E}_t(P_T - P_t). \]
$\int_0^t G(t-s) dX_s$ represents the cumulative effect of our trading activities on the current stock price, and $G$ is the decay kernel, which characterizes the resilience of price impact between trades. Our performance criterion is to maximize the expected profit/loss at $T$:

\[
V(u) = \mathbb{E}(\int_0^T (P_t - \int_0^t G(t-s) u_s ds) u_t dt + P_T X_T)
\]
\[
= \mathbb{E}(\int_0^T (P_t - \int_0^t G(t-s) u_s ds) u_t dt + P_T (X_0 - \int_0^T u_t dt))
\]
\[
= \mathbb{E}(P_T X_0 + \int_0^T (P_t - P_T - \int_0^t G(t-s) u_s ds) u_t dt)
\]
\[
= \mathbb{E}(P_T X_0 + \int_0^T (-\xi_t - \int_0^t G(t-s) u_s ds) u_t dt)
\]

over $u \in \mathcal{U}_x^0$ i.e. we must liquidate all inventory by time $T$ (where $\xi_t := \mathbb{E}_t(P_T - P_t)$ which means $V(u)$ simplifies to

\[
V(u) = X_0 \mathbb{E}(P_T) - \mathbb{E}(\int_0^T (\xi_t + \int_0^t G(t-s) u_s ds) u_t dt)
\]
A sufficient condition for $u$ to be an optimal trading strategy is that $u$ satisfies

$$\xi_t + \mathbb{E}_t\left(\int_0^T (G(|t - \nu|) - G(T - \nu))u_\nu d\nu\right) = 0$$  \hspace{1cm} (2)$$

with $X_T = 0$.

Follows from similar arguments to Lemma 5.3 in Bank, Soner & Voß [BSV17].
Trading Gaussian signals

We now assume that $\xi_t$ is a **Gaussian Volterra process** of the form

$$\xi_t = \bar{\xi}(t) + \int_0^t K_{\xi}(u, t) dW_u$$

for some deterministic function $\bar{\xi}(t)$, where $W$ is a standard Brownian motion and $\int_0^t K_{\xi}(u, t)^2 du < \infty$ for all $t \in [0, T]$ and $\mathcal{F}_t = \mathcal{F}_t^W$. Since $\xi_T = 0$ has zero mean and zero variance, we see that

$$\bar{\xi}(T) = K_{\xi}(u, T) = 0.$$ (3)

**Theorem.** The optimal trading strategy $X^*$ is given by

$$dX^*_t = dX^0_t - \hat{u}(t) dt,$$

where $\hat{u}(t) = \bar{u}(t) + \int_0^t k(v, t) dW_v$ and $k(u, .)$ and $\bar{u}(t)$ satisfy the following **Fredholm integral equations** of the first kind:

$$-K_{\xi}(u, t) = \int_u^T k(u, \nu)(G(|t - \nu|) - G(T - \nu))d\nu$$ (4)

$$-\bar{\xi}(t) = \int_0^T (G(|t - \nu|) - G(|T - \nu|))\bar{u}(\nu)d\nu$$ (5)

where the first equation holds for each $u \in [0, T]$ fixed and all $t \in [u, T]$. 

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Here $X^0(t) = X_0 - \int_0^t u_s^0 \, ds$ is the (deterministic) solution to the same problem but with no signal which (by (2)) satisfies

$$\int_0^T (G(|t - \nu|) - G(T - \nu)) \, dX^0_\nu = 0.$$ 

$d\hat{X}(t) = -\hat{u}(t) \, dt$ by itself is the optimal solution to the round trip problem, i.e. for the case $X_0 = 0$, and we can re-write (4) as

$$-K_\xi(u, t) = \int_u^T G(|t - \nu|) k(u, \nu) \, d\nu + \lambda(u)$$

(6)

$$-\bar{\xi}(t) = \int_0^T G(|t - \nu|) \bar{u}(\nu) \, d\nu + \lambda_2$$

(7)

where the function $\lambda(u)$ and the constant $\lambda_2$ are chosen to ensure that $\mathbb{E}(X_T^2) = 0$, for which the following two conditions are necessary and sufficient

$$\int_{u_T}^T k(u, t) \, dt = 0 \quad \text{for all} \quad u \in [0, T], \quad \int_0^T \bar{u}(\nu) \, d\nu = 0.$$ 

(8)
Outline Proof. We initially set \( X_0 = 0 \) i.e. a round trip until the end of the proof, at which point we explain how to deal with the case \( X_0 \neq 0 \). Since \( \hat{u} \) has to be adapted, we guess that \( \hat{u}_t = \bar{u}(t) + \int_0^t k(v, t) dW_v \), so
\[
\mathbb{E}_t(\hat{u}_v) = \hat{u}_t + \int_0^{t\wedge v} k(u, v) dW_u.
\]
Then we see that
\[
0 = \xi_t + \mathbb{E}_t \left( \int_0^T (G(|t - v|) - G(|T - v|)) \hat{u}_v dv \right)
\]
\[
= \bar{\xi}(t) + \int_0^t K_\xi(u, t) dW_u + \int_0^T (G(|t - v|) - G(|T - v|)) \bar{u}(v) dv
\]
\[
+ \int_0^T (G(|t - v|) - G(T - v)) \int_u^t k(u, v) dW_u dv
\]
\[
= \int_0^t (K_\xi(u, t) + \int_u^T k(u, v)(G(|t - v|) - G(T - v)) dv) dW_u
\]
\[
+ \bar{\xi}(t) + \int_0^T (G(|t - v|) - G(|T - v|)) \bar{u}(v) dv.
\]
Then we see that this is zero for all \( t \in [0, T] \) a.s. if and only if (4) and (5) are satisfied for all \( u, t \) with \( 0 \leq u \leq t \leq T \). For a fixed \( u \), (4) is a Fredholm equation of the first kind for \( k(u, .) \) where we have dropped the dependence of \( u \) here to emphasize this point. We can re-write this as (6), where the function \( \lambda(u) \) is chosen to ensure that \( \mathbb{E}(X_T^2) = 0 \).
Setting $\hat{u}(t) = \bar{u}(t) + \int_0^t k(\nu, t) dW_{\nu}$ we see that

$$X_T = - \int_0^T \bar{u}(\nu) d\nu - \int_0^T \int_0^t k(\nu, t) dW_{\nu} dt$$

$$- \int_0^T \bar{u}(\nu) d\nu - \int_0^T \int_0^T k(\nu, t) dt dW_{\nu}.$$

To enforce $\mathbb{E}(X_T^2) = 0$ we see that (8) holds, and this Eq determines $\lambda(u)$ (below we will show that $\lambda(u)$ is uniquely determined + we give an explicit formula in (10)). Setting $t = T$ in (6) and using that $K_\xi(u, T) = 0$ (from (3)), we see that

$$0 = \int_u^T G(|T - \nu|)k(u, \nu)d\nu + \lambda(u)$$

so (4) is also satisfied. Similarly using that $\bar{\xi}(T) = 0$ we find that

$$\int_0^T G(|T - \nu|)\bar{u}(\nu)d\nu + \lambda_2 = 0,$$

so (5) is also satisfied. We now transform (4) so the range of integration is $[0, 1]$. To this end, we first re-write (6) in the form

$$c \int_u^T \frac{g(\nu)}{|x - \nu|^{\gamma}}d\nu = \tilde{f}(x)$$
and let \( w = \frac{v-u}{T-u} \), so \( dw = \frac{dv}{T-u} \), then we can re-write this as

\[
c(T-u) \int_0^1 \frac{g((T-u)w + u)}{|x - (T-u)w - u|^\gamma} dw
= c(T-u) \int_0^1 \frac{g_1(w)}{|x - (T-u)w - u|^\gamma} dw = \tilde{f}(x).
\]
Now let \( x - u = (T - u)x' \) and \( g_1(w) = g((T - u)w + u) \), to obtain

\[
c(T - u) \int_0^1 \frac{g_1(w)}{|(T - u)x' - (T - u)w|^{\gamma}} dw = c|T - u|^{1-\gamma} \int_0^1 \frac{g_1(w)}{|x' - w|^{\gamma}} dw
\]

We now define the \textbf{operator} \( G \) by \( Gf = \int_0^1 f(s)G(t - s)ds \), then from Example 9.2 (see also Example 6.2) in [PS90] we know that \( G = \mathcal{T}\mathcal{T}^* \), where \( \mathcal{T} \) is the Volterra-type operator defined by

\[
(\mathcal{T}\phi)(t) = \int_0^t k(s, t)\phi(s)ds
\]

and \( k(s, t) = c_{\nu}(\frac{t}{s})^{(1-\gamma)/2}(t - s)^{-\frac{1}{2}(1+\gamma)} \) for some constant \( c_{\nu} \) depending on \( \nu \), and \( \mathcal{T}^* \) is its \textbf{adjoint} given by \( (\mathcal{T}^*\phi)(t) = \int_s^T k(s, t)\phi(t)dt \) (see e.g. the start of Appendix A of [FZ17] to see why \( \mathcal{T}^* \) takes this form). Moreover from Lemma 6.7 in [PS90], we know that \( \mathcal{T} \) is a \textbf{bounded operator} on \( L^2([0, T]) \) when \( T = 1 \), but we can easily show this result also holds for any \( T > 0 \) by making the simple linear transformation \( t \mapsto tT \). \( \mathcal{T} \) is bounded on \( L^2([0, T]) \) so \( \mathcal{T}^* \) is also bounded, and thus so is \( G \).
Then
\[ k(u, t) = g(t) = \frac{1}{c|T-u|^{1-\gamma}} G_1^{-1}(-K_\xi(u, u + (T-u)(.) - \lambda(u))(\frac{t-u}{T-u}) \]

where \( G_1 \) is the same operator as the \( G \) but with \( T = 1 \). Using that \( \int_u^T k(u, t)dt = 0 \) for all \( u \in [0, T] \) and moving the \( \lambda(u) \) term to the right hand side and cancelling terms, we see that
\[ \int_u^T G_1^{-1}(-K_\xi(u, u + (T-u)(.)))(\frac{t-u}{T-u})dt = \int_u^T G_1^{-1}(\lambda(u))(\frac{t-u}{T-u})dt \]

so by the linearity of \( G^{-1}_1 \), we see that
\[ \lambda(u) = \frac{-\int_u^T G_1^{-1}(K_\xi(u, u + (T-u)(.)))(\frac{t-u}{T-u})dt}{\int_u^T G_1^{-1}(1)(\frac{t-u}{T-u})dt}. \quad (10) \]

Moreover, from Example 2.30 in \([GSS12]\), we know that
\[ G_1^{-1}(1)(s) = \frac{c_\gamma}{(s(1-s))^{(1-\gamma)/2}} \]

for some constant \( c_\gamma > 0 \). Then
\[ \int_u^T G_1^{-1}(1)(\frac{t-u}{T-u})dt = c'_\gamma(T-u) \]
for some constant $c'_\gamma$, so $\lambda(u)$ simplifies to

$$
\lambda(u) = -\frac{1}{c'_\gamma}\frac{1}{T-u} \int_u^T G_1^{-1}(K_\xi(u, u + (T - u)(.)))(\frac{t-u}{T-u})dt
$$

$$
= -\frac{1}{c'_\gamma} \int_0^1 G_1^{-1}(K_\xi(u, u + (T - u)(.)))(s)ds.
$$

We now recall that $G = G_1 = T^*T$ (when $T = 1$). Then we can further re-write $T$ as $T = M^{-1}l_\nu M$, where $M$ is the bounded operator on $L^2$ which multiplies functions by $t^{-(1-\nu)/2}$ and $l_\nu$ is the Riemann-Liouville operator $(l_\nu \phi)(t) := \int_0^t (t-s)^{-\frac{1}{2}(1+\gamma)}\phi(s)ds = \frac{1}{\Gamma(1-r)}l^r$ where $r = \frac{1}{2} - \frac{1}{2}\gamma$ and $l^r$ is the fractional integral operator of order $r$, so $l_\nu^{-1} = \Gamma(1-r)D^r$ where $D^r$ is the fractional derivative operator of order $r$. Summing this up, we can re-write (9) as

$$
TT^*g_1 = h_1
$$

for some function $h_1$, which has solution

$$
g_1 = T^{-1}(T^{-1}h_1).
$$
To compute \((T^*)^{-1}\), we note that \((\phi, T\psi) = (\phi, M^{-1}I_\nu M\psi) = (M^{-1}\phi, I_\nu M\psi) = (I_\nu^* M^{-1}\phi, M\psi) = (MI_\nu^* M^{-1}\phi, \psi)\), so \(T^* = MI_\nu^* M^{-1}\), 

+ we know how to invert \(M\) and \(I_\nu^*\). We can read off the solution more explicitly from [CG94], with \(f(x_1) = \frac{\tilde{f}(x')}{|T-u|^{1-\gamma}}\) and their \(a = b = c\), for which the explicit solution is given in Eqs 3.14a and 3.14b in [CG94] which we can re-write in our variables as

\[
\begin{align*}
  k(u, t) &= -t^{\gamma+\mu-1} \frac{\sin^2(\pi \gamma)}{\pi^2} \frac{d}{dt} \int_t^1 \frac{1}{(s-t)^\gamma} \int_0^s \frac{v^{-\gamma} h(v)}{(s-v)^{1-\gamma}} dv \\
  h(t) &= \frac{t^\alpha}{b} \frac{d}{dt} \int_0^t \frac{f(y)}{(x-y)^\alpha} dy.
\end{align*}
\]

Finally for the general case with \(X_0 \neq 0\), we can easily verify that \(X^0(t) + \hat{X}_t\) satisfies (2), i.e. we can decompose the general solution as the (deterministic) no-signal solution plus the round trip solution.
Proposition Any solution to (6) is unique. Proof. Let \( \phi(t) = k(u, t) \) and assume \( \phi \in L^2([0, T]) \) for all \( u \) and set \( \varphi(t) = \phi(t)1_{t \in [0, T]} \).

\[
\int_{[0, T]} \int_{[0, T]} \phi(s) \phi(t) G(|t - s|) ds dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t) \varphi(s) G(|t - s|) ds dt
\]

\[
= \int_{-\infty}^{\infty} \varphi(t) (\varphi * G)(t) dt
\]

\[
= \text{const.} \int_{-\infty}^{\infty} \hat{\varphi}(\xi) \hat{G}(\xi) \hat{\varphi}(t) d\xi
\]

(using Plancherel’s theorem)

\[
= \int_{-\infty}^{\infty} |\hat{\varphi}(\xi)|^2 \hat{G}(\xi) d\xi . \quad (11)
\]

But \( \phi \neq 0 \) implies that \( \hat{\varphi} \neq 0 \) and \( \hat{G}(\xi) > 0 \) (since \( G \) is positive definite), so (11) is strictly positive.
If $\bar{\xi}_t \equiv 0$, the expected profit/loss from the optimal trading strategy is

$$V(\hat{u}) = \mathbb{E}(P_T X_0) - \mathbb{E}(\int_0^T (\xi_t + \int_0^t G(t-s)\hat{u}_s ds)\hat{u}_t dt)$$

$$= \mathbb{E}(P_T X_0) - \mathbb{E}(\int_0^T \int_0^t K_{\xi}(s, t)dW_s \int_0^t k(u, t)dW_u)$$

$$- \mathbb{E}(\int_0^t G(t-s)\int_0^s k(u, s)dW_u)(\int_0^t k(v, t)dW_v)ds dt)$$

$$= \mathbb{E}(P_T X_0) - \mathbb{E}(\int_0^T \int_0^t K_{\xi}(u, t)k(u, t)dudt)$$

$$- \mathbb{E}(\int_0^T \int_0^t G(t-s)\int_0^s k(u, s)k(u, t)duds dt)$$

(12)
If we add a temporary price impact term $-\eta \dot{u}_t$ into the right hand side of (4), then (2) changes to

$$\xi_t + \eta(u_t - \mathbb{E}_t(u_T)) + \mathbb{E}_t(\int_0^T (G(|t - v|) - G(T - v))u^*_v dv) = 0$$

and we can readily verify that (4) changes to

$$K_\xi(u, t) + \eta(k(u, t) - k(u, T)) + \int_u^T k(u, v)(G(|t - v|) - G(T - v))dv = 0$$

which is now a Fredholm equation of the second kind, for $u \in [0, T]$ fixed.
The zero-signal case

For the case of power-law impact where \( G(t) = ct^{-\gamma} \) for \( \gamma \in (0, 1) \), the optimal selling speed with no-signal is given by

\[
u_0(t) = \frac{c_1}{(t(T-t))^{\frac{1}{2}(1-\gamma)}}\]

(see Example 2.30 in [GSS12]), where the constant \( c_1 \) is chosen to ensure \( X_T = 0 \) and can be computed explicitly.
At the moment our optimal selling speed is expressed as 
\[ u_t = \int_0^t k(u, t)dW_t, \]
but it is more natural and useful to re-express \( u_t \) in terms of \( P \) itself. To this end, let \( Z_t = \int_0^t g(s, t)dW_s \), and we seek a function \( h \in L^2 \) with \( h(t, t) = 0 \) such that \( W_t = \int_0^t h(u, s)dZ_s \). Then using integration by parts and stochastic Fubini we see that

\[
\int_0^t h(s, t)dZ_s = h(t, t)Z_t - \int_0^t h_s(s, t)Z_s ds
\]

\[ = - \int_0^t h_s(s, t) \int_0^s g(u, s)dW_u ds \]

\[ = - \int_0^t \int_u^t h_s(s, t)g(u, s)dsdW_u. \]

To find an inversion formula, we need to solve the integral equation

\[- \int_u^t h_s(s, t)g(u, s)ds = 1. \]
If \( g(s, t) = g(t - s) \) and we guess that \( h(s, t) = h(t - s) \), then the equation takes the special form
\[
\int_{u}^{t} h'(t - s)g(s - u)ds = 1.
\]
Setting \( \tilde{s} = s - u \), we can re-write as
\[
\int_{0}^{t-u} h'(t - (u + \tilde{s}))g(\tilde{s})d\tilde{s} = 1
\]
and replacing \( t - u \) with \( t \) we can further re-write as
\[
\int_{0}^{t} h'(t - \tilde{s})g(\tilde{s})d\tilde{s} = h * g = 1.
\]
Then taking the Laplace transform, we have
\[
\hat{(h')}\hat{g} = \lambda \hat{h} \hat{g} = \frac{1}{\lambda}
\]
so we see that
\[
\hat{h} = \frac{1}{\lambda^2 \hat{g}}. \quad (13)
\]
Then if $P_t = \int_0^t g(t-u) dW_u$ for some $g \in L^2$, then from (13) we have the inversion formula

\[
W_t = \int_0^t h(u-t) dP_u \tag{14}
\]

so

\[
u_t = \int_0^t k(u,t) d\left( \int_0^t h(u-t) dP_u \right)
\]

so we now see how $\hat{u}$ depends solely on the (unaffected) stock price history $(P_u)_{0 \leq u \leq t}$, which gives us our signal-adaptive optimal selling speed.

We can compute $\dagger g$ explicitly for the case when $g(t) = t^{H - \frac{1}{2}} e^{-\theta t}$ for $H > 0$, $\theta > 0$ (this includes the Riemann-Liouville and OU processes as special cases).
Figure: Round trip case: for a single simulation of \( W \), on the top left we have plotted the optimal buying speed \(-\hat{u}_t\) (blue) and the signal \( \xi_t \) (red), and (on the top right) we see the optimal inventory \( X_t^* \), for \( c = 1.5, \gamma = .5, T = 1 \) and
\( P_t = \frac{4}{3} t W_t \) so \( \bar{\xi}(t) = 0 \). On the bottom left we have plotted the expected profit/loss as a function of \( \alpha \) times the optimal trading speed, as a function of \( \alpha \) (which we see is maximized close to \( \alpha = 1 \), the small error is there because we have to estimate the triple integral in (12) with Monte Carlo), and on the bottom right we have plotted \( k(u, t) \).

**Figure:** Non-Round trip case: from left to right (with \( X_0 = .25 \) and the same parameters as above) we see (i) the optimal buying speed with no-signal (ii) \( X_t^* \) with no signal (iii) the optimal buying speed with non-zero signal (blue) with \( \xi_t \) (red) and the no-signal optimal speed (grey) (iv) \( X_t^* \) with non-zero signal, for the same parameters as above.


