# Sharp tail estimates for the correlated SABR model

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**Abstract**

SABR model is a stochastic volatility model that is ubiquitously used by practitioners in derivatives markets for fitting to the volatility smile. By using known results for the Brownian exponential functional given in Yor et al. [MY05], [Yor92], we compute a sharp right tail estimate for the stock price density under the SABR model with $\beta = 1$ and correlation coefficient $\rho < 0$, This extends the result in Gulisashvili&Stein [GS10] to the case of non-zero correlation and sharpens the crude tail estimate obtained in Lions&Musiela [LM07]. We demonstrate the application of our result by deriving a large-strike asymptotic expansion for the implied volatility.

**Keywords:** Asymptotics; implied volatility; SABR model; saddlepoint expansion.
**AMS Classification:** 91G20; 65C50.

## 1 Introduction

The last ten years has seen a growing strand of literature on large-strike asymptotics for call options and implied volatility in model-independent settings and for specific models, see for example, the moment formula of Lee [Lee04], and the subsequent work of Benaim&Friz[BF08],[BF09] using regular variation theory. More recently, Gulisashvili et al. [Gul10],[GS10],[GS10II] have computed asymptotic expansions for the tail behaviour of the stock price density under the uncorrelated Heston, Hull-White and Stein-Stein stochastic volatility models. [FGGS10] exploit the affine structure of the Heston moment generating function to compute a saddlepoint right tail estimate for the Heston model; more specifically, they express the density of $S_T$ as a Mellin inversion, for which the associated Mellin transform is analytic in a strip. They then shift the contour to the right (close to the where the moment generating function blows up) so it passes through the saddlepoint of the integrand in the Mellin inversion. For large $x$, the integral is then concentrated around this saddlepoint and a saddelpoint approximation is applied. For the case of non-zero correlation, large-strike asymptotics are also obtained for the Stein-Stein model in Deuschel et al.[DFJV14] using Laplace's method on Wiener space and in a very recent article, [GHJ15] have also computed small-time asymptotics for the probability of absorption at zero for the SABR model with $\beta < 1$ and small-strike, small-time asymptotics.

The SABR model is used ubiquitously on equity and interest rate derivatives trading desks. For $\beta = 1$, the model is a special case of the Hull-White model when the drift of the volatility process is zero, and it is well known that for $m > 1$, $\mathbb{E}(S_t^m) < \infty$ if and only if $m \leq \frac{1}{1-\rho^2}$ (see e.g. Theorem 2.3 in [LM07]), so in particular for $\rho = 0$, the critical moment $m^* = 1$ so in this case the model is extreme in some sense. In this article, we consider the SABR model with $\beta = 1$ and in Theorem 2.1 we compute a sharp right tail estimate for the stock density when $\rho < 0$.[1] This expansion leads immediately to an asymptotic expansions for implied volatility using a result in [Gul10].

Our approach utilizes exponential functionals of Brownian motion and contour integrals in a similar spirit to [GS10]. In particular, we rely on results in [MY05] and [Yor92] to obtain an integral representation

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[1] Note that $\rho > 0$ is uninteresting because the stock price process is no longer a martingale

for the SABR stock price density. Unlike the uncorrelated case, a two-dimensional (not one-dimensional) mixing density is needed, and we are initially faced with a triple integral. To alleviate this intractability, we exploit the well-known asymptotics of the modified Bessel functions and Laplace's method for an oscillating integral. The main result is then obtained via delicate estimates of the leading term and the tail integrals. Compared with the existing literature [GS10], we also use an Laplace expansion but at around an implicitly defined critical point. Moreover, our novel approach reduces the problem to a sharp estimate of an integral over an asymptotically small interval near zero, which is not considered in [GS10]. Finally, due to the non-zero correlation, we obtain a qualitatively different leading term in the asymptotic expansion than the uncorrelated case.

Throughout the paper, we adopt the following notations. The set of all nonnegative integers is denoted $\mathbb{N}$. For any complex number $z$, $\Im(z)$ and $\overline{z}$ denote the imaginary part and the conjugate of $z$ respectively. For a function $f(\cdot)$ and a positive function $g(\cdot)$ defined on for sufficiently large (small, resp.) $x > 0$, we write $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$ (as $x \downarrow 0$, resp.) if $\lim_{x\to\infty} |f(x)|/g(x) = 0$ ($\lim_{x\downarrow 0} |f(x)|/g(x) = 0$, resp.). We write $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$ (as $x \downarrow 0$, resp.) if $\limsup_{x\to\infty} |f(x)|/g(x) < \infty$ ($\limsup_{x\downarrow 0} |f(x)|/g(x) < \infty$, resp.). Whenever $\lim_{x\to\infty} f(x)/g(x) = 1$ ($\lim_{x\downarrow 0} f(x)/g(x) = 1$, resp.), we write $f(x) \sim g(x)$, as $x \to \infty$ (as $x \downarrow 0$, resp.) .

The remainder of the paper is structured as follows. In Section 2, we present the SABR model and state the main results of this article on density tail expansions. We also translate the expansion into a sharp asymptotic estimate for implied volatility for large-strike call options. The proofs of the main result are given in Section 3.

# 2 The model and the main result

In a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$, we consider the well known SABR stochastic volatility model for a stock or forward price process $S_t$ under a risk neutral measure when the $\beta$ parameter is 1:

$$\begin{cases} dS_t = S_t Y_t(\rho dB_t + \hat{\rho} dW_t), \quad S_0 = 1, \\ dY_t = \sigma Y_t dB_t, \quad Y_0 = y_0 > 0. \end{cases} \tag{1}$$

where $\hat{\rho} = \sqrt{1-\rho^2}$ and $W, B$ are independent standard $\mathbb{F}$-Brownian motions, and $\sigma > 0$ is known as the "volatility of volatility" or "vol of vol".

We note that the system of stochastic differential equations (1) can be solved explicitly. Indeed, from Itō's lemma we see that

$$\begin{aligned} \log S_t =& -\frac{1}{2}\int_0^t Y_s^2 ds + \frac{\rho}{\sigma}(Y_t - y_0) + \bar{\rho}\int_0^t Y_s dW_s \overset{d}{=} -\frac{1}{2}T_t + \frac{\rho}{\sigma}(Y_t - y_0) + \bar{\rho} W_{T_t}\,, \\ Y_t =& y_0 \exp(\sigma B_t - \frac{1}{2}\sigma^2 t). \end{aligned}$$

where $T_t = \int_0^t Y_s^2 ds$. Conditioning on $T_t$ and $Y_t$, we see that $\log S_t$ follows a normal distribution with mean $-\frac{1}{2}T_t + \frac{\rho}{\sigma}(Y_t - y_0)$ and variance $\hat{\rho}^2 T_t$. In other words, if we let $p_t(x)$ denote the density of $X_t$, i.e., $\mathbb{P}(S_t \in dx) = p_t(x)dx$, then we have

$$p_t(x) \quad = \quad \mathbb{E}(\frac{1}{x T_t^{\frac{1}{2}} \hat{\rho}\sqrt{2\pi}}\, e^{-[\log x + \frac{1}{2}T_t - \frac{\rho}{\sigma}(Y_t - y_0)]^2/2T_t\hat{\rho}^2}), \qquad \forall x > 0. \tag{2}$$

As seen from (2), one has to compute an expectation that involves $(T_t, Y_t)$ before one can evaluate the transition density of $S_t$ or price options on $S_t$, which turns out to be a non-trivial issue. Monte Carlo simulation can be used to estimate the density for moderate values of $x$, but an analytical characterization of the density $p_t(x)$ is often favorable in extreme regimes. [GS10] consider the case $\rho = 0$ and derive a sharp asymptotic estimate for the tail density $p_t(x)$ when $x$ is large. The methodology used there is based on properties of exponential functionals of Brownian motion $T_t$. A sharp estimate of $p_t(x)$ is obtained by analyzing the asymptotic behaviour of a certain contour integral. The main contribution of this paper is then to generalize the result in [GS10] to correlated cases. In particular, for a fixed $\rho \in (-1, 0)$, we obtain an asymptotic expansion of $p_t(x)$ in large $x$, fixed $t$ regime.

**Theorem 2.1.** *If $\rho \in (-1,0)$, then as $x \to \infty$,*

$$\begin{aligned} p_t(x) =& \frac{y_0}{2\sigma\sqrt{\pi}} \frac{e^{-\frac{\rho}{\hat{\rho}^2}\frac{y_0}{\sigma} - \frac{\sigma^2 t}{8} - \frac{\log^2 2}{2\sigma^2 t}}}{C(x)^{\frac{3}{2}}} x^{-1-\frac{1}{\hat{\rho}^2}} [\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0} D(x)]^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}} [\log(\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0} D(x))]^{-\frac{1}{2}} \\ & \times \exp(-\frac{1}{2\sigma^2 t}[\log(\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0} D(x))]^2)(1 + \mathcal{O}([\log(|\rho|\sigma D(x)/\hat{\rho}^2 y_0]^{-\frac{1}{2}})). \end{aligned} \tag{3}$$

*where*

$$C(x) := \sqrt{(\log x + \frac{y_0}{\sigma}\rho)^2 + \frac{y_0^2}{\sigma^2}\hat{\rho}^2} > 0, \quad D(x) := \sqrt{(\log x + \frac{y_0}{\sigma}\rho)^2 + \frac{y_0^2}{\sigma^2}\hat{\rho}^2} + \log x + \frac{\rho y_0}{\sigma} > 0. \tag{4}$$

**Remark 2.1.** *Using the facts that $C(x) = \log x + \frac{\rho y_0}{\sigma} + \mathcal{O}(\frac{1}{\log x})$ and $D(x) = 2\log x + \frac{2\rho y_0}{\sigma} + \mathcal{O}(\frac{1}{\log x})$ as $x \to \infty$, we obtain from Theorem 2.1 that*

$$\begin{aligned} p_t(x) =& \frac{|\rho|}{\hat{\rho}^2} \frac{e^{-\frac{\rho}{\hat{\rho}^2}\frac{y_0}{\sigma} - \frac{\sigma^2 t}{8} - \frac{\log^2 2}{2\sigma^2 t}}}{\sigma\sqrt{\pi}} (\frac{\hat{\rho}^2}{2|\rho|}\frac{y_0}{\sigma})^{\frac{1}{2}+\frac{\log 2}{\sigma^2 t}} \cdot x^{-1-\frac{1}{\hat{\rho}^2}} [\log x]^{-1-\frac{\log 2}{\sigma^2 t}} [\log\log x]^{-\frac{1}{2}} \\ & \times \exp(-\frac{1}{2\sigma^2 t}[\log(\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0} D(x))]^2)(1 + \mathcal{O}([\log\log x]^{-\frac{1}{2}})). \end{aligned}$$

*From this explicit asymptotic formula we know that, for $m > 1$, the $m$-th moment for of $S$ is finite if and only if $m$ and $\rho$ satisfies $m \leq \frac{1}{1-\rho^2}$. This is consistent with Theorem 2.3 in Lions&Musiela[LM07].*

**Remark 2.2.** *We comment that the result for $\rho \in (-1,0)$ in Theorem 2.1 is qualitative different from that for $\rho = 0$ appeared in Gulisashvili&Stein [GS10]. That is, the asymptotic tail expansion is not left continuous in $\rho$ at 0.*

## 2.1 Asymptotic behaviour of the Implied volatility

In this section, we convert the asymptotic expansion we obtained in Theorem 2.1 into asymptotic expansion of the implied volatility for large strikes. To that end, we notice that for any fixed $\rho \in (-1, 0)$, equation (61) in [Gul10] is satisfied, i.e. $\mathbb{E}(X_t^{1+\varepsilon}) < \infty$ for $\varepsilon$ sufficiently small, so taking the log of the density expansion in Theorem 2.1 and setting $\log x \mapsto x$, we see $p_t(x)$ satisfies Theorem 5.4, part 3. in [Gul10] with

$$h(x) = const. - \frac{1}{\hat{\rho}^2}x + (-1 - \frac{\log 2}{\sigma^2 t})\log x - \frac{1}{2}\log\log x - \frac{1}{2\sigma^2 t}[\log(\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0}\hat{D}(x))]^2$$

where $\hat{D}(x) = D(e^x)$, and we can readily verify that $h(x) \in R_1$ (we say that $h \in R_\alpha$ if $\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = \lambda^\alpha$), because all terms except the $-(1 + \frac{1}{\hat{\rho}^2})x$ here are slowly varying at infinity (i.e. are in $R_0$). Thus we have the following asymptotic behaviour for the implied volatility $I(K)$ at strike $K$:

**Corollary 2.1.** *As $K \to \infty$, for any fixed maturity $T > 0$, we have*

$$I(K)^2 \sim \frac{\log K}{T}\,\psi(-\frac{\log(K^2 p_t(K))}{\log K})$$

*where $\psi(u) := 2 - 4(\sqrt{u^2 + u} - u)$.*

# 3 Proof of Theorem 2.1

We present the proofs for Theorem 2.1 in this section. To that end, we first express the density (2) using results on exponential functionals of Brownian motion, and then estimate an associated integral using Laplace approximation-like estimation with the help of contour integrals over the complex plane.

### 3.1 Brownian functionals of Brownian motion and SABR density

Let us denote by $A_t = \int_0^t e^{2B_s - t} ds$, where $B$ is a standard Brownian motion starting at 0. Recall from Theorem 4.1 of Matsumoto&Yor[MY05] that, for $a, y > 0$, we have

$$\mathbb{P}(A_t \in da, e^{B_t - \frac{1}{2}t} \in dy) = \frac{e^{\frac{\pi^2}{2t} - \frac{t}{8}}}{\sqrt{2\pi^3 t}} \frac{1}{a^2\sqrt{y}} \exp(-\frac{1+y^2}{2a}) \psi_{\frac{y}{a}}(t) da dy \,, \tag{5}$$

where

$$\psi_r(t) = \int_0^\infty e^{-\frac{z^2}{2t} - r\cosh z} \sinh z \sin \frac{\pi z}{t} dz \;\; \forall r, t > 0 \,.$$

Recall that $T_t = \int_0^t \exp(-\sigma^2 s + 2\sigma B_s) ds = \int_0^t Y_s^2 ds$. From the usual space-time scaling properties of Brownian motion, we immediately have

$$(T_t, Y_t) \;\; \stackrel{d}{=} \;\; (\frac{y_0^2}{\sigma^2} A_{\sigma^2 t} \,,\, y_0 e^{B_{\sigma^2 t} - \frac{1}{2}\sigma^2 t}) \tag{6}$$

Thus, by setting $\hat{T}_t := \frac{\sigma^2}{y_0^2} T_t, \hat{Y}_t := \frac{Y_t}{y_0}$, we obtain from (5) and (6) that

$$\begin{aligned}\mathbb{P}(\hat{T}_t \in da, \hat{Y}_t \in dy) =& \mathbb{P}(A_{\sigma^2 t} \in da, e^{B_{\sigma^2 t} - \frac{1}{2}\sigma^2 t} \in dy) \\ =& \frac{e^{\frac{\pi^2}{2\sigma^2 t} - \frac{\sigma^2 t}{8}}}{\sigma\sqrt{2\pi^3 t}} \frac{1}{a^2\sqrt{y}} \exp(-\frac{1+y^2}{2a}) \psi_{\frac{y}{a}}(\sigma^2 t) da dy\end{aligned} \tag{7}$$

Returning to the original expectation in (2), and using $(T_t, Y_T) = (\frac{y_0^2}{\sigma^2}\hat{T}_t, y_0 \hat{Y}_t)$ and (7) we have

$$\begin{aligned}p_t(x) =& \mathbb{E}(\frac{\sigma}{x y_0 \hat{T}_t^{\frac{1}{2}} \hat{\rho}\sqrt{2\pi}} \, e^{-[\log x + \frac{1}{2}\frac{y_0^2}{\sigma^2}\hat{T}_t - \rho\frac{y_0}{\sigma}(\hat{Y}_t - 1)]^2 / 2(\frac{y_0^2}{\sigma^2}\hat{T}_t)\hat{\rho}^2}) \\ =& \frac{e^{\frac{\pi^2}{2\sigma^2 t} - \frac{\sigma^2 t}{8} - \frac{\rho}{2\hat{\rho}^2}\frac{y_0}{\sigma}}}{2\pi^2 \hat{\rho}\sqrt{t} y_0 \, x^{1+\frac{1}{2\hat{\rho}^2}}} \int_0^\infty \int_0^\infty \frac{1}{\sqrt{ay}} g(a,y) \psi_{\frac{y}{a}}(\sigma^2 t) \frac{da}{a^2} dy,\end{aligned}$$

where

$$g(a,y) := \exp(-\frac{\log^2 x}{2\frac{y_0^2}{\sigma^2}\hat{\rho}^2 a} - \frac{1}{8}\frac{y_0^2}{\sigma^2}\frac{a}{\hat{\rho}^2} - \frac{\rho^2}{2\hat{\rho}^2}\frac{(y-1)^2}{a} + \frac{\rho}{\frac{y_0}{\sigma}\hat{\rho}^2}\frac{y-1}{a}\log x + \frac{\rho}{2\hat{\rho}^2}\frac{y_0}{\sigma} y - \frac{1+y^2}{2a}), \;\; \forall a, y > 0.$$

It will be more convenient for asymptotic analysis to express $p_t(x)$ using a "single" integral instead. To that end, we make the change of variables $u/(\sigma^2 t) = y/a, w = 1/y$, so $a = \sigma^2 t/(wu), y = 1/w$. Then one gets

$$\begin{aligned}p_t(x) =& \frac{e^{\frac{\pi^2}{2\sigma^2 t} - \frac{\sigma^2 t}{8} - \frac{\rho}{2\hat{\rho}^2}\frac{y_0}{\sigma}}}{2\pi^2 \hat{\rho}\sqrt{t}\, y_0 x^{1+\frac{1}{2\hat{\rho}^2}}} \int_0^\infty \int_0^\infty \sqrt{\frac{u}{\sigma^2 t}} g(\sigma^2 t (wu)^{-1}, w^{-1}) \psi_{\frac{u}{\sigma^2 t}}(\sigma^2 t) \sqrt{\frac{u}{\sigma^2 t}} \frac{du}{\sigma^2 t} dw \\ =& \frac{e^{\frac{\pi^2}{2\sigma^2 t} - \frac{\sigma^2 t}{8} - \frac{\rho}{2\hat{\rho}^2}\frac{y_0}{\sigma}}}{2\pi^2 \hat{\rho}\sqrt{t}\, y_0 x^{1+\frac{1}{2\hat{\rho}^2}}} \int_0^\infty \frac{du}{\sigma^2 t} \sqrt{\frac{u}{\sigma^2 t}} \exp(\frac{\rho^2}{\hat{\rho}^2}\frac{u}{\sigma^2 t} + \frac{\sigma}{y_0}\frac{\rho}{\hat{\rho}^2}\frac{u}{\sigma^2 t}\log x) \psi_{\frac{u}{\sigma^2 t}}(\sigma^2 t) \\ &\times \int_0^\infty \exp(-(\frac{\sigma^2}{y_0^2}\log^2 x + 2\frac{\rho\sigma}{y_0}\log x + 1)\frac{uw}{2\sigma^2 t \hat{\rho}^2} - (\frac{y_0^2}{\sigma^2} - 4\frac{\rho y_0}{\sigma}\frac{u}{\sigma^2 t} + 4(\frac{u}{\sigma^2 t})^2)\frac{\sigma^2 t}{8\hat{\rho}^2 uw}) dw \,.\end{aligned} \tag{8}$$

For the integral in $w$ in (8), we recall that

$$\int_0^\infty \exp(-\frac{\beta}{4w} - \gamma w) dw = \sqrt{\frac{\beta}{\gamma}} K_1(\sqrt{\beta\gamma}), \forall \gamma, \beta > 0, \tag{9}$$

where $K_1(.)$ is the modified Bessel function of the second kind of order 1. By setting $\beta = (\frac{y_0^2}{\sigma^2} - 4\frac{\rho y_0}{\sigma}\frac{u}{\sigma^2 t} + 4(\frac{u}{\sigma^2 t})^2)\sigma^2 t/(2u\hat{\rho}^2)$, $\gamma = (\frac{\sigma^2}{y_0^2}\log^2 x + 2\frac{\rho\sigma}{y_0}\log x + 1)u/(2\sigma^2 t\hat{\rho}^2)$ and

$$C(x) = \hat{\rho}\frac{y_0}{\sigma}\sqrt{\frac{2\sigma^2 t\gamma}{u}} \equiv \sqrt{(\log x + \frac{y_0}{\sigma}\rho)^2 + \frac{y_0^2}{\sigma^2}\hat{\rho}^2} > 0, \tag{10}$$

we obtain from (8) and (9) that

$$\begin{aligned} p_t(x) =& \frac{y_0}{\sigma^2}\frac{e^{\frac{\pi^2}{2\sigma^2 t} - \frac{\sigma^2 t}{8} - \frac{\rho}{2\hat{\rho}^2}\frac{y_0}{\sigma}}}{2\pi^2\hat{\rho}\sqrt{t}}\frac{x^{-1-\frac{1}{2\hat{\rho}^2}}}{C(x)}\int_0^\infty \sqrt{\frac{1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}}{\frac{u}{\sigma^2 t}}}\exp(\frac{\rho^2}{\hat{\rho}^2\sigma^2}\frac{u}{t} + \frac{1}{y_0\sigma}\frac{\rho}{\hat{\rho}^2}\frac{u}{t}\log x)\psi_{\frac{u}{\sigma^2 t}}(\sigma^2 t) \\ & \times K_1(\frac{C(x)}{2\hat{\rho}^2}\sqrt{1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}})\frac{du}{\sigma^2 t}. \end{aligned} \tag{11}$$

### 3.2 Asymptotic expansion of the Bessel function $K_1$ and the SABR density

We begin the sharp asymptotic expansion of $p_t(x)$ by first recalling that for sufficiently large $z > 0$, $K_1(z) = \sqrt{\frac{\pi}{2z}}e^{-z}(1 + \mathcal{O}(z^{-1}))$. Also notice that $1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2} \geq 1$ for all $u \geq 0$ and $\rho \leq 0$. Because $C(x) \to \infty$ as $x \to \infty$, we have,

$$\begin{aligned} & K_1(\frac{C(x)}{2\hat{\rho}^2}\sqrt{1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}}) \\ =& \hat{\rho}\sqrt{\frac{\pi}{\sqrt{1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}}\,C(x)}}\exp(-\frac{C(x)}{2\hat{\rho}^2}\sqrt{1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}})(1 + \mathcal{O}(C(x)^{-1})). \end{aligned}$$

As will be seen later, when a large $x > 0$ is fixed, only the behavior of the above expression for small $u > 0$ is relevant to us. By rationalizing the numerator, we obtain that

$$\sqrt{1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}} = 1 + (\sqrt{1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}} - 1) = 1 + \frac{4(\frac{1}{y_0^2\sigma^2}\frac{u^2}{t^2} - \frac{\rho}{y_0\sigma}\frac{u}{t})}{\sqrt{1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}} + 1}.$$

It follows that

$$\begin{aligned} & K_1(\frac{C(x)}{2\hat{\rho}^2}\sqrt{1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}}) \\ =& \frac{\exp(-\frac{C(x)}{2\hat{\rho}^2})}{C(x)^{\frac{1}{2}}}\frac{\hat{\rho}\sqrt{\pi}}{(1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2})^{\frac{1}{4}}}\exp(-\frac{C(x)}{2\hat{\rho}^2}\frac{(\frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2} - \frac{4\rho}{y_0\sigma}\frac{u}{t})}{\sqrt{1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}} + 1})(1 + \mathcal{O}(C(x)^{-1})) \\ =& \frac{x^{-\frac{1}{2\hat{\rho}^2}}e^{-\frac{\rho}{2\hat{\rho}^2}\frac{y_0}{\sigma}}}{C(x)^{\frac{1}{2}}}\frac{\hat{\rho}\sqrt{\pi}}{(1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2})^{\frac{1}{4}}}\exp(-\frac{2C(x)}{\hat{\rho}^2}\frac{(\frac{1}{y_0^2\sigma^2}\frac{u^2}{t^2} - \frac{\rho}{y_0\sigma}\frac{u}{t})}{\sqrt{1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}} + 1})(1 + \mathcal{O}([\log x]^{-1})), \end{aligned} \tag{12}$$

uniformly for all $u \geq 0$, where we have we used that $C(x) = \log x + \frac{y_0}{\sigma}\rho + \mathcal{O}(\frac{1}{\log x})$ for the final equality, and that $\exp(-\frac{C(x)}{2\hat{\rho}^2}) = x^{-\frac{1}{2\hat{\rho}^2}}e^{-\frac{\rho}{2\hat{\rho}^2}\frac{y_0}{\sigma}}(1 + \mathcal{O}([\log x]^{-1}))$.

Substituting (12) into the density $p_t(x)$ in (11), we obtain the following result.

**Proposition 3.1.** *As $x \to \infty$, we have*

$$p_t(x) = q_t(x) \cdot (1 + \mathcal{O}([\log x]^{-1})),$$

*where the leading term $q_t(x)$ is given as*

$$q_t(x) := \frac{y_0}{2\sigma^2\sqrt{\pi^3 t}} \frac{x^{-1-\frac{1}{\hat\rho^2}} e^{-\frac{\rho}{\hat\rho^2}\frac{y_0}{\sigma}+\frac{\pi^2}{2\sigma^2 t}-\frac{\sigma^2 t}{8}}}{C(x)^{\frac{3}{2}}} \int_0^\infty f^{(\rho)}(u,x)\psi_{\frac{u}{\sigma^2 t}}(\sigma^2 t)\frac{du}{\sigma^2 t}, \tag{13}$$

*with*

$$f^{(\rho)}(u,x) := \frac{(1-\frac{4\rho}{y_0\sigma}\frac{u}{t}+\frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2})^{\frac{1}{4}}}{(\frac{u}{\sigma^2 t})^{\frac{1}{2}}} \exp(\frac{\rho^2}{\hat\rho^2\sigma^2}\frac{u}{t}+\frac{\log x}{y_0\sigma}\frac{\rho}{\hat\rho^2}\frac{u}{t}-\frac{2C(x)}{\hat\rho^2}\frac{(\frac{1}{y_0^2\sigma^2}\frac{u^2}{t^2}-\frac{\rho}{y_0\sigma}\frac{u}{t})}{\sqrt{1-\frac{4\rho}{y_0\sigma}\frac{u}{t}+\frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}}+1}).$$

**Remark 3.1.** *We point out that the integral in* (13) *is finite for all $x > 0$. Indeed, notice that for any fixed $u > 0$,*

$$0 \le |\psi_{\frac{u}{\sigma^2 t}}(\sigma^2 t)| \le \int_0^\infty e^{-\frac{z^2}{2\sigma^2 t}-\frac{u}{\sigma^2 t}\cosh(z)}\sinh(z)dz \le \int_0^\infty e^{-\frac{u}{\sigma^2 t}\cosh(z)}d\cosh(z) = \int_1^\infty e^{-\frac{u}{\sigma^2 t}w}dw \le \frac{\sigma^2 t}{u}. \tag{14}$$

*So the exponential decay of $f^{(\rho)}$ for large $u$ makes the integral in* (13) *converge at $\infty$. Moreover, from [Yor92] we know that*

$$\psi_{\frac{u}{\sigma^2 t}}(\sigma^2 t) = \mathcal{O}(u^k), \quad \text{as } u \downarrow 0, \quad \text{for any } k \in \mathbb{N}.$$

*Thus, the integral in* (13) *converges at $0+$. The remaining task is thus to obtain sharp asymptotic estimate of the integral in* (13) *for large $x > 0$.*

In light of Proposition 3.1, the asymptotic expansion of $p_t(x)$ for large $x$ is now reduced to that of $q_t(x)$. More specifically, sharp estimate of (13) for large $x > 0$ will be essential to prove our main result. Similar to [GS10], we will show that for $x$ large, the main contribution to the integral in (13) comes from small values of $u$. However, due to the more complicated integrand (than that in Theorem 5.1 of [GS10]) resulted from correlation $\rho$, our estimate is technically more involved. In particular, the $u$-value that maximizes the exponential exponent in (13) is no longer available explicitly, and the standard Laplace approximation argument does not work anymore. To alleviate these inconveniences, we split the integral over $(0,\infty)$ into two parts, namely,

$$I_1 := \int_0^{C(x)^{-\frac{1}{3}}} f^{(\rho)}(u,x)\psi_{\frac{u}{\sigma^2 t}}(\sigma^2 t)\frac{du}{\sigma^2 t}, \quad I_2 := \int_{C(x)^{-\frac{1}{3}}}^\infty f^{(\rho)}(u,x)\psi_{\frac{u}{\sigma^2 t}}(\sigma^2 t)\frac{du}{\sigma^2 t}. \tag{15}$$

Then we can re-write (13) as

$$q_t(x) = \frac{y_0}{2\sigma^2\sqrt{\pi^3 t}} \frac{x^{-1-\frac{1}{\hat\rho^2}} e^{-\frac{\rho}{\hat\rho^2}\frac{y_0}{\sigma}+\frac{\pi^2}{2\sigma^2 t}-\frac{\sigma^2 t}{8}}}{C(x)^{\frac{3}{2}}} \times (I_1 + I_2). \tag{16}$$

In the sequel we separately derive sharp asymptotic estimates of $I_1$ and $I_2$.

## 3.3 Estimate of $I_2$

Before studying the main term $I_1$, we will first derive a upper bound for $|I_2|$. Indeed, using (14) and the inequality $\sqrt{1-\frac{4\rho}{y_0\sigma}\frac{u}{t}+\frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}} < \frac{2}{y_0\sigma}\frac{u}{t}+1$ for all $u \ge 0$ (because $\rho \in (-1,0]$), we have

$$\begin{aligned}
|I_2| <& \int_{C(x)^{-\frac{1}{3}}}^\infty \sqrt{\frac{\frac{2}{y_0\sigma}\frac{u}{t}+1}{\frac{u}{\sigma^2 t}}} \exp(\frac{\rho^2}{\hat\rho^2\sigma^2}\frac{u}{t}-\frac{2C(x)}{\hat\rho^2}\frac{\frac{1}{y_0^2\sigma^2}\frac{u^2}{t^2}}{\frac{2u}{y_0\sigma t}+1+1})\frac{\sigma^2 t}{u}\frac{du}{\sigma^2 t} \\
\le& C(x)^{\frac{1}{3}}\sqrt{\frac{2\sigma}{y_0}+\sigma^2 t C(x)^{\frac{1}{3}}} \int_{C(x)^{-\frac{1}{3}}}^\infty \exp(\frac{\rho^2}{\hat\rho^2\sigma^2}\frac{u}{t}-\frac{C(x)}{\hat\rho^2}\frac{\frac{1}{y_0^2\sigma^2}\frac{u^2}{t^2}}{\frac{u}{y_0\sigma t}+1})du \\
=& C(x)^{\frac{1}{3}}\sqrt{\frac{2\sigma}{y_0}+\sigma^2 t C(x)^{\frac{1}{3}}} \int_{C(x)^{-\frac{1}{3}}}^\infty \exp(\frac{1}{y_0\sigma\hat\rho^2}\frac{u}{t}(\frac{y_0}{\sigma}\rho^2-\frac{C(x)}{1+y_0\sigma\frac{t}{u}}))du \\
\le& C(x)^{\frac{1}{3}}\sqrt{\frac{2\sigma}{y_0}+\sigma^2 t C(x)^{\frac{1}{3}}} \int_{C(x)^{-\frac{1}{3}}}^\infty \exp(\frac{1}{y_0\sigma\hat\rho^2}\frac{u}{t}(\frac{y_0}{\sigma}\rho^2-\frac{C(x)}{1+y_0\sigma t C(x)^{\frac{1}{3}}}))du
\end{aligned}$$

On the other hand, by the fact that $C(x) \to \infty$ as $x \to \infty$, we see that $0 \le \frac{2\sigma}{y_0} + \sigma^2 t C(x)^{\frac{1}{3}} < 2\sigma^2 t C(x)^{\frac{1}{3}}$ and $\frac{y_0}{\sigma}\rho^2 - \frac{C(x)}{1+y_0\sigma t\, C(x)^{\frac{1}{3}}} \le -\frac{C(x)}{2(1+y_0\sigma t\, C(x)^{\frac{1}{3}})} < -\frac{C(x)^{\frac{2}{3}}}{4y_0\sigma t} < 0$ for $x > 0$ sufficiently large, so we have, as $x \to \infty$,

$$\begin{aligned}|I_2| \le& C(x)^{\frac{1}{3}}\sqrt{2\sigma^2 t C(x)^{\frac{1}{3}}}\int_{C(x)^{-\frac{1}{3}}}^{\infty} \exp(-\frac{C(x)^{\frac{2}{3}}}{4y_0\sigma t}\frac{1}{y_0\sigma\hat{\rho}^2}\frac{u}{t})du\\ =& 4\sqrt{2\sigma^2 t}\; C(x)^{-\frac{1}{6}} y_0^2\sigma^2 t\hat{\rho}^2 t\, \exp(-\frac{C(x)^{\frac{1}{3}}}{4y_0^2\sigma^2\hat{\rho}^2 t}).\end{aligned}$$

Thanks to the exponential factor, from $C(x) \to \infty$ as $x \to \infty$ we have proved the following result.

**Proposition 3.2.** *As $x \to \infty$, for any $k \in \mathbb{N}$, we have*

$$|I_2| \le \mathcal{O}(\exp(-\frac{C(x)^{\frac{1}{3}}}{4y_0^2\sigma^2\hat{\rho}^2 t^2})) = \mathcal{O}(C(x)^k).$$

## 3.4 Estimate of $I_1$

We now proceed to develop an asymptotic expansion of integral $I_1$ for large $x > 0$, whose leading term will be shown to have a polynomial order of $C(x)$. Hence, $I_2$ is negligible when compared with $I_1$. To estimate $I_1$, an accurate asymptotic expansion of function $\psi_{\frac{u}{s}}(s)$ for small $u$ is crucial to our purpose.

**Proposition 3.3.** *As $u \downarrow 0$, for any fixed $s > 0$ we have*

$$\begin{aligned}\psi_{\frac{u}{s}}(s) = \sqrt{\frac{\pi s}{2}} e^{-\frac{\log^2 2}{2s} - \frac{\pi^2}{2s}} u^{-1-\frac{1}{s}+\frac{\log 2}{s}} [\log(1/u)]^{\frac{1}{2}-\frac{\log 2}{s}} \exp(-\frac{1}{2s}(\log(1/u) + \log\log(1/u))^2)\\ \times (1 + \mathcal{O}([\log(1/u)]^{-\frac{1}{2}})).\end{aligned}$$

*Proof.* See Appendix A. ☐

Clearly Proposition 3.3 also provides an asymptotic estimate for $\psi_{\frac{u}{\sigma^2 t}}(\sigma^2 t)$ for small $u$, if we replace $s$ by $\sigma^2 t$. One can exploit this result to obtain an asymptotic expansion for $q_t(x)$ for large $x$. To that end, fix a large $x > 0$. Then as $u \downarrow 0$, we have that $(1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2})^{\frac{1}{4}}(1 + \mathcal{O}([\log(1/u)]^{-\frac{1}{2}})) = 1 + \mathcal{O}([\log(1/u)]^{-\frac{1}{2}})$. As a consequence, for $u > 0$ sufficiently small,

$$\begin{aligned}\exp(-\frac{2C(x)}{\hat{\rho}^2}\frac{(\frac{1}{y_0^2\sigma^2}\frac{u^2}{t^2} - \frac{\rho}{y_0\sigma}\frac{u}{t})}{\sqrt{1 - \frac{4\rho}{y_0\sigma}\frac{u}{t} + \frac{4}{y_0^2\sigma^2}\frac{u^2}{t^2}} + 1}) &= \exp((\frac{\rho}{y_0\sigma\hat{\rho}^2}\frac{u}{t} - \frac{1}{y_0^2\sigma^2}\frac{u^2}{t^2} + \mathcal{O}(u^3))C(x))\\ &= \exp((\frac{\rho}{y_0\sigma\hat{\rho}^2}\frac{u}{t} - \frac{1}{y_0^2\sigma^2}\frac{u^2}{t^2})C(x))(1 + \mathcal{O}(u^3 C(x))).\end{aligned}$$

Thus, from Proposition 3.3 and (15) we obtain that,

$$\begin{aligned}I_1 =& \sqrt{\frac{\pi\sigma^2 t}{2}} e^{-\frac{\log^2 2}{2\sigma^2 t} - \frac{\pi^2}{2\sigma^2 t}} \int_0^{C(x)^{-\frac{1}{3}}} \frac{1}{\sqrt{\sigma^2 t}} \exp(\frac{\rho^2}{\hat{\rho}^2\sigma^2}\frac{u}{t} + \frac{\log x}{y_0\sigma}\frac{\rho}{\hat{\rho}^2}\frac{u}{t} + (\frac{\rho}{y_0\sigma\hat{\rho}^2}\frac{u}{t} - \frac{1}{y_0^2\sigma^2}\frac{u^2}{t^2})C(x))\\ &\times u^{-\frac{3}{2}-\frac{1}{\sigma^2 t}+\frac{\log 2}{\sigma^2 t}} [\log(1/u)]^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}} \exp(-\frac{[\log(1/u) + \log\log(1/u)]^2}{2\sigma^2 t})(1 + \mathcal{O}([\log(1/u)]^{-\frac{1}{2}} + u^3 C(x)))du\\ =& \sqrt{\frac{\pi}{2}} e^{-\frac{\log^2 2}{2\sigma^2 t} - \frac{\pi^2}{2\sigma^2 t}} \int_0^{C(x)^{-\frac{1}{3}}} u^{-\frac{3}{2}-\frac{1}{\sigma^2 t}+\frac{\log 2}{\sigma^2 t}} [\log(1/u)]^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}} \exp(-\frac{[\log(1/u) + \log\log(1/u)]^2}{2\sigma^2 t})\\ &\times \exp(\frac{\rho}{\hat{\rho}^2}\frac{D(x)}{y_0\sigma}\frac{u}{t} - \frac{C(x)}{y_0^2\sigma^2}\frac{u^2}{t^2})(1 + \mathcal{O}([\log(1/u)]^{-\frac{1}{2}}) + \mathcal{O}(u^3 C(x)))du, \qquad (17)\end{aligned}$$

where $D(x)$ is given in (4).

We notice that the last exponential factor in (17) monotonically decreases in $u$ over $[0,\infty)$, while the exponential factor $\exp(-\frac{1}{2\sigma^2 t}[\log(1/u)+\log\log(1/u)]^2)$ is vanishing at 0. We thus look for a critical point to balance off the leading factors in (17). While the situations will be quite different for $\rho=0$ versus $\rho\in(-1,0)$, we will only focus on the later case here to avoid any repetition of estimates in [GS10].

### 3.4.1 The case $-1<\rho<0$

In this case, we denote by $u_x$ the unique value that maximizes the leading order terms $-\frac{1}{2\sigma^2 t}[\log(1/u)]^2+\frac{\rho}{\hat\rho^2}\frac{D(x)}{y_0\sigma}\frac{u}{t}$ in (17) (note that $u^2$ is $\mathcal{O}(u)$). Differentiating this expression with respect to $u$, we find that $u_x$ is the unique positive solution to the following equation:

$$\frac{\rho}{\hat\rho^2}\frac{\sigma}{y_0}D(x)=\frac{\log u_x}{u_x},\ \text{ or equivalently }\ \exp(\frac{\rho}{\hat\rho^2}\frac{D(x)}{y_0\sigma}\frac{u_x}{t})=u_x^{\frac{1}{\sigma^2 t}}. \tag{18}$$

Some useful asymptotic properties of $u_x$ are presented in Lemma 3.1 below.

**Lemma 3.1.** *As $x\to\infty$, for $u_x$ defined in* (18)*, we have,*

$$\log(1/u_x)+\log\log(1/u_x)=\log(\frac{|\rho|}{\hat\rho^2}\frac{\sigma}{y_0}D(x)), \tag{19}$$

$$\log(1/u_x)=\log(\frac{|\rho|}{\hat\rho^2}\frac{\sigma}{y_0}D(x))(1+\mathcal{O}(\frac{\log\log(|\rho|\sigma D(x)/\hat\rho^2 y_0)}{\log(|\rho|\sigma D(x)/\hat\rho^2 y_0)})), \tag{20}$$

$$C(x)u_x^2=\mathcal{O}(\frac{[\log(|\rho|\sigma D(x)/\hat\rho^2 y_0)]^2}{D(x)}). \tag{21}$$

*Proof.* See Appendix B. □

From (20), we see that for $x>0$ sufficiently large, we have $\log(1/u_x)\geq\frac{1}{2}\log(|\rho|\sigma D(x)/\hat\rho^2 y_0)$, so equivalently, $u_x\leq(|\rho|\sigma D(x)/\hat\rho^2 y_0)^{-\frac{1}{2}}$. It follows that, for large $x$ such that $(|\rho|\sigma D(x)/\hat\rho^2 y_0)^{-\frac{1}{2}}<C(x)^{-\frac{1}{3}}$, the critical value $u_x$ defined in (18) will stay inside the integral domain of $I_1$, $(0,C(x)^{-\frac{1}{3}})$.

We now continue from (17) to obtain

$$\begin{aligned}I_1=&\sqrt{\frac{\pi}{2}}e^{-\frac{\log^2 2}{2\sigma^2 t}-\frac{\pi^2}{2\sigma^2 t}}\int_0^{C(x)^{-\frac{1}{3}}}u^{-\frac{3}{2}-\frac{1}{\sigma^2 t}+\frac{\log 2}{\sigma^2 t}}[\log(1/u)]^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}}\exp(-\frac{[\log(1/u_x)+\log\log(1/u_x)]^2}{2\sigma^2 t})\\&\times\exp(\frac{|\rho|}{\hat\rho^2}\frac{\sigma}{y_0}D(x))(1+\mathcal{O}([\log(1/u)]^{-\frac{1}{2}})+\mathcal{O}(u^2C(x)))du.\end{aligned} \tag{22}$$

Let us denote by $g^{(\rho)}(u,x)$ the integrand (without error terms) in (22). That is,

$$g^{(\rho)}(u,x)=u^{-\frac{3}{2}-\frac{1}{\sigma^2 t}+\frac{\log 2}{\sigma^2 t}}[\log(1/u)]^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}}\exp(-\frac{[\log(1/u_x)+\log\log(1/u_x)]^2}{2\sigma^2 t}+\frac{|\rho|}{\hat\rho^2}\frac{\sigma}{y_0}D(x)).$$

The quantity $u_x$ defined in (18) will serve as a saddle point in our estimate. And we will show that the main contribution to (22) comes from a small neighborhood of $u_x$. To that end, consider a change of variable, $z:=\sqrt{u/u_x}$, then the integral domain $u\in(0,C(x)^{-\frac{1}{3}})$ is mapped to $z\in(0,(C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}})\supset(0,1]$, and

$$\begin{aligned}&\int_0^{C(x)^{-\frac{1}{3}}}g^{(\rho)}(u,x)(1+\mathcal{O}([\log(1/u)]^{-\frac{1}{2}})+\mathcal{O}(u^2C(x)))du\\=&2g^{(\rho)}(u_x,x)u_x\cdot\int_0^{(C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}}}\frac{g^{(\rho)}(u_x\cdot z^2,x)}{g^{(\rho)}(u_x,x)}z(1+\mathcal{O}([\log(1/z^2u_x)]^{-\frac{1}{2}})+\mathcal{O}(C(x)u_x^2z^4))dz\\=:&2g^{(\rho)}(u_x,x)u_x\cdot\int_0^{(C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}}}H(x,z)dz,\end{aligned} \tag{23}$$

where, after some algebra, we have

$$\frac{g^{(\rho)}(u_x \cdot z^2, x)}{g^{(\rho)}(u_x, x)} z = z^{-2-\frac{2}{\sigma^2 t}+\frac{2\log 2}{\sigma^2 t}}(1+\frac{2\log z}{\log u_x})^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}} \exp(-\frac{1}{\sigma^2 t}\Lambda(z,u_x)+\frac{z^2-1}{\sigma^2 t}\log u_x),$$

with

$$\Lambda(z,u_x)=\frac{1}{2}([\log(1/z^2u_x)+\log\log(1/z^2u_x)]^2-[\log(1/u_x)+\log\log(1/u_x)]^2).$$

To estimate (23) for $x>0$ large, we fix an arbitrary $\epsilon \in (0,(C(x)^{\frac{1}{3}}u_x)^{-1}-1)$, and introduce three integrals:

$$I_3=\int_{1-\epsilon}^{1+\epsilon} H(x,z)dz, \quad I_4=\int_0^{1-\epsilon} H(x,z)dz, \quad I_5=\int_{1+\epsilon}^{(C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}}} H(x,z)dz.$$

Then (22) becomes

$$I_1 \quad = \quad \sqrt{\frac{\pi}{2}} e^{-\frac{\log^2 2}{2\sigma^2 t}-\frac{\pi^2}{2\sigma^2 t}} g^{(\rho)}(u_x,x)(I_3+I_4+I_5). \tag{24}$$

In the sequel we will establish sharp estimate for $I_3$ and sharp bounds for $I_4, I_5$, which will yield a sharp estimate for $I_1$. To that end, the following estimate for the function $\Lambda(z,u_x)$ will be useful.

**Lemma 3.2.** *For any $1+w \in (0,(C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}})$, $\epsilon \in (0,(C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}}-1)$, $\epsilon' \in (0,1)$, and large enough $x>0$ such that $u_x<e^{-1}$, we have*

$$\Lambda(1+\varepsilon,u_x)=\begin{cases} (2\log u_x+E(x))w+(\log(1/u_x)+F(x))w^2+\mathcal{O}(G(x)\varepsilon^3), & w\in[-\epsilon',\epsilon] \\ \geq -2w\log(1/u_x)+2\log^2(1+w), & w\in(-1,-\epsilon') \\ \geq -2(w+1)[\log(1/u_x)+\log\log(1/u_x)],, & w\in(\epsilon,(C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}}-1). \end{cases}$$

*where*

$$\begin{aligned} E(x)=&-2\log\log(1/u_x)-2-2\frac{\log\log(1/u_x)}{\log(1/u_x)}, \\ F(x)=&\log\log(1/u_x)-(\frac{1}{\log u_x}+\frac{2}{[\log u_x]^2})(\log(1/u_x)+\log\log(1/u_x))+2(-1+\frac{1}{\log u_x})^2, \\ G(x)=&\log(1/u_x)+\log\log(1/u_x)+(-2+\frac{2}{\log u_x})(1-\frac{1}{\log u_x}-\frac{2}{[\log u_x]^2}). \end{aligned}$$

*Proof.* The proof can be found in Appendix C. ☐

**Proposition 3.4.** *For $x>0$ large enough, we have*

$$I_3=\sqrt{\frac{\pi\sigma^2 t}{2\log(1/u_x)}}(1+\mathcal{O}([\log(1/u_x)]^{-\frac{1}{2}})),\ I_4=\mathcal{O}(1/\log(1/u_x)),\ I_5=\mathcal{O}(1/\log(1/u_x)).$$

*Proof.* We begin with the estimate for $I_3$. Notice that for $w\in(-\epsilon',\epsilon)$, we have $(1+w)^{-2-\frac{2}{\sigma^2 t}+\frac{2\log 2}{\sigma^2 t}}[1+\frac{2\log(1+w)}{\log u_x}]^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}}=1+\mathcal{O}(w)$. Hence, by Lemma 3.2 we have that

$$\begin{aligned} I_3=&\int_{-\epsilon'}^{\epsilon}(1+\mathcal{O}(w))\exp(\frac{w}{\sigma^2 t}[2\log(1/u_x)-E(x)]-\frac{w^2}{\sigma^2 t}[\log(1/u_x)+F(x)]) \\ &\times(1+\mathcal{O}(G(x)w^3))\exp(\frac{w(2+w)}{\sigma^2 t}\log u_x)(1+\mathcal{O}([\log(1/u_x(1+\varepsilon)^2)]^{-\frac{1}{2}})+\mathcal{O}(C(x)u_x^2(1+\varepsilon)^4))dw \\ =&\int_{-\epsilon'}^{\epsilon}\exp(-\frac{w^2}{\sigma^2 t}[2\log(1/u_x)+F(x)]) \\ &\times(1+\mathcal{O}(w)+\mathcal{O}(w|E(x)|)+\mathcal{O}(|G(x)|w^3)+\mathcal{O}([\log(1/u_x(1+w)^2)]^{-\frac{1}{2}})+\mathcal{O}(C(x)u_x^2(1+w)^4))dw \end{aligned}$$

Using Lemma 3.1 and Lemma D.2 (with $c(x,u) = [\log(1/u_x(1+u)^2)]^{-\frac{1}{2}} + C(x)u_x^2(1+u)^4)$, we know that

$$I_3 = \sqrt{\frac{\pi\sigma^2 t}{2\log(1/u_x)}}(1 + \mathcal{O}([\log(1/u_x)]^{-\frac{1}{2}})).$$

Moreover, for $w \in (-1, -\epsilon')$, by applying Lemma 3.2 we have that

$$\begin{aligned}
|I_4| \leq &\int_{-1}^{-\epsilon'} (1+w)^{-2-\frac{2}{\sigma^2 t}+\frac{2\log 2}{\sigma^2 t}}(1+\frac{2\log(1+w)}{\log u_x})^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}}\exp(\frac{2w}{\sigma^2 t}\log(1/u_x) - \frac{2\log^2(1+w)}{\sigma^2 t} + \frac{w(2+w)}{\sigma^2 t}\log u_x)\\
&\times (1 + \mathcal{O}([\log(1/(1+w)^2 u_x)]^{-\frac{1}{2}}) + \mathcal{O}(C(x)u_x^2(1+w)^4))dw\\
= &\int_{-1}^{-\epsilon'} (1+w)^{-2-\frac{2}{\sigma^2 t}+\frac{2\log 2}{\sigma^2 t}}(1+\frac{2\log(1+w)}{\log u_x})^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}}\exp(-\frac{2}{\sigma^2 t}\log^2(1+w) - \frac{w^2}{\sigma^2 t}\log(1/u_x))\\
&\times (1 + \mathcal{O}([\log(1/(1+w)^2 u_x)]^{-\frac{1}{2}}) + \mathcal{O}(C(x)u_x^2(1+w)^4)dw.
\end{aligned}$$

On the other hand, by setting $\delta = 1+w$, it is easily seen that

$$\begin{aligned}
m(w) :=&(1+w)^{-2-\frac{2}{\sigma^2 t}+\frac{2\log 2}{\sigma^2 t}}(1+\frac{2\log(1+w)}{\log u_x})^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}}\exp(-\frac{2}{\sigma^2 t}\log^2(1+w))\\
=&\delta^{-2-\frac{2}{\sigma^2 t}+\frac{2\log 2}{\sigma^2 t}}(1+\frac{2\log\delta}{\log u_x})^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}}\exp(-\frac{2}{\sigma^2 t}\log^2\delta),
\end{aligned}$$

which tends to 0 as $w \downarrow -1$ (i.e. as $\delta \downarrow 0$). Thus, the expression $m(w)$ is bounded for all $w \in (-1, -\epsilon')$. Moreover, we also notice that, uniformly for all $w \in (-1, -\epsilon')$, $[\log 1/(1+w)^2 u_x]^{-\frac{1}{2}} = \mathcal{O}([\log(1/u_x)]^{-\frac{1}{2}})$ and $C(x)u_x^2(1+w)^4 = \mathcal{O}(C(x)u_x^2)$, both of which, by Lemma 3.1, are $\mathcal{O}(1)$ as $x \to \infty$. Hence,

$$|I_4| = \mathcal{O}(\int_{-1}^{-\epsilon'} \exp(-\frac{w^2}{\sigma^2 t}\log(1/u_x))dw) = \mathcal{O}(\frac{1}{\epsilon'\log(1/u_x)}u_x^{\frac{(\epsilon')^2}{\sigma^2 t}}) = \mathcal{O}(1/\log(1/u_x)),$$

using Lemma D.2 in Appendix D.

For $I_5$, for all $w \in (\epsilon, (C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}} - 1)$, by applying Lemma 3.1 we have that,

$$\begin{aligned}
\frac{1}{3}\log C(x) < \log(1/(1+w)^2 u_x) < \log(1/(1+\epsilon)^2 u_x) = \log(1/u_x) - 2\log(1+\epsilon)\\
< \log(1/u_x) = (1+\mathcal{O}(1))\log(|\rho|D(x)/\hat{\rho}^2) = (1+\mathcal{O}(1))\log C(x).
\end{aligned}$$

Hence, uniformly for all $w \in (\epsilon, (C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}} - 1)$, we have $\mathcal{O}([\log(1/(1+w)^2 u_x)]^{-\frac{1}{2}}) = \mathcal{O}([\log C(x)]^{-\frac{1}{2}}) = \mathcal{O}(1)$, and $0 < C(x)u_x^2(1+w)^4 < C(x)(C(x)^{-\frac{1}{3}})^2 = C(x)^{\frac{1}{3}}$, as $x \to \infty$. Lastly, recall that

$$1 > 1 + \frac{2\log(1+w)}{\log u_x} \geq \frac{1}{3}\frac{\log C(x)}{\log(1/u_x)},$$

which converges to $\frac{1}{3}$ as $x \to \infty$, due to the fact that $C(x) \sim D(x)$ and Lemma 3.1. Hence, $(1 + \frac{2\log(1+w)}{\log u_x})^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}}$ is uniformly bounded for all $w \in (\epsilon, (C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}} - 1)$. As a consequence, by Lemma 3.2 we have

$$\begin{aligned}
|I_5| \leq &\int_{\epsilon}^{(C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}}-1} (1+w)^{-2-\frac{2}{\sigma^2 t}+\frac{2\log 2}{\sigma^2 t}}(1+\frac{2\log(1+w)}{\log u_x})^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}}\exp(\frac{2w+2}{\sigma^2 t}[\log(1/u_x) + \log\log(1/u_x)])\\
&\times \exp(\frac{w(2+w)}{\sigma^2 t}\log u_x)(1 + \mathcal{O}([\log(1/(1+w)^2 u_x)]^{-\frac{1}{2}}) + \mathcal{O}(C(x)u_x^2(1+w)^4))dw\\
=&\mathcal{O}(C(x)^{\frac{1}{3}}\exp(\frac{2}{\sigma^2 t}[\log(1/u_x) + \log\log(1/u_x)])\\
&\times \int_{\epsilon}^{(C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}}-1} (1+w)^{-2-\frac{2}{\sigma^2 t}+\frac{2\log 2}{\sigma^2 t}}\exp(-\frac{w^2}{\sigma^2 t}\log(1/u_x) + \frac{2w}{\sigma^2 t}\log\log(1/u_x))dw)\\
=&\mathcal{O}(C(x)^{\frac{1}{3}}u_x^{-\frac{2}{\sigma^2 t}}[\log(1/u_x)]^{\frac{2}{\sigma^2 t}}\int_{\epsilon}^{\infty}(1+w)^{-2-\frac{2}{\sigma^2 t}+\frac{2\log 2}{\sigma^2 t}}\exp(-\frac{w^2}{\sigma^2 t}\log(1/u_x) + \frac{2w}{\sigma^2 t}\log\log(1/u_x))dw).
\end{aligned}$$

To get a upper bound for the above expression, we will use a part of the exponential factor to kill the polynomial preceding it. More precisely, we notice that $(1+w)^{-2-\frac{2}{\sigma^2 t}+\frac{2\log 2}{\sigma^2 t}}\exp(-\frac{w^2}{2\sigma^2 t}\log(1/u_x)+\frac{2w}{\sigma^2 t}\log\log(1/u_x))$ is uniformly bounded for all $w > \epsilon$, so we have

$$
\begin{aligned}
|I_5| \leq & \mathcal{O}(C(x)^{\frac{1}{3}} u_x^{-\frac{2}{\sigma^2 t}} [\log(1/u_x)]^{\frac{2}{\sigma^2 t}} \int_\epsilon^\infty \exp(-\frac{w^2}{2\sigma^2 t}\log(1/u_x))dw) \\
= & \mathcal{O}(C(x)^{\frac{1}{3}} (\frac{|\rho| D(x)}{\hat{\rho}^2})^{\frac{2}{\sigma^2 t}} [\log(1/u_x)]^{-1} \epsilon^{-1} u_x^{\frac{\epsilon^2}{2\sigma^2 t}}),
\end{aligned}
$$

where we used the definition of $u_x$ in (18) for the last equality. Finally, we notice that, for $x$ large enough, by (21) in Lemma 3.1, we have $(C(x)^{\frac{1}{3}} u_x)^{-\frac{1}{2}} - 1 > \sqrt{\frac{4}{3}\sigma^2 t + 8}$ so we can set $\epsilon = \sqrt{\frac{4}{3}\sigma^2 t + 8}$ in the last inequality to obtain

$$
|I_5| \leq \mathcal{O}(C(x)^{\frac{1}{3}} D(x)^{\frac{2}{\sigma^2 t}} [\log(1/u_x)]^{-1} u_x^{\frac{2}{3}+\frac{4}{\sigma^2 t}}) = \mathcal{O}((C(x) u_x^2)^{\frac{1}{3}+\frac{2}{\sigma^2 t}} [\log(1/u_x)]^{-1}) = \mathcal{O}([\log(1/u_x)]^{-1}).
$$

using that $C(x) \sim D(x)$ and Lemma 3.1. □

We are ready to give a sharp estimate for $I_1$. To that end, recall that

$$
\begin{aligned}
g^{(\rho)}(u_x, x) = & u_x^{-\frac{3}{2}-\frac{1}{\sigma^2 t}+\frac{\log 2}{\sigma^2 t}} [\log(1/u_x)]^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}} \exp(-\frac{[\log(1/u_x)+\log\log(1/u_x)]^2}{2\sigma^2 t}) \exp(\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0} D(x)) \\
= & u_x^{-\frac{3}{2}+\frac{\log 2}{\sigma^2 t}} [\log(1/u_x)]^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}} \exp(-\frac{[\log(1/u_x)+\log\log(1/u_x)]^2}{2\sigma^2 t}),
\end{aligned} \tag{25}
$$

where we used (18) in the second equality. By (24), Lemma 3.1 and Proposition 3.4, we obtain the following result.

**Proposition 3.5.** *As $x \to \infty$, we have*

$$
\begin{aligned}
I_1 = & \pi\sigma\sqrt{t}\, e^{-\frac{\log^2 2}{2\sigma^2 t}-\frac{\pi^2}{2\sigma^2 t}} [\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0} D(x)]^{\frac{1}{2}-\frac{\log 2}{\sigma^2 t}} [\log(\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0} D(x))]^{-\frac{1}{2}} \exp(-\frac{1}{2\sigma^2 t}[\log(\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0} D(x))]^2) \\
& \times (1 + \mathcal{O}([\log(|\rho|\sigma D(x)/\hat{\rho}^2 y_0]^{-\frac{1}{2}})).
\end{aligned} \tag{26}
$$

Overall, by combining results in Proposition 3.1, (16), Proposition 3.2 and Proposition 3.5, we obtain the main result in Theorem 2.1.

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# A Proof of Proposition 3.3

To be able to use contour integral trick as in [GS10], we rewrite the integral $\psi_{\frac{u}{s}}(s)$ as an integral on $\mathbb{R}$ using the even function property of its integrand:

$$\psi_{\frac{u}{s}}(s) = \frac{1}{2i}\int_{-\infty}^{\infty} F(z)dz = \frac{1}{2}\Im(\int_{-\infty}^{\infty} F(z)dz),$$

where $F(z) := e^{-\frac{z^2}{2s}}\exp(-\frac{u}{s}\cosh z)\sinh z e^{i\frac{\pi z}{s}}$, and $\Im(z)$ takes the imaginary part of complex number $z$. Straightforward calculation shows that $\overline{F}(z) = F(z+i2\pi)$ and $F(-x+iy) = \overline{F(x+iy)}$ for all $x, y, z \in \mathbb{R}$, where the overline stands for complex conjugate. Thus, we have

$$\psi_{\frac{u}{s}}(s) = \frac{1}{4}\Im(\int_{-\infty}^{\infty} F(z)dz + \int_{\infty+i2\pi}^{-\infty+i2\pi} F(z)dz). \tag{A-1}$$

Using Cauchy's theorem, we know that

$$\int_{-\infty}^{\infty} F(z)dz + \int_{\infty+i2\pi}^{-\infty+i2\pi} F(z)dz = \int_{\gamma^+} F(z)dz + \int_{\gamma^-} F(z)dz, \tag{A-2}$$

where $\gamma^+$ and $\gamma^-$ are respectively the union of directed path segments $(\infty + i2\pi, N_u + i2\pi] \cup [N_u + i2\pi, N_u] \cup [N_u, \infty)$ and $(-\infty, -N_u] \cup [-N_u, -N_u + i2\pi] \cup [-N_u + i2\pi, -\infty + i2\pi)$, and $N_u > 0$ is the unique positive solution to

$$\frac{1}{u} = \frac{\sinh N_u}{N_u}, \tag{A-3}$$

which tends to $\infty$ as $u$ tends to 0. In fact, one can easily obtain that $N_u = (\log(1/u) + \log 2)(1 + \mathcal{O}(1))$ for small $u > 0$. We comment that, like in [GS10], we choose $N_u$ to minimize the leading exponential term for the norm of $F(z + i\pi)$ for $z > 0$. More precisely, notice that

$$|F(z+i\pi)| = \exp(-\frac{z^2 - \pi^2}{2s} + \frac{u}{s}\cosh(z))\sinh(z)e^{-\frac{\pi^2}{s}}, \ \ \forall z \geq 0.$$

The leading exponential term $\Phi(z) = -\frac{z^2}{2s} + \frac{u}{s}\cosh z$ is minimized at $N_u$.

We now analyze the horizontal contour integrals on the right hand side of (A-2). Using that $F(-x+iy) = \overline{F(x+iy)}$ and $\overline{F}(z) = F(z+i2\pi)$ for all $x, y, z \in \mathbb{R}$, we obtain the following

$$\int_{-\infty}^{-N_u} F(z)dz = \int_{N_u}^{\infty} F(-z)dz = \int_{N_u}^{\infty} \overline{F(z)}dz = \int_{N_u+i2\pi}^{\infty+i2\pi} F(z)dz, \tag{A-4}$$

and similarly

$$\int_{\infty}^{N_u} F(z)dz = \int_{-N_u}^{-\infty} F(-z)dz = \int_{-N_u}^{-\infty} \overline{F(z)}dz = \int_{-N_u+i2\pi}^{-\infty+i2\pi} F(z)dz \tag{A-5}$$

Thus we see that the four horizontal contour integrals in (A-2) cancel out; specifically, from (A-4) we see that the upper right horizontal contour integral cancels the lower left integral, and from (A-5) the upper left cancels the lower right.

For the vertical contour integrals, we use the following

$$\begin{aligned}\int_{-N_u}^{-N_u+i2\pi} F(z)dz =& i\int_0^{2\pi} F(-N_u+iy)dy = i\int_0^{2\pi} \overline{F(N_u+iy)}dy \\ =& \overline{\int_0^{2\pi} F(N_u+iy)(-idy)} = \overline{\int_{2\pi}^{0} F(N_u+iy)idy}\end{aligned} \tag{A-6}$$

Thus, from (A-2), if we let $\gamma_r$ and $\gamma_l$ denote the left and right vertical contours, we have

$$\begin{aligned}\Im\Big(\int_{\gamma^l} F(z)dz + \int_{\gamma^r} F(z)dz\Big) =& \Im\Big(\int_{2\pi}^{0} F(N_u+iy)idy + \overline{\int_{2\pi}^{0} F(N_u+iy)idy}\Big) \\ =& 2\Im\Big(\int_{2\pi}^{0} F(N_u+iy)idy\Big) = 2\Re\Big(\int_{2\pi}^{0} F(N_u+iy)dy\Big),\end{aligned} \tag{A-7}$$

The integral on the right hand side of (A-7) can be estimated using the Laplace method. To this end, use the fact that $\sinh(z) = \frac{e^z - e^{-z}}{2}$ we have

$$\begin{aligned}\int_{2\pi}^{0} F(N_u+iy)dy =& \int_{2\pi}^{0} \exp\Big(-\frac{u}{s}\cosh(N_u+iy)\Big)\sinh(N_u+iy)e^{-\frac{(N_u+iy)^2}{2s}+i\frac{\pi(N_u+iy)}{s}}dy \\ =& \frac{e^{N_u-\frac{N_u^2}{2s}}}{2}\int_{2\pi}^{0} \exp\Big(-\frac{u}{s}\cosh(N_u+iy)\Big)e^{iy-\frac{iN_u y}{s}+\frac{y^2}{2s}+i\frac{\pi(N_u+iy)}{s}}dz \\ &- \frac{e^{-N_u-\frac{N_u^2}{2s}}}{2}\int_{2\pi}^{0} \exp\Big(-\frac{u}{s}\cosh(N_u+iy)\Big)e^{-iy-\frac{iN_u y}{s}+\frac{y^2}{2s}+i\frac{\pi(N_u+iy)}{s}}dy\end{aligned} \tag{A-8}$$

To estimate the above integrals, let $P(y) := -\frac{u}{s}\cosh(N_u+iy) - \frac{iN_u y}{s} + i\frac{\pi N_u}{s} - \frac{u}{s}\cosh(N_u)$ and $Q(y) := e^{iy+\frac{y^2}{2s}-\frac{\pi y}{s}+\frac{u}{s}\cosh(N_u)}$. Setting $y = \pi - z$, where $z \in [-\pi, \pi]$. Then

$$\begin{aligned}P(y) =& -\frac{u}{s}\cosh(N_u)\cosh(i(\pi-z)) - \frac{u}{s}\sinh(N_u)\sinh(i(\pi-z)) - \frac{N_u}{s}i(\pi-z) + i\frac{\pi N_u}{s} - \frac{u}{s}\cosh(N_u) \\ =& \frac{u}{s}\cosh(N_u)\cos(z) - \frac{u}{s}\sinh(N_u)i\sin(z) + i\frac{N_u z}{s} - \frac{u}{s}\cosh(N_u) \\ =& \frac{u}{s}\cosh(N_u)[\cos(z)-1] + i\frac{u}{s}\sinh(N_u)[z-\sin(z)] \\ =& -2\frac{u}{s}\cosh(N_u)\sin^2\Big(\frac{z}{2}\Big) + i\frac{u}{s}\sinh(N_u)[z-\sin(z)],\end{aligned} \tag{A-9}$$

where we used the definition of $N_u$ in (A-3) in the third equality. Similarly,

$$
\begin{aligned}
Q(y) =& \exp(i(\pi - z) + \frac{(\pi - z)^2}{2s} - \frac{\pi}{s}(\pi - z) + \frac{u}{s}\cosh(N_u)) \\
=& e^{\frac{z^2-\pi^2}{2s} + \frac{u}{s}\cosh(N_u)}[-\cos(z) + i\sin(z)] =: q(z).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_{2\pi}^{0} e^{P(y)}Q(y)dy - \int_{-\pi}^{\pi} e^{-2\frac{u}{s}\cosh(N_u)\sin^2(\frac{z}{2})} e^{i\frac{u}{s}\sinh(N_u)[z-\sin(z)]} q(z)ds \\
&\qquad = - \int_{-1}^{1} e^{-2\frac{u}{s}\cosh(N_u)w^2} \exp(i\frac{u}{s}\sinh(N_u)(z - \sin(z)))q(2\arcsin(w))\frac{2}{\sqrt{1-w^2}}dw \\
&\qquad = - 2q(0)[\int_{-1}^{1} e^{-2\frac{u}{s}\cosh(N_u)w^2}dw + \mathcal{O}(\frac{s}{v\cosh(N_u)})] \\
&\qquad =2e^{-\frac{\pi^2}{2s}+\frac{u}{s}\cosh(N_u)}[\sqrt{\frac{\pi s}{2u\cosh(N_u)}} + \mathcal{O}(\frac{s}{u\cosh(N_u)})]
\end{aligned}
\tag{A-10}
$$

where we have used the substitution $w = \sin(\frac{z}{2})$ in the second line, and the Laplace's method in the penultimate line (or we can use the direct argument given on page 460 in [GS10]). We point out that the second integral in (A-8) is negligible when compared with the first integral, because of the extra factor $e^{-2N_u} = u^2(1 + \mathcal{O}(1))$ and $u$ is small. Thus (A-7) is equal to

$$
\begin{aligned}
&2e^{N_u - \frac{N_u^2}{2s}} e^{-\frac{\pi^2}{2s}+\frac{u}{s}\cosh(N_u)}[\sqrt{\frac{\pi s}{2u\cosh(N_u)}} + \mathcal{O}(\frac{s}{u\cosh(N_u)})] \\
=&4\frac{N_u}{u}[1 - e^{-2N_u}]^{-1} e^{-\frac{\pi^2}{2s} - \frac{N_u^2}{2s} + \frac{N_u}{s}} \sqrt{\frac{\pi s}{2}}\frac{1}{N_u^{\frac{1}{2}}}\sqrt{\frac{\sinh(N_u)}{\cosh(N_u)}}(1 + \mathcal{O}(\sqrt{\frac{s}{u\sinh(N_u)}})) \\
&\text{(using the definition of } N_u\text{, i.e. that } 1/u = \frac{\sinh N_u}{N_u} \text{ and that } \cosh N_u = \sinh N_u\,[1 + \mathcal{O}(e^{-2N_u})]\,) \\
=&\frac{4}{u}\sqrt{\frac{\pi s}{2}} e^{-\frac{\pi^2}{2s}} N_u^{\frac{1}{2}} \exp(-\frac{N_u^2}{2s} + \frac{N_u}{s})(1 + \mathcal{O}(N_u^{-\frac{1}{2}})).
\end{aligned}
\tag{A-11}
$$

By Lemma D.1, we know that (A-11) is equal to

$$
\frac{4}{u}\sqrt{\frac{\pi s}{2}} e^{-\frac{\pi^2}{2s}} N_u^{\frac{1}{2}} \exp(-\frac{N_u^2}{2s} + \frac{N_u}{s})(1 + \mathcal{O}(N_u^{-\frac{1}{2}})) = \frac{4}{u}\sqrt{\frac{\pi s}{2}} e^{-\frac{\pi^2}{2s}} N_u^{\frac{1}{2}} \exp(-\frac{N_u^2}{2s} + \frac{N_u}{s})(1 + \mathcal{O}([\log(1/u)]^{-\frac{1}{2}})).
\tag{A-12}
$$

Furthermore, using Lemma D.1 we also have,

$$
\begin{aligned}
e^{\frac{N_u}{s}} =& (\frac{2}{u})^{\frac{1}{s}}(\log(1/u))^{\frac{1}{s}}(1 + \mathcal{O}(\frac{\log\log(1/u)}{\log(u)})), \\
e^{-\frac{N_u^2}{2s}} =& 2^{-\frac{1}{s}} e^{-\frac{\log^2 2}{2s}} u^{\frac{\log 2}{s}} [\log(1/u)]^{-\frac{1+\log 2}{s}} \exp(-\frac{1}{2s}[\log(1/u) + \log\log(1/u)]^2)(1 + \mathcal{O}(\frac{[\log\log(1/u)]^2}{\log(1/u)})).
\end{aligned}
$$

Thus, (A-12) is equal to

$$
4\sqrt{\frac{\pi s}{2}} e^{-\frac{\log^2 2}{2s}} e^{-\frac{\pi^2}{2s}} u^{-1-\frac{1}{s}+\frac{\log 2}{s}} [\log(1/u)]^{\frac{1}{2}-\frac{\log 2}{s}} \exp(-\frac{1}{2s}(-\log(u) + \log\log(1/u))^2)(1 + \mathcal{O}([\log(1/u)]^{-\frac{1}{2}})).
\tag{A-13}
$$

Finally, we obtain the result from (A-1) and (A-13).

# B Proof of Lemma 3.1

First, by re-writing (18) as $\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0}D(x) = \frac{1}{u_x}\log\frac{1}{u_x}$ and taking logs, we obtain (19). Taking the logarithm of both sides of (19) again, and using the equality $\log(A+B) = \log A + \mathcal{O}(\frac{A}{B})$ for $A, B > 0$, we have

$$\log\log(1/u_x) + \mathcal{O}(\frac{\log\log(1/u_x)}{\log(1/u_x)}) = \log\log(\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0}D(x)).$$

Using (19), it follows that

$$\log(1/u_x) + \mathcal{O}(\frac{\log\log(1/u_x)}{\log(1/u_x)}) = \log(\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0}D(x)) - \log\log(\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0}D(x)).$$

This proves (20). Similarly, we also have

$$-\log(1/u_x) + \log\log(1/u_x) + \mathcal{O}(\frac{\log\log(1/u_x)}{\log(1/u_x)}) = -\log(\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0}D(x)) + 2\log\log(\frac{|\rho|}{\hat{\rho}^2}\frac{\sigma}{y_0}D(x)),$$

and exponentiating both sides we see that

$$u_x\log(1/u_x)(1+\mathcal{O}(1)) = \frac{[\log(|\rho|\sigma D(x)/\hat{\rho}^2 y_0)]^2}{|\rho|\sigma D(x)/\hat{\rho}^2 y_0}.$$

Finally, recall that

$$C(x)u_x^2 = \frac{C(x)}{D(x)}u_x \cdot D(x)u_x = \frac{C(x)}{D(x)}u_x \cdot \frac{\hat{\rho}^2}{|\rho|}\frac{y_0}{\sigma}\log(1/u_x) = \mathcal{O}(u_x\log(1/u_x)),$$

since $C(x)/D(x) = \mathcal{O}(1)$ as $x \to \infty$. This proves (21).

# C Proof of Lemma 3.2

It is straightforward to verify that

$$\Lambda(1+\varepsilon, u_x) = \zeta\left[\log(1/u_x) + \log\log(1/u_x)\right] + \frac{1}{2}\zeta^2 \tag{A-14}$$

where $\zeta = [-\log 2(1+\varepsilon) + \log(1+\frac{2\log(1+\varepsilon)}{\log u_x})]$. We now deal with the three cases separately:

- For $\varepsilon \in (-\epsilon', \epsilon)$ and large $x$ (so $u_x$ is small), we have $\frac{\log(1+\varepsilon)}{\log u_x} = \mathcal{O}(1)$. Thus we can use that $\log(1+\varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ to obtain

  $$-2\log(1+\varepsilon) + \log(1+\frac{2\log(1+\varepsilon)}{\log u_x}) = (-2+\frac{2}{\log u_x})\varepsilon + (1 - \frac{1}{\log u_x} - \frac{2}{[\log u_x]^2})\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

  This proves the asymptotic expansion of $\Lambda(1+\varepsilon, u_x)$ round 0.

- $\varepsilon \in (-1, \epsilon')$. In this case, we note that $\log(1+\varepsilon) < \log(1-\epsilon') < 0$ for all $\epsilon \in (-1, -\epsilon')$ and $\log u_x < 0$, so

  $$-2\log(1+\varepsilon) + \log(1+\frac{2\log(1+\varepsilon)}{\log u_x}) \geq -2\log(1+\varepsilon) + \log 1 = -2\log(1+\varepsilon) > 0.$$

  On the other hand, when $x$ is sufficiently large, $u_x < C(x)^{-\frac{1}{3}} < e^{-1}$, so we have $\log(1/u_x), \log\log(1/u_x) > 0$. It follows that

  $$\Lambda(1+\varepsilon, u_x) \geq -2\log(1+\varepsilon)[\log(1/u_x) + \log\log(1/u_x)] + 2\log^2(1+\varepsilon) \geq -2\varepsilon\log(1/u_x) + 2\log^2(1+\varepsilon),$$

  where we used the inequality $\log(1+\varepsilon) \leq \varepsilon$ for all $\varepsilon > -1$ and $\log\log(1/u_x) > 0$ in the last inequality.

- $\varepsilon \in (\epsilon, (C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}}-1)$. In this, we have $0 < \log(1+\varepsilon) < \log((C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}}) = -\frac{1}{6}\log C(x) - \frac{1}{2}\log u_x$ for all $\varepsilon \in (\epsilon, (C(x)^{\frac{1}{3}}u_x)^{-\frac{1}{2}}-1)$. Because $\log u_x < 0$,

$$0 > \log(1 + \frac{2\log(1+\varepsilon)}{\log u_x}) \geq \log(\frac{1}{3}\frac{\log C(x)}{\log(1/u_x)}) \overset{\text{as } x\to\infty}{\to} -\log 3,$$

by Lemma 3.1 below. Recall that $\log(1+\varepsilon) \leq \varepsilon$ for all $\varepsilon > -1$, we have (by $\log 3 < 2$)

$$-2\log(1+\epsilon) + \log(1 + 2\frac{\log(1+\epsilon)}{\log u_x}) \geq -2\varepsilon - 2$$

The proof is complete by using the fact that $\log(1/u_x) + \log\log(1/u_x) > 0$ for large enough $x$.

# D Auxiliary Lemmas

**Lemma D.1.** *As $u \downarrow 0$, the quantity $N_u$ defined in* (A-3) *satisfies*

$$N_u = \log(1/u) + \log 2 + \log\log(1/u) + \frac{\log\log(1/u) + \log 2}{\log(1/u)} + \mathcal{O}((\frac{\log\log(1/u)}{\log(1/u)})^2), \tag{A-15}$$

$$N_u^2 = [\log(1/u) + \log\log(1/u)]^2 + \log^2 2 + 2\log 2[\log(1/u) + \log\log(1/u)] + 2[\log\log(1/u) + \log 2] + \mathcal{O}(\frac{[\log\log(1/u)]^2}{\log(1/u)}) \tag{A-16}$$

*Proof.* From $N_u = \frac{u}{2}e^{N_u}(1 - e^{-2N_u})$ we see that

$$N_u = \log(1/u) + \log 2 + \log N_u + \mathcal{O}(e^{-2N_u}).$$

Since $N_u \to \infty$ as $u \to 0$ and $\log N_u = \mathcal{O}(N_u)$, we use the asymptotic relation $\log(A+B) = \log A + \frac{B}{A} + \mathcal{O}(\frac{B^2}{A^2})$ recursively to obtain

$$N_u = \log(1/u) + \log 2 + \log\log(1/u) + \mathcal{O}(\frac{\log\log(1/u)}{\log(1/u)}),$$

$$N_u = \log(1/u) + \log 2 + \log\log(1/u) + \frac{\log\log(1/u) + \log 2}{\log(1/u)} + \mathcal{O}((\frac{\log\log(1/u)}{\log(1/u)})^2).$$

Then equation (A-16) follows immediately. □

**Lemma D.2.** *Let $a(\cdot) > 0, b(\cdot)$ be functions on $\mathbb{R}_+$ such that $\lim_{x\to\infty} a(x) = \infty, \lim_{x\to\infty} \frac{b(x)}{a(x)^{\frac{n+1}{2}}} = 0$ for some constant $n \geq 0$. Let $c(x,\cdot)$ be a positive continuous function defined on $(-1,\infty)$, such that, for any fixed $\epsilon > 0$ and $\epsilon' \in (0,1)$, there are positive constants $C_1 > C_1 > 0$ such that*

$$\lim_{x\to\infty} c(x,0) = 0, \quad C_2 \cdot c(x,0) \leq c(x,u) \leq C_1 \cdot c(x,0), \quad \forall u \in (-\epsilon', \epsilon),$$

*and all sufficiently large $x > 0$. Then for any fixed $\epsilon \in (0,1)$, we have*

$$\int_{-\epsilon'}^{\epsilon} e^{-a(x)u^2}(1 + \mathcal{O}(u) + |u|^n b(x) + c(x,u))du = \sqrt{\frac{\pi}{a(x)}}(1 + \mathcal{O}([a(x)]^{-\frac{1}{2}}) + \mathcal{O}(\frac{b(x)}{a(x)^{\frac{n}{2}}}) + \mathcal{O}(c(x,0))),$$

$$\int_{-\infty}^{-\epsilon'} e^{-a(x)u^2}du = \mathcal{O}(\frac{1}{a(x)}e^{-a(x)(\epsilon')^2}) \qquad (x \to \infty).$$

*Proof.* For all $n \geq 0$, we consider the integral on the positive half domain only. The integral on the negative half domain can be treated similarly.

$$\begin{aligned}\int_0^{\epsilon} e^{-a(x)u^2}u^n du &\overset{v=u^2}{=} \int_0^{\epsilon^2} e^{-a(x)v}v^{\frac{n}{2}} \cdot \frac{1}{2}v^{-\frac{1}{2}}dv = \frac{1}{2}\int_0^{\epsilon^2} e^{-a(x)v}v^{\frac{n-1}{2}}dv \\ &= \frac{1}{2}\int_0^{\infty} e^{-a(x)v}v^{\frac{n+1}{2}-1}dv - \frac{1}{2}\int_{\epsilon^2}^{\infty} e^{-a(x)v}v^{\frac{n+1}{2}-1}dv \\ &= \frac{1}{2}\Gamma(\frac{n+1}{2})a(x)^{-\frac{n+1}{2}} - \frac{1}{2}\int_{\epsilon^2}^{\infty} e^{-a(x)v}v^{\frac{n-1}{2}}dv,\end{aligned}$$

by Gamma integrals (Note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$). Furthermore, using the fact that, uniformly for all $x > 0$, $e^{-\frac{a(x)}{2}v} v^{\frac{n-1}{2}} = \mathcal{O}(1)$ for all $v \geq \epsilon^2$, we have,

$$|\int_0^\epsilon e^{-a(x)u^2} u^n du - \frac{1}{2}\Gamma(\frac{n+1}{2})a(x)^{-\frac{n+1}{2}}| = \frac{1}{2}\int_{\epsilon^2}^\infty e^{-a(x)v} v^{\frac{n-1}{2}} dv \leq \mathcal{O}(\int_{\epsilon^2}^\infty e^{-\frac{a(x)}{2}v} dv) = \mathcal{O}(a(x)^{-1} e^{-\frac{a(x)}{2}\epsilon^2}),$$

which is of order $\mathcal{O}(a(x)^{-\frac{n-1}{2}})$ as $x \to \infty$. On the other hand, from the conditions on $c(x, u)$ one gets

$$C_2 \cdot c(x,0) \int_{-\epsilon'}^{\epsilon} e^{-a(x)u^2} du \leq \int_{-\epsilon'}^{\epsilon} e^{-a(x)u^2} c(x,u) du \leq C_1 \cdot c(x,0) \int_{\epsilon'}^{\epsilon} e^{-a(x)u^2} du.$$

But we already know that $\int_{-\epsilon'}^{\epsilon} e^{-a(x)u^2} du = \mathcal{O}(\sqrt{\frac{\pi}{a(x)}})$. Finally, using $\int_{-\infty}^{-A} e^{-v^2} dv = \mathcal{O}(A^{-1} e^{-A^2})$ as $A \to \infty$, we have $\int_{-\infty}^{-\epsilon'} e^{-a(x)u^2} du = \frac{1}{\sqrt{a(x)}} \int_{-\infty}^{-\sqrt{a(x)}\epsilon'} e^{-v^2} dv = \mathcal{O}(\frac{1}{a(x)} e^{-a(x)(\epsilon')^2})$. $\square$