Duality for three robust hedging problems - finite strikes and maturities with tradeable exotics, or given joint/ marginal laws for the minimum and the terminal price

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Abstract: We establish (quasi-sure) duality for three robust hedging problems, by adapting the proof of the main result in Guo,Tan&Touzi[GTT15]. In all cases we consider superhedging for a general path-dependent claim; we first deal with the case when there is only a finite number of tradeable options at multiple maturities which can also include exotic options; we exclude price processes which can go negative but allow price processes which can absorb at zero to allow for the possibility of default, and we incorporate an investor's belief that the quadratic variation $\langle X\rangle$ of the underlying is bounded. The duality is obtained using tightness arguments and the Fenchel-Moreau theorem combined with standard properties of the Snell envelope as in [GTT15], and we show that the supremum in the primal problem is attained by some model. In the second problem we remove the condition on $\langle X\rangle$ and we now assume we have a given joint law for the terminal level and its minimum at a single maturity, which corresponds to tradeable barrier option prices at all strike and barrier levels; in this setting the duality is established using the Wasserstein topology $\mathcal{W}^1$ and a re-formulation of the Rogers characterization for the admissible joint laws. In the final problem, we analyze the case when we just have the marginals for the terminal level and minimum at a single maturity, which is tantamount to tradeable European call options at all strikes and One-Touch options at all barrier levels less than the initial stock price.

1. Introduction

The robust superhedging problem has attracted a huge amount of interest in recent years. [GTT15] consider the problem for a path-dependent payoff when we have marginals μ_k at a finite number of maturities $t_1 < ... < t_n = 1$, by reformulating the problem as a Skorokhod embedding problem for Brownian motion evaluated a finite number of stopping times $T_1 \leq ... \leq T_n$ such that $B_{T_k\wedge .}$ is uniformly integrable (U.I.) with the constraint that $B_{T_k} \sim \mu_k$ for $1 \leq k \leq n$. Kellerer's theorem states that we find such a sequence of stopping times if and only if the marginals are centred and increasing in the convex order with finite first absolute moments i.e. the marginals form a *peacock* (see e.g. [BMS15] for further details on this, in particular the role of Strassen's theorem). [GTT15] use the Wasserstein topology $\mathcal{W}^1$ (on the space of peacocks $\mathbf{P}^{\preceq}$ (as opposed to the usual weak topology); this allows them to prove that $\mathbf{P}^{\preceq}$ is a closed, convex subspace of the space of probability measures on $\mathbb{R}^n$, and (in Lemma 4.2 in [GTT15]) if we have a sequence of peacocks $\mu^m \to \mu^0$ under $\mathcal{W}^1$, then any sequence of admissible "models" $\bar{\mathbb{P}}^m$ corresponding to μ^m is tight and thus (by Prokhorov's theorem) has a convergent subsequence tending to some $\bar{\mathbb{P}}^0$ under the weak topology on the (enlarged) path space. The use of the Wasserstein topology here is crucial in establishing that the stopping times are U.I. under $\bar{\mathbb{P}}^0$. In Lemma 4.7 in [GTT15] they use this to prove that the primal problem is a concave and upper semicontinuous function of the target measures under $\mathcal{W}^1$, and using this they establish the first duality result by re-writing the primal as an infimum of classical (i.e. no marginal constraints) stopping time problems over the space of admissible European option portfolios, using the bi-conjugate theorem (also known as the Fenchel-Moreau theorem) for concave USC functions defined on a locally convex Hausdorff space. The full duality (i.e. showing that the infimum of the cost of all admissible superhedging strategies is equal to supremum of the expected value of the claim over calibrated martingale measures) is then obtained (in a quasi-sure formulation) using properties of the Snell envelope applied to the aforementioned family of classical stopping problems combined with the Dambis-Dubins-Schwarz theorem and the usual aggregation method using the pathwise definition of the stochastic integral given in Karandikar[Kar95]. This

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duality also allows one to reproduce the geometric characterization of the optimal embedding introduced in [BCH14]; see [GTT15c] for further results in this direction.

[GTT15b] consider the robust hedging problem on the Skorokhod space of càdlàg paths, which allows for jump processes. The set of martingale measures $\mathcal{M}(\mu)$ consistent with a finite set of marginals μ is not tight with respect to the standard topologies, which makes it difficult to adapt the duality results in discrete-time settings. Dolinsky&Soner[DS14] and Hou&Obloj[HO15] circumvent these issues by discretizing the paths and they take a different approach where the superhedge has to hold *pathwise* for *all* possible continuous price paths and they only allow trading strategies with bounded variation so the Stieltjes integration-by-parts formula holds. In particular, [HO15] also assume that the market prices are in the interior of the no-arbitrage region. [GTT15b] get round this issue by using the S-topology on $\mathcal{M}(\mu)$ introduced in Jakubowski[Jak97]. The S-topology is induced by the notion of S-convergence, and we can then define S^*-convergence as the convergence induced by the S-topology. Rather than use the usual weak topology on the space of probability measures, they use another notion of convergence, which allows for a variant of the standard Prokhorov theorem to hold under S-tightness, i.e. where tightness yields sequential compactness (by S-tightness they are just replacing the usual notion of tightness using compact sets with a set which is compact under the S-topology).

Kallblad et al.[KTT15] look at the case when we have full marginals i.e. marginals at all maturities, by taking the limit as $n \to \infty$ of the results in [GTT15]. In particular, [KTT15] look at the Azéma-Yor embedding with full marginals as a limiting case of the n-marginals case covered in [HOST14], and they compute an elegant variational representation for the optimal strike function as a function of time. [HOST14] use a recursive argument to extend the solution for the two-marginals case which is covered in Brown et al.[BHR01]. Beiglbock et al.[BCH14] consider the robust hedging problem with a single marginal; in this setting they establish a duality result using pseudo-randomized stopping times and dual optional projections and the Monge-Kantorovich duality, and they also provide a geometric characterization of the optimal embedding which includes all previous known embeddings, e.g. Azéma-Yor, Perkins, Root and Rost.

In another recent article, [DS15] consider two different super-replication problems for a continuous time market where the investor can engage in dynamic trading (with bounded variation) of the underlying subject to proportional transaction costs and take a static long position in n options (with possibly path-dependent payoffs) with ask prices $\mathcal{L}_i$ for $1 \leq i \leq n$ (they also have to impose that one of these options has a super quadratic payoff). The first problem considers model-independent hedging where the superhedge is required to hold pathwise for all continuous paths; in the second problem, the market model is given via a probability measure $\mathbb{P}$ on path space (e.g. a specific model like Black-Scholes or Heston) and the superhedge just has to work $\mathbb{P}$-almost surely. Remarkably, the superhedging cost is shown to be the same in both settings, if $\mathbb{P}$ satisfies the conditional full support property, which precludes degenerate models. Thus, for example, even if we know the true model is Black-Scholes with zero drift with proportional transaction costs where n options can be bought at time-zero at prices $\mathcal{L}_i$ for $1 \leq i \leq n$, this extra knowledge is of no help in reducing the superhedging cost of the exotic claim.

In this article we prove duality results for three different problems, using similar techniques to Guo,Tan&Touzi[GTT15]. In all cases we consider a general path-dependent claim; we first consider the scenario when there is only a finite number of options at multiple maturities which can also include exotic options which are available for static trading at time zero; we preclude price processes which can go negative but allow price processes which can absorb at zero to allow for default, and we incorporate an investor's belief that the quadratic variation of the underlying $\langle X \rangle \leq N$ which will reduce the superhedging cost if an admissible model exists, and we show that the supremum in the primal problem is attained by some model. In the second problem we remove the condition that $\langle X \rangle \leq N$ and we now assume we have a given joint law for the terminal level and its minimum at a single maturity, which equates to having observed barrier option prices at all strikes and barrier levels; for this problem the duality is proved using the Wasserstein topology $\mathcal{W}^1$ and a re-formulation of the Rogers[Rog93] characterization for the admissible joint laws where we essentially take the Laplace transform of the original condition. The Wasserstein topology is the same as the topology induced by the Wasserstein distance (see e.g. [GTT15b] for further details). In the final problem, we analyze the case when we just have the marginals for the terminal level and minimum at a single maturity, which is tantamount to tradeable European call options at all strikes and One-Touch options at all barrier levels less than the initial stock price.

2. Set up and notation

Notations. We first introduce some notation similar to [GTT15]:

- We say that a stopping time T for a Brownian motion B is uniformly integrable if $B_{T \wedge t}$ is a uniformly integrable family of random variables for all $t \in [0, \infty)$.
- Let $\Omega_{x_0} = C(\mathbb{R}^+, \mathbb{R})$ denote be the space of all continuous paths ω on $\mathbb{R}^+$ such that $\omega_0 = x_0 > 0$ and let B be the canonical process, $\mathbb{P}_0$ the Wiener measure on Ω_{x_0}, $\mathbb{F}$ the canonical filtration generated by B and $\mathbb{F}^a := (\mathcal{F}_t)_{t \geq 0}$ the augmented filtration under $\mathbb{P}_0$.

- For some fixed $n \geq 1$, let $\bar{\Omega}_{x_0} = \Omega_{x_0} \times \Theta$, where $\Theta = \{(\theta_1, .., \theta_n) : 0 \leq \theta_1 \leq ... \leq \theta_n\}$. Elements of $\bar{\Omega}_{x_0}$ are denoted by $\bar{\omega} = (\omega, \theta)$ with $\theta = (\theta_1, ..., \theta_n)$. Let (B, T) (with $T = (T_1, ..., T_n)$) denote the canonical element on $\bar{\Omega}_{x_0}$, i.e. $B_t(\bar{\omega}) = \omega_t$ and $T(\bar{\omega}) = \theta$. The enlarged canonical filtration is denoted by $\bar{\mathbb{F}} := (\bar{\mathcal{F}}_t)_{t \geq 0}$ where $\bar{\mathcal{F}}_t$ is generated by $(B_s)_{0 \leq s \leq t}$ and all sets of the form $\{T_k \leq s\}$ for all $s \in [0, t]$, so $T_1, ..., T_n$ are $\bar{\mathbb{F}}$-stopping times.
- We furnish Ω_{x_0} with the compact convergence topology, and Θ with the classical Euclidean topology, so Ω_{x_0} and $\bar{\Omega}_{x_0}$ are both Polish spaces.

We work in a similar framework to [GTT15] but we assume there is only a finite set $\mathcal{X}$ of tradeable options with (possibly exotic) payoffs $\mathcal{X}_i$ for $1 \leq i \leq M$ which all expire at one of n maturities $t_1 < ... < t_n = 1$ (these are the maturities in the real world, see below for technical conditions on $\mathcal{X}_i$), which can be bought or sold at price $\mathcal{L}_i$ at time zero only. We also incorporate an investor's belief that the quadratic variation of the underlying is bounded by N, and we exclude price processes which can go negative but include price processes which absorb at zero which allows for the possibility of default.

We now set

$$\begin{aligned} \bar{\mathcal{P}}_N &:= \{\bar{\mathbb{P}} \in \bar{\mathcal{P}}(\bar{\Omega}_{x_0}) : B \text{ is an } \bar{\mathbb{F}}\text{-Brownian motion and } T_n \leq N \quad \bar{\mathbb{P}}\text{-a.s. }\} \\ \bar{\mathcal{P}}_N(\mathcal{L}) &:= \{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_N \,:\, \mathbb{E}^{\bar{\mathbb{P}}}(\mathcal{X}_i(\omega, \theta)) \,=\, \mathcal{L}_i\,,\ \ 1 \leq i \leq M\} \end{aligned}$$

and we assume that $\bar{\mathcal{P}}_N(\mathcal{L})$ is non-empty (this is a non-trivial issue which we do not address in this article) and for all $i \leq M_e \leq M$ we have $\mathcal{X}_i = (B_{T_{k_i}} - K_i)^+$ for some k_i with $1 \leq k_i \leq n$ (i.e. the first M_e options in the calibration set are European call options with strike K_i and maturity t_{k_i}) and for $M_E \leq i \leq M$ we specify that $\mathcal{X}_i : \bar{\Omega}_{x_0} \to \mathbb{R}$ is a non-anticipative (i.e. $\mathcal{X}_i(\omega, \theta) = \mathcal{X}_i(\omega_{\theta_n \wedge .}, \theta)$), bounded continuous function (this corresponds to the non-European and possibly path-dependent options in the calibration set).

We set the final instrument in the calibration set to be an **artificial contract** with payoff $\mathcal{X}_M = \psi(\underline{\omega}_{T_n})$ with price $\mathcal{L}_M = 0$, where ψ is a smooth bounded function with $\psi(x) = 0$ for $x \geq 0$ and $\psi(x) > 0$ for $x < 0$, and $\underline{\omega}_t = \inf_{0 \leq s \leq t} \omega_t$. This contract does not exist in reality (and hence is not tradeable); rather we include it at this stage to enforce that the price process cannot go negative (we will see later that this contract can be removed from the set of tradeable instruments in the dual problem by restricting to non-negative price processes so the final duality result is natural). Any $\mathcal{L}$ for which $\bar{\mathcal{P}}_N(\mathcal{L})$ is non-empty is known as an *admissible* price vector, and we denote the space of admissible price vectors by $\mathcal{A} \subset \mathbb{R}^M$.

Let $\Phi : \bar{\Omega}_{x_0} \to \mathbb{R}$ and assume that Φ is non-anticipative, bounded and upper semicontinuous function (under the compact convergence topology). Then the primal problem is

$$P(\mathcal{L}) \;:=\; \sup_{\bar{\mathcal{P}}_N(\mathcal{L})} \mathbb{E}^{\bar{\mathbb{P}}}(\Phi(B, T)) \;=\; \sup_{\mathcal{P}^+_N(\mathcal{L})} \mathbb{E}^{\bar{\mathbb{P}}}(\Phi(B, T))$$

where $\bar{\mathcal{P}}^+_N(\mathcal{L})$ is the set of all elements of $\mathcal{P}_N(\mathcal{L})$ for which $\tau_0 \geq T_n$ where $\tau_0 = \inf\{s : B_s = 0\}$. The first dual problem is now given by

$$D_0(\mathcal{L}) \;:=\; \inf_{\lambda \in \Lambda} \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}_N} \mathbb{E}^{\bar{\mathbb{P}}}\big[\Phi \,-\, \sum_{i=1}^{M} \lambda_i(\mathcal{X}_i - \mathcal{L}_i)\big]$$

where $\Lambda = \mathbb{R}^M$, where Φ and $\mathcal{X}_i$ are shorthand for $\Phi(B, T)$ and $\mathcal{X}_i(B, T)$ respectively.

In addition to the original assumptions on Φ and $\mathcal{X}_i$, we make the following additional assumption (similar to Assumption 2.3 in [GTT15]).

Assumption 2.1. *$\Phi = \sum_{k=1}^n \Phi_k(\omega, \theta_k)$, $\mathcal{X}_i = \sum_{k=1}^n \mathcal{X}_{i,k}(\omega, \theta_k)$ with $1 \leq i \leq n$, and the maps $\theta_k \mapsto \Phi_k(\omega, \theta_k)$, $\theta_k \mapsto \mathcal{X}_{i,k}(\omega, \theta_k)$ are càdlàg.*

3. Duality results

3.1. Technical lemmas

In this subsection, we prove some simple technical lemmas which are needed for the main duality results in the next subsection.

Lemma 3.1. *Let $\mathcal{L}^m \in \mathcal{A}$ be a sequence (with $\mathcal{L}^m_N = 0$) which converges to $\mathcal{L}^0 \in \mathbb{R}^M$ under the usual Euclidean topology. Let $\bar{\mathbb{P}}^m$ be a sequence of probability measures with $\bar{\mathbb{P}}^m \in \mathcal{P}_N(\mathcal{L}^m)$. Then the sequence $(\bar{\mathbb{P}}^m)_{m\geq 1}$ is tight, $\mathcal{L}^0 \in \mathcal{A}$ and any limit point of $\bar{\mathbb{P}}^m$ lies in $\mathcal{P}_N(\mathcal{L}^0)$.*

Remark 3.1. *The lemma implies that $\mathcal{A}$ is closed under the Euclidean topology.*

Proof. For all $\varepsilon > 0$, there exists a compact set $D \subset \Omega$ such that $\bar{\mathbb{P}}^m(D^c \times \Theta) \leq \varepsilon$, because $\bar{\mathbb{P}}^m$ induces the same measure on Ω_{x_0} for all m (i.e. the Wiener measure). Moreover

$$\bar{\mathbb{P}}^m(T_n > N) \quad = \quad 0\,.$$

Then it follows that

$$\bar{\mathbb{P}}^m(\bar{\omega} \notin (D \times \{\theta_n \leq N\})) \quad \leq \quad \varepsilon$$

so we have tightness as required, and thus by Prokhorov's theorem $\bar{\mathbb{P}}^m$ has a convergent subsequence under the weak topology. Let $\bar{\mathbb{P}}^0$ be any limit point and let $\bar{\mathbb{P}}^m$ now denote the convergent subsequence. To verify that $\bar{\mathbb{P}}^0 \in \mathcal{P}_N(\mathcal{L}^0)$, we first use the same argument as in the proof of Lemma 4.3 in [GTT15] to show that B is an $\bar{\mathbb{F}}$-Brownian motion under $\bar{\mathbb{P}}^0$. We also know that

$$\bar{\mathbb{P}}^m(T_n > N) \quad = \quad 0$$

so $\mathbb{E}^{\bar{\mathbb{P}}^m}(\phi(T_n)) = 0$ for any smooth bounded function ϕ with $\phi(x) = 0$ for $x \leq N$ and $\phi(x) > 0$ for $x > N$. Hence

$$\lim_{m\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^m}(\phi(T_n)) \quad = \quad \mathbb{E}^{\bar{\mathbb{P}}^0}(\phi(T_n)) \quad = \quad 0$$

so $T_n \leq N$ $\bar{\mathbb{P}}^0$-a.s. as required, which implies that $B_{T_n \wedge \cdot}$ is U.I. under $\bar{\mathbb{P}}^0$ so in particular

$$\mathbb{E}^{\bar{\mathbb{P}}^0}(B_{T_n}) \quad = \quad x_0\,. \tag{3.1}$$

Finally, we have to show that $\mathbb{E}^{\bar{\mathbb{P}}^0}(\mathcal{X}_i) = \mathcal{L}^0_i$ for $1 \leq i \leq M$, which will also show that $\mathcal{L}^0 \in \mathcal{A}$:

- For $M_e \leq i \leq M$ we have

$$\mathcal{L}^0_i \quad = \quad \lim_{m\to\infty} \mathcal{L}^m_i \quad = \quad \lim_{m\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^m}(\mathcal{X}_i) \quad = \quad \mathbb{E}^{\bar{\mathbb{P}}^0}(\mathcal{X}_i)$$

  because the $\mathcal{X}_i$'s are bounded and continuous, so in particular $\mathbb{E}^{\bar{\mathbb{P}}^0}(\mathcal{X}_M) = 0$, hence B cannot go negative before T_n $\bar{\mathbb{P}}^0$-a.s.

- For $1 \leq i \leq M_e$ we have

$$\lim_{m\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^m}(K_i - B_{T_{k_i}})^+ \quad = \quad \mathbb{E}^{\bar{\mathbb{P}}^0}(K_i - B_{T_{k_i}})^+$$

  because under $\bar{\mathbb{P}}^m$ and $\bar{\mathbb{P}}^0$, B cannot go negative due to the presence of the artificial contract ψ_M so the put option payoff here is bounded. Hence, we have

$$\begin{aligned}
\lim_{m\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^m}(\mathcal{X}_i) \quad &= \quad \lim_{m\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^m}(B_{T_{k_i}} - K_i)^+ \\
&= \quad \lim_{m\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^m}(B_{T_{k_i}} - K_i + (K_i - B_{T_{k_i}})^+) \\
&= \quad x_0 - K_i \, + \mathbb{E}^{\bar{\mathbb{P}}^0}(K_i - B_{T_{k_i}})^+ \\
&= \quad \mathbb{E}^{\bar{\mathbb{P}}^0}(B_{T_{k_i}}) - K_i \, + \mathbb{E}^{\bar{\mathbb{P}}^0}(K_i - B_{T_{k_i}})^+ \\
&= \quad \mathbb{E}^{\bar{\mathbb{P}}^0}(B_{T_{k_i}} - K_i)^+
\end{aligned}$$

  where we have used (3.1) in the penultimate line.

□

Lemma 3.2. *The map $\mathcal{L} \in \mathcal{A} \mapsto P(\mathcal{L}) \in \mathbb{R}$ is concave and upper semicontinuous in the usual Euclidean topology and for all $\mathcal{L} \in \mathcal{A}$, the supremum is attained by some $\bar{\mathbb{P}}^* \in \bar{\mathcal{P}}_N(\mathcal{L})$.*

Proof. We can easily verify that $\mathcal{A}$ is convex. Now consider $\mathcal{L}^1, \mathcal{L}^2 \in \mathcal{A}$. Then we can find some $\bar{\mathbb{P}}_1 \in \mathcal{P}_N(\mathcal{L}^1)$, $\bar{\mathbb{P}}_2 \in \mathcal{P}_N(\mathcal{L}^2)$ such that $P(\mathcal{L}^1) \leq \mathbb{E}^{\bar{\mathbb{P}}_1}(\Phi) + \varepsilon$ and $P(\mathcal{L}^2) \leq \mathbb{E}^{\bar{\mathbb{P}}_2}(\Phi) + \varepsilon$. Since $\alpha\bar{\mathbb{P}}_1 + (1-\alpha)\bar{\mathbb{P}}_2 \in \bar{\mathcal{P}}_N(\alpha\mathcal{L}_1 + (1-\alpha)\mathcal{L}_2)$ for $\alpha \in [0,1]$, then we have

$$\begin{aligned} P(\alpha\mathcal{L}_1 + (1-\alpha)\mathcal{L}_2) \quad \geq \quad \mathbb{E}^{\alpha\bar{\mathbb{P}}_1 + (1-\alpha)\bar{\mathbb{P}}_2}(\Phi) \quad &= \quad \alpha\mathbb{E}^{\bar{\mathbb{P}}_1}(\Phi) + (1-\alpha)\mathbb{E}^{\bar{\mathbb{P}}_2}(\Phi) \\ &\geq \quad \alpha P(\mathcal{L}^1) + (1-\alpha)P(\mathcal{L}^2) - 2\varepsilon \end{aligned}$$

and the required concavity follows on sending $\varepsilon \to 0$. To establish upper semicontinuity, let $\mathcal{L}^m \in \mathcal{A}$ with $\mathcal{L}^m \to \mathcal{L}^0$. By definition of $P(.)$, there exists some $\bar{\mathbb{P}}^m$ such that

$$\mathbb{E}^{\bar{\mathbb{P}}^m}(\Phi) \quad \leq \quad P(\mathcal{L}^m) \quad \leq \quad \mathbb{E}^{\bar{\mathbb{P}}^m}(\Phi) + \frac{1}{m}\,.$$

Then

$$\limsup_{m\to\infty} P(\mathcal{L}^m) \quad = \quad \limsup_{m\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^m}(\Phi) \quad = \quad \lim_{k\to\infty} \mathbb{E}^{\bar{\mathbb{P}}_{m_k}}(\Phi)$$

for some subsequence m_k and $\bar{\mathbb{P}}_{m_k} \overset{w}{\to} \bar{\mathbb{P}}^0$ for some probability measure $\bar{\mathbb{P}}^0$ on $\bar{\Omega}_{x_0}$, where we have used the tightness of the sequence $(\bar{\mathbb{P}}^m)$ from the previous lemma and Prokhorov's theorem. Using the upper semi-continuity of Φ we have

$$\limsup_{m\to\infty} P(\mathcal{L}^m) \quad = \quad \lim_{k\to\infty} \mathbb{E}^{\bar{\mathbb{P}}_{m_k}}(\Phi) \quad \leq \quad \mathbb{E}^{\bar{\mathbb{P}}_0}(\Phi) \quad \leq \quad P(\mathcal{L}^0)$$

where the final inequality follows because $\bar{\mathbb{P}}^0 \in \mathcal{P}_N(\mathcal{L}^0)$ (by Lemma 3.1). To show that the sup is attained, we just replace $\mathcal{L}^m$ with $\mathcal{L}$ (for all m) and repeat the arguments above to get

$$P(\mathcal{L}) \quad = \quad \lim_{k\to\infty} \mathbb{E}^{\bar{\mathbb{P}}_{m_k}}(\Phi) \quad \leq \quad \mathbb{E}^{\bar{\mathbb{P}}_0}(\Phi) \quad \leq \quad P(\mathcal{L})\,.$$

□

3.2. *The first duality*

We now prove the first duality result.

Proposition 3.2. *We have the duality*

$$P(\mathcal{L}) \quad = \quad D_0(\mathcal{L})\,.$$

Proof. Following the proof of Theorem 2.4. in [GTT15], we first extend the definition of $P(.)$ to the linear space $\mathbb{R}^M$, setting $P(\mathcal{L}) = -\infty$ if $\mathcal{L} \notin \mathcal{A}$. Then P is still concave and USC; thus by the Fenchel-Moreau theorem (see e.g. Theorem 4.1 in [GTT15b]), we have that

$$P(\mathcal{L}) \quad = \quad P^{**}(\mathcal{L})\,.$$

But we also have

$$\begin{aligned} D_0(\mathcal{L}) \quad = \quad \inf_{\lambda\in\Lambda}[\sup_{\bar{\mathbb{P}}\in\bar{\mathcal{P}}_N} \mathbb{E}^{\bar{\mathbb{P}}}(\Phi - \lambda.\mathcal{X} + \lambda.\mathcal{L})] \quad &= \quad \inf_{\lambda\in\Lambda}[\lambda.\mathcal{L} + \sup_{\mathcal{L}'\in\mathbb{R}^M} \sup_{\bar{\mathbb{P}}\in\bar{\mathcal{P}}_N(\mathcal{L}')} \mathbb{E}^{\bar{\mathbb{P}}}(\Phi - \lambda.\mathcal{X})] \\ &= \quad \inf_{\lambda\in\Lambda}[\lambda.\mathcal{L} + \sup_{\mathcal{L}'\in\mathbb{R}^M}[-\lambda.\mathcal{L}' + P(\mathcal{L}')]] \\ &= \quad \inf_{\lambda\in\Lambda}[\lambda.\mathcal{L} - \inf_{\mathcal{L}'\in\mathbb{R}^M}[\lambda.\mathcal{L}' - P(\mathcal{L}')]] \\ &= \quad \inf_{\lambda\in\Lambda}[\lambda.\mathcal{L} - P^*(\lambda)] \quad = \quad P^{**}(\mathcal{L})\,. \end{aligned}$$

□

As in [GTT15], we let $\mathcal{T}_N^a$ denote the collection of all families of $\mathbb{F}^a$-stopping times $\tau = (\tau_1, .., \tau_n)$ with $\tau_1 \leq ... \leq \tau_n$ such that $\tau_n \leq N$, and $\mathcal{T}_N^0$ the collection of all $\mathbb{F}^a$-stopping times τ such that $\tau \leq N$.

Lemma 3.3. *We have*

$$\sup_{\bar{\mathbb{P}}\in\bar{\mathcal{P}}_N} \mathbb{E}^{\bar{\mathbb{P}}}\big[\Phi - \sum_{i=1}^M \lambda_i(\mathcal{X}_i - \mathcal{L}_i) \quad = \quad \sup_{\tau\in\mathcal{T}_N^a} \mathbb{E}^{\mathbb{P}_0}\big[\Phi(B,\tau) - \sum_{i=1}^M \lambda_i(\mathcal{X}_i(B,\tau) - \mathcal{L}_i)\big]\,.$$

Proof. Follows from Eq 4.20 in [GTT15] which in turn follows from Lemma 4.5 in [GTT15], because we can absorb our exotic $\mathcal{X}_i$ claims into their Φ function, and the European options in our calibration set are continuous with linear growth and therefore fall in their Λ set. □

3.3. The second duality

Following section 4.3.2 in [GTT15], we let $\mathcal{H}$ denote the collection of all $\mathbb{F}^a$-predictable processes $H : \mathbb{R}^+ \times \Omega_{x_0} \to \mathbb{R}$ such that $\int_0^N H_s^2 ds < \infty$ $\mathbb{P}_0$-a.s. and $(H.B) := \int_0^{\cdot} H_s dB_s$ is an $\mathbb{F}^a$-martingale under $\mathbb{P}_0$. Let

$$\begin{aligned} \mathcal{D} \quad := \quad & \{(\lambda, z, H^{)} \in \Lambda \times \mathbb{R} \times (\mathcal{H})^n : z + \sum_{i=1}^{M} \lambda_i \mathcal{X}_i(B, \theta_1, \ldots, \theta_n) + \sum_{k=1}^{n} \int_{\theta_{k-1}}^{\theta_k} H_s^k dB_s \quad \geq \quad \Phi(B, \theta_1, \ldots, \theta_n) \\ & \mathbb{P}_0 - a.s. \text{ for all } 0 \leq \theta_1 \leq \cdots \leq \theta_n.\} \end{aligned} \tag{3.2}$$

Proposition 3.3. *If $\mathcal{P}(\mathcal{L})$ is non-empty, we have the second duality*

$$P(\mathcal{L}) \quad = \quad D(\mathcal{L}) \quad := \quad \inf_{(\lambda,z,H)\in\mathcal{D}} (z + \sum_{i=1}^{M} \lambda_i \mathcal{L}_i)\,.$$

Proof. We first need to prove the easier inequality

$$P(\mathcal{L}) \quad \leq \quad D(\mathcal{L})$$

and for this we use a similar argument to Eq 3.1 in [HO15]. $\mathcal{P}(\mathcal{L})$ is non-empty by assumption and $\mathcal{D}$ is clearly non-empty because Φ is bounded, and by definition of $\mathcal{D}$, for any triplet $(\lambda, z, H^1, \ldots, H^n) \in \mathcal{D}$

$$z + \sum_{i=1}^{M} \lambda_i \mathcal{X}_i(B, \theta_1, \ldots, \theta_n) + \sum_{k=1}^{n} \int_{\theta_{k-1}}^{\theta_k} H_s^k dB_s \quad \geq \quad \Phi(B, \theta_1, \ldots, \theta_n) \quad , \quad \mathbb{P}_0 - a.s. \text{ and for all } 0 \leq \theta_1 \leq \cdots \leq \theta_n.$$

Hence the above inequality also holds

$$z + \sum_{i=1}^{M} \lambda_i \mathcal{X}_i(B, T) + \sum_{k=1}^{n} \int_{\theta_{k-1}}^{\theta_k} H_s^k dB_s \quad \geq \quad \Phi(B, T) \ , \quad \bar{\mathbb{P}} - \text{a.s.}$$

for **any** $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_N$ (recall that the measure $\bar{\mathbb{P}}(B \in (.))$ is just the Wiener measure $\mathbb{P}^0$).

If we choose any $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_N(\mathcal{L}) \subset \bar{\mathcal{P}}_N$ and take the expectation of the above expression under $\bar{\mathbb{P}}$, we see that

$$z + \sum_{i=1}^{M} \lambda_i \mathcal{L}_i \quad \geq \quad \mathbb{E}^{\bar{\mathbb{P}}}(\Phi(B, T)).$$

Since this holds for any arbitrary triplet $(\lambda, z, H) \in \mathcal{D}'$ and any $\bar{\mathbb{P}} \in \bar{\mathcal{P}}_N(\mathcal{L})$, we see that

$$D(\mathcal{L}) \quad = \quad \inf_{(\lambda,z,H)\in\mathcal{D}'} [z + \sum_{i=1}^{M} \lambda_i \mathcal{L}_i] \quad \geq \quad \sup_{\bar{\mathbb{P}}\in\bar{\mathcal{P}}_N(\mathcal{L})} \mathbb{E}^{\bar{\mathbb{P}}} \Phi(B, T) \quad = \quad \mathcal{P}(\mathcal{L}). \tag{3.3}$$

The reverse inequality is more involved, but will essentially follow from similar arguments to the proof of Proposition 4.11 in [GTT15] before they take the limit as $N \to \infty$. We give a sketch of the proof below.

The payoffs for Φ and the exotic options $\mathcal{X}_i$, $M_e + 1 \leq i \leq M$ are bounded by assumption and the European options $\mathcal{X}_i$, $1 \leq i \leq M_e$ have linear growth, so we have the bound

$$|\Phi^\lambda(\omega, \theta)| \quad \leq \quad C(1 + \sum_{i=1}^{n} |\omega_{\theta_i}|)$$

for some constant $C > 0$, where $\Phi^\lambda(\omega, \theta) := \Phi(\omega, \theta) - \lambda(\omega, \theta)$ and $\lambda(\omega, \theta) = \sum_{i=1}^{M} \lambda_i \mathcal{X}_i(\omega, \theta)$.

We now break the proof into three parts, first dealing with the case $n = 1$, then $n = 2$, and then the general case using recursion.

- The case $n = 1$. We work under the augmented Brownian filtration $\mathbb{F}^a$ throughout and recall that

$$|\Phi^\lambda(B,t)| \quad \leq \quad C(1+|B_t|) \tag{3.4}$$

Consider the family of random variables $|B_{\tau\wedge N}|$ for all stopping times $\tau < \infty$. This family is U.I. (because $\mathbb{E}^{\mathbb{P}_0}(|B_{\tau\wedge N}|1_{|B_{\tau\wedge N}|>R}) \leq \frac{1}{R}\mathbb{E}^{\mathbb{P}_0}(B^2_{\tau\wedge N}) = \frac{1}{R}\mathbb{E}^{\mathbb{P}_0}(\tau\wedge N) \leq \frac{N}{R}$), and hence so is $\Phi^\lambda(B,\tau\wedge N)$ (from (3.4)), so the process $\Phi^\lambda(B,t\wedge N)$ is of class D and from Assumption 2.1 we know that $\Phi^\lambda(B,t)$ is a càdlàg process; hence using Lemma 4.6 in [GTT15] with $Y_t = \Phi^\lambda(B,\tau\wedge N)$, there is a cádlág supermartingale $Z^{1,N}_t$ which is the Snell envelope for the optimal stopping problem $\sup_{\tau\in\mathcal{T}^0}\mathbb{E}^{\mathbb{P}_0}(Y_t) = \sup_{\tau\in\mathcal{T}^0_N}\mathbb{E}^{\mathbb{P}_0}(\Phi^\lambda(B,\tau))$.

For all $\tau\in\mathcal{T}^0_N$ we know that $\mathbb{E}^{\mathbb{P}_0}(\Phi^\lambda(B,\tau)) \leq \mathbb{E}^{\mathbb{P}_0}(C(1+|B_\tau|) \leq \mathbb{E}^{\mathbb{P}_0}(C(1+|B_N|)$, so $Z^{1,N}_0 = \sup_{\tau\in\mathcal{T}^0_N}\mathbb{E}^{\mathbb{P}_0}(\Phi^\lambda(B,\tau)) \leq \mathbb{E}^{\mathbb{P}_0}(C(1+|B_N|) < \infty$. Moreover, from (3.4) we see that

$$\begin{aligned} Z^{1,N}_t \quad &= \quad \sup_{\tau\in\mathcal{T}^0_N:\tau\geq t}\mathbb{E}^{\mathbb{P}_0}(\Phi^\lambda(B,\tau)|\mathcal{F}_t) \quad \leq \quad \mathbb{E}^{\mathbb{P}_0}(C(1+|B_N|)|\mathcal{F}_t) \\ &\leq \quad \mathbb{E}^{\mathbb{P}_0}(C(1+|B_t|+|B_N-B_t|)|\mathcal{F}_t) \\ &= \quad C(1+|B_t|) + \mathbb{E}^{\mathbb{P}_0}(C(|B_{N-t}|) \\ &\leq \quad C(1+|B_t|) + \mathbb{E}^{\mathbb{P}_0}(C(|B_N|) \\ &\leq \quad C_1(1+|B_t|) \end{aligned} \tag{3.5}$$

for some constant C_1; thus we can easily verify that $Z^{1,N}$ is of class D using the same argument as above and hence (by the Doob-Meyer decomposition - see e.g. Theorem 1.4.10 in [KS91]) we have that

$$Z^{1,N}_t \quad = \quad M_t - A_t \quad \geq \quad \Phi^\lambda(B,t)$$

for $t \leq N$, where M is an U.I. right continuous martingale, and A_t is an increasing process with $A_0 = 0$. Furthermore, by the martingale representation theorem in Theorem V.3.4 on page 200 in [RY99], we have

$$M_t \quad = \quad M_0 + \int_0^t H_s dB_s \quad = \quad Z^{1,N}_0 + \int_0^t H_s dB_s$$

for some predictable process H (hence H is also progressively measurable, see e.g. Proposition 2.42 in [CB12]) and $\int_0^N H_s^2 ds < \infty$ $\mathbb{P}_0$-a.s. Putting this together, we see that

$$Z^{1,N}_0 + (H\cdot B)_t \quad \geq \quad Z^{1,N}_t \quad \geq \quad \Phi(B,t) - \lambda(B,t)$$

for all $t\in[0,N]$, $\mathbb{P}_0$-a.s. (note that the $Z^{1,N}_0$ term was missing here in [GTT15]). Hence we see that $Z^{1,N}_0$ is a sufficient initial amount of capital to superhedge $\Phi(B,T) - \sum_{i=1}^M \lambda_i\mathcal{X}_i(B,T)$. But from Lemma 3.3 we know that $D_0(\mathcal{L}) = Z^{1,N}_0$, hence the minimal superhedging cost $D(\mathcal{L}) \leq D_0(\mathcal{L}) = P(\mathcal{L})$.

- The case $n = 2$. For $n = 2$, we first consider the optimal stopping problem

$$\sup_{\tau\in\mathcal{T}^0_N}\mathbb{E}^{\mathbb{P}_0}(\Phi_2(B,\tau) - \lambda_2(B,\tau)) \tag{3.6}$$

where $\lambda_2(.,.)$ is the sum of all static positions in the calibration set whose terminal payoffs are not $\mathcal{F}^a_{T_1}$-measurable (these are the contracts which expire after time t_1 in the real world), and we let λ_1 denote the remaining static positions. Similar to above, we know that

$$|\Phi_2(B,t) - \lambda_2(B,t)| \quad \leq \quad C(1+|B_t|)$$

hence (again by Lemma 4.6 in [GTT15]) (3.6) has a Snell envelope which we denote by $Z^{2,N}$, and $|Z^{2,N}_t| \leq C(1+|B_t|)$ and

$$Z^{2,N}_t \quad \geq \quad \Phi_2(B,t) - \lambda_2(B,t)$$

for all $t\leq N$, $\mathbb{P}_0$-a.s. $\Phi_2(B,t) - \lambda_2(B,t)$ is not bounded from below, but

$$\Phi_2(B,t) - \lambda_2(B,t) \quad \geq \quad -C(1+|B_t|) \quad \geq \quad -\sup_{0\leq s\leq t} C(1+|B_s|) \quad \geq \quad M_t$$

for all $t\leq N$, where $M_t := -\mathbb{E}^{\mathbb{P}_0}(\sup_{0\leq s\leq N} C(1+|B_s|)\,|\,\mathcal{F}_t)$ and M_t is a $\mathcal{F}^a_t$-martingale. Hence we can replace $\Phi_2(B,t) - \lambda_2(B,t)$ with the non-negative process $\Phi_2(B,t) - \lambda_2(B,t) - M_t$ and use that $\sup_{\tau\in\mathcal{T}^N_0}\mathbb{E}^{\mathbb{P}_0}(\Phi_2(B,\tau) -$

$\lambda_2(B,\tau) - M_\tau) = \sup_{\tau\in\mathcal{T}_0^N} \mathbb{E}^{\mathbb{P}_0}(\Phi_2(B,\tau) - \lambda_2(B,\tau)) - M_0 < \infty$. Hence we can apply formula D.5 and D.7 in Proposition D.2 in [KS01] [1] we take the sup over τ_2 inside the expectation to get

$$\begin{aligned}\sup_{\tau\in\mathcal{T}_N^a} \mathbb{E}^{\mathbb{P}_0}(\Phi(B,\tau) - \lambda(B,\tau)) &= \sup_{\tau_1\in\mathcal{T}_N^0} \mathbb{E}^{\mathbb{P}_0}(\sup_{\tau_2\in\mathcal{T}_N^0:\tau_2\geq\tau_1} \mathbb{E}^{\mathbb{P}_0}(\Phi_2(B,\tau_2) - \lambda_2(B,\tau_2)|\mathcal{F}^a_{\tau_1}) + \Phi_1(B,\tau_1) - \lambda_1(B,\tau_1)) \\ &= \sup_{\tau_1\in\mathcal{T}_N^0} \mathbb{E}^{\mathbb{P}_0}(Z^{2,N}_{\tau_1} + \Phi_1(B,\tau_1) - \lambda_1(B,\tau_1))\end{aligned}$$

and let $Z^{1,N}$ denote the cádlág supermartingale which is the Snell envelope associated with the expression on the right hand side, which also satisfies $|Z^{2,N}_t + \Phi_1(B,t) - \lambda_1(B,t)| \leq C(1+|B_t|)$. Then $Z^{1,N}, Z^{2,N}$ are both right continuous supermartingales of class (D). From the definition of the Snell envelope, then we have

$$\begin{aligned}Z^{1,N}_{\theta_1} &\geq Z^{2,N}_{\theta_1} + \Phi_1(B,\theta_1) - \lambda_1(B,\theta_1)\,, \\ Z^{2,N}_{\theta_2} &\geq \Phi_2(B,\theta_2) - \lambda_2(B,\theta_2)\end{aligned}$$

for all $\theta_1 \leq \theta_2$. Combining these two expressions we see that

$$Z^{1,N}_0 + Z^{1,N}_{\theta_1} - Z^{1,N}_0 + Z^{2,N}_{\theta_2} - Z^{2,N}_{\theta_1} \geq \Phi(B,\theta_1,\theta_2) - \lambda(B,\theta_1,\theta_2)\,.$$

By similar arguments to before, we may consider the right continuous modifications of $Z^{1,N}, Z^{2,N}$. Moreover, $Z^{1,N}_0 = \sup_{\tau\in\mathcal{T}_N^a} \mathbb{E}^{\mathbb{P}_0}(\Phi(B,\tau) - \lambda(B,\tau))$. After applying Doob-Meyer and the the martingale representation theorem as above, we get $\mathbb{F}^a$-predictable processes H^1, H^2 such that

$$Z^{1,N}_0 + \int_0^{\theta_1} H^1_s dB_s + \int_{\theta_1}^{\theta_2} H^2_s dB_s \geq \Phi(B,\theta_1,\theta_2) - \lambda(B,\theta_1,\theta_2), \qquad \mathbb{P}_0 - a.s., \text{ for all } 0\leq\theta_1\leq\theta_2\leq N \tag{3.7}$$

and we see that $Z^{1,N}_0$ is a sufficient amount of cash to superhedge the claim $\Phi - \lambda$.

- For $n > 2$ we just use the same recursive argument as above.

□

4. Financial application and the martingale transport problem

In this section, we show how the duality results of the previous section are applied to compute the minimal superhedging cost of path-dependent option in the real world, given a finite set of tradeable options as specified in the introduction.

Let $X = (X)_{0\leq t\leq 1}$ denote the asset price of interest with $X_0 = x_0$, and natural filtration $\tilde{\mathbb{F}} := (\tilde{\mathcal{F}}_t)_{0\leq t\leq 1}$. Let $\mathcal{M}_N$ denote the collection of all continuous martingale measures $\tilde{\mathbb{P}}$ (i.e. probability measures under which X is an continuous $\tilde{\mathcal{F}}_t$-martingale) such that $\langle X\rangle_1 \leq N$, $\tilde{\mathbb{P}}$-a.s. From e.g. Theorem 2 in Karandikar[Kar95], there is a non-decreasing $\tilde{\mathbb{F}}$-progressive process $\langle X\rangle$ which coincides with the quadratic variation of X, $\tilde{\mathbb{P}}$-a.s., for every martingale measure $\tilde{\mathbb{P}}$. Let

$$\langle X\rangle_t^{-1} = \inf\{s\geq 0 : \langle X\rangle_s > t\} \quad \text{and} \quad W_t := X_{\langle X\rangle_t^{-1}}\,.$$

Now let $\xi(X) = \Phi(W, \langle X\rangle_{t_1}, .., \langle X\rangle_{t_n})$ and $\Gamma_i(X) = \mathcal{X}_i(W, \langle X\rangle_{t_1}, .., \langle X\rangle_{t_n})$ for $1 \leq i \leq M$ where Φ and $\mathcal{X}_i$ are defined as in section 1 and $0 < t_1 < .. < t_n = 1$) denote the set of (real-world) maturities that are discussed in the introduction of the article. We set

$$\mathcal{M}_N(\mathcal{L}) := \{\tilde{\mathbb{P}} \in \mathcal{M}_N : \mathbb{E}^{\tilde{\mathbb{P}}}(\Gamma_i(X)) = \mathcal{L}_i \quad , \quad 1\leq i\leq M\}\,.$$

Then the martingale transport problem associated with these constraints is given by

$$\tilde{P}(\mathcal{L}) := \sup_{\tilde{\mathbb{P}}\in\mathcal{M}_N(\mathcal{L})} \mathbb{E}^{\tilde{\mathbb{P}}}(\xi(X))\,.$$

The real-world financial application here is the following: we are computing the supremum of the expected value of the claim $\xi(X)$ over all martingale models which are consistent with the market prices of the tradeable calibration set; we now wish to prove the duality which shows that this supremum is equal to the minimum superhedging cost of the

[1] specifically, we set $v = 0$ in D.5 and we sup over all $\tau \in S_v$ in D.7, and for us $\tau = \tau^1$ and $\rho = \tau_2 \geq \tau_1$

exotic derivative with payoff $\xi(X)$ at maturity 1, using just cash, dynamic trading in X and a static position in the tradeable options.

Let $\tilde{\mathcal{H}}_N$ denote the collection of all $\tilde{\mathbb{F}}$-progressive processes such that

$$\int_0^1 \tilde{H}_s^2 d\langle X\rangle_s < +\infty \quad \mathcal{M}_N - \text{q.s. and } (\tilde{H}.X) \text{ is a martingale } \forall \tilde{\mathbb{P}} \in \mathcal{M}_N\,.$$

Then the corresponding two dual problems are

$$\tilde{D}_0(\mathcal{L}) \;:=\; \inf_{\lambda\in\Lambda} \sup_{\tilde{\mathbb{P}}\in\mathcal{M}_N} \mathbb{E}^{\tilde{\mathbb{P}}}\big[\xi(X) - \sum_{i=1}^M \lambda_i(\Gamma_i - \mathcal{L}_i)\big]$$

and $\tilde{D}(\mathcal{L}) = \inf_{(\lambda,\tilde{H}^1,\ldots,\tilde{H}^n)\in\tilde{D}} \sum_{i=1}^M \lambda_i\mathcal{L}_i$, where

$$\tilde{D} \;:=\; \{(\lambda, z, \tilde{H}^1,\ldots,\tilde{H}^n) \in \Lambda\times\mathbb{R}\times(\tilde{\mathcal{H}}_N)^n : \sum_{i=1}^M \lambda_i\mathcal{X}_i(X) + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \tilde{H}_s^k dX_s \;\geq\; \xi \quad \mathcal{M}_N - \text{ q.s. }\}\,.$$

Theorem 4.1. *We have the duality*

$$\tilde{P}(\mathcal{L}) \;=\; \tilde{D}(\mathcal{L})\,.$$

Proof. As in the proof of Theorem 3.2 in [GTT15], combining the dualities we have already obtained: $P(\mathcal{L}) = D_0(\mathcal{L}) = D(\mathcal{L})$, and the weak dualities $\tilde{P}(\mathcal{L}) \leq \tilde{D}(\mathcal{L})$, it suffices to prove that $P(\mathcal{L}) \leq \tilde{P}(\mathcal{L})$ and $D(\mathcal{L}) \geq \tilde{D}(\mathcal{L})$, because this will imply that

$$D(\mathcal{L}) \;=\; P(\mathcal{L}) \;\leq\; \tilde{P}(\mathcal{L}) \;\leq\; \tilde{D}(\mathcal{L}) \;\leq\; D(\mathcal{L}).$$

- $P(\mathcal{L}) \leq \tilde{P}(\mathcal{L})$. Let

$$M_t \;:=\; B_{T_k + \frac{t-t_k}{t_{k+1}-t_k} \wedge T_{k+1}}$$

for $t \in [t_k, t_{k+1})$ and $0 \leq k \leq n-1$ and $M_1 = B_{T_n}$; then M is a continuous martingale for all $\mathbb{P} \in \bar{\mathcal{P}}_N$ and $M_{t_k} = B_{T_k}$. Let $\bar{\mathbb{P}} \in \mathcal{P}_N(\mathcal{L})$ be arbitrary, then $\tilde{\mathbb{P}} = \bar{\mathbb{P}}\circ M^{-1} \in \mathcal{M}_N(\mathcal{L})$ and $\langle M\rangle_{t_k} = T_k$ for $1 \leq k \leq n$ $\bar{\mathbb{P}}$-a.s., so we have

$$\begin{aligned}\xi(M) &= \Phi(B, \langle M\rangle_{t_1}, .., \langle M\rangle_{t_n}) = \Phi(B,T)\,,\\ \Gamma_i(M) &= \mathcal{X}_i(B, \langle M\rangle_{t_1}, .., \langle M\rangle_{t_n}) = \mathcal{X}_i(B,T)\end{aligned}$$

and $\mathbb{E}^{\bar{\mathbb{P}}}(\Gamma_i(M)) = \mathcal{L}_i$ for $1 \leq i \leq M$. Thus we have $\tilde{P}(\mathcal{L}) \geq P(\mathcal{L})$.

- $\tilde{D}(\mathcal{L}) \leq D(\mathcal{L})$. Let $(\lambda, z, \bar{H}^1, \ldots, \bar{H}^n) \in \mathcal{D}$, i.e.

$$z + \sum_{i=1}^M \lambda_i\mathcal{X}_i(B,\theta_1,\ldots,\theta_n) + \sum_{k=1}^n \int_{\theta_{k-1}}^{\theta_k} \bar{H}_s^k dB_s \quad\geq\quad \Phi(B,\theta_1,\ldots,\theta_n) \quad \forall 0\leq\theta_1\leq\cdots\leq\theta_n\leq N\;,\; \mathbb{P}_0 - \text{a.s.}$$

For every $\tilde{\mathbb{P}} \in \mathcal{M}_N$ it follows from the Dambis-Dubins-Schwarz theorem that the time-changed process $W_t = X_{\langle X\rangle_t^{-1}}$ is a Brownian motion under $\tilde{\mathcal{F}}_{\langle X\rangle_t^{-1}}$ under $\tilde{\mathbb{P}}$ and

$$X_t \;=\; W_{\langle X\rangle_t}$$

for all $t \in [0,1]$ $\tilde{\mathbb{P}}$-a.s., and $\langle X\rangle_{t_k}$ are $\tilde{\mathcal{F}}_{\langle X\rangle_t^{-1}}$-stopping times. Then under $\tilde{\mathbb{P}}$

$$z + \sum_{i=1}^M \lambda_i\mathcal{X}_i(W, \langle X\rangle_{t_1},...,\langle X\rangle_{t_n}) + \sum_{k=1}^n \int_{\langle X\rangle_{t_{k-1}}}^{\langle X\rangle_{t_k}} \bar{H}_s^k dW_s \quad\geq\quad \Phi(W, \langle X\rangle_{t_1},...,\langle X\rangle_{t_n})\,.$$

Define

$$\tilde{H}_s^k \;:=\; \bar{H}^k_{\langle X\rangle_s}\,.$$

Then it follows from Propositions V.1.4 and V.1.5 of [RY99] that $\tilde{H}^k$ are $\tilde{\mathbb{F}}$-progressively measurable such that

$$\sum_{k=1}^{n}\int_{\langle X\rangle_{t_{k-1}}}^{\langle X\rangle_{t_k}} (\tilde{H}^k_s)^2 d\langle X\rangle_s \quad = \quad \sum_{k=1}^{n}\int_{\langle X\rangle_{t_{k-1}}}^{\langle X\rangle_{t_k}} (\bar{H}^k_s)^2 ds \quad \leq \quad \int_0^N (\bar{H}^k_s)^2 ds \quad < \quad +\infty, \quad \tilde{\mathbb{P}}-\text{a.s.}$$

and

$$\sum_{i=1}^{M}\lambda_i\Gamma_i(X) + \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \tilde{H}^k_s dX_s \quad \geq \quad \xi(X), \qquad \tilde{\mathbb{P}}-\ \text{a.s.}$$

$\bar{H}^k \in \mathcal{H}$ and hence $(\bar{H}^k \cdot W)$ is a martingale under $\tilde{\mathbb{P}}$, which implies by the time-change argument that the stochastic integral $(\bar{H}^k.W)_{\langle X\rangle}$ is a martingale under $\tilde{\mathbb{P}}$ and hence so is $\tilde{H}^k \cdot X$. Hence $\tilde{H}^k \in \tilde{H}$ and $(\lambda, z, \tilde{H}^1, \ldots, \tilde{H}^n) \in \tilde{D}$. It follows that $\tilde{D}(\mathcal{L}) \leq D(\mathcal{L})$, as required.

□

4.1. The final duality - removing the artificial contract

Note that at the moment, the second duality formula takes an inf over Λ which includes λ_M, i.e. a position in the artificial contract, but we can re-write $D(\mathcal{L})$ as

$$\begin{aligned}\tilde{D}(\mathcal{L}) \quad &= \inf_{\lambda_1,\ldots\lambda_{M-1}\in\mathbb{R}} \inf_{z\in\mathbb{R}} \inf_{\lambda_M\in\mathbb{R}} \big[z + \sum_{i=1}^{M}\lambda_i\mathcal{L}_i \,:\, \exists(\tilde{H}^1,\ldots,\tilde{H}^n) \in (\tilde{\mathcal{H}})^n : \\ &\qquad z + \sum_{i=1}^{M}\lambda_i\Gamma_i(X) + \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\tilde{H}^k_s dX_s \;\geq\; \xi(X) \qquad \mathcal{M}_N-\text{q.s.}\big]\end{aligned} \tag{4.1}$$

where we use the convention that the inf of an empty set is $+\infty$. We now define $D^+(\mathcal{L})$ as

$$\begin{aligned}D^+(\mathcal{L}) \quad &:= \inf_{\lambda_1,\ldots\lambda_{M-1}\in\mathbb{R}} \inf_{z\in\mathbb{R}} \big[z + \sum_{i=1}^{M-1}\lambda_i\mathcal{L}_i \,:\, \exists(\tilde{H}^1,\ldots,\tilde{H}^n) \in (\tilde{\mathcal{H}})^n : \\ &\qquad z + \sum_{i=1}^{M-1}\lambda_i\Gamma_i(X) + \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\tilde{H}^k_s dX_s \quad \geq \quad \xi(X) \quad \mathcal{M}^+_N-\text{q.s.}\big]\end{aligned} \tag{4.2}$$

Note that the infs in both expressions are either $\sum_{i=1}^{M}\lambda_i\mathcal{L}_i$ if an admissible superhedging strategy exists (recall that $\mathcal{L}_M = 0$), or $+\infty$ otherwise. We first fix $\lambda_1, ..., \lambda_{M-1} \in \mathbb{R}^+$. Then if the innermost inf in (4.1) is $z + \sum_{i=1}^{M}\lambda_i\mathcal{L}_i$, then there must exist a $\lambda_M \in \mathbb{R}$ such that the superhedge works $\mathcal{M}_N$-q.s, and hence also $\mathcal{M}^+_N$-q.s (where $\mathcal{M}^+_N$ denotes the non-negative elements of $\mathcal{M}^+$) , hence $\mathcal{D}^+(\mathcal{L}) \leq \tilde{\mathcal{D}}(\mathcal{L})$ because the artificial contract $\mathcal{X}_M$ pays nothing for paths which go negative. But from the first duality we also know that

$$\tilde{\mathcal{D}}(\mathcal{L}) \quad = \sup_{\tilde{\mathbb{P}}\in\mathcal{M}_N(\mathcal{L})} \mathbb{E}^{\tilde{\mathbb{P}}}(\xi(X)) \quad = \sup_{\tilde{\mathbb{P}}\in\mathcal{M}^+_N(\mathcal{L})} \mathbb{E}^{\tilde{\mathbb{P}}}(\xi(X)) \quad \leq \quad \mathcal{D}^+(\mathcal{L}) \tag{4.3}$$

where $\mathcal{M}^+_N(\mathcal{L})$ is the set of non-negative elements of $\mathcal{M}_N(\mathcal{L})$. Hence $\tilde{\mathcal{D}}(\mathcal{L}) = \mathcal{D}^+(\mathcal{L})$, so $P(\mathcal{L}) = \mathcal{D}^+(\mathcal{L})$ which is the final duality result we wanted.

4.2. Examples

If $\Phi(B,T) = 1_{\bar{B}_{T_n}\geq b}$ where $\bar{B}_t = \sup_{0\leq s\leq t} B_s$ and $b > X_0$, then $\Phi : \bar{\Omega}_{x_0} \to \mathbb{R}$ is non-anticipative, USC and $\Phi(B,T)$ is càdlàg and thus satisfies our Assumptions (this corresponds to a One-Touch option in the real world which pays $1_{\bar{X}_{t_n}\geq b}$ when the barrier is greater than the initial asset price). Similarly $\Phi = 1_{\underline{B}_{T_n}\leq b}$ where $\underline{B}_t = \inf_{0\leq s\leq t} B_s$ also satisfies our assumptions (this is a One-Touch with barrier below X_0). Our set of tradeable instruments $\mathcal{X}$ has to be continuous, and hence can include payoffs of the form $\mathcal{X}_i = (\bar{B}_{T_{k_i}} - K)^+$ or $\mathcal{X}_i = (\underline{B}_{T_{k_i}} - K)^+$ (i.e. lookback options) for $1 \leq k_i \leq n$ or $\mathcal{X}_i = T_n - V$ (which corresponds to a variance swap) or $\mathcal{X}_i = (T_n - K)^+$ (which corresponds to a variance call option), because these payoffs are bounded and continuous because we are working under $\bar{\mathbb{P}}_N$ (see also Example 3.3 in [GTT15]).

5. Robust hedging given a joint law for the terminal level and minimum at a single maturity

We now return to the set up in [GTT15] where there is no restriction that T_n is bounded, and we set $n = 1$ and $T := T_1$ and we replace their sample space with $\Omega_{x_0} = C_{x_0}(\mathbb{R}^+, \mathbb{R})$ as before. We define

$$\begin{aligned} \bar{\mathcal{P}} &:= \{\bar{\mathbb{P}} \in \bar{\mathcal{P}}(\bar{\Omega}_{x_0}) : B \text{ is an } \bar{\mathbb{F}}\text{-Brownian motion with } B_0 = x \text{ and and } B_{.\wedge T} \text{ is U.I.}\} \\ \hat{\mathcal{P}} &:= \{\bar{\mathbb{P}} \in \bar{\mathcal{P}}(\bar{\Omega}_{x_0}) : B \text{ is an } \bar{\mathbb{F}}\text{-Brownian motion with } B_0 = x \text{ and} \\ &\qquad B_{.\wedge T} \text{ is U.I. and } T \le \tau_0\}\,. \end{aligned} \tag{5.1}$$

Let $\Phi : \bar{\Omega}_{x_0} \to \mathbb{R}$ and assume that Φ satisfies the same conditions as in section 2 with the non-anticipative condition that $\Phi(\omega, \theta) = \Phi(\omega_{\theta\wedge\tau_0\wedge.}, \theta)$, where $\tau_0 = \inf\{s : B_s = 0\}$. We now consider the primal problem

$$\underline{P}(\mu) \;:=\; \sup_{\underline{\mathcal{P}}(\mu)} \mathbb{E}^{\bar{\mathbb{P}}}(\Phi(B, T))$$

where

$$\underline{\mathcal{P}}(\mu) \;:=\; \{\bar{\mathbb{P}} \in \hat{\mathcal{P}} \,:\, (B_T, \underline{B}_T) \sim \mu\}$$

and $\underline{B}_t = \inf_{0\le s\le t} B_s$ is the running minimum of B and μ is a given probability measure on $\mathbb{R}\times(-\infty, x_0]$. In subsection 5.2 below, we adapt Rogers's well known result, which gives an if and only if condition on μ for $\underline{\mathcal{P}}(\mu)$ to be non-empty.

5.1. Financial Application

Let X denote the asset price process in the real world as in section 4, and consider a down-and-out call option on X which pays $(X_t - K)^+ 1_{\underline{X}_t > b}$ at time t (in the real world) for some $K \ge b \ge 0$. Then the price of such an option should be given by

$$P(K, b) \;:=\; \mathbb{E}^{\mathbb{Q}}((X_t - K)^+ 1_{\underline{X}_t > b})$$

for some martingale measure $\mathbb{Q}$, and the computing the right-derivative of P with respect to K, we have

$$P'_+(K, b) \;:=\; -\mathbb{Q}(X_t > K, \underline{X}_t > b)\,.$$

Thus if we are given $P(K, b)$ for all $K \ge b \ge 0$, we can extract the complementary joint distribution function of $(X_t, \underline{X}_t)$ and hence $\mu = \mu(dx, dy) = \mathbb{Q}(X_t \in dx, \underline{X}_t \in dy)$; this will then be the target law μ for $(B_T, \underline{B}_T)$ in the previous subsection, using the same time-change arguments as in section 4.

5.2. Re-formulating the Rogers characterization for the admissible joint laws

For a given Brownian motion B we let $\bar{B}_t = \sup_{0\le s\le t} B_s$ denote its running maximum process. Then by Theorem 3.1 and Eq 3.8 in Duembgen&Rogers[DR14] (see also [Rog12] and Theorem 3.1 in [Rog93]), we have the following result:

Proposition 5.1. *A probability measure μ on $\mathbb{R}\times\mathbb{R}^+$ is the joint law of $(B_\tau, \bar{B}_\tau)$ for a Brownian motion B with $B_0 = 0$ and some U.I. stopping time τ if and only if*

$$\begin{cases} \int\int 1_{y\ge a}\,(x-a)\,\mu(dx, dy) \;=\; 0 \qquad \forall a \ge 0 \\[2ex] \int\int |x|\,\mu(dx, dy) \;<\; \infty\,. \end{cases} \tag{5.2}$$

Remark 5.2. *For $a = x_0$ the first condition reduces to the centering condition $\int\int x\,\mu(dx, dy) = 0$. Moreover, applying the dominated convergence theorem to the sequence $\int 1_{y\ge a+\frac{1}{m}}\,(x - a - \frac{1}{m})$, we see that the first condition implies that $\int\int 1_{y>a}\,(x-a)\,\mu(dx, dy) = 0$.*

Making trivial amendments to Proposition 5.1, we have the following.

Proposition 5.3. *A probability measure μ on $\mathbb{R}\times(-\infty,x_0]$ is the joint law of $(B_\tau,\underline{B}_\tau)$ for a Brownian motion B with $B_0=x_0>0$ and some U.I. stopping time τ if and only if*

$$\begin{cases} \int\int 1_{y-x_0\leq a}\,(x-x_0-a)\,\mu(dx,dy) \;=\; 0 \qquad \forall a\leq 0 \\ \int\int |x|\,\mu(dx,dy)\;<\;\infty\,. \end{cases} \tag{5.3}$$

Proposition 5.4. (5.3) *is satisfied if and only if the following two conditions hold:*

$$\begin{cases} \hat{g}(p) \;=\; \int\int G_p(x-x_0,y-x_0)\,\mu(dx,dy)\;=\;0\,, \\ \int\int |x|\,\mu(dx,dy)\;<\;\infty \end{cases} \tag{5.4}$$

for all $p>0$, where $G_p(x,y)=\frac{1}{p^2}[1+px+e^{py}(p(y-x)-1)]$.

Proof. See Appendix A. □

Remark 5.5. *Proposition 5.4 is useful because the functions $G_p(.,.)$ for $p>0$ are continuous with linear growth (unlike $1_{y-x_0\leq a}\,(x-x_0-a)$), and this is the space of functions that we use when working with the Wasserstein topology below in Lemma 5.2.*

5.3. Technical lemmas

Henceforth, we make the natural assumption that μ satisfies the two conditions in (5.4) and we further impose that $\mu\{y<0\}=0$ to impose that $T\leq\tau_0$. This ensures that $\underline{\mathcal{P}}(\mu)$ is non-empty, and we let $\underline{\mathbf{P}}$ denote the space of probability measures μ on $\mathbb{R}\times(-\infty,x_0]$ which satisfy these three conditions.

Let $\mathbf{M}$ denote the space of all finite signed measures ν on $\mathbb{R}^2$ such that $\int(1+|x|+|y|)|\nu|(dx,dy)<\infty$. We will require the following lemma when we apply the bi-conjugate theorem in Proposition 5.6.

Lemma 5.1. ***M*** *is a locally convex Hausdorff space under $\mathcal{W}^1$, and its dual space can be identified by $\boldsymbol{M}^*=\Lambda$, where $\Lambda=\mathcal{C}^1$ (the space of continuous functions with linear growth on $\mathbb{R}^2$).*

Proof. Follows from trivial modifications to Lemma 4.2 in [GTT15b] which deals with the case when we have measures on $\mathbb{R}$ not $\mathbb{R}^2$. □

The first dual problem is now given by

$$\begin{aligned} \underline{D}_0(\mu) \quad &:= \quad \inf_{\lambda\in\Lambda}\sup_{\bar{\mathbb{P}}\in\hat{\mathcal{P}}}\mathbb{E}^{\bar{\mathbb{P}}}\big[\Phi(B,T)-\lambda(B,T)\;+\;\mu(\lambda)\big] \\ &= \quad \inf_{\lambda\in\Lambda}\sup_{\bar{\mathbb{P}}\in\bar{\mathcal{P}}}\mathbb{E}^{\bar{\mathbb{P}}}\big[\Phi(B,T\wedge\tau_0)-\lambda(B,T\wedge\tau_0)\;+\;\mu(\lambda)\big] \\ &= \quad \inf_{\lambda\in\Lambda}\sup_{\tau\in\mathcal{T}^0}\mathbb{E}^{\mathbb{P}_0}\big[\Phi(B,\tau\wedge\tau_0)-\lambda(B_{\tau\wedge\tau_0},\underline{B}_{\tau\wedge\tau_0})\;+\;\mu(\lambda)\big] \end{aligned} \tag{5.5}$$

(recall that $\Phi(\omega,\theta)=\Phi(\omega_{\theta\wedge\tau_0\wedge .},\theta)$), where $\Phi^\lambda:=\Phi-\lambda$ and $\mathcal{T}^0$ is the collection of all $\mathbb{F}^a$-stopping times such that $B_{\tau\wedge .}$ is U.I. for any $\tau\in\mathcal{T}^0$, and the second equality again follows from Lemma 4.5 in [GTT15] and B denotes standard Brownian motion with $B_0=x_0$ and an absorbing barrier at zero.

Lemma 5.2. *Under the Wasserstein topology $\mathcal{W}^1$ (see [GTT15],[GTT15b] for a definition), $\underline{\boldsymbol{P}}$ is a closed convex subspace of $\boldsymbol{M}$.*

Proof. Convexity is obvious. Now let $\mu^m\in\underline{\mathbf{P}}$ be a sequence such that $\mu^m\overset{\mathcal{W}^1}{\to}\mu$. Then $G_p\in\mathcal{C}^1$ so we have

$$0\;=\;\lim_{m\to\infty}\int G_p(x,y)\mu^m(dx,dy)\;=\;\int G_p(x,y)\mu(dx,dy)\,.$$

□

Lemma 5.3. *Let $\mu^m \in \underline{\boldsymbol{P}}$ be a sequence which tends to $\mu^0 \in \underline{\boldsymbol{P}}$ under $\mathcal{W}^1$. Let $\bar{\mathbb{P}}^m$ be a sequence of probability measures with $\bar{\mathbb{P}}^m \in \underline{\mathcal{P}}(\mu^m)$. Then $(\bar{\mathbb{P}}^m)_{m\geq 1}$ is tight under the weak topology, and any limit point of $\bar{\mathbb{P}}^m$ lies in $\underline{\mathcal{P}}(\mu^0)$.*

Proof. We proceed using similar steps (with the same numbered bullet points) as in Lemma 4.3 in [GTT15]:

1. For all $\varepsilon > 0$, there exists a compact set $D \subset \Omega$ such that $\bar{\mathbb{P}}^m(D^c \times \Theta) \leq \varepsilon$, because $\bar{\mathbb{P}}^m$ induces the same measure on Ω_{x_0} for all m (i.e. the Wiener measure). Moreover, from Proposition B.1 in Appendix B, we have
$$\bar{\mathbb{P}}^m(T \geq C) \quad \leq \quad \frac{x_0^2+1}{C^{1/3}} - \frac{x_0^2}{C} \quad \leq \quad \varepsilon$$
for $C = C(\varepsilon)$ sufficiently large, where the middle equality follows because we are assuming that $\mu\{y < 0\} = 0$ so $\mu\{x < 0\} = 0$ as well. Then it follows that
$$\bar{\mathbb{P}}^m(\omega \in D^c \cup \{T > C\}) \quad \leq \quad 2\varepsilon$$
so
$$\bar{\mathbb{P}}^m(K_\varepsilon) \quad \geq \quad 1 - 2\varepsilon$$
where $K_\varepsilon = D \times \{T \leq C\}$ is compact, so we have tightness, hence we can apply Prokhorov's theorem as in bullet point (i) in Lemma 4.3 in [GTT15b] to show that $\bar{\mathbb{P}}^m$ has a limit point $\bar{\mathbb{P}}^0$ under the weak topology.
2. To verify that B is an $\mathbb{F}$-Brownian motion under $\bar{\mathbb{P}}^0$, we again proceed as in [GTT15] .
3. To show that $\bar{\mathbb{P}}^0(T \leq \tau_0) = 1$, let ϕ be a bounded continuous function such that $\phi(x) = 0$ if $x \geq 0$ and $\phi(x) > 0$ when $x < 0$. Then $\mathbb{E}^{\bar{\mathbb{P}}^m}(\phi(\underline{B}_T)) = 0$ for all m and hence $\mathbb{E}^{\bar{\mathbb{P}}^0}(\phi(\underline{B}_T)) = 0$ which gives us $T \leq \tau_0$ $\mathbb{P}^0$-a.s.
4. We prove that $B_{T\wedge .}$ is U.I. using the same argument as [GTT15]. Finally, we have to verify that $(B_T, \underline{B}_T) \sim \mu^0$ under $\bar{\mathbb{P}}^0$. But
$$\mathbb{E}^{\bar{\mathbb{P}}^0}(e^{ikB_T+iu\underline{B}_T}) \quad = \quad \lim_{m\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^m}(e^{ikB_T+iu\underline{B}_T}) \quad = \quad \lim_{m\to\infty} \mu^m(\Upsilon) \quad = \quad \mu^0(\Upsilon)$$
where $\Upsilon(x, y) = e^{ikx+iuy}$, because the map $(\omega, \theta) \mapsto \omega_\theta$ is continuous and Υ is continuous and bounded.

□

Lemma 5.4. *The map $\mu \in \underline{\boldsymbol{P}} \mapsto P(\mu) \in \mathbb{R}$ is concave and upper semicontinuous under $\mathcal{W}^1$ and the supremum is attained by some $\bar{\mathbb{P}}^* \in \underline{\mathcal{P}}(\mu)$.*

Proof. Follows by almost identical argument to Lemma 4.7 in [GTT15]. □

5.4. The first duality

We now prove the first duality result.

Proposition 5.6. *We have the duality*
$$P(\mu) \quad = \quad \underline{D}_0(\mu)\,.$$

Proof. Following the proof of Theorem 2.4 in [GTT15], we first extend the definition of $P(.)$ to the linear space of finite signed measures on $\mathbb{R}^2$, setting $P(\mu) = -\infty$ if $\mu \notin \underline{\mathbf{P}}$. Then P is still concave and USC; thus by the Fenchel-Moreau theorem (see e.g. Theorem 4.1 in [GTT15b]), we have that
$$P(\mu) \quad = \quad P^{**}(\mu)\,.$$

But we also have
$$\begin{aligned}
\underline{D}_0(\mu) \quad &= \quad \inf_{\lambda\in\Lambda}[\sup_{\bar{\mathbb{P}}\in\hat{\mathcal{P}}} \mathbb{E}^{\bar{\mathbb{P}}}(\Phi - \lambda + \mu(\lambda))] \quad = \quad \inf_{\lambda\in\Lambda}[\mu(\lambda) + \sup_{\nu\in\underline{\mathbf{P}}} \sup_{\bar{\mathbb{P}}\in\hat{\mathcal{P}}(\nu)} \mathbb{E}^{\bar{\mathbb{P}}}(\Phi - \nu(\lambda))] \\
&= \quad \inf_{\lambda\in\Lambda}[\mu(\lambda) + \sup_{\nu\in\underline{\mathbf{P}}}[-\nu(\lambda) + P(\nu)] \\
&= \quad \inf_{\lambda\in\Lambda}[\mu(\lambda) - \inf_{\nu\in\underline{\mathbf{P}}}[\,\nu(\lambda) - P(\nu)] \\
&= \quad \inf_{\lambda\in\Lambda}[\mu(\lambda) - P^*(\lambda)] \quad = \quad P^{**}(\mu)\,.
\end{aligned}$$

□

5.5. Removing the negative elements of Λ

Recall that B is Brownian motion and $T \leq \tau_0$ so $\underline{B}_T \in [0, x_0]$. We also know that any $\lambda \in \Lambda$ satisfies $\lambda(x,y) \leq K_\lambda(1+|x|+|y|)$ for some constant K_λ, but from the previous sentence we can effectively assume that $\lambda(x,y) \leq K_\lambda(1+x)$ so $-\lambda(x,y) \geq -K_\lambda(1+x)$. This means that

$$\begin{aligned}\sup_{\tau\in\mathcal{T}^0} \mathbb{E}^{\mathbb{P}_0}(\Phi(B,\tau\wedge\tau_0) - \lambda(B_{\tau\wedge\tau_0}, \underline{B}_{\tau\wedge\tau_0}) + \mu(\lambda)) &= \sup_{\tau\in\mathcal{T}^0} \mathbb{E}^{\mathbb{P}_0}(\Phi(B,\tau\wedge\tau_0) - \lambda(B_{\tau\wedge\tau_0}, \underline{B}_{\tau\wedge\tau_0}) + K_\lambda(1 + B_{\tau\wedge\tau_0}) + \mu(\lambda')) \\ &= \sup_{\tau\in\mathcal{T}^0} \mathbb{E}^{\mathbb{P}_0}(\Phi(B,\tau\wedge\tau_0) - \lambda'(B_{\tau\wedge\tau_0}, \underline{B}_{\tau\wedge\tau_0}) + \mu(\lambda')) \qquad (5.6)\end{aligned}$$

where $\lambda'(x,y) = \lambda(x,y) - K_\lambda(1+x)$ so $-\lambda'(x,y) \geq 0$. Hence we can replace the Λ with Λ^+, the set of non-negative elements of Λ.

5.6. The second duality

We let $\mathcal{H}$ denote the collection of all $\mathbb{F}^a$-predictable processes $H : \mathbb{R}^+ \times \Omega_{x_0} \to \mathbb{R}$ such that $\int_0^t H_s^2 ds < \infty$ for every $t \geq 0$ $\mathbb{P}^0$-a.s. and $(H.B) := \int_0^{\cdot} H_s dB_s$ is an $\mathbb{F}^a$-strong supermartingale (see [GTT15] for a definition) under $\mathbb{P}^0$. Let

$$\underline{\mathcal{D}} \;:=\; \{(\lambda, H) \in \Lambda \times \mathcal{H} : \lambda(B_t, \hat{\underline{B}}_t) + \int_0^t H_s dB_s \;\geq\; \Phi(B,t) \text{ for every } t \geq 0 \text{ and } \mathbb{P}^0 - a.s.\} \qquad (5.7)$$

Proposition 5.7. *We have the second duality*

$$\underline{P}(\mu) \;=\; \underline{D}(\mu) \;:=\; \inf_{(\lambda,H)\in\underline{\mathcal{D}}} \mu(\lambda)\,.$$

Proof. The weak duality

$$\underline{P}(\mu) \;\leq\; \underline{D}(\mu)$$

follows by the same argument used at the beginning of the proof of Proposition 3.3.

The reverse inequality will again follow from the same arguments as the proof of Proposition 4.11 in [GTT15] and we now have to carefully take the limit as $N \to \infty$. We again work under the augmented Brownian filtration $\mathbb{F}^a$ throughout. Φ is bounded and λ has at most linear growth and from the previous section it suffices to consider non-negative λ contracts, so we have

$$-C(1 + |B_t|) \;\leq\; \Phi^\lambda(B,t) \;\leq\; C \qquad (5.8)$$

where $\Phi^\lambda(B,t) := \Phi(B,t) - \lambda(B_t, \underline{B}_t)$. For N fixed, the family of random variables $\Phi^\lambda(B, \tau\wedge N)$ for all stopping times $\tau < \infty$ is U.I. as before, so the process $\Phi^\lambda(B, t\wedge N)$ is of class D and from Assumption 2.1 we know that $\Phi^\lambda(B,t)$ is a càdlàg process; hence by Lemma 4.6 in [GTT15] there is a supermartingale $Z_t^{1,N}$ which is the Snell envelope for the optimal stopping problem $\sup_{\tau\in\mathcal{T}^a} \mathbb{E}^{\mathbb{P}_0}[\Phi^\lambda(B,\tau\wedge\tau_0\wedge N)] = \sup_{\tau\in\mathcal{T}_N^a} \mathbb{E}^{\mathbb{P}_0}[\Phi^\lambda(B,\tau\wedge\tau_0)]$. Setting $Z_t^1 = \sup_N Z_t^{1,N}$, we first note that $\sup_N Z_t^{1,N} = \lim_{N\to\infty} Z_t^{1,N} \leq C$ because $Z_t^{1,N}$ is increasing in N. But we also know that $|Z_t^{1,N}| \leq C(1+|B_t|)$, and $\mathbb{E}^{\mathbb{P}_0}(C(1+|B_t|)) < \infty$ for t fixed. Using the supermartingale property of $Z^{1,N}$ and the conditional dominated convergence theorem letting $N \to \infty$ with t fixed, we see that

$$\mathbb{E}^{\mathbb{P}_0}(Z_t^1 \,|\, \mathcal{F}_s^a) \;=\; \lim_{N\to\infty} \mathbb{E}^{\mathbb{P}_0}(Z_t^{1,N} \,|\, \mathcal{F}_s^a) \;\leq\; \lim_{N\to\infty} Z_s^{1,N} \;=\; Z_s^1 \qquad (5.9)$$

so Z_t^1 is also a supermartingale. By the same argument as page 16 in [GTT15] we can show that Z^1 is right continuous in expectation and hence admits a right continuous modification (we note that they should be taking $\sigma_\varepsilon \vee \tau_n$ not $\sigma_\varepsilon \wedge \tau_n$ in their proof).

From (5.8) we can also easily verify that $Z^{1,N}$ is of class D (using the same argument as in (3.5)) and hence Z^1 is of class DL; hence by the Doob-Meyer decomposition (Theorem 1.4.10 in [KS91]) we have that

$$Z_t^1 \;=\; M_t - A_t \;\geq\; \Phi^\lambda(B, t\wedge\tau_0) \qquad (5.10)$$

for $t \geq 0$, where M is a right continuous martingale, and A_t is an increasing process with $A_0 = 0$. Furthermore, by the martingale representation theorem in Theorem V.3.4 on page 200 in [RY99], we have

$$M_t \;=\; M_0 + \int_0^t H_s dB_s \;=\; Z_0^1 + \int_0^t H_s dB_s$$

for some predictable process H and $\int_0^t H_s^2 ds < \infty$ $\mathbb{P}_0$-a.s. Putting this together, we see that

$$Z_0^1 + (H\cdot B)_t \;\geq\; Z_t^1 \;\geq\; \Phi(B, t\wedge\tau_0) - \lambda(B_{t\wedge\tau_0}, \hat{\underline{B}}_{t\wedge\tau_0})$$

$\mathbb{P}^0$-a.s, and hence

$$Z_0^1 + (H\cdot B)_T \;\geq\; Z_T^1 \;\geq\; \Phi(B,T) - \lambda(B_T, \underline{B}_T)$$

$\hat{\mathcal{P}}$-q.s. Hence we see that $Z_0^1 = \underline{D}_0(\mu) = P(\mu)$ is a sufficient initial amount of capital to superhedge $\Phi(B,T) - \lambda(B_T, \hat{\underline{B}}_T)$, so the minimal superhedging cost $\underline{D}(\mu) \leq \underline{D}_0(\mu)$. □

Remark 5.8. *We can translate this into a duality result in the "real world" where the stock price is a continuous non-negative martingale, in a similar spirit to Theorem 4.1; we omit the details for the sake of brevity.*

6. Robust hedging given marginal laws for the terminal level and minimum at a single maturity

In this subsection we consider the modified primal problem

$$\underline{P}(\mu, \underline{\mu}) \;:=\; \sup_{\bar{\mathcal{P}}(\mu,\underline{\mu})} \mathbb{E}^{\bar{\mathbb{P}}}(\Phi(B,T))$$

where

$$\bar{\mathcal{P}}(\mu, \underline{\mu}) \;:=\; \{\bar{\mathbb{P}} \in \hat{\mathcal{P}} \,:\, B_T \sim \mu\,,\, \underline{B}_T \sim \underline{\mu}\}$$

where $\hat{\mathcal{P}}$ is defined as (5.1), and we assume there exists a $\nu \in \underline{\mathbf{P}}$ (defined above) such that $\mu(dx) = \int_{y\in[0,x\wedge x_0]} \nu(dx,dy)$, $\underline{\mu}(dy) = \int_{x\in[y,\infty)} \nu(dx,dy)$ which ensures that $\bar{\mathcal{P}}(\mu,\underline{\mu})$ is non-empty. We let $\mathbf{P}_2$ denote the collection of ordered pairs $(\mu,\underline{\mu})$ of probability measures which satisfy this property.

6.1. Financial Application

Let X denotes the asset price process in the real world as in section 4, and consider a digital call option on X which pays $1_{X_1>x}$ at time 1 and a One-Touch option on X which pays $1_{\underline{X}_t<b}$ at time t, for $b < x_0$. Then the price of these two options should be given by

$$D(x) \;=\; \mathbb{Q}(X_t > x) \quad , \quad \mathcal{O}(b) \;=\; \mathbb{Q}(\underline{X}_t \leq b)$$

for some martingale measure $\mathbb{Q}$. Thus if we given $D(x)$ for all $x \geq 0$ and $\mathcal{O}(b)$ for all $b \in [0, x_0]$, we can extract the target laws $(\mu, \underline{\mu})$ for $(B_T, \hat{\underline{B}}_T)$ in the previous subsection, using the same time-change arguments as in section 4.

6.2. Technical Lemmas

The first dual problem is now given by

$$\begin{aligned}
\underline{D}_0(\mu) \;&:=\; \inf_{\lambda,\underline{\lambda}\in\Lambda} \sup_{\bar{\mathbb{P}}\in\hat{\mathcal{P}}} \mathbb{E}^{\bar{\mathbb{P}}}\big[\Phi(B,T) - \lambda(B,T) - \underline{\lambda}(B,T) + \mu(\lambda) + \underline{\mu}(\underline{\lambda})\big] \\
&=\; \inf_{\lambda,\underline{\lambda}\in\Lambda} \sup_{\bar{\mathbb{P}}\in\hat{\mathcal{P}}} \mathbb{E}^{\bar{\mathbb{P}}}\big[\Phi(B,T\wedge\tau_0) - \lambda(B,T\wedge\tau_0) - \underline{\lambda}(B,T\wedge\tau_0) + \mu(\lambda) + \underline{\mu}(\underline{\lambda})\big] \\
&=\; \inf_{\lambda,\underline{\lambda}\in\Lambda} \sup_{\tau\in\mathcal{T}^0} \mathbb{E}^{\mathbb{P}_0}\big[\Phi(B,\tau\wedge\tau_0) - \lambda(B_{\tau\wedge\tau_0}) - \underline{\lambda}(B_{\tau\wedge\tau_0}) + \mu(\lambda) + \underline{\mu}(\underline{\lambda})\big] \qquad (6.1)
\end{aligned}$$

where now $\Lambda = \mathcal{C}^1$ (the space of continuous functions with linear growth on $\mathbb{R}$) and B denotes standard Brownian motion with $B_0 = x_0$.

Lemma 6.1. *Let $(\mu^m, \underline{\mu}^m) \in \boldsymbol{P}_2$ be a sequence which tends to $(\mu^0, \underline{\mu})$ under $\mathcal{W}^1$. Let $\bar{\mathbb{P}}^m$ be a sequence of probability measures with $\bar{\mathbb{P}}^m \in \bar{\mathcal{P}}(\underline{\mu}^m, \underline{\mu}^m)$. Then $(\bar{\mathbb{P}}^m)_{m\geq 1}$ is tight under the weak topology, $(\mu^0, \underline{\mu}^0) \in \boldsymbol{P}_2$ and any limit point of $\bar{\mathbb{P}}^m$ lies in $\bar{\mathcal{P}}(\mu^0, \underline{\mu}^0)$.*

Proof. We proceed using similar steps to the proof of Lemma 5.3. Steps 1,2 and the U.I. property follow by the same arguments. We just have to verify that $B_T \sim \mu$ and $\hat{\underline{B}}_T \sim \underline{\mu}$ under $\bar{\mathbb{P}}^0$. To this end, we see that

$$\begin{aligned}
\mathbb{E}^{\bar{\mathbb{P}}^0}(e^{ikB_T}) &= \lim_{m\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^m}(e^{ikB_T}) = \lim_{m\to\infty} \mu^m(e^{ik(.)}) = \mu^0(e^{ik(.)}), \\
\mathbb{E}^{\bar{\mathbb{P}}^0}(e^{ik\hat{\underline{B}}_T}) &= \lim_{m\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^m}(e^{ik\hat{\underline{B}}_T}) = \lim_{m\to\infty} \underline{\mu}^m(e^{ik(.)}) = \underline{\mu}^0(e^{ik(.)})
\end{aligned}$$

because the map $(\omega, \theta) \mapsto \omega_\theta$ is continuous. □

Lemma 6.2. $(\mu, \underline{\mu}) \in \boldsymbol{P}_2 \mapsto \underline{P}(\mu) \in \mathbb{R}$ *is concave and upper semicontinuous under $\mathcal{W}^1$ and the supremum is attained by some $\bar{\mathbb{P}}^* \in \bar{\mathcal{P}}(\mu, \underline{\mu})$.*

Proof. Follows by almost identical argument to Lemma 4.7 in [GTT15]. □

6.3. The first duality

We now prove the first duality result.

Proposition 6.1. *We have the duality*

$$P(\mu, \underline{\mu}) = D_0(\mu, \underline{\mu}).$$

Proof. Following the proof of Theorem 2.4. in [GTT15], we first extend the definition of $P(.)$ to the linear space of pairs of finite signed measures on $\mathbb{R}$, setting $P(\mu, \underline{\mu}) = -\infty$ if $(\mu, \underline{\mu}) \notin \mathbf{P}_2$. Then P is still concave and USC; thus by the Fenchel-Moreau theorem (see e.g. Theorem 4.1 in [GTT15b]), we have that

$$P(\mu, \underline{\mu}) = P^{**}(\mu, \underline{\mu}).$$

But we also have

$$\begin{aligned}
D_0(\mu, \underline{\mu}) &= \inf_{\lambda, \underline{\lambda} \in \Lambda} [\sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}} \mathbb{E}^{\bar{\mathbb{P}}}(\Phi - \lambda - \underline{\lambda} + \mu(\lambda) + \underline{\mu}(\underline{\lambda}))] = \inf_{\lambda, \underline{\lambda} \in \Lambda} [\mu(\lambda) + \underline{\mu}(\underline{\lambda}) + \sup_{(\nu, \underline{\nu}) \in \mathbf{P}_2} \sup_{\bar{\mathbb{P}} \in \bar{\mathcal{P}}(\nu, \underline{\nu})} \mathbb{E}^{\bar{\mathbb{P}}}(\Phi - \nu(\lambda) - \underline{\nu}(\underline{\lambda}))] \\
&= \inf_{\lambda, \underline{\lambda} \in \Lambda} [\mu(\lambda) + \underline{\mu}(\underline{\lambda}) + \sup_{(\nu, \underline{\nu}) \in \mathbf{P}_2} [-\nu(\lambda) - \underline{\nu}(\underline{\lambda}) + P(\nu, \underline{\nu})] \\
&= \inf_{\lambda, \underline{\lambda} \in \Lambda} [\mu(\lambda) + \underline{\mu}(\underline{\lambda}) - \inf_{(\nu, \underline{\nu}) \in \mathbf{P}_2} [\nu(\lambda) + \underline{\nu}(\underline{\lambda}) - P(\nu, \underline{\nu})] \\
&= \inf_{\lambda, \underline{\lambda} \in \Lambda} [\mu(\lambda) + \underline{\mu}(\underline{\lambda}) - P^*(\lambda, \underline{\lambda})] = P^{**}(\mu, \underline{\mu}).
\end{aligned}$$

□

6.4. Removing the negative elements of $\boldsymbol{\Lambda}$

Recall that B is Brownian motion with $T \leq \tau_0$ so $\hat{\underline{B}}_T \in [0, x_0]$. We also know that any $\lambda, \underline{\lambda} \in \Lambda$ satisfies $\lambda(x) \leq K_\lambda(1 + |x|), \underline{\lambda}(y) \leq K_\lambda(1 + |y|)$ for some constant K_λ, but as before, from the bounds on $\hat{\underline{B}}$ we can effectively assume that $\underline{\lambda}$ is bounded, so we can use the same argument as before to replace Λ with Λ^+, the set of non-negative elements of Λ.

6.5. The second duality

Recall the definition of $\mathcal{H}$ in subsection 5.6 and let

$$\begin{aligned}
\underline{\mathcal{D}} := \{&(\lambda, \underline{\lambda}, H) \in \Lambda \times \Lambda \times \mathcal{H} : \lambda(B_t) + \underline{\lambda}(\underline{B}_t) + \int_0^t H_s dB_s \geq \Phi(B, t) \\
&\text{for every } t \geq 0 \text{ and } \mathbb{P}^0 - a.s.\}
\end{aligned} \tag{6.2}$$

Proposition 6.2. *We have the second duality*

$$\underline{P}(\mu,\underline{\mu}) \quad = \quad \underline{D}(\mu,\underline{\mu}) \quad := \quad \inf_{(\lambda,\underline{\lambda},H)\in\underline{\mathcal{D}}}[\mu(\lambda)+\underline{\mu}(\underline{\lambda})]\,.$$

Proof. Follows by a a very similar argument to the proof of Proposition 5.7. □

Remark 6.3. *As in the previous section, it should be possible to translate this into a duality result in the "real world", and we omit the tedious details.*

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Appendix A: Proof of Proposition 5.4

We first assume that (5.4) holds and that $x_0 = 0$. Let (X, Y) denote two random variables with $(X, Y) \sim \mu$. Then for $p > 0$ we have

$$\begin{aligned}\hat{c}^+(u,p) &= \int_{-\infty}^{\infty} e^{-iua} e^{pa}\, \mathbb{E}(1_{Y\leq a} X)\, 1_{a\leq 0} da = \int_{-\infty}^{\infty} e^{iua} e^{-pa}\, \mathbb{E}(1_{Y\leq -a} X)\, 1_{a\geq 0}\, da \\ &= \mathbb{E}\Big(\int_{-\infty}^{\infty} e^{iua} e^{-pa}\, 1_{Y\leq -a} X\, 1_{a\geq 0}\, da\Big) \\ &= \mathbb{E}(G^+_{p-iu}(X,Y)) \\ &= \frac{1}{p-iu}\mathbb{E}(X(1-e^{(p-iu)Y})) \qquad \text{(A-1)}\end{aligned}$$

for $u \in \mathbb{R}$, where

$$G^+_p(x,y) = \int_0^{\infty} e^{-pa}\, 1_{y\leq -a}\, x\, da = \frac{x}{p}(1-e^{py})$$

(we can apply Fubini in (A-1) because $|1_{Y\leq a}X| \leq |X|$ and $\mathbb{E}(|X|) < \infty$ by assumption). Using a similar argument we find that

$$\begin{aligned}\hat{c}^-(u,p) &= \int_{-\infty}^{\infty} e^{-iua} e^{pa}\, \mathbb{E}(1_{Y\leq a}\, a)\, 1_{a\leq 0}\, da = -\int_{-\infty}^{\infty} e^{iua} e^{-pa}\, a\, \mathbb{E}(1_{Y\leq -a})\, 1_{a\geq 0}\, da \\ &= -\mathbb{E}\Big(\int_{-\infty}^{\infty} e^{iua} e^{-pa}\, 1_{Y\leq -a}\, a)\, 1_{a\geq 0}\, da \\ &= -\mathbb{E}(G^-_{p-iu}(X,Y)) \\ &= -\frac{1}{(p-iu)^2}\mathbb{E}(1-e^{-(p-iu)Y}(1+(p-iu)Y)) \qquad \text{(A-2)}\end{aligned}$$

where

$$G^-_p(x,y) = \int_0^{\infty} e^{-pa} a\, 1_{y\leq -a}\, da = \frac{1}{p^2}[1+e^{py}(py-1)]\,.$$

$\hat{c}^+(.,p)$ may not be in L^1, so we cannot compute its inverse Fourier transform directly. However, from e.g. Theorem 8.3 in [Cha87] we have the following inversion formulae:

$$\begin{aligned}e^{-pa}\mathbb{E}(1_{Y\geq a}\, X) &= \lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-iua}\, e^{-|u|/n}\, \hat{c}^+(u,p)\, du\,, \\ e^{-pa}\mathbb{E}(1_{Y\geq a}\, a) &= \lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-iua}\, e^{-|u|/n}\, \hat{c}^-(u,p)\, du\end{aligned}$$

for almost all (a.a.) $a \geq 0$. Subtracting the second equation from the first and multiplying by e^{pa}, we see that

$$c(a) = \mathbb{E}(1_{Y\geq a}(X-a)) = e^{pa}\lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-iua}\, e^{-|u|/n}\,(\hat{c}^+(u,p) - \hat{c}^-(u,p))\, du$$

for almost all (a.a.) $a \geq 0$.

Subtracting (A-2) from (A-1) we find that $\hat{c}^+(u,p) - \hat{c}^-(u,p) = \hat{g}(p - iu)$, where

$$\hat{g}(p) = \int_0^{\infty} e^{-pa}\, \mathbb{E}(1_{Y\leq a}(X-a))\, da = \mathbb{E}(G_p(X,Y))$$

because $G_p(x,y) = G^+_p(x,y) + G^-_p(x,y)$. But from e.g. Theorem 5a on page in 57 in Widder[Wid46], $\hat{g}$ is analytic in the right half plane and from (5.4), $\hat{g} = 0$ on the positive real axis, thus $\hat{g} = 0$ for all $p \in \mathbb{C}, \Re(p) > 0$. Thus

$$c(a) = 0$$

for a.a. $a > 0$. But from the dominated convergence theorem, $c(.)$ is left continuous, so $c(a) = 0$ for all $a \geq 0$ as required.

To go the other way, we now assume that the Rogers condition in (5.3) holds, and set

$$g(p) \quad := \quad \int_{-\infty}^{0} e^{pa}\,\mathbb{E}(1_{Y\le a}(X-a))da, \quad \text{for } p>0. \tag{A-3}$$

Then by Fubini's theorem, we can interchange the order of integration to get

$$\mathbb{E}(G_p(X,Y)) \quad = \quad 0 \quad \forall p>0$$

as required.

Appendix B: Monroe's result

Proposition B.1. *(see Proposition 7 in [Mon72]). Let W be a Brownian motion with an absorbing barrier at 0 and $W_0 = x_0 > 0$ and let T be a minimal stopping time with $\mathbb{E}|W_T| = x_0$. Then we have*

$$\mathbb{P}(T\ge\lambda) \quad \le \quad \frac{x_0^2+1}{\lambda^{1/3}} - \frac{x_0^2}{\lambda}\,.$$

for $\lambda > 1$.

Proof. Let $T_K = \inf\{t : |W_t| \ge Kx_0\}$ with $K > 1$. Then we have

$$Kx_0\mathbb{P}(T>T_K) \quad \le \quad \mathbb{E}(|W|_{T\wedge T_K}) \quad \le \quad \mathbb{E}(|W|_T) \quad = \quad x_0$$

which we can re-arrange as

$$\mathbb{P}(T>T_K) \quad \le \quad \frac{1}{K}\,.$$

But from the optional stopping theorem we also know that

$$\mathbb{E}(W_{T_K}^2 - T_K) \quad = \quad W_0^2 - 0 \quad = \quad x_0^2\,.$$

This gives us $\mathbb{E}T_K = K^2x_0^2 - x_0^2$ and so

$$\mathbb{P}(T_K>\lambda) \quad \le \quad \frac{K^2x_0^2 - x_0^2}{\lambda}. \tag{B-1}$$

Then if $T > \lambda$ this implies that either $T_K > \lambda$ or $T > \lambda \ge T_K$; hence we have

$$\mathbb{P}(T>\lambda) \quad \le \quad \mathbb{P}(T>T_K) + \mathbb{P}(T_K>\lambda) \quad \le \quad \frac{1}{K} + \frac{K^2x_0^2 - x_0^2}{\lambda}. \tag{B-2}$$

Setting $K = \lambda^{\frac{1}{3}}$ (recall that $K > 1$) and so we must have $\lambda > 1$ we have

$$\mathbb{P}(T>\lambda) \quad \le \quad \frac{x_0^2+1}{\lambda^{1/3}} - \frac{x_0^2}{\lambda}\,.$$

□