Rough volatility and CGMY jumps with a finite history and the Rough Heston model - small-time asymptotics in the $k\sqrt{t}$ regime

Martin Forde Benjamin Smith* Lauri Viitasaari†

Abstract

A small-time Edgeworth expansion is established for near-the-money European options under a general rough stochastic volatility (RSV) model driven by a Riemann-Liouville (RL) process plus an additional generalized tempered stable Lévy process with $Y \in (1,2)$ when $H \in (1-\frac{1}{2}Y, 2(1-\frac{1}{2}Y)\wedge\frac{1}{2})$ [1], in the regime where log-moneyness $\log\frac{K}{S_0} \sim z\sqrt{t}$ as $t \to 0$ for z fixed, conditioned on a finite volatility history. This can be viewed as a more practical variant of Theorem 3.1 in Fukasawa[Fuk17] ([Fuk17] does not allow for jumps or a finite history and uses the somewhat opaque Muravlev representation for fractional Brownian motion), or if we turn off the rough stochastic volatility, the expansion is a variant of the main result in Mijatovic&Tankov[MT16] and Theorem 3.2 in Figueroa-López et al.[FGH17]. The $z\sqrt{t}$ regime is directly applicable to FX options where options are typically quoted in terms of delta (.10, .25 and .50) not absolute strikes, and we also compute a new prediction formula for the Riemann-Liouville process, which allows us to express the history term for the Edgeworth expansion in a more useable form in terms of the volatility process itself. We later relax the assumption of bounded volatility, and we also compute a formal small-time expansion for implied volatility in the Rough Heston model in the same regime (without jumps) which includes an additional at-the-money, convexity and fourth order correction term, and we outline how one can go to even higher order in the three separate cases $H > \frac{1}{6}$, $H = \frac{1}{6}$ and $H < \frac{1}{6}$.[2]

1 Introduction

[Fuk17] derives a small-time Edgeworth expansion for European options under a rough stochastic volatility model where the volatility is a function of the solution to an SDE with an unspecified drift and a linear noise term given by two-sided fractional Brownian motion(fBM), in the asymptotic regime where log-moneyness $\log\frac{K}{S_0} = z\sqrt{t}$, for z fixed. [Fuk17] uses the Muravlev representation for fBM, but using the stochastic Fubini theorem, one can show that the Muravlev representation is equivalent to more commonly used Mandelbrot-van Ness representation. The $z\sqrt{t}$ regime is directly relevant to FX options where options are quoted in deltas not absolute strikes, because fixed strike option prices move around too much. [Fuk17] also show how to include the history of the fBM into the asymptotics, which has been ignored elsewhere in the literature except in the context of the Rough Bergomi model where the history can be inferred from the forward variance term structure, see e.g. [BFG16]. El Euch et al.[EFGR18] go to the next order in this expansion for a class of models which includes the Rough Bergomi model with a non-flat initial variance curve term structure, using a Fourier transform approach with an asymptotic expansion of the characteristic function of the log stock price.

[BFGHS19] derive small-time asymptotics for European options under a rough stochastic volatility in the so-called "moderate deviations" regime, using a stochastic Taylor series with a Laplace approximation inspired by the earlier work of Ben-Arous[BA88], which involves using the Cameron-Martin theorem for (fBM) to switch to a measure under which the large deviation event is no longer atypical, using the solution to the associated unperturbed control ODE which minimizes the rate function (the cheapest admissible control in their terminoogy), [BFGHS19] also compute a Taylor series expansion for the rate function in [FZ17] for the large deviations regime.

[JR16] introduced the Rough Heston stochastic volatility model and show that the model arises naturally as the large-time limit of a market microstructure model driven by two nearly unstable self-exciting Poisson processes (otherwise known as Hawkes process) with a kernel containing a Mittag-Leffler function which drives buy and sell orders (a Hawkes process is a point process where the intensity is itself stochastic and depends on the jump history via the kernel). The microstructure model captures the effects of endogeneity of the market, no-arbitrage, buying/selling asymmetry and the presence of metaorders. [ER19] show that the characteristic function of the log stock price for the

*Dept. Mathematics, King's College London, Strand, London, WC2R 2LS (Benjamin.Smith@kcl.ac.uk)

†Department of Mathematics and Statistics, University of Helsinki, Helsinki, P.O. Box 68, FIN-00014 Finland. (lauri.viitasaari@iki.fi), Department of Information and Service Management, Aalto University School of Business, P.O. Box 21210, 00076 Aalto

[1]this relaxes the more complicated and restrictive condition which appeared in an earlier version of the article.

[2]We thank Masaaki Fukasawa and Hongzhong Zhang for fruitful discussions, Adam Hesse for help with Matlab computations and Vian Dinh for help with IT issues.

Rough Heston model is the solution to a fractional Riccati equation which is non-linear (see also [EFR18] and [ER18]), and the variance curve for the model evolves as $d\xi_u(t)=\kappa(u-t)\sqrt{V_t}dW_t$, where $\kappa(t)$ is the kernel for the V_t process itself multiplied by a *Mittag-Leffler* function. Theorem 2.1 in [ER18] shows that a Rough Heston model conditioned on its history is still a Rough Heston model, but with a time-dependent mean reversion level $\theta(t)$ which depends on the history of the V process.

[GK19] consider a more general class of affine forward variance (AFV) models of the form $d\xi_u(t)=\kappa(u-t)\sqrt{V_t}dW_t$ (for which the Rough Heston model is a special case). They show that AFV models arise naturally as the weak limit of a so-called affine forward intensity (AFI) model, where order flow is driven by two generalized Hawkes-type process with an arbitrary jump size distribution, and we exogenously specify the evolution of the conditional expectation of the intensity at different maturities in the future, in a similar fashion to a conventional variance curve model. The weak limit here involves letting the jump size tends to zero as the jump intensity tends to infinity in a certain way. Using martingale arguments (which do not require considering a Hawkes process as in the aforementioned El Euch&Rosenbaum articles) they show that the mgf of the log stock price for the AFV model satisfies a convolution Riccati equation, or equivalently is a non-linear function of the solution to a VIE. [GGP19] use comparison principle arguments for VIEs to show that the moment explosion time for the Rough Heston model is finite if and only if it is finite for the standard Heston model, and they establish upper and lower bounds for the explosion time.

In this article, we derive a small-time Edgeworth expansion for near-the-money European option under a rough stochastic volatility (RSV) model driven by a Riemann-Liouville (RL) process plus an additional independent tempered stable (CGMY) Lévy process with $Y\in(1,2)$ (i.e. infinite variation), in the Central Limit Theorem-type regime where log-moneyness $\log\frac{K}{S_0}\sim z\sqrt{t}$ as $t\to 0$, for z fixed. Unlike [Fuk17], our result only requires a finite history and our history correction term is expressed in terms of history of the RL process itself (or equivalently the volatility) and thus easier to compute in practice. For the parameter range considered, we find that the jump component gives the first order correction to the call price and the Rough stoc vol component gives the 2nd order correction, and the CGMY component gives rise to a (re-scaled) short-maturity smile which is non-affine in z, in contrast to the pure rough stochastic volatility model in [Fuk17], which affords greater flexibility in fitting short-term smiles in FX options. The jump term is a variant of Theorem 1 in Mijatovic&Tankov[MT16] and Theorem 3.2 in Figueroa-López et al.[FGH17], and we corroborate our results numerically using the moment-matching Monte Carlo scheme described in Horvath et al.[HJM17], and changing to a measure under which the tempered stable process is a pure α-stable process. We also give a self contained proof of the inversion formula for the RL process (i.e. a formula for the driving Brownian motion expressed in terms of the history of the RL process), and use this to compute a new prediction formula for the RL process. We conclude by deriving a formal small-time Edgeworth expansion for implied volatility under the popular Rough Heston model (without jumps) by solving a nested sequence of linear Volterra integral equations, which gives a (higher order) at-the-money and convexity correction to the first order skew term. We also give a blueprint on how to go to higher order, for which there are three separate cases $H>\frac{1}{6}$, $H=\frac{1}{6}$ and $H<\frac{1}{6}$ to consider.

1.1 Model setup

We work on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ throughout, with filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. We assume interest rates are zero and consider the following rough stochastic volatility model with jumps for a log stock price process $X_t=\log S_t$:

$$\begin{cases} dX_t = -\frac{1}{2}v(Y_t)^2dt+v(Y_t)(\rho dW_t+\bar{\rho}dW_t^{\perp})-dL_t\,, \\ Y_t = Y_{t_0} + \sqrt{2H}\int_{t_0}^t(t-s)^{H-\frac{1}{2}}dW_s \end{cases} \tag{1}$$

for $H\in(0,\frac{1}{2})$ and $t_0\leq 0$, where $(W_t)_{t\in\mathbb{R}}$, $(W_t^{\perp})_{t\geq 0}$ are two independent Brownian motions, $|\rho|<1$, $\bar{\rho}=\sqrt{1-\rho^2}$, and $(L_t)_{t\geq 0}$ is a generalized tempered stable process with Lévy density $\nu(x)=\frac{C_+e^{-Mx}}{x^{1+Y}}1_{x>0}+\frac{C_-e^{-G|x|}}{|x|^{1+Y}}1_{x<0}$ independent of W and $W^{\perp}$ with $Y\in(1,2)$ and $\mathbb{E}(e^{-(L_t-L_0)})=1$ for all $t>0$. This implies that

$$\phi_t^L(u) \;:=\; \mathbb{E}(e^{-iu(L_t-L_0)}) \;=\; e^{iubt+C_+\Gamma(-Y)t[(M+iu)^Y-M^Y]+C_-\Gamma(-Y)t[(G-iu)^Y-G^Y]} \tag{2}$$

with b chosen so that $\phi_t^L(-i)=1$, i.e.

$$b \;=\; -C_+((M+1)^Y-M^Y)\Gamma(-Y)\;-\;C_-((G-1)^Y-G^Y)\Gamma(-Y) \tag{3}$$

which ensures that e^{-L_t} is an $\mathcal{F}_t^L$-martingale.

We assume for now that $v\in C_b^2$, with $0\leq v(y)\leq\bar{v}<\infty$; we will relax this assumption in Subsection 2.1.

From here on, we assume $X_0=0$ so $S_0=1$ and $Y_{t_0}=0$ without loss of generality (since otherwise we can just modify $v(.)$ so that $Y_{t_0}=0$) and $L_0=0$ (also without loss of generality, since all Lévy processes have stationary and independent increments).

Remark 1.1 Note that $Y_t=\sqrt{2H}\int_0^{t-t_0}(t-t_0-s)^{H-\frac{1}{2}}dB_s=Z_{t-t_0}^H$ where $B_s=W_{s+t_0}$ is also a Brownian motion and

$$Z_t^H \;=\; \sqrt{2H}\int_0^t(t-s)^{H-\frac{1}{2}}dB_s \tag{4}$$

is a standard Riemann-Liouville (RL) process, which is a Gaussian H-self-similar process like fBM (and Z^H has the same marginals as fBM), but Z^H does not have stationary increments. We interpret t_0 as a reference point in the past (e.g. the date of the company's IPO when the stock started trading) and $t=0$ as the current time, and we assume we know the history of Y over $[t_0,0]$, and we wish to incorporate this history into our asymptotic call/put option estimates.

1.2 Conditional decomposition of the Riemann-Liouville process

Our first important observation is that Y admits the conditional decomposition:

$$\begin{aligned} Y_{\theta u}-Y_0 &= \sqrt{2H}\int_{t_0}^{0}[(\theta u-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}]dW_s + \sqrt{2H}\int_0^{\theta u}(\theta u-s)^{H-\frac{1}{2}}dW_s \\ &= \zeta(\theta u) + \sqrt{2H}\int_0^{\theta u}(\theta u-s)^{H-\frac{1}{2}}dW_s \end{aligned} \tag{5}$$

for $\theta>0$ and $u\in[0,1]$, where $\zeta(u):=\mathbb{E}(Y_u|\mathcal{F}_0^W)-Y_0=\sqrt{2H}\int_{t_0}^0[(u-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}]dW_s$, which is a *Fredholm* Gaussian process in u (see [SV16] for more on these type of processes). We now define

$$Y_u^\theta \;=\; \theta^{-H}(Y_{\theta u}-\mathbb{E}(Y_{\theta u}|\mathcal{F}_0^W)) \;=\; \theta^{-H}\sqrt{2H}\int_0^{\theta u}(\theta u-s)^{H-\frac{1}{2}}dW_s \;\sim\; N(0,u^{2H})$$

and $\zeta(\theta u)$ and Y_u^θ are independent, since they solely depend on (respectively) W before $t=0$ and after $t=0$.

1.3 The inversion formula for the Riemann-Liouville process

Lemma 1.1 *(Inversion formula for the Riemann-Liouville process and equivalence of filtrations). We have the following inversion formula for the Riemann-Liouville process Z_t^H defined in Remark 1.1:*

$$B_t \;=\; \bar{c}_H\int_0^t (t-s)^{\frac{1}{2}-H}dZ_s^H \tag{6}$$

where $\bar{c}_H := [\sqrt{2H}(\frac{1}{2}-H)\pi\sec(H\pi)]^{-1} = [\sqrt{2H}\,\Gamma(\frac{3}{2}-H)\Gamma(\frac{1}{2}+H)]^{-1}$ and $\mathcal{F}_t^B=\mathcal{F}_t^{Z^H}$ (see also Remark 5.5 in [Jost06]).

Proof. See Appendix F. ■

Remark 1.2 See Figure 4 for numerical test of (6) using Monte Carlo.

In practice, we would observe the sample path of $v(Y_t)$ not W, but for $t\geq t_0$ we can re-write W_t in terms of Y_t as

$$W_t \;=\; B_{t-t_0} \;=\; \bar{c}_H\int_0^{t-t_0}(t-t_0-s)^{\frac{1}{2}-H}dZ_s^H \;=\; \bar{c}_H\int_{t_0}^t (t-u)^{\frac{1}{2}-H}dY_u$$

(where we have set $u=s+t_0$, and $B_t=W_{t+t_0}$ and $Z_t^H=Y_{t+t_0}$ as in Remark 1.1). But we know that $Y_t = Y_{t_0}+\sqrt{2H}\int_{t_0}^t(t-s)^{H-\frac{1}{2}}dW_s$, hence we see that $\mathcal{F}_t^W=\mathcal{F}_t^Y$.

1.4 Small-time tail estimate for the Lévy process in the $k_t=z\sqrt{t}$ regime

The following lemma will be needed in the sequel:

Lemma 1.2 [3] *Let L be a Lévy process with a Lévy density $\nu_L(x)$ which is bounded by $\frac{C}{|x|^{1+Y}}1_{\{x\neq 0\}}$ for some $C>0$. Then for any $\alpha>0$, and $0<\epsilon<\frac{1}{2}-\frac{Y}{4}$, there exists a constant $K>0$ such that for θ sufficiently small, we have*

$$\mathbb{P}(|L_\theta|>\alpha\theta^{\frac{1}{2}}) \;\leq\; \mathbb{P}(|L_\theta|>\alpha\theta^{\frac{1}{2}+\varepsilon}) \;\leq\; K\theta^{1-\frac{1}{2}Y-2\epsilon}.$$

Proof. See Appendix B. ■

Remark 1.3 For our two-sided tempered stable Lévy process L_t, we can easily verify that under the measure $\mathbb{P}^*(A):=\mathbb{E}(e^{-L_t}1_A)$ for $A\in\mathcal{F}_t$, if $G>1$ then L is still a two-sided tempered stable Lévy process (recall that we imposed that $\mathbb{E}(e^{-L_t})=1$), but with M replaced by $\bar{M}:=M+1$ and G replaced by $\bar{G}:=G-1$ (see also [FLF12] below Eq 36 and section 2 in [FGH17]). Thus from Lemma 1.2 we also have that

$$\mathbb{E}(e^{-L_\theta}1_{|L_\theta|>\alpha\theta^{\frac{1}{2}}}) \;\leq\; K_2\theta^{1-\frac{1}{2}Y-2\epsilon} \tag{7}$$

for some constant K_2, and θ sufficiently small.

[3] We thank Hongzhong Zhang for outlining the main arguments of this proof

2 The main result

We now state the main result of the article.

Theorem 2.1 *For the model in* (1), *if* $H \in (1-\frac{1}{2}Y, 2(1-\frac{1}{2}Y)\wedge\frac{1}{2})$, $G>1$ *and* $M>1$, *then we have the following asymptotic expansion for the price of European put option with strike* $S_0 e^{\sqrt{\theta}\,z}$ *(with* $z\in\mathbb{R}$*) at time* $t=0$*:*

$$\frac{1}{\sqrt{\theta}}\mathbb{E}((S_0 e^{\sqrt{\theta}\,z}-S_\theta)^+|\mathcal{F}_0^Y) \;=\; \mathbb{E}(z-X_1^0)^+ \,+\, \theta^{1-\frac{1}{2}Y}A_1(z) \,+\, \theta^H(\alpha+\beta_\theta)\phi(\frac{z}{v_0}) \,+\, o(\theta^H) \tag{8}$$

a.s. as $\theta\to 0$, *where* $\phi(z)$ *is the standard normal density,* $X_1^0\sim N(0,v_0^2)$ *is a Normal random variable independent of* L,

$$v_\theta \;:=\; v(Y_\theta)$$

and

$$\begin{aligned} A_1(z) &= \frac{1}{\pi}\int_0^\infty \mathrm{Re}[e^{-iuz}e^{-\frac{1}{2}v_0^2u^2}(C_+\Gamma(-Y)(iu)^{Y-2}+C_-\Gamma(-Y)(-iu)^{Y-2}]du \\ \alpha &= \rho z\frac{\sqrt{2H}}{(\frac{1}{2}+H)(\frac{3}{2}+H)}\frac{v'(Y_0)}{v(Y_0)} \\ \beta_\theta &= \theta^{-H}v'(Y_0)\int_0^1\zeta(\theta u)du \;=\; \theta^{-H}v'(Y_0)\int_{t_0}^0 a_H(s,\theta)Y_s ds \end{aligned} \tag{9}$$

where $a_H(s,\theta)=\frac{1}{\pi}\cos(H\pi)\int_0^1(-\frac{\theta u}{s})^{\frac{1}{2}+H}\frac{1}{\theta u-s}du$. β_θ *is a family of* $\mathcal{F}_0^W$*-measurable Normal random variables with zero expectation, and* $\mathrm{Var}(\beta_\theta)\to v'(Y_0)^2\ell$ *as* $\theta\to 0$, *where*

$$\ell \;:=\; 2H\int_{-\infty}^0[\int_0^1[(u-z)^{H-\frac{1}{2}}-(-z)^{H-\frac{1}{2}}]du]^2dz \;<\; \infty$$

(note that ℓ *is independent of* θ*) and*

$$\theta^{-H}|\int_0^1\zeta(\theta u)du| \;\leq\; (c_1+\varepsilon)\int_0^1(\log\frac{1}{\theta u})^{\frac{1}{2}}u^H du$$

a.s. for all $\varepsilon>0$ *for some deterministic constant* c_1 *and all non-negative* θ *less than some (*$\mathcal{F}_0^Y$*-measurable)* $\theta^*(\varepsilon,\omega)>0$, *so in particular* $\theta^\varepsilon\beta_\theta\to 0$ *a.s. as* $\theta\to 0$.

Remark 2.1 If $t_0=0$ then $\beta_\theta=0$, i.e. there is no history term.

Remark 2.2 (9) shows that for a RSV model with a general non-linear v function, knowing the forward variance curve $\mathbb{E}(v(Y_{\theta u})^2|\mathcal{F}_0)$ is not directly relevant to the small-time Edgeworth asymptotics, rather it shows that what really matters is $\zeta(\theta u)$. Thus if we evolve the model to a later time t, looking at the updated forward variance curve $\mathbb{E}(v(Y_{t+\theta u})^2|\mathcal{F}_t)$ will not help us determine the updated small-time Edgeworth asymptotics (see also Eq (18) in the proof below which is the starting point for this Theorem before we let $\theta\to 0$, and also shows why the variance curve itself is not directly relevant). We only know of explicit relationships between the variance curve and the history $\xi(.)$ for specific parametric models like Rough Bergomi, fractional Stein-Stein and the Rough Heston model (see Theorem 2.1 in [ER18] for the latter). Blind use of variance curves also frequently ignores the regularity properties that variance curves should have as the maturity T approaches the current time t, which is closely related to observation in Eq (38) below that $\zeta(\theta)$ has law-of-the-iterated log-type behaviour at the front end as $\theta\to 0$.

Remark 2.3 By Theorem 1 in Gassiat[Gas19]), S is a martingale, so in particular put-call parity is preserved, and using the put-call parity we see that

$$\begin{aligned} \frac{1}{\sqrt{\theta}}\mathbb{E}((S_\theta-S_0e^{\sqrt{\theta}\,z})^+|\mathcal{F}_0^Y) &= \frac{1}{\sqrt{\theta}}S_0(1-e^{\sqrt{\theta}\,z})+\frac{1}{\sqrt{\theta}}\mathbb{E}((S_0e^{\sqrt{\theta}\,z}-S_\theta)^+|\mathcal{F}_0^Y) \\ &= -z+\frac{1}{\sqrt{\theta}}\mathbb{E}((e^{\sqrt{\theta}\,z}-S_\theta)^+|\mathcal{F}_0^Y)+O(\sqrt{\theta}) \end{aligned} \tag{10}$$

as $\theta\to 0$, where we have also used that $S_0=1$.

From here until the end of Subsection 2.1, the expectation symbol $\mathbb{E}(.)$ means $\mathbb{E}((.)|\mathcal{F}_0^W)=\mathbb{E}((.)|\mathcal{F}_0^Y)$ unless otherwise stated.

Proof. Now let $\tilde{S}_t = S_t e^{L_t}$, i.e. the S process with the tempered stable Lévy process removed (i.e. essentially the same process that [Fuk17] but with zero mean reversion for Y). Then we see that

$$\mathbb{E}((S_0 e^{\sqrt{\theta}\, z} - \tilde{S}_\theta e^{-L_\theta})^+) \quad = \quad \mathbb{E}(e^{-L_\theta}(S_0 e^{\sqrt{\theta}(z+\frac{L_\theta}{\sqrt{\theta}})} - \tilde{S}_\theta)^+)\,. \tag{11}$$

Thus we can price a put option under the model with jumps as a put option under the model without jumps but with an adjusted (stochastic) strike and an additional e^{-L_θ} factor inside the expectation. Now let

$$X_u^\theta \quad = \quad \theta^{-\frac{1}{2}}(\tilde{S}_{\theta u} - S_0)/S_0 \tag{12}$$

i.e. the same as X_u^θ in [Fuk17]. Then

$$dX_u^\theta \quad = \quad \theta^{-\frac{1}{2}}\frac{d\tilde{S}_{\theta u}}{S_0} \quad = \quad \theta^{-\frac{1}{2}}\frac{1}{S_0}\tilde{S}_{\theta u}v(Y_{\theta u})(\rho dW_{\theta u} + \bar{\rho} dW_{\theta u}^\perp) \quad = \quad \theta^{-\frac{1}{2}}\frac{1}{S_0}\tilde{S}_{\theta u}v(Y_{\theta u})d\tilde{B}_{\theta u} = (1+\sqrt{\theta}X_u^\theta)v(Y_{\theta u})d\hat{B}_u \tag{13}$$

where $\tilde{B}_t := \rho W_t + \bar{\rho} W_t^\perp$ and $\hat{B}_t = \theta^{-\frac{1}{2}}\tilde{B}_{\theta t}$. Setting $\hat{X}_u^\theta := v_0\theta^{-\frac{1}{2}}(\rho W_{\theta u} + \bar{\rho} W_{\theta u}^\perp) \sim N(0, v_0^2 u)$ we see that

$$d(X_u^\theta - \hat{X}_u^\theta) \quad = \quad (v(Y_{\theta u}) - v_0)d\hat{B}_u \,+\, \sqrt{\theta}X_u^\theta v(Y_{\theta u})d\hat{B}_u$$

and applying the Itô isometry to this expression, we see that

$$\begin{aligned}\mathbb{E}(X_1^\theta - \hat{X}_1^\theta)^2 \quad &= \quad \int_0^1 \mathbb{E}((v(Y_{\theta u}) - v_0)^2)du \,+\, 2\int_0^1 \mathbb{E}((v(Y_{\theta u}) - v_0)\sqrt{\theta}X_u^\theta v(Y_{\theta u}))du \,+\, \theta\int_0^1 \mathbb{E}((X_u^\theta)^2 v(Y_{\theta u})^2)du \\ &\leq \quad \int_0^1 \mathbb{E}((v(Y_{\theta u}) - v_0)^2)du \,+\, 2\,\mathbb{E}((v(Y_{\theta u}) - v_0)^2)^{\frac{1}{2}}\,\bar{v}\,\mathbb{E}(((\frac{\tilde{S}_{\theta u} - S_0}{S_0})^2)^{\frac{1}{2}} \,+\, \int_0^1 \mathbb{E}((\frac{\tilde{S}_{\theta u} - S_0}{S_0})^2 v(Y_{\theta u})^2)du\,.\end{aligned} \tag{14}$$

Clearly $\mathbb{E}((\frac{\tilde{S}_{\theta u} - S_0}{S_0})^2 v(Y_{\theta u})^2) \leq \bar{v}^2\mathbb{E}((\frac{\tilde{S}_{\theta u} - S_0}{S_0})^2) \to 0$ as $\theta \to 0$ uniformly for $u \in [0,1]$, using that $\mathbb{E}(\tilde{S}_t^2) \leq S_0^2 e^{\bar{v}^2 t}$. Then applying the bounded convergence theorem to all three terms in (14) we see that

$$\lim_{\theta\to 0}\mathbb{E}((X_u^\theta - \hat{X}_u^\theta)^2) \quad = \quad 0 \tag{15}$$

i.e. X_u^θ tends to $\hat{X}_u^\theta$ in L^2.

We now let $\hat{W}_u^\theta := \theta^{-\frac{1}{2}}W_{\theta u} = \theta^{-\frac{1}{2}}\int_0^{\theta u} dW_s$, and recall that $Y_u^\theta = \frac{Y_{\theta u} - \mathbb{E}(Y_{\theta u}|\mathcal{F}_0^W)}{\theta^H} \sim Y_u^0$ where $Y_u^0 \sim N(0, u^{2H})$, i.e. independent of θ, so trivially $Y_u^\theta \xrightarrow{w} Y_u^0$. We also note that

$$\begin{aligned}\mathbb{E}(\hat{W}_u^\theta Y_u^\theta) \quad &= \quad \mathbb{E}(\theta^{-\frac{1}{2}}W_{\theta u}\cdot\theta^{-H}\sqrt{2H}\int_0^{\theta u}(\theta u - r)^{H-\frac{1}{2}}dW_r) \\ &= \quad \mathbb{E}(\theta^{-\frac{1}{2}}\int_0^{\theta u}dW_s\cdot\theta^{-H}\sqrt{2H}\int_0^{\theta u}(\theta u - r)^{H-\frac{1}{2}}dW_r) \quad = \quad \eta u^{H+\frac{1}{2}}\end{aligned} \tag{16}$$

where $\eta := \frac{\sqrt{2H}}{\frac{1}{2}+H}$ (this calculation also appears in [Fuk17]), hence

$$\mathbb{E}(\hat{X}_u^\theta Y_u^\theta) \quad = \quad \rho v_0 \eta u^{H+\frac{1}{2}} \tag{17}$$

and $(\hat{X}_1^\theta, Y_1^\theta)$ has a bivariate Normal law (independent of θ) with $\mathbb{E}((\hat{X}_u^\theta)^2) = v_0^2 u$, $\mathbb{E}(\hat{X}_u^\theta Y_u^\theta) = \rho v_0\eta u^{H+\frac{1}{2}}$ and $\mathbb{E}((Y_u^\theta)^2) = u^{2H}$.

Let $p(x,u) = p(x,u;z) := \mathbb{E}((\Delta - v_0 W_1)^+\,|\,v_0 W_u = x)$ for $u < 1$ (page 4 in [Fuk17] gives explicit formulae for $p(x,u)$, $p_x(x,u)$ and $p_{xx}(x,u)$), which satisfies the heat eq: $p_u + \frac{1}{2}v_0^2 p_{xx} = 0$ with terminal condition $p(x,1) = (\Delta - x)^+$, where $\Delta = \Delta(z)$ is defined as on page 3 in [Fuk17] as

$$\Delta(z) \quad := \quad \frac{e^{z\sqrt{\theta}} - 1}{\sqrt{\theta}}\,.$$

Note that

$$\begin{aligned}\mathbb{E}(p(X_1^\theta, 1)) \quad &= \quad \mathbb{E}((\Delta - X_1^\theta)^+) \quad = \quad \mathbb{E}((\frac{e^{z\sqrt{\theta}} - 1}{\sqrt{\theta}} - X_1^\theta)^+) \quad = \quad \frac{1}{\sqrt{\theta}}\mathbb{E}((e^{z\sqrt{\theta}} - (1+\sqrt{\theta}\,X_1^\theta))^+) \quad = \quad \frac{1}{\sqrt{\theta}}\mathbb{E}((e^{z\sqrt{\theta}} - \frac{\tilde{S}_\theta}{S_0})^+) \\ p(0,0) \quad &= \quad \Delta\Phi(\frac{\Delta}{v_0}) + v_0\phi(\frac{\Delta}{v_0}) \quad = \quad z\Phi(\frac{z}{v_0}) + v_0\phi(\frac{z}{v_0}) \,+\, O(\sqrt{\theta}) \quad = \quad \mathbb{E}((z - X_1^0)^+) \,+\, O(\sqrt{\theta})\end{aligned}$$

and recall that X_1^0 is just any $N(0,v_0^2)$ random variable which is independent of L. Applying Itô's lemma to $p(X_u^\theta,u)$ and using (13) and integrating over $u\in[0,1]$ and taking expectations as in [Fuk17], we have

$$\begin{aligned}\mathbb{E}(p(X_1^\theta,1)) &= p(X_0^\theta,0) + \int_0^1 \mathbb{E}[\frac{1}{2}(1+\sqrt{\theta}X_u^\theta)^2 p_{xx}(X_u^\theta,u)v(Y_{\theta u})^2 + p_u(X_u^\theta,u))]du\\ &= p(0,0) + \frac{1}{2}\int_0^1 \mathbb{E}(p_{xx}(X_u^\theta,u)[(1+\sqrt{\theta}X_u^\theta)^2 v(Y_{\theta u})^2 - v_0^2])du\\ &= \mathbb{E}((z-X_1^0)^+) + O(\sqrt{\theta}) + \frac{1}{2}\int_0^1 \mathbb{E}(p_{xx}(X_u^\theta,u)[(1+\sqrt{\theta}X_u^\theta)^2 v(Y_{\theta u})^2 - v_0^2])du\end{aligned}$$

using the heat equation above. Thus we have

$$\frac{1}{\sqrt{\theta}}\mathbb{E}((e^{z\sqrt{\theta}}-\frac{\tilde{S}_\theta}{S_0})^+) - \mathbb{E}((\Delta(z)-X_1^0)^+) = \frac{1}{2}\int_0^1 \mathbb{E}(p_{xx}(X_u^\theta,u)[(1+\sqrt{\theta}X_u^\theta)^2 v(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)^2 - v_0^2])du \tag{18}$$

(recall again that $X_1^0\sim N(0,v_0^2)$); this is also the first main equation in step 3 on page 9 in [Fuk17].

Lemma 2.2 *$L_\theta/\sqrt{\theta}$ tends weakly to 0 as $\theta\to 0$.*

Proof. Using (2) we can easily verify that $\mathbb{E}(e^{\frac{iu}{\sqrt{\theta}}L_\theta})\to 1$ as $\theta\to 0$; the result then follows from the Lévy convergence theorem (see e.g. chapter 18 in [Wil91]). ■

To incorporate the additional independent Lévy process L, we set $\hat{L}_\theta := L_\theta/\sqrt{\theta}$ and combine (18) with (11) to obtain

$$\begin{aligned}&\frac{1}{\sqrt{\theta}}\mathbb{E}((e^{z\sqrt{\theta}}-\frac{S_\theta}{S_0})^+)\\ =\;& \mathbb{E}(e^{-L_\theta}p(0,0;z+\hat{L}_\theta)) + \frac{1}{2}\int_0^1 \mathbb{E}(e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\hat{L}_\theta)[(1+\sqrt{\theta}X_u^\theta)^2 v(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)^2 - v_0^2])du\\ =\;& \mathbb{E}(e^{-L_\theta}(\Delta(z+\hat{L}_\theta)-X_1^0)^+)\\ +\;& \frac{1}{2}\int_0^1 \mathbb{E}(e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\hat{L}_\theta)[(1+\sqrt{\theta}X_u^\theta)^2 v(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)^2 - v_0^2])du\\ =\;& \mathbb{E}[e^{-L_\theta}(z-(X_1^0-\frac{L_\theta}{\sqrt{\theta}}))^+] + O(\sqrt{\theta}) + \frac{1}{2}\int_0^1 \mathbb{E}(e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\hat{L}_\theta)[(1+\sqrt{\theta}X_u^\theta)^2 v(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)^2 - v_0^2])du\end{aligned} \tag{19}$$

where we have used Lemma 2.3 below in the final line, and recall that $X_1^0\sim N(0,v_0^2)$.

Lemma 2.3

$$\mathbb{E}(e^{-L_\theta}(\Delta(z+\hat{L}_\theta)-X_1^0)^+) = \mathbb{E}(e^{-L_\theta}(\frac{e^{z\sqrt{\theta}+L_\theta}-1}{\sqrt{\theta}}-X_1^0)^+) = \mathbb{E}(e^{-L_\theta}(z+\frac{L_\theta}{\sqrt{\theta}}-X_1^0)^+) + O(\theta^{\frac{1}{2}}).$$

Proof. Recall that $\hat{L}_\theta := L_\theta/\sqrt{\theta}$. Then by the Taylor remainder theorem applied to the exponential function, we see that

$$\begin{aligned}\mathbb{E}(e^{-L_\theta}(\frac{e^{z\sqrt{\theta}+L_\theta}-1}{\sqrt{\theta}}-X_1^0)^+) &= \mathbb{E}(e^{-L_\theta}(\frac{e^{\sqrt{\theta}(z+\hat{L}_\theta)}-1}{\sqrt{\theta}}-X_1^0)^+)\\ &= \mathbb{E}(e^{-L_\theta}(\frac{\sqrt{\theta}\,(z+\hat{L}_\theta)+\frac{1}{2}\theta(z+\hat{L}_\theta)^2e^\xi}{\sqrt{\theta}}-X_1^0)^+)\\ &= \mathbb{E}(e^{-L_\theta}(z+\hat{L}_\theta+\frac{1}{2}\sqrt{\theta}\,(z+\hat{L}_\theta)^2e^\xi-X_1^0)^+)\end{aligned}$$

for some $\xi\in(0,\sqrt{\theta}(z+\hat{L}_\theta))=c\sqrt{\theta}(z+\hat{L}_\theta)$ for some (random) constant $c\in[0,1]$ (note ξ may be negative). Then

$$\begin{aligned}&\mathbb{E}(e^{-L_\theta}(\frac{e^{z\sqrt{\theta}+L_\theta}-1}{\sqrt{\theta}}-X_1^0)^+) - \mathbb{E}(e^{-L_\theta}(z+\hat{L}_\theta-X_1^0)^+)\\ =\;& \frac{1}{2}\sqrt{\theta}\,\mathbb{E}(e^{-L_\theta}(z+\hat{L}_\theta)^2e^{c\sqrt{\theta}(z+\hat{L}_\theta)}1_{z+\hat{L}_\theta-X_1^0>0})\\ +\;& \mathbb{E}(e^{-L_\theta}(z+\hat{L}_\theta+\frac{1}{2}\sqrt{\theta}\,(z+\hat{L}_\theta)^2e^{c\sqrt{\theta}(z+\hat{L}_\theta)}-X_1^0)^+1_{z+\hat{L}_\theta<X_1^0<z+\hat{L}_\theta+\frac{1}{2}\theta(z+\hat{L}_\theta)^2e^{c\sqrt{\theta}(z+\hat{L}_\theta)}}).\end{aligned} \tag{20}$$

The first term here is less than or equal to

$$\frac{1}{2}\sqrt{\theta}\,\mathbb{E}(e^{-L_\theta}(z+\hat{L}_\theta)^2 e^{c\sqrt{\theta}(z+\hat{L}_\theta)}\,(1_{z+\hat{L}_\theta>0}+1_{-X_1^0>0})) \tag{21}$$

and the first term in (21) is less than or equal to

$$\sqrt{\theta}\,e^{z\theta}\,\mathbb{E}((z+\hat{L}_\theta)^2 e^{c\sqrt{\theta}(z+\hat{L}_\theta)}) \;\leq\; \sqrt{\theta}\,e^{z\theta}\mathbb{E}((z+\frac{L_\theta}{\sqrt{\theta}})^2 e^{\sqrt{\theta}(z+\hat{L}_\theta)}) + \sqrt{\theta}\,e^{z\theta}\mathbb{E}((z+\frac{L_\theta}{\sqrt{\theta}})^2) \tag{22}$$

Using (2), we now recall the following relations:

$$\begin{aligned}
\mathbb{E}(L_\theta e^{pL_\theta}) &= \frac{d}{dp}\mathbb{E}(e^{pL_\theta}) = \theta\Lambda'(p)e^{\theta\Lambda(p)} = O(\theta)\\
\mathbb{E}(L_\theta^2 e^{pL_\theta}) &= \frac{d^2}{dp^2}\mathbb{E}(e^{pL_\theta}) = (\theta\Lambda''(p)+\theta^2\Lambda'(p)^2)e^{\theta\Lambda(p)} = O(\theta)
\end{aligned}$$

for $p\in(-G,M)$, where $\Lambda(p) = -pb\theta + C_+\Gamma(-Y)\theta((M-p)^Y - M^Y) + C_-\Gamma(-Y)\theta((G+p)^Y - G^Y)$. Using these asymptotic relations and using that 0 and 1 lie in $(-G,M)$, we find that (22) is $O(\sqrt{\theta})$.

Similarly, since L_θ and X_1^0 are independent, the second term in (21) can be broken up as

$$\begin{aligned}
\frac{1}{2}\sqrt{\theta}\,\mathbb{E}(e^{-L_\theta}(z+\hat{L}_\theta)^2 e^{c\sqrt{\theta}(z+\hat{L}_\theta)})\,\mathbb{P}(-X_1^0>0) &\leq \frac{1}{2}\sqrt{\theta}\,\mathbb{E}(e^{-L_\theta}(z+\hat{L}_\theta)^2 e^{c\sqrt{\theta}(z+\hat{L}_\theta)})\\
&= \frac{1}{2}\sqrt{\theta}\,\mathbb{E}(e^{-L_\theta}(z+\hat{L}_\theta)^2 e^{c\sqrt{\theta}(z+\hat{L}_\theta)}(1_{\hat{L}_\theta>-z}+1_{\hat{L}_\theta<-z}))\\
&= \frac{1}{2}\sqrt{\theta}\,\mathbb{E}(e^{-L_\theta}(z+\hat{L}_\theta)^2(e^{\sqrt{\theta}(z+\hat{L}_\theta)}+1))
\end{aligned}$$

and the final line is $O(\sqrt{(t)})$ using the same argument as above.

The final term in (20) can be bounded as

$$\mathbb{E}(e^{-L_\theta}(\frac{1}{2}\sqrt{\theta}\,(z+\hat{L}_\theta)^2 e^{\xi})^+ 1_{z+\hat{L}_\theta - X_1^0<0<z+\hat{L}_\theta)+\frac{1}{2}\theta(z+\hat{L}_\theta)^2 e^{\xi}}) \;\leq\; \frac{1}{2}\sqrt{\theta}\,\mathbb{E}(e^{-L_\theta}(z+\hat{L}_\theta)^2 e^{\xi})$$

and again use the same arguments as above to show this is $O(\sqrt{\theta})$. ■

We can trivially decompose the final integral term in (19) as

$$\begin{aligned}
&\int_0^1 \mathbb{E}(e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\hat{L}_\theta)[(1+\sqrt{\theta}X_u^\theta)^2 v(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)^2 - v_0^2)]du\\
=&\int_0^1 \mathbb{E}(e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\hat{L}_\theta)[v(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)^2((1+\sqrt{\theta}X_u^\theta)^2-1)])du\\
+&\int_0^1 \mathbb{E}(e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\hat{L}_\theta)[(v(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)^2 - v(Y_0+\zeta(\theta u))^2)])du\\
+&\int_0^1 \mathbb{E}(e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\hat{L}_\theta)[(v(Y_0+\zeta(\theta u))^2 - v_0^2)])du
\end{aligned} \tag{23}$$

and (as in [Fuk17]) we deal with the three terms separately in the analysis which follows.

We first compute an asymptotic expansion as $\theta\to 0$ for the $\mathbb{E}[e^{-L_\theta}(z-(X_1^0-\frac{L_\theta}{\sqrt{\theta}}))^+]$ term which appears in (19), using a put-call parity argument. Let $\tilde{W}$ denote a Brownian motion independent of L. Then

$$\frac{1}{\sqrt{\theta}}\mathbb{E}[e^{-L_\theta}(v_0\tilde{W}_\theta - L_\theta - z\sqrt{\theta})^+] \;=\; \mathbb{E}[e^{-L_\theta}(X_1^0-\frac{L_\theta}{\sqrt{\theta}}-z)^+]\,. \tag{24}$$

Moreover we know that

$$(X_1^0-\frac{L_\theta}{\sqrt{\theta}}-z)^+ + z \;=\; (z-(X_1^0-\frac{L_\theta}{\sqrt{\theta}}))^+ + X_1^0 - \frac{L_\theta}{\sqrt{\theta}}$$

Then multiplying by e^{-L_θ} taking expectations and using that $\mathbb{E}(e^{-L_\theta})=1$ and $\mathbb{E}(e^{-L_\theta}\frac{L_\theta}{\sqrt{\theta}}) = O(\sqrt{\theta})$ we see that

$$\mathbb{E}(e^{-L_\theta}(X_1^0-\frac{L_\theta}{\sqrt{\theta}}-z)^+) + z \;=\; \mathbb{E}(e^{-L_\theta}(z-(X_1^0-\frac{L_\theta}{\sqrt{\theta}}))^+) + \mathbb{E}(e^{-L_\theta}\frac{L_\theta}{\sqrt{\theta}}) \;=\; \mathbb{E}(e^{-L_\theta}(z-(X_1^0-\frac{L_\theta}{\sqrt{\theta}}))^+) + O(\sqrt{\theta}) \tag{25}$$

Proposition 2.4 *Under the assumptions on Y and H in Theorem 2.1, we have*

$$\mathbb{E}[e^{-L_\theta}(X_1^0 - \frac{L_\theta}{\sqrt{\theta}} - z)^+] - \mathbb{E}[e^{-L_\theta}(X_1^0 - z)^+] = \theta^{1-\frac{1}{2}Y}\frac{1}{\pi}\int_0^\infty e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}(iu)^Y\Gamma(-Y)(C_+ + C_-)]\frac{du}{u^2} + O(\theta^{2-Y\wedge\frac{1}{2}}).$$

Remark 2.4 Note that $O(\theta^{2-2Y}) = o(H)$ under the assumption in the main Theorem 2.1.

Remark 2.5 From (25), we see that we have the same asymptotic behaviour for $\mathbb{E}(e^{-L_\theta}(z - (X_1^0 - \frac{L_\theta}{\sqrt{\theta}}))^+)$ at this order, so this yields the first correction term in the main Theorem 2.1 on the right hand side of (8).

Proof. See Appendix C. ■

We now recall the well known general result that if $\bar{Z}_n$ and Z_n are two sequences of random variables in $\mathbb{R}^n$ with $|\bar{Z}_n - Z_n| \overset{p}{\to} 0$ (convergence in probability) and $Z_n \overset{w}{\to} Z$ then $\bar{Z}_n \overset{w}{\to} Z$. From (15) we know that $|X_u^\theta - \hat{X}_u^\theta| \to 0$ in L^2 and hence $|X_u^\theta - \hat{X}_u^\theta| \overset{p}{\to} 0$, which (by the result in the previous sentence) also implies that X_u^θ tends weakly to a random variable X_u^0 which has the same law as $\hat{X}_u^\theta \sim N(0, v_0^2 u)$. Moreover

$$\mathbb{E}((X_1^\theta - \hat{X}_1^\theta)^2 + (Y_1^\theta - Y_1^\theta)^2) = \mathbb{E}((X_1^\theta - \hat{X}_1^\theta)^2) \to 0. \tag{26}$$

Hence $(X_1^\theta, Y_1^\theta) \to (\hat{X}_1^\theta, Y_1^\theta)$ in L^2 so $|(X_1^\theta, Y_1^\theta) - (\hat{X}_1^\theta, Y_1^\theta)| \overset{p}{\to} 0$, and $(\hat{X}_1^\theta, Y_1^\theta)$ has a bivariate Normal distribution which is independent of θ (see (see eq (17) for details), so (trivially) $(\hat{X}_1^\theta, Y_1^\theta)$ converges weakly to this bivariate Normal law, and again using the result in the previous sentence we have that $(X_1^\theta, Y_1^\theta) \overset{w}{\to} (\hat{X}_1^0, Y_1^0)$, where $(\hat{X}_1^0, Y_1^0)$ has a bivariate Normal law with the same joint law as $(\hat{X}_1^\theta, Y_1^\theta)$.

We can further simplify the first term on the right hand side of (23) to the following expression:

$$\sqrt{\theta}\int_0^1 \mathbb{E}(e^{-L_\theta}p_{xx}(X_u^\theta, u; z + \hat{L}_\theta)X_u^\theta(2 + \sqrt{\theta}\,X_u^\theta)v(Y_0 + \theta^H Y_u^\theta + \zeta(\theta u))^2)du \tag{27}$$

and our first task is to verify that this term tends to zero as $\theta \to 0$, by showing that it has a finite limit when divided by $\sqrt{\theta}$. Performing this division and first concentrating on the tail event $|L_\theta| > \sqrt{\theta}$ and using that $p_{xx}(x, u; z) = \frac{1}{v_0\sqrt{1-u}}\phi(\frac{\Delta - x}{v_0\sqrt{1-u}})$ (see also page 4 in [Fuk17]), we have

$$\begin{aligned}
&\mathbb{E}(e^{-L_\theta}p_{xx}(X_u^\theta, u; z + \frac{L_\theta}{\sqrt{\theta}})|X_u^\theta|(2 + \sqrt{\theta}\,|X_u^\theta|)v(Y_0 + \theta^H Y_u^\theta + \zeta(\theta u))^2 1_{|L_\theta| > \sqrt{\theta}}) \\
\leq\ & \frac{c_1}{\sqrt{1-u}}\bar{v}^2\mathbb{E}(e^{-L_\theta}1_{|L_\theta| > \sqrt{\theta}}(2|X_u^\theta| + \sqrt{\theta}\,(X_u^\theta)^2)) \\
=\ & \frac{c_1}{\sqrt{1-u}}\bar{v}^2\mathbb{E}(e^{-L_\theta}1_{|L_\theta| > \sqrt{\theta}})\mathbb{E}(2|X_u^\theta| + \sqrt{\theta}\,(X_u^\theta)^2) \quad \text{(using the independence of } L_\theta \text{ and } X_u^\theta) \\
=\ & \frac{c_1}{\sqrt{1-u}}\bar{v}^2\mathbb{E}(e^{-L_\theta}1_{L_\theta < -\sqrt{\theta}} + 1_{L_\theta > \sqrt{\theta}})[2\mathbb{E}((X_u^\theta)^2)^{\frac{1}{2}} + \sqrt{\theta}\,\mathbb{E}((X_u^\theta)^2)] \\
\leq\ & \frac{c_1}{\sqrt{1-u}}\bar{v}^2 K_2\theta^{1-\frac{1}{2}Y-2\varepsilon}(2v_0\sqrt{u} + \sqrt{\theta}\,v_0^2 u + o(1))
\end{aligned} \tag{28}$$

as $\theta \to 0$, where $c_1 = \frac{1}{v_0\sqrt{2\pi}}$, and we have used (7) and that $\mathbb{E}((X_u^\theta)^2) \to v_0^2$ in the final line.

From basic calculus we find that $p_{xx}(x, u)x^2 = p_{xx}(x, u; z)x^2$ attains its extrema at $x_2^\pm = \frac{1}{2}(\Delta \pm q_2)$ where $q_2 = q_2(u, \Delta) := \sqrt{8v_0^2(1-u) + \Delta^2}$ (where we are now showing the explicit dependence of p on z as well) and recall that $\Delta(z) = \frac{e^{z\sqrt{\theta}} - 1}{\sqrt{\theta}}$. $p_{xx}(x, u)x$ attains its maximum at $x_1^\pm = \frac{1}{2}(\Delta \pm q_1)$ where $q_1 = q_1(u) := \sqrt{4v_0^2(1-u) + \Delta(z)^2}$; hence for $|z| \leq 1$ we have

$$\begin{aligned}
p_{xx}(x, u; z)x^2 &\leq p_{xx}(x, u; z)x^2|_{x = x_2^+} + p_{xx}(x, u; z)x^2|_{x = x_2^-} \leq \frac{1}{4}c_1\frac{1}{\sqrt{1-u}}[(\Delta(z+1) + \bar{q}_2)^2 + (\Delta(z-1) - \bar{q}_2)^2] \\
p_{xx}(x, u; z)x &\leq p_{xx}(x, u; z)x|_{x = x_1^+} + p_{xx}(x, u; z)x|_{x = x_1^-} \leq \frac{1}{2}c_1\frac{1}{\sqrt{1-u}}[(\Delta(z+1) + \bar{q}_1) + |\Delta(z-1) - \bar{q}_1)|]
\end{aligned} \tag{29}$$

where $\bar{q}_2 = \bar{q}_2 := \sqrt{8v_0^2 + \Delta(z+1)^2}$, $\bar{q}_1 := \sqrt{4v_0^2 + \Delta(z+1)^2}$, and recall that z may be negative.

The following simple lemmas will be needed:

Lemma 2.5 *Let (X_n, Y_n, Z_n) be a sequence of random variables which converges weakly to $(X, Y, 0)$. Then for $f \in C_b$ and $R > 0$ we have*

$$\lim_{n\to\infty}\mathbb{E}(f(X_n, Y_n, Z_n)1_{Z_n \leq R}) = \mathbb{E}(f(X, Y, 0)).$$

Proof. See Appendix D. ■

Lemma 2.6 *Let (X_n, Y_n, Z_n) be a sequence of random variables which converges weakly to $(X, Y, 0)$ and assume that Y has a density. Then for $f \in C_b$ and $R, K > 0$ we have*

$$\lim_{n\to\infty} \mathbb{E}(f(X_n, Y_n, Z_n)1_{Y_n\le R}1_{Z_n\le K}) \quad = \quad \mathbb{E}(f(X, Y, 0)1_{Y\le R}) .$$

Proof. See Appendix E. ■

Then using Lemma 2.2 and the weak convergence of (X^θ_u, Y^θ_u), we know that $(X^\theta_u, Y^\theta_u, L_\theta/\sqrt{\theta}) \stackrel{w}{\to} (X^0_u, Y^0_u, 0)$ so

$$\begin{aligned}
&\lim_{\theta\to 0} \mathbb{E}(e^{-L_\theta}p_{xx}(X^\theta_u, u; z + \frac{L_\theta}{\sqrt{\theta}})X^\theta_u(2 + \sqrt{\theta}\, X^\theta_u)v(Y_0 + \theta^H Y^\theta_u + \zeta(\theta u))^2 1_{|L_\theta|\le\sqrt{\theta}})\\
= \;& 2v_0^2\mathbb{E}(p_{xx}(X^0_u, u; z)X^0_u)
\end{aligned}$$

using (29) and Lemma 2.5 in the final line. Combining this with (28) (which deals with the contribution from $1_{\frac{L_\theta}{\sqrt{\theta}}>1}$) we see that

$$\lim_{\theta\to 0} \mathbb{E}(p_{xx}(X^\theta_u, u; z + \frac{L_\theta}{\sqrt{\theta}})X^\theta_u(2 + \sqrt{\theta}\, X^\theta_u)v(Y_0 + \theta^H Y^\theta_u + \zeta(\theta u))^2) \quad = \quad 2v_0^2\mathbb{E}(p_{xx}(X^0_u, u; z)X^0_u) \tag{30}$$

and we have the bound

$$\begin{aligned}
&\mathbb{E}(e^{-L_\theta}p_{xx}(X^\theta_u, u; z + \frac{L_\theta}{\sqrt{\theta}})X^\theta_u(2 + \sqrt{\theta}\, X^\theta_u)v(Y_0 + \theta^H Y^\theta_u + \zeta(\theta u))^2)\\
\le \;& v_0^2\,[2\cdot\frac{1}{2}c_1\frac{1}{\sqrt{1-u}}[(\Delta(z+1) + \bar{q}_1) \;+\; |\Delta(z-1) - \bar{q}_1)|] \;+\; \frac{1}{4}c_1\sqrt{\theta}\frac{1}{\sqrt{1-u}}[(\Delta(z+1) + \bar{q}_2)^2 \;+\; (\Delta(z-1) - \bar{q}_2)^2]\\
+ \;& \frac{c_1 K}{\sqrt{1-u}}\theta^{1-\frac{1}{2}Y-2\epsilon}(2v_0\sqrt{u} \;+\; \sqrt{\theta}\, v_0^2 u + o(1))] \,.
\end{aligned} \tag{31}$$

Moreover, using that $\int_0^1 \frac{1}{\sqrt{1-u}}du < \infty$, and the pointwise convergence of the integrand in (30), from the dominated convergence theorem we have

$$\lim_{\theta\to 0}\int_0^1 \mathbb{E}[e^{-L_\theta}p_{xx}(X^\theta_u, u; z + \hat{L}_\theta)X^\theta_u(2 + \sqrt{\theta}\, X^\theta_u)v(Y_0 + \theta^H Y^\theta_u + \zeta(\theta u))^2]du \quad = \quad 2v_0^2\int_0^1 \mathbb{E}(p_{xx}(X^0_u, u)X^0_u)du \quad = \quad v_0 z\phi(\frac{z}{v_0}) .$$

where we have used that $p_{xx}(x, u) \to \frac{1}{v_0\sqrt{1-u}}\phi(\frac{z-x}{v_0\sqrt{1-u}})$ (see page 4 in [Fuk17]) and $X^0_u \sim N(0, v_0^2 u)$.

Thus we have finally shown that the expression in (27) is $O(\theta^{\frac{1}{2}}) = o(\theta^H)$ (since we assuming $H \in (0, \frac{1}{2})$), and hence will not show up at the order we are interested in. Recall that this is also the first term in (23).

We now analyze the second term in (23):

$$\frac{1}{2}\int_0^1 \mathbb{E}[e^{-L_\theta}p(X^\theta_u, u; z + \frac{L_\theta}{\sqrt{\theta}})(v(Y_0 + \zeta(\theta u) + \theta^H Y^\theta_u)^2 - v(Y_0 + \zeta(\theta u))^2)]du \,.$$

Let $V(y) = v(y)^2$. Then using Taylor's remainder theorem, we have

$$\begin{aligned}
V(Y_0 + \zeta(\theta u) + \theta^H Y^\theta_u) \quad &= \quad v_0^2 \;+\; V'(Y_0)(\zeta(\theta u) + \theta^H Y^\theta_u) + \frac{1}{2}V''(\xi)(\zeta(\theta u) + \theta^H Y^\theta_u)^2\\
V(Y_0 + \zeta(\theta u)) \quad &= \quad v_0^2 \;+\; V'(Y_0)\zeta(\theta u) + \frac{1}{2}V''(\xi_2)\zeta(\theta u)^2
\end{aligned}$$

for some $\xi \in (Y_0, Y_0 + \zeta(\theta u) + \theta^H Y^\theta_u)), \xi_2 \in (Y_0, Y_0 + \zeta(\theta u))$. Thus

$$\begin{aligned}
\theta^{-H}[V(Y_0 + \zeta(\theta u) + \theta^H Y^\theta_u) - V(Y_0 + \zeta(\theta u))] \quad &= \quad \theta^{-H}[V'(Y_0)\theta^H Y^\theta_u \;+\; \frac{1}{2}V''(\xi)(\theta^H\hat{\zeta}_\theta + \theta^H Y^\theta_u)^2 \;-\; \frac{1}{2}V''(\xi_2)\theta^{2H}\hat{\zeta}^2_\theta]\\
&= \quad V'(Y_0)Y^\theta_u \;+\; \theta^H[\frac{1}{2}V''(\xi)(\hat{\zeta}_\theta + Y^\theta_u)^2 \;-\; \frac{1}{2}V''(\xi_2)\hat{\zeta}^2_\theta]
\end{aligned}$$

where $\hat{\zeta}_\theta = \theta^{-H}\zeta(\theta u)$, and we know that the law of $\hat{\zeta}_\theta$ is independent of θ as $\theta \to 0$ (as is the law of Y^θ_u).

Using similar arguments as above, we see that

$$
\begin{aligned}
& \mathbb{E}(e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\frac{L_\theta}{\sqrt{\theta}})|V(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)-V(Y_0+\zeta(\theta u))|\,1_{|L_\theta|>\sqrt{\theta}}) \\
\leq\; & \frac{c_1}{\sqrt{1-u}}\mathbb{E}(e^{-L_\theta}1_{|L_\theta|>\sqrt{\theta}}|V(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)-V(Y_0+\zeta(\theta u))|) \\
=\; & \frac{c_1}{\sqrt{1-u}}\mathbb{E}(e^{-L_\theta}1_{|L_\theta|>\sqrt{\theta}})\mathbb{E}(|V(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)-V(Y_0+\zeta(\theta u))|)\ \text{(using the independence of } L_\theta \text{ and } X_u^\theta) \\
\leq\; & \frac{c_1}{\sqrt{1-u}}K_2\theta^{1-\frac{1}{2}Y-2\varepsilon}\mathbb{E}(|V(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)-V(Y_0+\zeta(\theta u))|) \\
=\; & \frac{c_1}{\sqrt{1-u}}K_2\theta^{1-\frac{1}{2}Y-2\varepsilon}\theta^H\mathbb{E}(|V'(Y_0)Y_u^\theta\;+\;\theta^H[\frac{1}{2}V''(\xi)(\hat{\zeta}_\theta+Y_u^\theta)^2\;-\;\frac{1}{2}V''(\xi_2)\hat{\zeta}_\theta^2]|) \\
=\; & \frac{c_1}{\sqrt{1-u}}K_2\theta^{1-\frac{1}{2}Y-2\varepsilon}O(\theta^H)
\end{aligned} \tag{32}
$$

and we have

$$
\begin{aligned}
\mathcal{I} \;:=\; & \theta^{-H}\mathbb{E}[e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\frac{L_\theta}{\sqrt{\theta}})(V(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)-V(Y_0+\zeta(\theta u)))1_{|L_\theta|\leq\sqrt{\theta}}] \\
\leq\; & \mathbb{E}[e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\frac{L_\theta}{\sqrt{\theta}})|V'(Y_0)|Y_u^\theta 1_{Y_u^\theta\leq R}1_{|L_\theta|\leq\sqrt{\theta}}] \\
+\; & \theta^H\mathbb{E}[e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\frac{L_\theta}{\sqrt{\theta}})(\frac{1}{2}V''(\xi)(\hat{\zeta}_\theta+Y_u^\theta)^2\;-\;\frac{1}{2}V''(\xi_2)\hat{\zeta}_\theta^2)1_{Y_u^\theta\leq R}1_{|L_\theta|\leq\sqrt{\theta}}] \\
\leq\; & \frac{1}{v_0\sqrt{2\pi(1-u)}}\mathbb{E}[(|V'(Y_0)|\,|Y_u^\theta|\;+\;\theta^H(\frac{1}{2}|V''(\xi)|(\hat{\zeta}_\theta+Y_u^\theta)^2\;+\;\frac{1}{2}|V''(\xi_2)|\hat{\zeta}_\theta^2))1_{Y_u^\theta>R}]\,.
\end{aligned}
$$

Then using the weak convergence of $(X_u^\theta, Y_u^\theta, \frac{L_\theta}{\sqrt{\theta}})$ to $(X_u^0, Y_u^0, 0)$ and Lemma 2.6 for the first line of the final expression, weak convergence of the integrand to zero of the second term, we see that for all $\varepsilon, R > 0$ there exists a $\theta^*(\varepsilon, R) > 0$ such that for $\theta \in (0, \theta^*(\varepsilon, R))$ we have

$$
\begin{aligned}
\mathcal{I} \;\leq\; & V'(Y_0)\mathbb{E}[p_{xx}(X_u^0,u;z)Y_u^0 1_{Y_u^0\leq R})]\;+\;\varepsilon \\
+\; & \frac{c_1}{\sqrt{1-u}}\mathbb{E}[|V'(Y_0)|\frac{1}{R}(Y_u^\theta)^2+\theta^H(\frac{1}{2}|V''(\xi)|(\frac{1}{R^2}\hat{\zeta}_\theta^2(Y_u^\theta)^2+2\hat{\zeta}_\theta\frac{1}{R}(Y_u^\theta)^2+\frac{1}{R^2}(Y_u^\theta)^4)+\frac{1}{2}|V''(\xi_2)|\frac{1}{R^2}\hat{\zeta}_\theta^2(Y_u^\theta)^2)]
\end{aligned}
$$

using simple Chebychev bounds. Now choose $R = \frac{1}{\varepsilon}$ and $\theta \leq \theta^*(\varepsilon, \frac{1}{\varepsilon})$. Then the law of Y_u^θ is independent of θ and $\hat{\zeta}_\theta$ is $\mathcal{F}_0^W$-measurable, so the final term tends to zero as well as $\theta \to 0$. Thus

$$
\begin{aligned}
& \lim_{\theta\to 0}\frac{1}{2}\theta^{-H}\int_0^1\mathbb{E}[p_{xx}(X_u^\theta,u;z+\frac{L_\theta}{\sqrt{\theta}})(v(Y_0+\zeta(\theta u)+\theta^H Y_u^\theta)^2-v(Y_0+\zeta(\theta u))^2)]du \\
=\; & v_0 v'(Y_0)\int_0^1\mathbb{E}(p_{xx}(X_u^0,u;z)Y_u^0)du \\
=\; & v_0 v'(Y_0)\int_0^1\mathbb{E}(p_{xx}(X_u^0,u;z)\mathbb{E}(Y_u^0|X_u^0))du\,.
\end{aligned} \tag{33}
$$

But for any centred bivariate Normal random variable (X, Y) with standard deviations σ_X, σ_Y and correlation ρ_1, $\mathbb{E}(Y|X) = \frac{\mathbb{E}(XY)}{\mathbb{E}(X^2)}X = \frac{\rho_1\sigma_X\sigma_Y}{\sigma_X^2}X = \rho_1\frac{\sigma_Y}{\sigma_X}X$ and for us $\rho_1 = \frac{\sqrt{2H}}{\frac{1}{2}+H}\rho$. Hence we can re-write (33) as

$$
v_0 v'(Y_0)\int_0^1\mathbb{E}(p_{xx}(X_u^0,u;z)X_u^0)\frac{\rho v_0\eta u^{H+\frac{1}{2}}}{v_0^2 u}du \;=\; \rho z\frac{\sqrt{2H}}{(\frac{1}{2}+H)(\frac{3}{2}+H)}\frac{v'(Y_0)}{v_0}\phi(\frac{z}{v_0}) \tag{34}
$$

using (17).

The final term in (23) is the history correction term (which is independent of ρ):

$$
\frac{1}{2}\int_0^1\mathbb{E}[e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\hat{L}_\theta)(v(Y_0+\zeta(\theta u))^2-v_0^2)]du \;=\; \frac{1}{2}\int_0^1\mathbb{E}[e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\frac{L_\theta}{\sqrt{\theta}})(v(Y_0+\zeta(\theta u))^2-v_0^2)]du \tag{35}
$$

Using Taylor's remainder theorem again, we have

$$
v(Y_0+\zeta(\theta u))^2\;-\;v_0^2 \;=\; 2v_0 v'(Y_0)\zeta(\theta u)\;+\;(v'(\xi_1)^2+v(\xi_1)v''(\xi_1))\zeta(\theta u)^2
$$

for some $\xi_1 \in (0, \zeta(\theta u))$, so (35) can be written as

$$
v_0 v'(Y_0)\int_0^1\mathbb{E}[e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\frac{L_\theta}{\sqrt{\theta}})]\zeta(\theta u)du\;+\;\frac{1}{2}\int_0^1\mathbb{E}[e^{-L_\theta}p_{xx}(X_u^\theta,u;z+\frac{L_\theta}{\sqrt{\theta}})](v'(\xi_1)^2+v(\xi_1)v''(\xi_1))\zeta(\theta u)^2du\,. \tag{36}
$$

Using that $\mathbb{E}[e^{-L_\theta}p_{xx}(X^\theta_u,u;z+\frac{L_\theta}{\sqrt{\theta}})1_{|L_\theta|>\sqrt{\theta}} \le \frac{c_1}{\sqrt{1-u}}K_2\theta^{1-\frac{1}{2}Y-2\varepsilon}$ as in (32) and Lemma 2.6 as before, we know that $\lim_{\theta\to 0}\mathbb{E}[e^{-L_\theta}p_{xx}(X^\theta_u,u;z+\frac{L_\theta}{\sqrt{\theta}})] = \mathbb{E}(p_{xx}(X^0_u,u;z))$, and we can also easily check that $\mathbb{E}(p_{xx}(X^0_u,u;z)) = \frac{1}{v_0}\phi(\frac{z}{v_0})$, which is independent of u.

Moreover if $\Upsilon_\theta := \theta^{-H}\int_0^1\zeta(\theta u)du$, then $(\Upsilon_\theta)_{\theta>0}$ is a family of Gaussian random variables with zero expectation, and using the stochastic Fubini theorem we see that

$$\begin{aligned}\int_0^1\zeta(\theta u)du &= \sqrt{2H}\int_0^1\int_{t_0}^0[(\theta u-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}]dW_s du\\ &= \sqrt{2H}\int_{t_0}^0\int_0^1[(\theta u-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}]du dW_s\end{aligned}$$

which implies that

$$\mathbb{E}[\int_0^1\zeta(\theta u)du]^2 = 2H\int_{t_0}^0[\int_0^1[(\theta u-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}]du]^2ds.$$

Setting $s=\theta z$ in the outer integral, we see that

$$\int_{t_0}^0[\int_0^1[(\theta u-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}]du]^2ds = \theta^{2H}\int_{\frac{t_0}{\theta}}^0[\int_0^1[(u-z)^{H-\frac{1}{2}}-(-z)^{H-\frac{1}{2}}]du]^2dz.$$

Hence

$$\theta^{-2H}\mathbb{E}[\int_0^1\zeta(\theta u)du]^2 = 2H\int_{\frac{t_0}{\theta}}^0[\int_0^1[(u-z)^{H-\frac{1}{2}}-(-z)^{H-\frac{1}{2}}]du]^2dz.$$

Now the integrand is positive, so the limit exists with respect to θ exists and equals

$$\ell := 2H\int_{-\infty}^0[\int_0^1[(u-z)^{H-\frac{1}{2}}-(-z)^{H-\frac{1}{2}}]du]^2dz.$$

Finally we need to verify that $\ell<\infty$. Indeed, for any $N>0$ and $z\in[-N,0]$ and $u\in[0,1]$, since $H<\frac{1}{2}$ we see that

$$|(u-z)^{H-\frac{1}{2}}-(-z)^{H-\frac{1}{2}}| \le 2(-z)^{H-\frac{1}{2}}$$

which is square integrable with respect to z over $[-N,0]$. For $z>0$, the mean value theorem implies that $(1+z)^{H-\frac{1}{2}}-z^{H-\frac{1}{2}} = (H-\frac{1}{2})x^{H-\frac{3}{2}}$ for some $x\in(z,z+1)$. Thus

$$z^{H-\frac{1}{2}}-(1+z)^{H-\frac{1}{2}} = (\frac{1}{2}-H)x^{H-\frac{3}{2}} \le (\frac{1}{2}-H)z^{H-\frac{3}{2}}. \tag{37}$$

Then setting $z\mapsto -z$, we see that for $u\in[0,1]$ we have

$$|(u-z)^{H-\frac{1}{2}}-(-z)^{H-\frac{1}{2}}| \le (-z)^{H-\frac{1}{2}}-(1-z)^{H-\frac{1}{2}} \le (\frac{1}{2}-H)(-z)^{H-\frac{3}{2}}$$

which is also square integrable with respect to z over $(-\infty,-N]$. Thus $\mathrm{Var}(\Upsilon_\theta)\to\ell$, as claimed in the theorem. Moreover

$$\begin{aligned}\mathbb{E}((\zeta_t-\zeta_s)^2) = \mathbb{E}[(\mathbb{E}(Y_t|\mathcal{F}_0)-\mathbb{E}(Y_s|\mathcal{F}_0))^2] &= \mathbb{E}[\mathbb{E}(Y_t-Y_s|\mathcal{F}_0)^2]\\ &\le \mathbb{E}[\mathbb{E}((Y_t-Y_s)^2|\mathcal{F}_0)]\\ &\le \mathbb{E}((W^H_t-W^H_s)^2) = |t-s|^{2H}.\end{aligned}$$

where W^H is a two-sided fBM, and the final line follows by a simple comparison of the covariance function of Y vs that of W^H. Thus by the result on from page 220 (see also page 216) in [Lif95], we know that for all $\delta>0$, there exists a $\theta^*(\delta,\omega)$ such that for $\theta\in(0,t^*(\delta,\omega))$ we have

$$|\zeta(\theta)| = |\mathbb{E}(Y_\theta|\mathcal{F}_0)| \le (c_1+\delta)(\log\frac{1}{\theta})^{\frac{1}{2}}\theta^H \tag{38}$$

for θ less than some $\theta^*(\delta,\omega)>0$, where the constant c_1 can be chosen to be deterministic. Moreover, $(\zeta(t))_{t\in\mathbb{R}}$ is known at time zero, so $\theta^*(\delta,\omega)$ is $\mathcal{F}^Y_0$-measurable. Hence $\Upsilon(\theta)=\theta^{-H}\int_0^1\zeta(\theta u)du$ satisfies $\theta^\varepsilon\Upsilon_\theta\to 0$ a.s. for $\varepsilon>0$, which means that $(v'(\xi_1)^2+v(\xi_1)v''(\xi_1))\zeta(\theta u)^2$ in (36) is $o(\theta^{2H-2\varepsilon})=o(\theta^H)$. Thus we see that

$$\begin{aligned}\frac{1}{2}\int_0^1\mathbb{E}[e^{-L_\theta}p_{xx}(X^\theta_u,u;z+\frac{L_\theta}{\sqrt{\theta}})(v(Y_0+\zeta(\theta u))^2-v_0^2)]du &= v_0v'(Y_0)\int_0^1\mathbb{E}[p_{xx}(X^0_u,u;z)]\zeta(\theta u)du + o(\theta^H)\\ &= v'(Y_0)\phi(\frac{z}{v_0})\int_0^1\zeta(\theta u)du + o(\theta^H).\end{aligned}$$

Moreover, from Proposition 2.11 below, we know that

$$\begin{aligned}\zeta(\theta u) &= \mathbb{E}(Y_{\theta u}|\mathcal{F}_0^Y) \;=\; \mathbb{E}(Z^H_{\theta u - t_0}|\mathcal{F}^{Z^H}_{-t_0}) \;=\; \frac{1}{\pi}\cos(H\pi)\int_0^{-t_0}(\frac{\theta u}{-t_0-s})^{\frac{1}{2}+H}\frac{1}{\theta u - t_0 - s}Z^H_s ds\\ &= \frac{1}{\pi}\cos(H\pi)\int_{t_0}^0(-\frac{\theta u}{s})^{\frac{1}{2}+H}\frac{1}{\theta u - s}Y_s ds\end{aligned}$$

where $Y_t = Z^H_{t-t_0}$ (as in Remark 1.1), and thus

$$\int_0^1 \zeta(\theta u)du \;=\; \int_{t_0}^0 a_H(s,\theta)\,Y_s ds$$

where $a_H(s,\theta) := \frac{1}{\pi}\cos(H\pi)\int_0^1(-\frac{\theta u}{s})^{\frac{1}{2}+H}\frac{1}{\theta u - s}du$, and we have used Fubini's theorem in the final line. ∎

2.1 Extending to unbounded volatility

We now replace the assumption that v is C_b^2 with the milder assumption that v is C_b^2 on $(-\infty, y]$ for any $y > 0$, which (by Theorem 1 in Gassiat[Gas19]) ensures that S is still a martingale when $\rho \le 0$, so in particular put-call parity is preserved. For this reason we always assume $\rho \le 0$ in this subsection. We use the bar notation $\bar{X}_t := \max_{0\le s\le t} X_s$ to denote the running maximum of a generic process X (we are not referring to the X process in (1) here). Then (from the aforementioned condition in [Gas19]) we know that for some $w > v_0$ sufficiently large, if $A := \{\bar{v}_\theta \le w\}$ then

$$A^c \;=\; \{\bar{v}_\theta \ge w\} \;\subseteq\; B^c$$

where $B := \{\bar{Y}_\theta \le a\}$, for some $a = a(w) > Y_0$.

Now recall from (5) that

$$Y_{\theta u} - Y_0 \;=\; \zeta(\theta u) + Z_{\theta u}$$

where $Z_t = \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}}dW_s$. It is well known that $Z_{\theta(.)}$ satisfies the large deviation principle on $C_0[0,1]$ with speed $1/\theta^{2H}$ and good rate function

$$I(\hat{f}) \;:=\; \begin{cases} \frac{1}{2}\int_0^1 f(s)^2 ds & \text{if } \hat{f}(t) = \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}}f(s)ds \\ \infty & \text{otherwise}\end{cases}$$

(see e.g. [FZ17] and [BFGHS19]), and we let $\mathbf{K}_{\mathrm{RL}}$ denote the operator acting on $L^2([0,1])$ defined by $(\mathbf{K}_{\mathrm{RL}}f)(t) = \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}}f(s)ds$.

From the law of the iterated logarithm-type estimate in (38) and the fact that $\zeta(.)$ is $\mathcal{F}_0$-measurable, we know that $\mathbb{P}(\bar{\zeta}(\theta) > \delta\,|\,\mathcal{F}_0) = 1_{\bar{\zeta}(\theta)>\delta} = 0$ a.s. for $\theta \in (0,\theta^*)$ for some $\mathcal{F}_0$-measurable $\theta^* = \theta^*(\delta,\omega) > 0$, so $\limsup_{\theta\to 0}\theta^{2H}\log\mathbb{P}(\bar{\zeta}(\theta) > \delta\,|\,\mathcal{F}_0) = -\infty$ a.s. for $\delta > 0$. Thus $Z_{\theta(.)}$ and $\zeta(\theta(.)) + Z_{\theta(.)}$ are *exponentially equivalent* on $C_0[0,1]$ under the sup norm metric in the sense of Definition 4.2.10 in [DZ98], so (by Theorem 4.2.13 in [DZ98]), $Y_{\theta(.)} - Y_0 = \zeta(\theta(.)) + Z_{\theta(.)}$ also satisfies the same LDP as $Z_{\theta(.)}$, a.s. (note the a.s. qualifier is needed here, because $\zeta(.)$ is random so the value of θ^* is not known until time zero and (38) is only known to hold a.s.).

Then (from the contraction principle from large deviations theory), since the maximum of a function is a continuous functional under the sup norm metric, $\bar{Y}_\theta - Y_0$ satisfies the LDP with speed θ^{-2H} and good rate function:

$$J(y) \;=\; \inf_{\hat{f}\in\mathbf{K}_{\mathrm{RL}}L^2([0,1]):\bar{\hat{f}}(1)=y} I(\hat{f}) \;=\; \inf_{f\in L^2([0,1]):\overline{\mathbf{K}_{\mathrm{RL}}f}(1)=y}\frac{1}{2}\|f\|^2_{L^2([0,1])}\,.$$

Moreover, from the Cauchy-Schwarz inequality we know that for $t \in [0,1]$

$$(\mathbf{K}_{\mathrm{RL}}f)(t)^2 \;=\; 2H(\int_0^t (t-s)^{H-\frac{1}{2}}f(s)ds)^2 \;\le\; t^{2H}\cdot\|f\|^2_{L^2([0,1])} \;\le\; \|f\|^2_{L^2([0,1])}$$

so

$$\overline{\mathbf{K}_{\mathrm{RL}}f}(t) \;\le\; \|f\|_{L^2([0,1])}\,.$$

Thus if $\overline{(\mathbf{K}_{\mathrm{RL}}f)}(t) \ge a$ for some $t \in [0,1]$, then $\|f\|_{L^2([0,1])} \ge a$, and hence $I(f) \ge \frac{1}{2}a^2$. This means that $J(y) \ge \frac{1}{2}y^2$ for $y > 0$.

We now let $\hat{S}_t = e^{\hat{X}_t}$, where

$$d\hat{X}_t \;=\; -\frac{1}{2}\tilde{v}(Y_t)^2 dt + \tilde{v}(Y_t)(\rho dW_t + \bar{\rho}dW_t^\perp) - dL_t\,, \qquad (39)$$

and $\hat{X}_0 = 0$ i.e. the same stock price process (defined on the same probability space) but with our (new) unbounded v function replaced by a C_b^2 function $\tilde{v}$ such that $v(y) = \tilde{v}(y)$ for $y \leq a$. Then we see that

$$\begin{aligned}\mathbb{E}((e^{z\sqrt{\theta}} - S_\theta)^+) &= \mathbb{E}((e^{z\sqrt{\theta}} - \hat{S}_\theta)^+ 1_B) + \mathbb{E}((e^{z\sqrt{\theta}} - S_\theta)^+ 1_{B^c}) \\ &\leq \mathbb{E}((e^{z\sqrt{\theta}} - \hat{S}_\theta)^+) + e^{z\sqrt{\theta}}\mathbb{P}(B^c) \\ &\leq \mathbb{E}((e^{z\sqrt{\theta}} - \hat{S}_\theta)^+) + e^{z\sqrt{\theta}} e^{-\frac{1}{\theta^{2H}}(\inf_{y\geq a} J(y) - \varepsilon)}\end{aligned}$$

where we have used the upper bound implied by the aforementioned LDP in the final line, and we know that $\inf_{y\geq a} J(y) \geq \frac{1}{2}a^2 > 0$. Similarly

$$\begin{aligned}\mathbb{E}((e^{z\sqrt{\theta}} - S_\theta)^+) &\geq \mathbb{E}((e^{z\sqrt{\theta}} - \hat{S}_\theta)^+ 1_B) \\ &= \mathbb{E}((e^{z\sqrt{\theta}} - \hat{S}_\theta)^+) - \mathbb{E}((e^{z\sqrt{\theta}} - \hat{S}_\theta)^+ 1_{B^c}) \\ &\geq \mathbb{E}((e^{z\sqrt{\theta}} - \hat{S}_\theta)^+) - e^{z\sqrt{\theta}} e^{-\frac{1}{\theta^{2H}}(\inf_{y\geq a} J(y) - \varepsilon)}\,.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{\sqrt{\theta}}\mathbb{E}((e^{z\sqrt{\theta}} - \hat{S}_\theta)^+) - \frac{e^{z\sqrt{\theta}}}{\sqrt{\theta}} e^{-\frac{1}{\theta^{2H}}(\inf_{y\geq a} J(y) - \varepsilon)} &\leq \frac{1}{\sqrt{\theta}}\mathbb{E}((e^{z\sqrt{\theta}} - S_\theta)^+) \\ &\leq \frac{1}{\sqrt{\theta}}\mathbb{E}((e^{z\sqrt{\theta}} - \hat{S}_\theta)^+) + \frac{e^{z\sqrt{\theta}}}{\sqrt{\theta}} e^{-\frac{1}{\theta^{2H}}(\inf_{y\geq a} J(y) - \varepsilon)}\,.\end{aligned}$$

From the main Theorem 2.1, we have a small-θ expansion for $\frac{1}{\sqrt{\theta}}\mathbb{E}((e^{z\sqrt{\theta}} - \hat{S}_\theta)^+)$, and for θ small, $\frac{1}{\sqrt{\theta}} e^{z\sqrt{\theta}} e^{-\frac{1}{\theta^{2H}}(\inf_{y\geq a} J(y) - \varepsilon)}$ is higher order than the error term in Theorem 2.1, so the same expansion holds for $\frac{1}{\sqrt{\theta}}\mathbb{E}((e^{z\sqrt{\theta}} - S_\theta)^+)$.

2.2 Implied volatility

Lemma 2.7 *Let $C^{BS}(S, K, \sigma, T)$ denote the usual Black-Scholes formula for the price of a European call option when interest rates and dividends are zero, and let $\sigma_t = \sigma + at^{1-\frac{1}{2}Y} + bt^H$ where $0 < 1 - \frac{1}{2}Y < H < \frac{1}{2}$. Then*

$$\frac{1}{\sqrt{t}} C^{BS}(1, e^{z\sqrt{t}}, \sigma_t, t) = \mathbb{E}((X_1^0 - z)^+) + \phi(\frac{z}{\sigma})(at^{1-\frac{1}{2}Y} + bt^H) + o(t^H)\,. \tag{40}$$

By equating terms in (10) and (40), we obtain the following:

Corollary 2.8 *Under the assumptions on Y and H in the main Theorem 2.1, we have the following asymptotic behaviour for the implied volatility in the small-maturity limit in the $k_t = z\sqrt{t}$ regime:*

$$\sigma_{\text{impl}}(z\sqrt{\theta}, \theta) = v_0 + \frac{A_1(z)}{\phi(\frac{z}{v_0})}\theta^{1-\frac{1}{2}Y} + \theta^H(\alpha + \beta_\theta) + o(\theta^H) \tag{41}$$

where α, β_θ are defined as in (9).

Remark 2.6 The rough stochastic volatility skew correction term here (without the history term β_θ) agrees with the term obtained by [BFGHS19] for their *moderate deviations* regimes.

(41) shows that if the market observed skew in the $k_t = z\sqrt{t}$ regime is not affine in z, then the model cannot be pure fractional stochastic volatility. As a trivial corollary, we see that for $z > y$ we have the following asymptotic form for the implied vol skew:

$$\frac{\sigma_{\text{impl}}(z\sqrt{\theta}, \theta) - \sigma_{\text{impl}}(y\sqrt{\theta}, \theta)}{\sqrt{\theta}(z - y)} \sim \frac{1}{z - y}\left(\frac{A_1(z)}{\phi(\frac{z}{v_0})} - \frac{A_1(y)}{\phi(\frac{y}{v_0})}\right)\theta^{\frac{1}{2}(1-Y)} \tag{42}$$

as $\theta \to 0$. This shows that exploding power behaviour for the implied vol skew can also be caused by jumps and not just fractional models, and similar behaviour is reported in Corollary 6 in Gerhold et al.[GGP16] for a class of pure Lévy models.

2.3 Extracting the volatility history and H from the log stock price sample path

In practice, we can only directly observe the log stock price process X_t. But we also know that the quadratic variation $[L]_t$ of L is itself a non-decreasing Lévy process (i.e. a subordinator) with Lévy density $q(y) = \frac{\nu(\sqrt{y})}{2\sqrt{y}} + \frac{\nu(-\sqrt{y})}{2\sqrt{y}} = \frac{1}{2}e^{-M\sqrt{y}}y^{-1-\frac{1}{2}Y} + \frac{1}{2}e^{-G\sqrt{y}}|y|^{-1-\frac{1}{2}Y}$ for $y > 0$ (see e.g. [CGMY05]), which has finite variation a.s. Then

$$[X]_t = \int_0^t v(Y_s)^2 ds + \sum_{s\in[0,t]}^{\Delta[L]_s \neq 0} \Delta[L]_s = \int_0^t v(Y_s)^2 ds + \lim_{\varepsilon\to 0} \sum_{s\in[0,t]}^{\Delta[L]_s \geq \varepsilon} \Delta[L]_s$$

where $\Delta[L]_t = [L]_t - [L]_{t-}$. Hence given the whole path $(X_t)_{t\geq 0}$, we can extract Y_t as

$$Y_t \;=\; (v^2)^{-1}(\frac{d}{dt}([X]_t - \sum_{s\in[0,t]}^{\Delta[X]_s\neq 0} \Delta[X]_s)) \,.$$

Of course in practice one would only have access to discrete time data, but H can be then estimated from a time series using maximum likelihood methods (see e.g. [Cha14] for details). See also the many articles by Aït-Sahalia and Jacod for estimators of e.g. the Blumenthal-Getoor index of a jump process from discretely sampled data.

2.4 Monte Carlo simulation and numerical results

The following result allows for fast and exact simulation of the CGMY process in terms of an α-stable process, without having to truncate small jumps.

Lemma 2.9 *(see Theorem 3.1 in Poirot&Tankov[PT06]). Let X^+ be a tempered stable Lévy processes on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ with characteristic triplet $(0,\nu,\Gamma^+)$, where $\Gamma_+ = C_+\frac{e^{-Mx}}{|x|^{1+Y}}1_{x>0}$. Let $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} := e^{U_t}$ where $U_t := MX_t^+ + ct$, and c is chosen so that $e^{MX_t^+ + ct}$ is a $\mathbb{P}$-martingale. Then $(X_t^+)_{t\geq 0}$ is an α-stable process under $\mathbb{Q}$ with Lévy triple $(b,0,\tilde{\nu})$ for some $b\in\mathbb{R}$, where $\tilde{\nu}(x) = \frac{C}{x^{1+Y}}1_{x>0}$.*

Corollary 2.10 *(see page 11 in [PT06]). For any $\mathcal{F}_t$-measurable random variable H_t, we have*

$$\mathbb{E}^{\mathbb{P}}(H_t) \;=\; \mathbb{E}^{\mathbb{Q}}(H_t e^{-U_t})\,. \tag{43}$$

Remark 2.7 We can simulate an α-stable random variable using an independent uniform and exponential random variable, using the well known Chambers,Mallows&Stuck method (see [CMS76] and [PT06] for details).

Our Monte Carlo scheme uses (43) and the [CMS76] method to simulate L_t, and the moment-matching scheme for the Riemann-Liouville process described at the top of page 15 in [HJM17] (see Figures 2-4).

2.5 The prediction formula for the Riemann-Liouville process

Remark 1.1 shows that we can transform the Y process to a standard Riemann-Lioville process. The next proposition computes the conditional mean and covariance of the RL process given its history, similar to the prediction formula in Theorem 3.1 in [SV17] for one-sided fBM.

Proposition 2.11 *Let $Z_t^H = \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}}dB_s$ be a Riemann-Liouville process where B is a standard Brownian motion. Then Z^H has conditional mean and covariance given by*

$$\mathbb{E}(Z_u^H|\mathcal{F}_t^{Z^H}) \;=\; \int_0^t \bar{k}_H(s) Z_s^H ds \tag{44}$$

$$\mathrm{Cov}(Z_s^H, Z_u^H|\mathcal{F}_t^{Z^H}) \;=\; 2H\int_t^{s\wedge u} (u-v)^{H-\frac{1}{2}}(s-v)^{H-\frac{1}{2}}dv$$

for $u\geq t$, where

$$\bar{k}_H(s) \;=\; \bar{k}_H(s;t,u) \;=\; \sqrt{2H}\,\bar{c}_H(\frac{1}{2}-H)(\frac{u-t}{t-s})^{\frac{1}{2}+H}\frac{1}{u-s} \;=\; \frac{1}{\pi}\cos(H\pi)(\frac{u-t}{t-s})^{\frac{1}{2}+H}\frac{1}{u-s} \tag{45}$$

and $\bar{c}_H$ is defined as in Lemma 1.1.

Proof. See Appendix A ■

Remark 2.8 For a rough Bergomi model calibrated to the observed variance term structure at time zero, the instantaneous variance (i.e. volatility squared) at time $u\geq 0$ is given by

$$v_u \;=\; \xi_0(u)e^{\eta_H Z_u^H - \frac{1}{2}\eta_H^2 u^{2H}}\,.$$

Hence from Proposition 2.11 we have

$$\mathbb{E}(\log v_u|\mathcal{F}_t^v) \;=\; \log\xi_0(u) \,+\, \eta_H\int_0^t \bar{k}_H(s)Z_s^H ds \,-\, \frac{1}{2}\eta_H^2 u^{2H}$$

for $t\in[0,u]$, which we can view as method of forecasting log volatility at time u given the vol history up to time t.

3 Small-time Edgeworth expansions for the Rough Heston model

In this section, we consider the Rough Heston model (without jumps) for a log stock price process X_t introduced in [JR16]:

$$\begin{aligned} dX_t &= -\frac{1}{2}V_t dt + \sqrt{V_t} dB_t \\ V_t &= V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda(\theta - V_s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\nu\sqrt{V_s}dW_s \end{aligned}$$

for $\alpha \in (\frac{1}{2}, 1]$, where W and B are two correlated Brownian motions with $dW_t dB_t = \rho dt$ with $\rho \in (-1, 1)$. We assume $X_0 = 0$ and zero interest rate without loss of generality, since the law of $X_t - X_0$ is independent of X_0. The parameter α controls the roughness of the sample path of V, so V is rougher when α is smaller and vice versa.

3.1 The characteristic function of the log stock price

From Corollary 3.1 in [ER19] (see also Section 5 in [GGP19]), we know that for all $t > 0$

$$\mathbb{E}(e^{pX_t}) = e^{V_0 I^{1-\alpha} f(p,t) + \lambda\theta I^1 f(p,t)} \tag{46}$$

for p in some open interval $(p_-, p_+) \supset [0, 1]$, where $f(p,t)$ satisfies

$$D^\alpha f(p,t) = \frac{1}{2}(p^2 - p) + (p\rho\nu - \lambda)f(p,t) + \frac{1}{2}\nu^2 f(p,t)^2 \tag{47}$$

with initial condition $f(p, 0) = 0$, and D^α denotes the fractional derivative operator of order α (see page 17 in [ER19] for definition).

3.2 Small-time Edgeworth expansions

We now consider the following family of re-scaled Rough Heston models:

$$\begin{aligned} dX^\varepsilon_t &= -\frac{1}{2}\varepsilon V^\varepsilon_t dt + \sqrt{\varepsilon}\sqrt{V^\varepsilon_t} dB_t \\ V^\varepsilon_t &= V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{H-\frac{1}{2}}\varepsilon^\alpha\lambda(\theta - V^\varepsilon_s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{H-\frac{1}{2}}\varepsilon^H\nu\sqrt{V^\varepsilon_s}dW_s \end{aligned}$$

where $H = \alpha - \frac{1}{2} \in (0, \frac{1}{2}]$. We make the following assumption throughout this subsection:

Assumption 3.1 $H \in (0, \frac{1}{4})$.

We relax this assumption when we go to higher order in the next subsection.

From [FGS19] we know that $(X^\varepsilon_{(.)}, V^\varepsilon_{(.)}) \overset{(\mathrm{d})}{=} (X^1_{\varepsilon(.)}, Y^1_{\varepsilon(.)})$ and specifically from Eqs 15 and 16 in [FGS19], we know that

$$\mathbb{E}(e^{pX^\varepsilon_t}) = e^{V_0 I^{1-\alpha} f_\varepsilon(p,t) + \varepsilon^\alpha\lambda\theta f_\varepsilon(p,t)} \tag{48}$$

on some non-empty interval $[0, T^*_\varepsilon(p))$, where $f_\varepsilon(p,t)$ satisfies

$$D^\alpha f_\varepsilon(p,t) = \frac{1}{2}\varepsilon(p^2 - p) + \varepsilon^\alpha(p\rho\nu - \lambda)f_\varepsilon(p,t) + \frac{1}{2}\varepsilon^{2H}\nu^2 f_\varepsilon(p,t)^2 \tag{49}$$

and $f_\varepsilon(p, 0) = 0$. Setting $f_\varepsilon(\frac{p}{\sqrt{\varepsilon}}, t) = \phi_\varepsilon(p,t)$, we see that $\phi_\varepsilon(p,t)$ satisfies

$$D^\alpha\phi_\varepsilon(p,t) = \frac{1}{2}p^2 - \frac{1}{2}p\sqrt{\varepsilon} + \varepsilon^H p\rho\nu\phi_\varepsilon(p,t) + \frac{1}{2}\varepsilon^{2H}\nu^2\phi_\varepsilon(p,t)^2 - \lambda\varepsilon^\alpha\phi_\varepsilon(p,t) \tag{50}$$

with $\phi_\varepsilon(p, 0) = 0$. The quadratic function $G(p, w) := \frac{1}{2}p^2 - \frac{1}{2}p\sqrt{\varepsilon} + p\varepsilon^H\rho\nu w + \frac{1}{2}\varepsilon^{2H}\nu^2 w^2$ associated with the VIE for $\phi_\varepsilon(p,t)$ in (50) has purely imaginary roots with a positive minimizer (akin to case B in [GGP19]), and thus has a finite explosion time $\hat{T}_\varepsilon(p)$, but the linear and the non-linear terms in this VIE tend to zero as $\varepsilon \to 0$, so we can easily verify using the lower bound on the moment explosion time in Theorem 4.1 in [GGP19] that $\hat{T}_\varepsilon(p) \to \infty$ as $\varepsilon \to 0$, and in fact all we need here is that $\hat{T}_\varepsilon(p) > 1$ for ε sufficiently small.

Guessing an approximate asymptotic small-time solution for $\phi_\varepsilon(p,t)$ of the form:

$$\bar{\phi}_\varepsilon(p,t) = \phi(p,t) + \phi_1(p,t)\varepsilon^H + \phi_2(p,t)\varepsilon^{2H} \tag{51}$$

for $t \in [0,1]$, and (using that $H \in (0, \frac{1}{4})$) and equating like powers of ε, we find that

$$\begin{aligned} D^\alpha \phi(p,t) &= \frac{1}{2}p^2 \\ D^\alpha \phi_1(p,t) &= p\rho\nu\phi(p,t) \\ D^\alpha \phi_2(p,t) &= \frac{1}{2}\nu^2\phi(p,t)^2 + p\rho\nu\phi_1(p,t) \end{aligned}$$

so $\phi(p,t) = I^\alpha(\frac{1}{2}p^2) = \frac{1}{2}p^2 \cdot I^\alpha 1 = \frac{1}{2}p^2 \frac{t^\alpha}{\Gamma(\alpha+1)}$, where we have also used the identity $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$. Recall that $\mathbb{E}(e^{\frac{p}{\sqrt{\varepsilon}}X_t^\varepsilon}) = e^{V_0 I^{1-\alpha} f_\varepsilon(\frac{p}{\sqrt{\varepsilon}},t) + \lambda\theta I^1 f_\varepsilon(\frac{p}{\sqrt{\varepsilon}},t)} = e^{V_0 I^{1-\alpha}\phi_\varepsilon(p,t) + \lambda\theta I^1 \phi_\varepsilon(p,t)}$. Then

$$\begin{aligned} \log \mathbb{E}(e^{\frac{p}{\sqrt{\varepsilon}}X_t^\varepsilon}) &= V_0 I^{1-\alpha}(\frac{1}{2}p^2\frac{t^\alpha}{\Gamma(\alpha+1)} + \varepsilon^H p\rho\nu\phi(p,t) + \varepsilon^{2H}(\frac{1}{2}\nu^2\phi(p,t)^2 + p\rho\nu\phi_1(p,t)) + o(\varepsilon^{2H}) \\ &= V_0(\frac{1}{2}p^2 t + \varepsilon^H \frac{p^3\rho\nu\, t^{1+\alpha}}{2\Gamma(2+\alpha)} + \varepsilon^{2H}\frac{p^4 t^{1+2\alpha}\nu^2}{8(1+2\alpha)\Gamma(1+\alpha)^2} + \varepsilon^{2H}\frac{p^4 t^{1+2\alpha}\nu^2\rho^2}{2(1+2\alpha)\Gamma(1+2\alpha)} + o(\varepsilon^{2H}))\,. \end{aligned}$$

Then setting $t=1$ but then replacing ε with t, we see that

$$\begin{aligned} \mathbb{E}(e^{\frac{p}{\sqrt{t}}X_t}) &= e^{V_0(\frac{1}{2}p^2 + t^H\frac{p^3\rho\nu}{2\Gamma(2+\alpha)} + t^{2H}\frac{p^4\nu^2}{8(1+2\alpha)\Gamma(1+\alpha)^2} + t^{2H}\frac{p^4\nu^2\rho^2}{2(1+2\alpha)\Gamma(1+2\alpha)} + o(t^{2H}))} \\ &= e^{V_0(\frac{1}{2}p^2 + b\rho\nu p^3 t^H + \nu^2(c+d\rho^2)p^4 t^{2H} + o(t^{2H}))} \end{aligned}$$

where $b = \frac{1}{2\Gamma(2+\alpha)}$, $c = \frac{1}{8(1+2\alpha)\Gamma(1+\alpha)^2}$ and $d = \frac{1}{2(1+2\alpha)\Gamma(1+2\alpha)}$, and the effect of λ is contained in the $o(t^{2H})$ error term here.

Then we expect the density $p_t(x)$ of $\frac{X_t}{\sqrt{t}}$ to have the following expansion

$$\begin{aligned} p_t(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iax}\mathbb{E}(e^{i\frac{a}{\sqrt{t}}X_t})da \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iax}e^{V_0(-\frac{1}{2}a^2 - ib\rho\nu a^3 t^H + (c+d\rho^2)\nu^2 a^4 t^{2H} + o(t^{2H}))}da \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iax}e^{-\frac{1}{2}a^2\sigma^2}(1 - ib\rho\nu V_0 a^3 t^H + (c+d\rho^2)\nu^2 V_0 a^4 t^{2H} - \frac{1}{2}b^2\rho^2\nu^2 V_0^2 a^6 t^{2H} + o(t^{2H}))da \\ &= n(x) - b\rho\nu V_0 n^{(3)}(x)t^H + [(c+d\rho^2)\nu^2 V_0 n^{(4)}(x) + \frac{1}{2}b^2\rho^2\nu^2 V_0^2 n^{(6)}(x)]t^{2H} + o(t^{2H}) \end{aligned}$$

as $t \to 0$, where $\sigma = \sqrt{V_0}$, ϕ is the standard normal density and $n(x) = \frac{1}{\sigma}\phi(\frac{x}{\sigma})$. The $O(t^H)$ term is the vol skew adjustment, and the $O(t^{2H})$ term is the vol convexity adjustment. In particular, from Lévy's convergence theorem, we expect that $X_t/\sqrt{t}$ tends weakly to an $N(0,V_0)$ random variable.

3.3 Asymptotics for call options and implied volatility

By integrating over our asymptotic expression for $p_t(x)$, we obtain the following small-time behaviour for European call options:

Proposition 3.2

$$\begin{aligned} \frac{1}{\sqrt{t}}\mathbb{E}((e^{X_t} - e^{z\sqrt{t}})^+) &= \mathbb{E}((X_1^0 - z)^+) + \phi(\frac{z}{\sqrt{V_0}})(\frac{b\rho\nu z}{\sqrt{V_0}}t^H + [(c+d\rho^2)(z^2 - V_0)\frac{\nu^2}{V_0^{\frac{3}{2}}} + \frac{b^2\nu^2\rho^2}{2V_0^{\frac{5}{2}}}(z^4 - 6z^2V_0 + 3V_0^2)]\,t^{2H}) \\ &+ o(t^{2H}) \end{aligned} \tag{52}$$

where $X_1^0 \sim N(0,V_0)$.

Lemma 3.3 *Let $C^{BS}(S,K,\sigma,T)$ denote the usual Black-Scholes formula for the price of a European call option when interest rates and dividends are zero, and let $\sigma_t = \sigma + at^H + bt^{2H}$ where $2H \in (0,\frac{1}{2})$. Then*

$$\frac{1}{\sqrt{t}}C^{BS}(1, e^{z\sqrt{t}}, \sigma_t, t) = \mathbb{E}((X_1^0 - z)^+) + \phi(\frac{z}{\sigma})(at^H + (b + \frac{a^2z^2}{2\sigma^3})t^{2H}) + o(t^{2H}) \tag{53}$$

Proof. We calculated this using the `Series` command in Mathematica.[4] ■

Equating the $O(1)$, $O(t^H)$ and $O(t^{2H})$ terms in (52) and (53), we obtain the following:

[4]Mathematica workbook available on request.

Corollary 3.4 *We have the following asymptotic behaviour for the implied volatility in the small-maturity limit in the $k_t = z\sqrt{t}$ regime:*

$$\sigma_{\text{impl}}(z\sqrt{t},t) \;=\; \sqrt{V_0} + \frac{b\rho\nu}{V_0^{\frac{1}{2}}} z t^H + \Big(-\frac{\nu^2(2c+(3b^2+2d)\rho^2)}{2V_0^{\frac{1}{2}}} + \frac{\nu^2(c+(3b^2+d)\rho^2)}{V_0^{\frac{3}{2}}} z^2\Big) t^{2H} + o(t^{2H}) \tag{54}$$

Remark 3.1 The $O(z)$ term here is the vol skew term, the $O(z^2)$ term is the vol convexity term and the $O(t^{2H})$ term evaluated at $z = 0$ is the at-the-money small-time implied vol correction term (see also Theorem 3.2 in [EFGR18] for an expansion at the same order for the Rough Bergomi model). (54) implies the same type of exploding power behaviour for the implied vol skew as in (42), and could be very useful for calibrating the Rough Heston model or parametrizing the vol surface for short-maturity FX options.

3.4 Higher order expansions

If we go to higher order and now set $\bar{\phi}_\varepsilon(p,t) = \phi(p,t) + \phi_1(p,t)\varepsilon^H + \phi_2(p,t)\varepsilon^{2H} + \phi_3(p,t)\varepsilon^{\frac{1}{2}}$ then (assuming $H \in (\frac{1}{6}, \frac{1}{4})$ so $0 < H < 2H < \frac{1}{2} < 3H$)) and equating like powers of ε, we find that

$$D^\alpha \phi_3(p,t) \;=\; -\frac{1}{2}p$$

which now captures the effect of the log stock price drift.

Conversely, if $H \in (0, \frac{1}{6})$ so $0 < H < 2H < 3H < \frac{1}{2}$, and we set $\bar{\phi}_\varepsilon(p,t) = \phi(p,t) + \phi_1(p,t)\varepsilon^H + \phi_2(p,t)\varepsilon^{2H} + \phi_3(p,t)\varepsilon^{3H}$, then

$$D^\alpha \phi_3(p,t) \;=\; p\rho\nu\phi_2(p,t) + \frac{1}{2}\nu^2\phi(p,t)\phi_1(p,t)\,.$$

Finally, for $H = \frac{1}{6}$ we see that

$$D^\alpha \phi_3(p,t) \;=\; -\frac{1}{2}p + p\rho\nu\phi_2(p,t) + \frac{1}{2}\nu^2\phi(p,t)\phi_1(p,t)\,.$$

With a bit more pain, we can also translate these expansions into density, call option and implied volatility asymptotics (we omit the details as the implied volatility calculation at this order is rather lengthy and tedious).

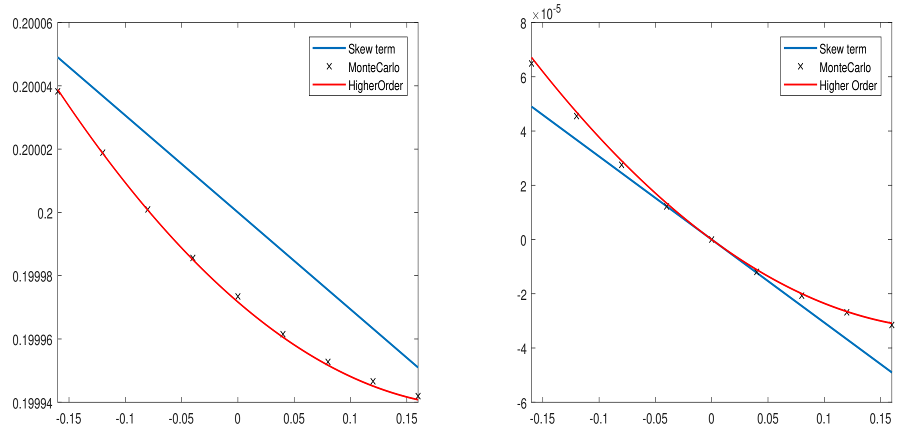

Figure 1: On the left we see the implied volatility for the Rough Heston model using Monte Carlo (black crosses) and the asymptotic implied volatility using the first order skew correction term (i.e. the first two terms in (54)) in blue, and the implied volatility using the 1st and 2nd order correction terms (i.e. all the terms in (54)) in red, $\alpha = .7$ (so $H = .2$), $\nu = .15$, $\rho = -.02$, $\lambda = 0$, $T = .000001$, 10^6 simulations and 400 time steps. The right plot shows the skew, in this case just the OTM implied vol minus the ATM implied vol, for the same simulation and parameters

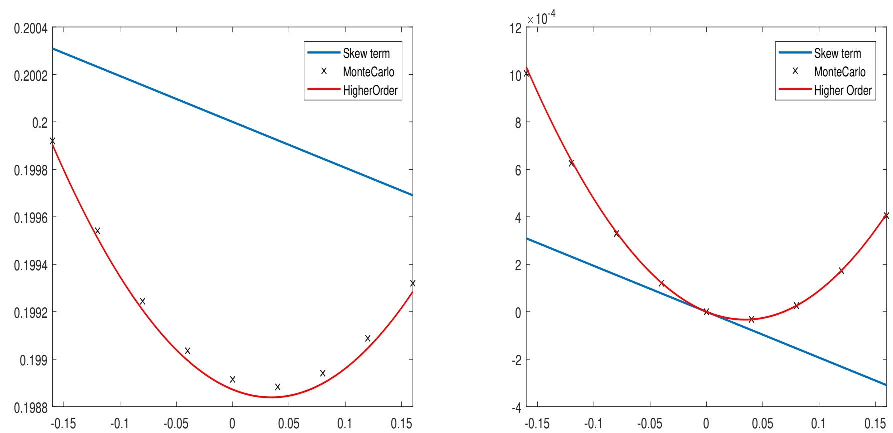

Figure 2: Here we have plotted the same quantities with the same parameters, but for the larger maturity $T = .01$.

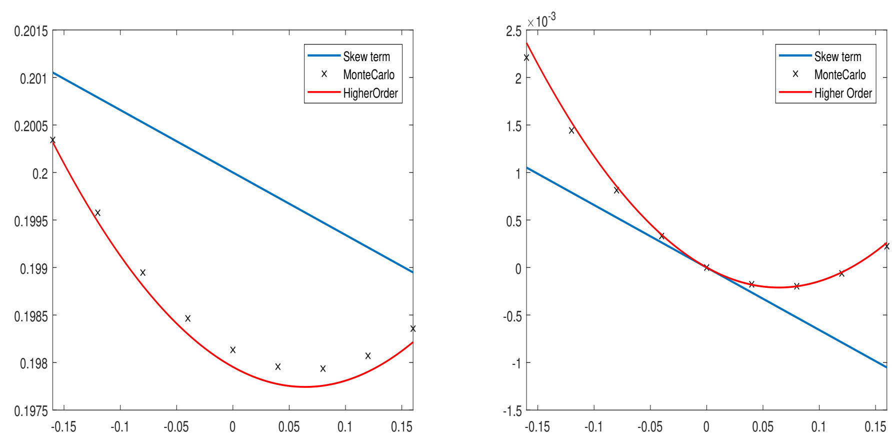

Figure 3: Here we have plotted the same quantities with $\alpha = .6$ (so $H = .1$), $\nu = .15$, $\rho = -.05$, $\lambda = 0$ and $T = .001$.

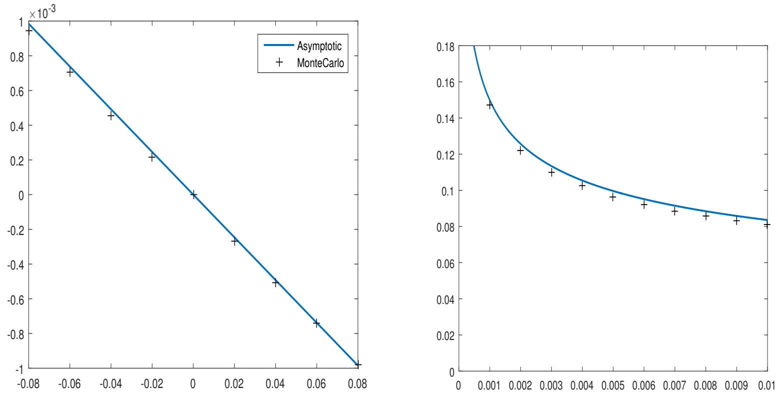

Figure 4: On the left we have plotted the Rough stoc vol correction term $\rho z \frac{\sqrt{2H}}{(\frac{1}{2}+H)(\frac{3}{2}+H)} \frac{v'(Y_0)}{v_0}$ (i.e. the α term in (41)) for our original general model in (1), against the answer implied for this term by Monte Carlo simulation of the model with the jumps included, when $Y_t = \sqrt{2H} \int_0^T (t-s)^{H-\frac{1}{2}} dW_s$. We use the moment-matching scheme described in section 3.3.1 in [HJM17] and the α-stable measure change and the Chambers,Mallows&Stuck[CMS76] method discussed in Subsection 2.4 with antithetic sampling, for $H = .25$, $\rho = -.1$, $v(y) = .913579(.2 + .05y)$, $C_+ = .0003$, $C_=0$, $Y = 1.6$, $M = 1.93$ with no history, and maturity t=.000001 using 2,000,000 simulations and 200 time steps. This maturity is clearly unrealistically small but the point here is to numerically verify that Rough stoc vol correction term is correct before testing larger maturities below. On the right we have plotted the implied vol skew (close to at-the-money) $\frac{\sigma_{\text{impl}}(z\sqrt{t},t) - \sigma_{\text{impl}}(y\sqrt{t},t)}{\sqrt{t}\,(z-y)}$ as a function of the maturity t, with $z = .02$, $y = -.02$ and the same parameters as above except now $C_+ = .00005$ (see also Figure 1 in [BFGHS19]).

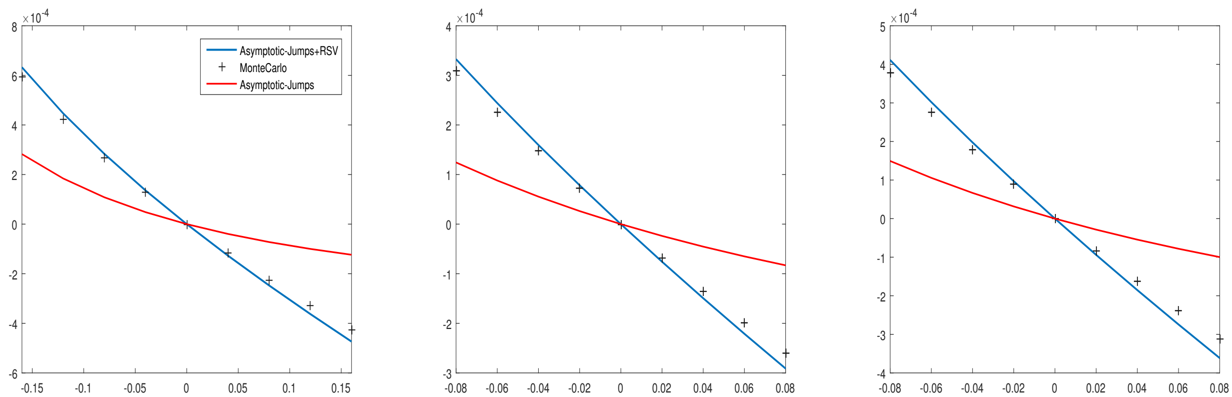

Figure 5: Here we have plotted $\sigma_{\text{impl}}(z\sqrt{t},t) - \sigma_{\text{impl}}(0,t)$ using the asymptotic expansion in (41) with (blue) and without (red crosses) the RSV correction term, and the answer obtained using Monte Carlo scheme (black crosses), for the same parameters but now $t = .001$ in the left plot, $t = .02$ in the middle plot and $t = .05$ in the right plot and we have used $1,000,000$ simulations and 100 time steps, and we now see the convexity i.e. smile effect caused by the jumps.

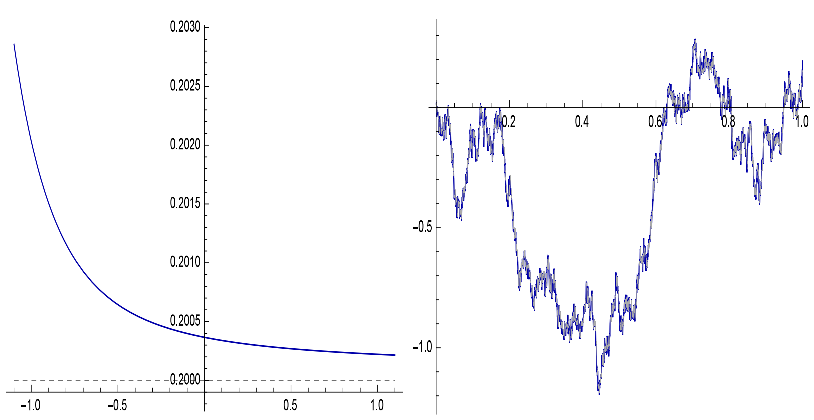

Figure 6: The left graph plots the leading order and first correction term in (41) (i.e. just jumps) for implied volatility for maturity $T = .001$ and same parameters as previous graph. The right graph plots the original Brownian signal W (blue) against the process obtained using the transformation formula in (6) (in grey) applied to Y where $Y_t = \sqrt{2H} \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ is the RL process generated by W (for $H = .4$ with 10,000 time steps), and we see that they are very close.

References

[BA88] Ben-Arous, G., "Methods de Laplace et de la phase stationnaire sur le space de Wiener", *Stochastics*, 25: 125-153, 1988.

[BFG16] Bayer, C., P.K.Friz and J.Gatheral, "Pricing Under Rough Volatility", *Quantitative Finance*, 16(6), 887-904, 2016.

[BFGHS19] Bayer, C., P.K.Friz, A.Gulisashvili, B.Horvath, B.Stemper, "Short-Time Near-The-Money Skew In Rough Fractional Volatility Models", *Quantitative Finance*, 19:5, 779-798, 2019.

[CGMY05] Carr, P., H.Geman, D.Madan and M.Yor, "Pricing Options on Realized Variance", *Finance and Stochastics*, 9, 453-475, 2005.

[Cha14] Chang, Y.C., "Efficiently Implementing the Maximum Likelihood Estimator for Hurst Exponent", Hindawi Publishing Corporation Mathematical Problems in Engineering Volume, 2014.

[CMS76] Chambers, J., C.Mallows, and B.Stuck, "A method for simulating stable random variables", 71 (1976), pp. 340-344.

[DZ98] Dembo, A. and O.Zeitouni, "Large deviations techniques and applications", Jones and Bartlet publishers, Boston, 1998.

[EFR18] El Euch, O., M.Fukasawa, and M.Rosenbaum, "The microstructural foundations of leverage effect and rough volatility", *Finance and Stochastics*, 12 (6), p. 241-280, 2018.

[EFGR18] El Euch, O., Fukasawa, M., Gatheral, J. and M.Rosenbaum, "Short-term at-the-money asymptotics under stochastic volatility models", to appear in *SIAM Journal on Financial Mathematics*.

[ER18] El Euch, O. and M.Rosenbaum, "Perfect hedging in rough Heston models", *The Annals of Applied Probability*, 28 (6), p. 3813-3856, 2018.

[ER19] El Euch, O. and M.Rosenbaum, "The characteristic function of rough Heston models", *Mathematical Finance*, 29(1), 3-38, 2019.

[EFR18] El Euch, O., Fukasawa, M., Gatheral, J. and M.Rosenbaum, "The microstructural foundations of leverage effect and rough volatility", *Finance and Stochastics*, 22 (2), p. 241-280, 2018.

[FGH17] Figueroa-López, J., R.Gong and C.Houdré, "Third-Order Short-Time Expansions for Close-to-the-Money Option Prices Under the CGMY Model", *Journal of Applied Mathematical Finance*, 24, p. 547-574, 2017.

[FLF12] Figueroa-López, J. and M.Forde, "The small-maturity smile for exponential Lévy models", with J.E.Figueroa-López, *SIAM J. Finan. Math.*, 3, 33-65, 2012.

[FGS19] Forde, M., S.Gerhold and B.Smith, "Small-time, large-time and $H \to 0$ asymptotics for the Rough Heston model", preprint, 2019.

[FKZ13] Fink, H., C.Klüppelberg and M.Zähle, "Conditional Distributions Of Processes Related To Fractional Brownian Motion", *J.Appl.Prob.*, 50, 166-183, 2013.

[FZ17] Forde, M. and H.Zhang, "Asymptotics for rough stochastic volatility models", *SIAM Journal on Financial Mathematics*, 8, 114-145, 2017.

[Fuk17] Fukasawa, M., "Short-time at-the-money skew and rough fractional volatility", *Quantitative Finance*, 17(2), 189-198, 2017.

[Gas19] Gassiat, P., "On the martingale property in the rough Bergomi model“, *Electron. Commun. Probab.*, Volume 24 (2019), paper no. 33, 9 pp.

[GK19] Gatheral, J. and M.Keller-Ressel, "Affine forward variance models", *Finance and Stochastics*, volume 23, pages 501?533, 2019.

[GGP16] Gerhold, S., I.Cetin Gülüm & A.Pinter, "Small-Maturity Asymptotics for the At-The-Money Implied Volatility Slope in Lévy Models", *Applied Mathematical Finance*, 23:2, 135-157, 2016.

[GGP19] Gerhold, S., C.Gerstenecker and A.Pinter, "Moment Explosions In The Rough Heston Model", Decisions in Economics and Finance 42(2), pp. 575-608, 2019.

[HJM17] Horvath, B., A.Jacquier and A. Muguruza, "Functional central limit theorems for rough volatility", preprint, 2017.

[JR16] Jaisson, T. and M.Rosenbaum, "Rough fractional diffusions as scaling limits of nearly unstable heavy tailed Hawkes processes", *The Annals of Applied Probability*, 26 (5), 2860-2882, 2016.

[Jost06] Jost, C., "Transformation formulas for fractional Brownian motion", *Stochastic Processes and their Applications*, 116, 1341-1357, 2006.

[Kyp06] Kyprianou, A., "Introductory Lectures on Fluctuations of Lévy Processes with Applications", Springer, 2006.

[Lee04] Lee, R., "Option Pricing by Transform Methods: Extensions, Unification, and Error Control", *Journal of Computational Finance*, 7(3), 51-86, 2004.

[MT16] Mijatovic, A. and P.Tankov, "A new look at short-term implied volatility in asset price models with jumps", *Mathematical Finance*, 26, 1, 149-183, 2016.

[MV68] Mandelbrot, B. and J.W.Van Ness, "Fractional Brownian motions, fractional noises and applications", *SIAM Review*, 10, 422-437, 1968

[NP00] Nuzman, C.J. and H.Vincent Poor, "Linear estimation of self-similar processes via Lamperti's transformation", *J. Appl. Probab.*, 37, 429-452, 2000.

[JPS18] Jacquier, A., M.S.Pakkanen and H.M.C.Stone, "Pathwise large deviations for the Rough Bergomi model", *Journal of Applied Probability*, 55(4): 1078-1092, 2018.

[Lif95] Lifshits, M.A., "Gaussian Random Functions", Springer, 1995.

[Pic11] Picard, J., "Representation formulae for the fractional Brownian motion", *Seminaire de Probabilités*, Springer-Verlag, 2011, XLIII, pp. 3-70.

[PT06] Poirot, J. and P.Tankov, "Monte Carlo option pricing for tempered stable (CGMY) processes", *Asia Pacific Financial Markets*, Vol. 13-4, 2006.

[PT02] Pipiras, V. and M.S.Taqqu, "Deconvolution of fractional Brownian motion", *Journal of Time Series Analysis* 23, 487-501, 2002.

[SV16] Sottinen, T. and L.Viitasaari, "Stochastic Analysis of Gaussian Processes via Fredholm Representation', *International Journal of Stochastic Analysis*, Volume 2016, Article ID 8694365, 15 pages

[SV17] Sottinen, T. and L.Viitasaari, "Prediction Law of Fractional Brownian Motion", *Statistics and Probability Letters*, 129, 155-166, 2017.

[ST02] Samorodnitsky, G. and M.S.Taqqu, "Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance", CRC press, 2002.

[Wil91] Williams, D., "Probability with Martingales", *Cambridge Mathematical Textbooks*, 1991.

A Proof of Proposition 2.11

Z^H has the conditional decomposition

$$Z^H_u \;=\; \mathbb{E}(Z^H_u|\mathcal{F}^B_t) + \sqrt{2H}\int_t^u (u-s)^{H-\frac{1}{2}}dB_s\,.$$

Then $\mathbb{E}((Z^H)^2_u|\mathcal{F}^B_t) = (u-t)^{2H}$. Using the inversion formula and integration by parts we see that

$$\begin{aligned}
\frac{1}{\sqrt{2H}}\mathbb{E}(Z^H_u|\mathcal{F}^B_t) &= \int_0^t (u-s)^{H-\frac{1}{2}}dB_s \\
&= (u-s)^{H-\frac{1}{2}}B_s|_{s=0}^{s=t} + (H-\frac{1}{2})\int_0^t (u-s)^{H-\frac{3}{2}}B_s ds \\
&= (u-t)^{H-\frac{1}{2}}B_t + (H-\frac{1}{2})\int_0^t (u-s)^{H-\frac{3}{2}}B_s ds \\
&= (u-t)^{H-\frac{1}{2}}\bar{c}_H\int_0^t (t-s)^{\frac{1}{2}-H}dZ^H_s \\
&+ (H-\frac{1}{2})\int_0^t (u-s)^{H-\frac{3}{2}}\bar{c}_H\int_0^s (s-r)^{\frac{1}{2}-H}dZ^H_r ds \qquad \text{(A-1)}
\end{aligned}$$

which we can further re-write as

$$
\begin{aligned}
\frac{1}{\sqrt{2H}}\mathbb{E}(Z_u^H|\mathcal{F}_t^B) &= \bar{c}_H\,(u-t)^{H-\frac{1}{2}}(\frac{1}{2}-H)\int_0^t (t-s)^{-\frac{1}{2}-H} Z_s^H ds \\
&+ \bar{c}_H(H-\frac{1}{2})(\frac{1}{2}-H)\int_0^t (u-s)^{H-\frac{3}{2}}\int_0^s (s-r)^{-\frac{1}{2}-H} Z_r^H drds \\
&= \bar{c}_H\,(u-t)^{H-\frac{1}{2}}(\frac{1}{2}-H)\int_0^t (t-s)^{-\frac{1}{2}-H} Z_s^H ds \\
&+ \bar{c}_H(H-\frac{1}{2})(\frac{1}{2}-H)\int_0^t\int_r^t (u-s)^{H-\frac{3}{2}}(s-r)^{-\frac{1}{2}-H} ds\, Z_r^H dr \\
&= \bar{c}_H\,(u-t)^{H-\frac{1}{2}}(\frac{1}{2}-H)\int_0^t (t-s)^{-\frac{1}{2}-H} Z_s^H ds \\
&+ \bar{c}_H(H-\frac{1}{2})(\frac{1}{2}-H)\int_0^t\int_s^t (u-r)^{H-\frac{3}{2}}(r-s)^{-\frac{1}{2}-H} dr\, Z_s^H ds \\
&= \bar{c}_H\,(u-t)^{H-\frac{1}{2}}(\frac{1}{2}-H)\int_0^t (t-s)^{-\frac{1}{2}-H} Z_s^H ds \\
&+ \bar{c}_H(H-\frac{1}{2})(\frac{1}{2}-H)\int_0^t \frac{2(t-s)^{\frac{1}{2}-H}(u-t)^{-\frac{1}{2}+H}}{(2H-1)(s-u)}\, Z_s^H ds \\
&= \bar{c}_H(\frac{1}{2}-H)\int_0^t (\frac{u-t}{t-s})^{\frac{1}{2}+H}\frac{1}{u-s}\, Z_s^H ds\,.
\end{aligned}
$$

B Proof of Lemma 1.2

Fix a $\theta \in (0,1]$, we consider a modified Lévy-Itô decomposition of the process L:

$$
X_t = (b_1 + \int_{\sqrt{\theta}<|x|\le 1} x\nu_L(x)dx)t + M_t + P_t
$$

for some constant b_1, where M is a square-integrable pure-jump Lévy process martingale with Lévy density $1_{|x|\le\sqrt{\theta}}\nu_L(x)$, and P is a compound Poisson process with jump intensity $\lambda_\theta := \int_{|x|>\sqrt{\theta}}\nu_L(x)dx$ and jump size density $1_{|x|>\sqrt{\theta}}\nu_L(x)/\lambda_\theta$.

From Lemma 2.9 in [Kyp06], we know that

$$
\mathbb{E}(M_\theta^2) = \theta\int_{0<|x|<\sqrt{\theta}} x^2\nu_L(x)dx \le \theta\int_{0<|x|<\sqrt{\theta}} x^2\frac{C}{|x|^{Y+1}}dx = \theta\frac{2C}{2-Y}(\sqrt{\theta})^{2-Y} = \frac{2C}{2-Y}\theta^{2-\frac{1}{2}Y},
$$

By the Markov inequality, we have

$$
\mathbb{P}(|M_\theta| > \frac{\alpha}{2}\theta^{\frac{1}{2}+\epsilon}) \le \frac{8C}{\alpha^2(2-Y)}\theta^{1-\frac{1}{2}Y-2\epsilon}. \tag{B-1}
$$

Now let N_t the counting process for the number of jumps for process P, i.e. a Poisson process with intensity λ_θ. Then we have

$$
\mathbb{P}(N_\theta\ge 1) = 1-e^{-\lambda_\theta\theta} \le \theta\lambda_\theta = \theta\int_{|x|>\sqrt{\theta}}\nu_L(x)dx \le C\theta\int_{|x|>\sqrt{\theta}}\frac{1}{|x|^{1+Y}}dx = \frac{2C}{Y}\theta\frac{1}{(\sqrt{\theta})^Y} = \frac{2C}{Y}\theta^{1-\frac{1}{2}Y} \tag{B-2}
$$

Assume θ is sufficiently small so that $|b_1|\theta + \frac{2C}{Y-1}\theta^{\frac{3}{2}-\frac{1}{2}Y} < \frac{\alpha}{2}\theta^{\frac{1}{2}+\epsilon}$. Then we have

$$
\begin{aligned}
|b_1 + \int_{\sqrt{\theta}<|x|\le 1} x\nu_L(x)dx|\,\theta \le |b_1|\theta + 2\theta\int_{\sqrt{\theta}}^1\frac{C}{x^Y}dx < |b_1|\theta + \frac{2C}{Y-1}(\sqrt{\theta})^{1-Y}\theta &= |b_1|\theta + \frac{2C}{Y-1}\theta^{\frac{3}{2}-\frac{1}{2}Y} \\
&< \frac{\alpha}{2}\theta^{\frac{1}{2}+\epsilon}.
\end{aligned}
$$

It follows that

$$\begin{aligned}
\mathbb{P}(|X_\theta| > \alpha\theta^{\frac{1}{2}+\epsilon}) &= \mathbb{P}(|(b_1 + \int_{\sqrt{\theta}<|x|\le 1} x\nu_L(x)dx)\theta + M_\theta + P_\theta| > \alpha\theta^{\frac{1}{2}+\epsilon}) \\
&\le \mathbb{P}(|(\, b_1 + \int_{\sqrt{\theta}<|x|\le 1} x\nu_L(x)dx)\theta + M_\theta + P_\theta| - |(b_1 + \int_{\sqrt{\theta}<|x|\le 1} x\nu_L(x)dx)\theta| > \frac{\alpha}{2}\theta^{\frac{1}{2}+\epsilon}) \\
&\le \mathbb{P}(|M_\theta \,+\, P_\theta| > \frac{\alpha}{2}\theta^{\frac{1}{2}+\epsilon}), \\
&= \mathbb{P}(|M_\theta| > \frac{\alpha}{2}\theta^{\frac{1}{2}+\epsilon}\,,\, N_\theta = 0) \,+\, \mathbb{P}(|M_\theta \,+\, P_\theta| > \frac{\alpha}{2}\,\theta^{\frac{1}{2}+\epsilon}, N_\theta \ge 1) \\
&\le \mathbb{P}(|M_\theta| > \frac{\alpha}{2}\theta^{\frac{1}{2}+\epsilon}) \,+\, \mathbb{P}(N_\theta \ge 1) \\
&\le (\frac{8C}{\alpha^2(2-Y)} + \frac{2C}{Y}\theta^{2\epsilon})\,\theta^{1-\frac{1}{2}Y-2\epsilon} \\
&\le (\frac{8C}{\alpha^2(2-Y)} + \frac{2C}{Y})\theta^{1-\frac{1}{2}Y-2\epsilon}.
\end{aligned}$$

Hence, the stated theorem holds with $K = \frac{8C}{\alpha^2(2-Y)} + \frac{2C}{Y}$.

C Proof of Proposition 2.4

Let $\phi_t(u) := \mathbb{E}(e^{iu(v_0\tilde{W}_t - L_t)})$ and b is defined as in (3) to enforce that $\mathbb{E}(e^{-L_t}) = 1$, and let $\phi_t^*(u) := \mathbb{E}^*(e^{iu(v_0\tilde{W}_t - L_t)})$ where $\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-L_t}$ and $\mathbb{E}^*(.)$ denotes expectation under $\mathbb{P}^*$. We henceforth set

$$\gamma_\pm \;=\; C_\pm\Gamma(-Y)\,. \tag{C-1}$$

From Remark 1.3 we recall that under $\mathbb{P}^*$, L is a Lévy process with parameters $\bar{M} = M+1$ and $\bar{G} = G-1$. Using that $\tilde{W}$ and L are independent, we see that

$$\mathbb{E}^*(e^{iu(v_0\tilde{W}_t - L_t)}) \;=\; \mathbb{E}(e^{iuv_0\tilde{W}_t})\,\mathbb{E}^*(e^{-iuL_t})\,.$$

Then using (24) and Theorem 5.2 in [Lee04] (which deals with the G_2 case with $\alpha = 0$ using the notation in [Lee04]), and decomposing $\mathbb{E}[(X_1^0 - \frac{L_t}{\sqrt{t}} - z)^+]$ as the sum of its leading order Gaussian contribution and the remainder term due to the generalized tempered stable process L, we have

$$\begin{aligned}
I &= \mathbb{E}[e^{-L_t}(X_1^0 - \frac{L_t}{\sqrt{t}} - z)^+] \,-\, \mathbb{E}[e^{-L_t}(X_1^0 - z)^+] \\
&= \frac{1}{\sqrt{t}}\mathbb{E}[e^{-L_t}(v_0\tilde{W}_t - L_t - z\sqrt{t})^+] \,-\, \frac{1}{\sqrt{t}}\mathbb{E}[e^{-L_t}(v_0\tilde{W}_t - z\sqrt{t})]^+] \\
&= \frac{1}{\sqrt{t}}\mathbb{E}^*[(v_0\tilde{W}_t - L_t - z\sqrt{t})^+] \,-\, \frac{1}{\sqrt{t}}\mathbb{E}^*((v_0\tilde{W}_t - z\sqrt{t})^+)] \\
&= \mathbb{E}^*[(\frac{v\tilde{W}_t}{\sqrt{t}} - \frac{L_t}{\sqrt{t}} - z)^+] \,-\, \mathbb{E}[(\frac{v_0\tilde{W}_t}{\sqrt{t}} - z)^+]
\end{aligned}$$

(where the final expectation is now computed under $\mathbb{P}$ since $\tilde{W}$ and L are independent)

$$= [\frac{1}{\pi}\int_0^\infty \mathrm{Re}[e^{-iuz}(\phi_t^*(\frac{u}{\sqrt{t}}) - e^{-\frac{1}{2}v_0^2u^2})\cdot -\frac{1}{u^2}]du \,-\, \frac{1}{2}i\frac{d}{du}\phi_t^*(\frac{u}{\sqrt{t}})|_{u=0}]$$

(the last term comes from the [Lee04] R_{α,G_2} residue term with $\alpha = 0$ and there is only 1 term here instead of 4 because one term has vanished and two have cancelled)
since $\mathbb{E}(\tilde{W}_t) = 0$)

$$= \frac{1}{\pi}\int_0^\infty \mathrm{Re}[e^{-iuz}(\phi_t^*(\frac{u}{\sqrt{t}}) - e^{-\frac{1}{2}v_0^2u^2})\cdot -\frac{1}{u^2}]du] \,+\, O(\sqrt{t}) \tag{C-2}$$

where we have used that $\frac{1}{2}i\frac{d}{du}\phi_t^*(\frac{u}{\sqrt{t}})|_{u=0} = O(\sqrt{t})$ to obtain the final term. Here

$$\begin{aligned}
\phi_t^*(\frac{u}{\sqrt{t}}) &= e^{-\frac{1}{2}v_0^2u^2}e^{i\frac{u}{\sqrt{t}}bt+\gamma_+t[(\bar{M}+i\frac{u}{\sqrt{t}})^Y - \bar{M}^Y]+\gamma_-t[(\bar{G}-i\frac{u}{\sqrt{t}})^Y - \bar{G}^Y]} \\
&= e^{-\frac{1}{2}v_0^2u^2}e^{iub\sqrt{t}}e^{\gamma_+t^{1-\frac{1}{2}Y}[(\bar{M}\sqrt{t}+iu)^Y - \bar{M}^Yt^{\frac{1}{2}Y}]+\gamma_-t^{1-\frac{1}{2}Y}[(\bar{G}t^{\frac{1}{2}}-iu)^Y - \bar{G}^Yt^{\frac{1}{2}Y}]}\,.
\end{aligned}$$

where $\gamma_\pm$ is defined in (C-1).

Denote

$$\begin{aligned} F(u,t) &= iub\sqrt{t} + \gamma_+ t^{1-\frac{1}{2}Y}[(\bar{M}\sqrt{t}+iu)^Y - \bar{M}^Y t^{\frac{1}{2}Y}] + \gamma_- t^{1-\frac{1}{2}Y}[(\bar{G}\sqrt{t}-iu)^Y - \bar{G}^Y t^{\frac{1}{2}Y}] \\ &= iub\sqrt{t} + \gamma_+ t^{1-\frac{1}{2}Y}\Upsilon_+ + \gamma_- t^{1-\frac{1}{2}Y}\Upsilon_+ \end{aligned}$$

where $\Upsilon_+ := (\bar{M}\sqrt{t}+iu)^Y - \bar{M}^Y t^{\frac{1}{2}Y}$ and $\Upsilon_- := (\bar{G}\sqrt{t}-iu)^Y - \bar{G}^Y t^{\frac{1}{2}Y}$. From here on we will frequently use F as shorthand $F(u,t)$, and let F_r and F_i denote the real and imaginary part of F. Expanding $\Upsilon_\pm$ we find that

$$\begin{aligned} \Upsilon_+ &= (\bar{M}\sqrt{t}+iu)^Y - \bar{M}^Y t^{\frac{1}{2}Y} \\ &= \bar{M}^Y t^{\frac{1}{2}Y}(1+\frac{iu}{\bar{M}\sqrt{t}})^Y - \bar{M}^Y t^{\frac{1}{2}Y} \;=\; \bar{M}^Y t^{\frac{1}{2}Y}[Y\frac{iu}{\bar{M}\sqrt{t}} - \frac{1}{2}(Y^2-Y)\frac{u^2}{\bar{M}^2 t} + O((\frac{iu}{\sqrt{t}})^3 + O((\frac{u}{\sqrt{t}})^4)] \\ \Upsilon_- &= \bar{G}^Y t^{\frac{1}{2}Y}[-Y\frac{iu}{\bar{G}\sqrt{t}} - \frac{1}{2}(Y^2-Y)\frac{u^2}{\bar{G}^2 t} + O((\frac{iu}{\sqrt{t}})^3 + O((\frac{u}{\sqrt{t}})^4)]\,. \end{aligned} \tag{C-3}$$

The remainder term is less than $c_1|\frac{u}{\sqrt{t}}|$ for some constant c_1 if $u \le \delta\sqrt{t}$ for some $\delta > 0$ sufficiently small which depends on c_1. This just follows from the series expansion of the analytic function $(1+z)^Y$:

$$(1+z)^Y = \sum_{k=0}^{\infty}\binom{Y}{k}z^k$$

which has radius of convergence 1, where $\binom{r}{k} = \frac{r(r-1)\dots(r-k+1)}{k!}$. From the above we find that

$$F_r = -\frac{1}{2}(\gamma_+\bar{M}^{Y-2} + \gamma_-\bar{G}^{Y-2})(Y^2-Y)u^2 + O(t\cdot(\frac{u}{\sqrt{t}})^4) \tag{C-4}$$

$$F_i = ub\sqrt{t} + Y(\gamma_+\bar{M}^{Y-1} - \gamma_-\bar{G}^{Y-1})u\sqrt{t} + O(t\cdot(\frac{u}{\sqrt{t}})^3) \tag{C-5}$$

for $u \le \delta\sqrt{t}$, and δ sufficiently small.

Similarly for $u \ge \delta^{-1}\sqrt{t}$ with $\delta > 0$ sufficiently small, we see that

$$\begin{aligned} \Upsilon_+ &= (iu)^Y(1-i\frac{\bar{M}\sqrt{t}}{u})^Y - \bar{M}^Y t^{\frac{1}{2}Y} \;=\; (iu)^Y[1+O(\frac{\sqrt{t}}{u})] - \bar{M}^Y t^{\frac{1}{2}Y} \\ \Upsilon_- &= (iu)^Y(1+i\frac{\bar{G}\sqrt{t}}{u})^Y - \bar{G}^Y t^{\frac{1}{2}Y} \;=\; (iu)^Y[1+O(\frac{\sqrt{t}}{u})] - \bar{G}^Y t^{\frac{1}{2}Y} \end{aligned}$$

and from this we see that

$$F = iub\sqrt{t} + t^{1-\frac{1}{2}Y}u^Y i^Y(\gamma_+ + \gamma_-) - t(\gamma_+\bar{M} + \gamma_-\bar{G}) + O(t^{\frac{3}{2}-\frac{1}{2}Y}u^{Y-1}) \tag{C-6}$$

for $u \ge \delta^{-1}\sqrt{t}$, where $\gamma_\pm = C_\pm\Gamma(-Y)$ as in (C-1). Since $\log(iu) = \log u + i\frac{\pi}{2}$, we have

$$i^Y = (e^{i\frac{\pi}{2}})^Y = \cos(\frac{1}{2}\pi Y) + i\sin(\frac{1}{2}\pi Y)$$

and recall that $Y \in (1,2)$ so $\frac{1}{2}\pi Y \in (\frac{1}{2}\pi, \pi)$. This leads to the decomposition

$$F_r = t^{1-\frac{1}{2}Y}u^Y(\gamma_+ + \gamma_-)\cos(\frac{1}{2}\pi Y) + O(t^{\frac{3}{2}-\frac{1}{2}Y}u^{Y-1}) = O(t^{1-\frac{1}{2}Y}u^Y) \tag{C-7}$$

$$F_i = ub\sqrt{t} + t^{1-\frac{1}{2}Y}u^Y(\gamma_+ + \gamma_-)\sin(\frac{1}{2}\pi Y) + O(t^{\frac{3}{2}-\frac{1}{2}Y}u^{Y-1}) = O(t^{1-\frac{1}{2}Y}u^Y)\,. \tag{C-8}$$

for $u \ge \delta^{-1}\sqrt{t}$. Now consider the integral in (C-2) given by

$$\int_0^\infty \mathrm{Re}[e^{-iuz}(\phi_t^*(\frac{u}{\sqrt{t}}) - e^{-\frac{1}{2}v_0^2u^2})\cdot -\frac{1}{u^2}]du] = \int_0^\infty e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}(e^F-1)]\frac{du}{u^2} \tag{C-9}$$

$$= (\int_0^{\delta\sqrt{t}} + \int_{\delta\sqrt{t}}^{\delta^{-1}\sqrt{t}} + \int_{\delta^{-1}\sqrt{t}}^1 + \int_1^\infty)e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}(e^F-1)]\frac{du}{u^2} \tag{C-10}$$

$$:= I_1 + I_2 + I_3 + I_4.$$

Note that $e^F = e^{F_r+iF_i} = e^{F_r}(\cos F_i + i\sin F_i)$, so we see that

$$\mathrm{Re}[e^{-iuz}(e^F-1)] = \mathrm{Re}[e^{-iuz}(e^{F_r}\cos(F_i) - 1 + ie^{F_r}\sin(F_i))] = \cos(uz)(e^{F_r}\cos(F_i)-1) + \sin(uz)e^{F_r}\sin(F_i). \tag{C-11}$$

Recall that

$$F(u,t) = iub\sqrt{t} + \gamma_+ t^{1-\frac{1}{2}Y}[(\bar{M}\sqrt{t}+iu)^Y - \bar{M}^Y t^{\frac{1}{2}Y}] + \gamma_- t^{1-\frac{1}{2}Y}[(\bar{G}\sqrt{t}-iu)^Y - \bar{G}^Y t^{\frac{1}{2}Y}].$$

Then for the I_2 term, since $\delta\sqrt{t} \leq u \leq \delta^{-1}\sqrt{t}$, substituting $u = \delta^{-1}\sqrt{t}$ into $F(u,t)$ and assuming $t < 1$, we find that

$$|F| \;\leq\; Cu^2$$

for some constant C which depends on δ, so $e^F - 1 = \sum_{k\geq 1} \frac{F^k}{k!}$ and

$$|e^F - 1| \;\leq\; |F| \sum_{k\geq 1} \frac{|F|^{k-1}}{k!} \;\leq\; |F| \sum_{k\geq 1} \frac{C^{k-1}}{k!} \quad \text{(since } u \text{ is bounded)} \;=\; |F|\, C^{-1} \sum_{k\geq 1} \frac{C^k}{k!} \;\leq\; |F|\, C^{-1} e^C$$

so $\text{Re}[e^{-iuz}(e^F - 1)] \leq \mathit{const.} \times u^2$ for $\delta\sqrt{t} \leq u \leq \delta^{-1}\sqrt{t}$. Thus given the $\frac{1}{u^2}$ factor that appears in (C-10), we see that

$$|I_2| \;\leq\; \int_{\delta\sqrt{t}}^{\delta^{-1}\sqrt{t}} du \;=\; O(\sqrt{t}) \tag{C-12}$$

We now consider the I_1 term. Since $u \leq \delta\sqrt{t}$, we can apply expansions (C-4) and (C-5). From (C-4) we get

$$e^{F_r} \;=\; 1 + F_r + O(F_r^2) \;=\; 1 - \frac{1}{2}[\gamma_+ \bar{M}^{Y-2} + \gamma_- \bar{G}^{Y-2}](Y^2 - Y)u^2 + O(\frac{u^4}{t}) \;=\; 1 + O(u^2).$$

Similarly (C-5) yields

$$\cos(F_i) \;=\; 1 + O(F_i^2) \;=\; 1 + O(u^2 t) \tag{C-13}$$

so the first term in (C-11) can be bounded as

$$|\cos(uz)(e^{F_r}\cos(F_i) - 1)| \;\leq\; Cu^2. \tag{C-14}$$

From this we see that

$$\int_0^{\delta\sqrt{t}} e^{-\frac{1}{2}v_0^2 u^2} |\cos(uz)(e^{F_r}\cos(F_i) - 1)| \frac{du}{u^2} \;\leq\; C \int_0^{\delta\sqrt{t}} du \;=\; O(\sqrt{t}). \tag{C-15}$$

From (C-4) and (C-5) we have $e^{F_r} \leq C$ and $|F_i| \leq Cu\sqrt{t}$, so that

$$|\sin(F_i)| \;\leq\; C|F_i| \;\leq\; Cu\sqrt{t} \tag{C-16}$$

and $|\sin(uz)| \leq |uz|$, which leads to

$$\int_0^{\delta\sqrt{t}} e^{-\frac{1}{2}v_0^2 u^2} |\sin(uz) e^{F_r} \sin(F_i)| \frac{du}{u^2} \;\leq\; C \int_0^{\delta\sqrt{t}} du \;=\; O(\sqrt{t}).$$

Hence, since $\frac{1}{2} > 1 - \frac{1}{2}Y$, we see that

$$I_1 \;=\; O(\sqrt{t}) \tag{C-17}$$

i.e. the same order as I_2.

We now consider the I_3 integral term for which $\delta^{-1}\sqrt{t} \leq u \leq 1$. Then we may apply (C-7)–(C-8), and similar computations as above to obtain

$$\begin{aligned} \cos F_i &= 1 + O(F_i^2) = 1 + O(t^{2-Y} u^{2Y}) \\ \sin F_i &= F_i + O(F_i^3) = F_i + O(t^{3-\frac{3}{2}Y} u^{3Y}). \end{aligned} \tag{C-18}$$

From (C-7), $|F_r|$ is bounded for $\delta^{-1}\sqrt{t} \leq u \leq 1$ and

$$e^{F_r} \;=\; 1 + F_r + O(|F_r|^2) \;=\; 1 + F_r + O(t^{2-Y} u^{2Y}). \tag{C-19}$$

Combining these estimates yield

$$e^{F_r}\cos(F_i) - 1 \;=\; F_r + O(t^{2-Y} u^{2Y})$$

and for the $O(t^{2-Y} u^{2Y})$ remainder term we have

$$\int_{\delta^{-1}\sqrt{t}}^1 t^{2-Y} u^{2Y} \frac{du}{u^2} \;\leq\; Ct^{2-Y}$$

so (recalling the first half of decomposition of $\text{Re}[e^{-iuz}(e^F - 1)]$ in (C-11)) we see that

$$\int_{\delta^{-1}\sqrt{t}}^1 e^{-\frac{1}{2}v_0^2 u^2} \cos(uz)(e^{F_r}\cos(F_i) - 1) \frac{du}{u^2} \;=\; \int_{\delta^{-1}\sqrt{t}}^1 e^{-\frac{1}{2}v_0^2 u^2} \cos(uz) F_r \frac{du}{u^2} + O(t^{2-Y}). \tag{C-20}$$

Similarly, using the boundedness of F_r on $u \le 1$ and the expansion $e^{F_r} = 1 + O(F_r)$, together with (C-7) and (C-18), we have

$$e^{F_r}\sin(F_i) \;=\; F_i + O(t^{3-\frac{3}{2}Y}u^{3Y}) + F_i O(t^{1-\frac{1}{2}Y}u^{Y}).$$

Here $O(t^{3-\frac{3}{2}Y}u^{3Y}) = O(t^{2-Y}u^{2Y})$ (to see this, divide the left hand side by $t^{2-Y}u^{2Y}$ to get $O(t^{1-\frac{1}{2}Y}u^{Y})$ which is $O(1)$ for t small and $u \le 1$) and (by (C-8)) we see that

$$F_i O(t^{1-\frac{1}{2}Y}u^{Y}) \;=\; O(t^{2-Y}u^{2Y})$$

so $e^{F_r}\sin(F_i) = F_i + O(t^{2-Y}u^{2Y}) = F_i + O(t^{2-Y}u^{2Y})$ for $u \le 1$, so (using the second half of decomposition in (C-11)) we see that

$$\int_{\delta^{-1}\sqrt{t}}^{1} e^{-\frac{1}{2}v_0^2u^2}\sin(uz)e^{F_r}\sin(F_i)\frac{du}{u^2} \;=\; \int_{\delta^{-1}\sqrt{t}}^{1} e^{-\frac{1}{2}v_0^2u^2}\sin(uz)F_i\frac{du}{u^2} + O(t^{2-Y}).$$

so using $\mathrm{Re}[e^{-iuz}F] = \cos(uz)F_r + \sin(uz)F_i$ and combining with (C-20) we see that

$$I_3 \;=\; \int_{\delta^{-1}\sqrt{t}}^{1} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}F]\frac{du}{u^2} + O(t^{2-Y}). \tag{C-21}$$

Next we prove that for the final term I_4 we have

$$I_4 \;=\; \int_{1}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}F]\frac{du}{u^2} + O(t^{2-Y}). \tag{C-22}$$

Now

$$I_4 - \int_{1}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}F]\frac{du}{u^2} \;=\; \int_{1}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}(e^F - 1 - F)]\frac{du}{u^2}$$

and we note that

$$|\mathrm{Re}(e^{-iuz}(e^F - 1 - F))| \;\le\; |e^{-iuz}(e^F - 1 - F)| \;\le\; |e^F - 1 - F| \;\le\; \sum_{k=2}^{\infty}\frac{|F|^k}{k!}.$$

Recall that

$$F(u,t) \;=\; iub\sqrt{t} + \gamma_+ t^{1-\frac{1}{2}Y}[(\bar{M}\sqrt{t} + iu)^Y - \bar{M}^Y t^{\frac{1}{2}Y}] + \gamma_- t^{1-\frac{1}{2}Y}[(\bar{G}\sqrt{t} - iu)^Y - \bar{G}^Y t^{\frac{1}{2}Y}].$$

Here $|F|$ can be bounded as

$$\begin{aligned} |F| &= |iub\sqrt{t} - (\gamma_+\bar{M} + \gamma_-\bar{G})t + \gamma_+ t^{1-\frac{1}{2}Y}(\bar{M}\sqrt{t} + iu)^Y + \gamma_- t^{1-\frac{1}{2}Y}(\bar{G}\sqrt{t} - iu)^Y| \\ &\le bu\sqrt{t} + Ct + Ct^{1-\frac{1}{2}Y}u^Y \\ &\le t^{1-\frac{1}{2}Y}(C + u + u^Y) \end{aligned}$$

for some constant $C > 0$. This leads to

$$\sum_{k=2}^{\infty}\frac{|F|^k}{k!} \;\le\; \sum_{k=2}^{\infty}\frac{(t^{2-Y})^k(C + u + u^Y)^k}{k!} \;\le\; \sum_{k=2}^{\infty}\frac{t^{2-Y}(C + u + u^Y)^k}{k!} \;\le\; t^{2-Y}e^{C+u+u^Y}.$$

Thus we see that

$$\int_{1}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\left|\mathrm{Re}(e^{-iuz}(e^F - 1 - F)\right|\frac{du}{u^2} \;\le\; t^{2-Y}\int_{1}^{\infty} e^{-\frac{1}{2}v_0^2u^2}e^{C+u+u^Y}du \;<\; \infty$$

where the finiteness follows since $Y < 2$, and (C-22) follows. Putting all this together (i.e. combining (C-17), (C-12), (C-21) and (C-22)), we obtain

$$I_1 + I_2 + I_3 + I_4 \;=\; \int_{\delta^{-1}\sqrt{t}}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}F]\frac{du}{u^2} + O(t^{2-Y}) + O(t^{\frac{1}{2}}).$$

From (C-6), we see that

$$\begin{aligned} \int_{\delta^{-1}\sqrt{t}}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}F]\frac{du}{u^2} &= t^{1-\frac{1}{2}Y}\int_{\delta^{-1}\sqrt{t}}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}t^{1-\frac{1}{2}Y}(iu)^Y(\gamma_+ + \gamma_-)]\frac{du}{u^2} \\ &+ \int_{\delta^{-1}\sqrt{t}}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}iub\sqrt{t}]\frac{du}{u^2} \\ &- \int_{\delta^{-1}\sqrt{t}}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}t(\gamma_+\bar{M} + \gamma_-\bar{G})]\frac{du}{u^2} \\ &+ \int_{\delta^{-1}\sqrt{t}}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}O(t^{\frac{3}{2}-\frac{1}{2}Y}u^{Y-1})]\frac{du}{u^2}. \end{aligned}$$

Since $\delta^{-1}\sqrt{t} \leq u$, for any $\varepsilon > 0$ we have that

$$\left|\int_{\delta^{-1}\sqrt{t}}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}iub\sqrt{t}]\frac{du}{u^2}\right| \leq \left|\int_{\delta^{-1}\sqrt{t}}^{\infty} e^{-\frac{1}{2}v_0^2u^2}ub\sqrt{t}\frac{du}{u^2}\right| = \frac{1}{2}b\sqrt{t}\,\Gamma(0,\frac{tv_0^2}{2\delta^2}) = O(\sqrt{t}\log|t|) = O(t^{\frac{1}{2}-\varepsilon})$$

for any $\varepsilon > 0$, where $\Gamma(.,.)$ denotes the incomplete Gamma function. Similarly, again ignoring the e^{-iuz} term we find that

$$\int_{\delta^{-1}\sqrt{t}}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}t(\gamma_+\bar{M} + \gamma_-\bar{G})]\frac{du}{u^2} = O(\sqrt{t})\,.$$

and

$$\int_{\delta^{-1}\sqrt{t}}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}O(t^{\frac{3}{2}-\frac{1}{2}Y}u^{Y-1})]\frac{du}{u^2} = O(\sqrt{t}).$$

Taking all this into account, we conclude that

$$\mathbb{E}[e^{-L_t}(X_1^0 - \frac{L_t}{\sqrt{t}} - z)^+] - \mathbb{E}[e^{-L_t}(X_1^0 - z)^+] = t^{1-\frac{1}{2}Y}\frac{1}{\pi}\int_{\delta^{-1}\sqrt{t}}^{\infty} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}(iu)^Y(\gamma_+ + \gamma_-)]\frac{du}{u^2} + O(t^{2-Y}).$$

Now the claim follows from simple observations

$$t^{1-\frac{1}{2}Y}\int_0^{\delta^{-1}\sqrt{t}} e^{-\frac{1}{2}v_0^2u^2}\mathrm{Re}[e^{-iuz}(iu)^Y(\gamma_+ + \gamma_-)]\frac{du}{u^2} = O(t^{\frac{1}{2}})\,.$$

This concludes the proof.

D Proof of Lemma 2.5

Let $h_{\delta,R}(u) = 1$ for $u \leq R$, $h_{\delta,R}(u) = 0$ for $u \geq R+\delta$ and $h_{\delta,R}(u) = 1 - \frac{1}{\delta}(u-R)$ for $u \in (R, R+\delta)$, so $h_{\delta,R} \in C_b$. Then

$$\mathbb{E}(f(X_n, Y_n, Z_n)1_{Z_n\leq R}) \leq \mathbb{E}(f(X_n, Y_n, Z_n)h_{\delta,R}(Z_n))$$

and from weak convergence $\lim_{n\to\infty}\mathbb{E}(f(X_n, Y_n, Z_n)h_{\delta,R}(Z_n)) = \mathbb{E}(f(X, Y, 0))$, so $\limsup_{n\to\infty}\mathbb{E}(f(X_n, Y_n, Z_n)1_{Z_n\leq R}) \leq \mathbb{E}(f(X, Y, 0))$ since $h_{\delta,R} \leq 1$. Similarly

$$\mathbb{E}(f(X_n, Y_n, Z_n)1_{Z_n\leq R}) \geq \mathbb{E}(f(X_n, Y_n, Z_n)h_{\delta,R}(Z_n+\delta))$$

for $\delta \in [0, R)$, and again from weak convergence we have $\lim_{n\to\infty}\mathbb{E}(f(X_n, Y_n, Z_n)h_{\delta,R}(Z_n+\delta)) = \mathbb{E}(f(X, Y, 0))$ since $h_{\delta,R}(\delta) = 1$, so

$$\liminf_{n\to\infty}\mathbb{E}(f(X_n, Y_n, Z_n)1_{Z_n\leq R}) \geq \mathbb{E}(f(X, Y, 0))\,.$$

E Proof of Lemma 2.6

Define $h_{\delta,R}(y)$ as in Appendix D. Then

$$\mathbb{E}(f(X_n, Y_n, Z_n)1_{Y_n\leq R}1_{Z_n\leq K}) \leq \mathbb{E}(f(X_n, Y_n, Z_n)h_{\delta,R}(Y_n)h_{\delta,K}(Z_n))$$

and from weak convergence of the right hand side and using that $Z_n \xrightarrow{w} 0$, we see that

$$\limsup_{n\to\infty}\mathbb{E}(f(X_n, Y_n, Z_n)1_{Y_n\leq R}1_{Z_n\leq K}) \leq \mathbb{E}(f(X, Y, 0)h_{\delta,R}(Y)) \leq \mathbb{E}(f(X, Y, 0)1_{Y\leq R+\delta})\,.$$

But the expression on the right hand side is clearly just a multiple of the distribution function of Y (evaluated at $R+\delta$) under the measure $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{f(X,Y,0)}{\mathbb{E}^{\mathbb{P}}(f(X,Y,0))}$, and hence is right continuous, so we can let $\delta \to 0$ to obtain

$$\limsup_{n\to\infty}\mathbb{E}(f(X_n, Y_n, Z_n)1_{Y_n\leq R}1_{Z_n\leq K}) \leq \mathbb{E}(f(X, Y, 0)1_{Y\leq R})\,.$$

Similarly

$$\mathbb{E}(f(X_n, Y_n, Z_n)1_{Y_n\leq R}1_{Z_n\leq K}) \geq \mathbb{E}(f(X_n, Y_n, Z_n)h_{\delta,R}(Y_n+\delta)h_{\delta,R}(Z_n+\delta))$$

for $\delta < R$, and again from weak convergence of the right hand side and using that $Z_n \to 0$ we have

$$\liminf_{n\to\infty}\mathbb{E}(f(X_n, Y_n, Z_n)1_{Y_n\leq R}1_{Z_n\leq K}) \geq \mathbb{E}(f(X, Y, Z)h_{\delta,R}(Y+\delta)h_{\delta,R}(\delta)) = \mathbb{E}(f(X, Y, 0)h_{\delta,R}(Y+\delta))$$

so

$$\begin{aligned}\mathbb{E}(f(X,Y,0)h_{\delta,R}(Y+\delta)) &\leq \liminf_{n\to\infty}\mathbb{E}(f(X_n,Y_n,Z_n)1_{Y_n\leq R}1_{Z_n\leq K}) \leq \limsup_{n\to\infty}\mathbb{E}(f(X_n,Y_n,Z_n)1_{Y_n\leq R}) \\ &\leq \mathbb{E}(f(X,Y,0)1_{Y\leq R})\,.\end{aligned}$$

Letting $\delta\to 0$ on the left hand side and using the monotone convergence theorem (since $h_{\delta,R}(Y+\delta)\nearrow 1_{Y<R}$) we get

$$\mathbb{E}(f(X,Y,0)1_{Y<R}) \leq \liminf_{n\to\infty}\mathbb{E}(f(X_n,Y_n,Z_n)1_{Y_n\leq R}1_{Z_n\leq K}) \leq \limsup_{n\to\infty}\mathbb{E}(f(X_n,Y_n,Z_n)1_{Y_n\leq R}) \leq \mathbb{E}(f(X,Y,0)1_{Y\leq R})$$

But Y has no atom at R by assumption, so the left hand side is equal to $\mathbb{E}(f(X,Y,0)1_{Y\leq R})$.

F Proof of Lemma 1.1

Using integration by parts and the stochastic Fubini theorem we see that

$$\begin{aligned}\int_0^t (t-s)^{\frac{1}{2}-H}dZ^H_s &= (\frac{1}{2}-H)\int_0^t (t-s)^{-\frac{1}{2}-H}Z^H_s\,ds \\ &= \sqrt{2H}\,(\frac{1}{2}-H)\int_0^t\int_0^s (t-s)^{-\frac{1}{2}-H}(s-u)^{H-\frac{1}{2}}dB_u ds \\ &= \sqrt{2H}\,(\frac{1}{2}-H)\int_0^t\int_u^t (t-s)^{-\frac{1}{2}-H}(s-u)^{H-\frac{1}{2}}ds dB_u\end{aligned}$$

and

$$\sqrt{2H}\,(\frac{1}{2}-H)\int_u^t (t-s)^{-\frac{1}{2}-H}(s-u)^{H-\frac{1}{2}}ds = \overline{c}_H^{-1}\,.$$

Hence $\overline{c}_H\int_0^t (t-s)^{\frac{1}{2}-H}dZ^H_s=\int_0^t dB_s=B_t$. Comparing (4) and (6) we see that $\mathcal{F}^B_t=\mathcal{F}^{Z^H}_t$.