Sub and super-critical GMC for the the Riemann-Liouville process as $H \to 0$, and skew flattening/blow up for the Rough Bergomi model

Martin Forde  Masaaki Fukasawa*  Stefan Gerhold†  Benjamin Smith‡

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Abstract

Following [NR18], we consider a re-scaled Riemann-Liouville (RL) process $Z^H_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$, and using Lévy’s continuity theorem for random fields we show that $Z^H$ tends weakly to an almost log-correlated Gaussian field as $H \to 0$, which lives in the fractional Sobolev space $H^{-\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$. As a corollary, we show that $\xi^H_t(dt) = e^{\gamma X - \frac{1}{2} \gamma^2 \text{Var}(X^\gamma)} dt$ tends to a Gaussian multiplicative chaos (GMC) random measure $\xi$, for $\gamma \in (0,1)$ as $H \to 0$. We later consider the full sub-critical phase $\gamma \in [0,\sqrt{2})$ using Strassen’s theorem and the Shamov[Sha16] approximation theorem, and the super-critical phase $\gamma > \sqrt{2}$, using an additional independent stable subordinator as in [BJRV14] to construct an atomic GMC with the correct (locally) multifractal exponent $\xi$. For the sub-critical regime, $\xi$ is non-atomic and locally multifractal away from zero. As a financial application of these results, we use Jacod’s stable convergence theorem to prove the surprising result that the martingale component $X_1$ of the log stock price for the popular Rough Bergomi stochastic volatility model (with zero or non-zero correlation) tends stably to $B_{\xi_{\gamma}(0,t)}$ as $H \to 0$ where $B$ is a Brownian motion independent of everything else, which means the implied volatility smile for the model with $\rho \leq 0$ is symmetric in the $H \to 0$ limit. We also derive a closed-form expression for the conditional third moment $E((X_{t+h} - X_t)^3|F_t)$ (for $H > 0$) given a finite history, and $E(X^2_\gamma)$ tends to zero (or blows up) exponentially fast as $H \to 0$ depending on whether $\gamma$ is less than or greater than a critical $\gamma \approx 1.61711$. We describe how to use this equation to calibrate a time-dependent correlation function $\rho(t)$ to the observed skew term structure ($\rho(t)$ satisfies an Abel integral equation for which we establish existence and uniqueness). Finally we also briefly discuss the pros and cons of a $H = 0$ model with non-zero skew for which $X_t/\sqrt{t}$ tends weakly to a non-Gaussian random variable $X_1$ with non-zero skewness as $t \to 0$.

1 Introduction

Originally pioneered by Kahane[Kah85], Gaussian multiplicative chaos (GMC) is a random measure on a domain of $\mathbb{R}^d$ that can be formally written as

$$M_\gamma(dx) = e^{\gamma X/d} 1_{\text{var}(X^\gamma)}(x) dx$$

where $X$ is a Gaussian field with zero mean and covariance $K(x,y) := \mathbb{E}(X_x X_y) = \log |y-x| + g(x,y)$ for some bounded continuous function $g$. $X$ is not defined pointwise because there is a singularity in its covariance, rather $X$ is a random tempered distribution, i.e. an element of the dual of the Schwartz space $S$ under the locally convex topology induced by the Schwartz space semi-norms. For this reason, making rigorous sense of (1) requires a regularizing sequence $X^\epsilon$ of Gaussian processes (with the singularity removed), (see e.g. [BBM13] and [BM03] for a description of such a regularization in 1d based on integrating a Gaussian white noise over truncated triangular region, which we summarize in Section 2.3 here, or page 17 in [RV10] and section 3.4 in [Sha16] for a general method in $\mathbb{R}^d$ using a convolution to smooth $X$). In most of the literature on GMC, the choice of $X^\epsilon$ is a martingale in $\epsilon$, from which we can then easily verify that $M_\gamma(A) = \int_A e^{\gamma X^\epsilon - \frac{1}{2} \gamma^2 \text{var}(X^\gamma)} dx$ is a martingale, and then obtain a.s. convergence of $M_\gamma(A)$ using the martingale convergence to a random variable $M_\gamma(A)$ with $\mathbb{E}(M_\gamma(A)) = \text{Leb}(A)$, and with a bit more work we can verify that $M_\gamma(.)$ defines a random measure (see the end of Section 4 on page 18 in [RV10]).

If $\gamma^2 < 2d$, $M_\gamma(dx) = e^{\gamma X^\epsilon - \frac{1}{2} \gamma^2 \text{var}(X^\gamma)} dx$ tends weakly to a multifractal random measure $M_\gamma$ with full support a.s. which satisfies the multifractal property

$$\mathbb{E}(M_\gamma([0,t]^d)) = c_q t^{\zeta(q)}$$

*Graduate School of Engineering Science, Osaka University 1-3 Machikaneyama, Toyonaka, Osaka, Japan
†TU Wien, Financial and Actuarial Mathematics, Wiedner Hauptstraße 8/105-1, A-1040 Vienna, Austria
‡Dept. Mathematics, King’s College London, Strand, London, WC2R 2LS
for \( q \in (1, q^*) \) for some constant \( c_q = E(M_\gamma([0, 1])^q) \), where \( q^* = \frac{2}{\gamma^2} \) and

\[
\zeta(q) = q - \frac{1}{2} \gamma^2(q^2 - q)
\]

and \( E(M_\gamma([0, t])) = \infty \) if \( q > \frac{2}{\gamma^2} \), see Theorem 2.13 in [RV14] and Lemma 3 in [BM03]). Moreover, we can show that the support of \( M_\gamma \) is a so-called \( \gamma \)-thick points of \( X \), i.e., points such that \( \lim_{\epsilon \to 0} \frac{X^\epsilon}{\log \frac{1}{\epsilon}} = \gamma \) (see e.g. section 2 in [Aru17], [Ber17b] and page 7 in [RV16] for more on this), and for \( \gamma \equiv 0 \), explicit expressions are known for the Mellin transform of the law of \( M_\gamma([0, 1]) \) (see e.g. [Ost09], [Ost13], [Ost18] and [RZ17]), which show that \( \log M_\gamma([0, 1]) \) has an infinitely divisible law. The explicit expression for the total mass of the GMC on a circle was conjectured by Fyodorov and Bouchaud [FB08], and rigorously proved by Remy [Rem17], whereas for the interval was conjectured by Ostrovsky [Ost09] and Fyodorov et al. [FDR09], and later proved in [RZ17]. Among other applications, relations to GMC proved instrumental for gaining insights into statistics of high values of the Riemann zeta function as conjectured by Fyodorov and Keating [FK14], and actively researched by many authors since, see e.g. the recent review article by Harper [Har19].

\( M_\gamma \) is the zero measure for \( \gamma^2 = 2d \) and \( \gamma^2 > 2d \); in these cases a different re-normalization is required to obtain a non-trivial limit. Specifically, for \( \gamma^2 = 2d \), we obtain a non-trivial limit by considering \( \sqrt{\log \frac{1}{\epsilon}} \cdot M_\gamma^\gamma \) as \( \epsilon \to 0 \) or the “derivative measure” \( d\gamma \! e^\gamma X^\epsilon - \frac{1}{\epsilon} \gamma^2 \text{Var}(X_\epsilon) |_{\gamma = \sqrt{2d}} \) [DRSV14] show that both these objects tend weakly to the same measure \( \mu' \) as \( \epsilon \to 0 \), and in 2d Aru et al. [APS19] have shown that \( \frac{M_\gamma}{\sqrt{2d}} \to 2 \mu' \) in probability as \( \gamma \) tends to the critical value of 2, and the critical \( \gamma \)-value is particularly important in Liouville quantum gravity (again see [DRSV14] for further discussion).

In the sub-critical case, using a limiting argument it can be shown that \( M_\gamma \) satisfies the “master equations”:

\[
M(X + f, dz) = e^{\gamma f(z)} M(X, dz)
\]

and

\[
E(\int_D F(X, z)M_\gamma(dz)) = E(\int_D F(X + \gamma^2 K(z, \cdot), z)dz)
\]

for any measurable function \( F \) and any interval \( D \), which comes from the Cameron-Martin theorem for Gaussian measures and the notion of rooted measures and the disintegration theorem (see section 2.1 in [Aru17] for a nice discussion on this). Moreover, either of these two can equations can be taken as the definition of GMC, and they uniquely determine \( M_\gamma \) as a measurable function of \( X \), and hence also uniquely fix its law (see also Eq 6, Eq 7 and Theorem 6 in Shamov [Sha16], and we discuss this paper in more detail below which will be fundamental to the construction of our Riemann-Liouville GMC for \( \gamma \in [1, \sqrt{2}) \)).

Junnilla [Jun16] consider a non-Gaussian extension of standard GMC as the limit of the martingale

\[
\mu_n(A) := \int_A \frac{e^{\gamma \sum_{k=1}^n X_k(t)}}{E(e^{\gamma \sum_{k=1}^n X_k(t)})} dt
\]

as \( n \to \infty \), where \( X_k(t) = \frac{1}{\sqrt{e}} B_k \sin(2\pi kt) \) for some i.i.d. sequence of random variables \( B_k \) with variance 1 and \( E(e^{\rho B_k}) < \infty \) for all \( \rho > 0 \), for which the limiting covariance is \( \lim_{n \to \infty} \text{cov}(\sum_{k=1}^n X_k(s) \cdot \sum_{k=1}^n X_k(t)) = \frac{1}{2} \log(\frac{\sin[(\pi + (s + t))]}{\sin(\pi (s + t))}) \).

GMC also has a natural and important application in Liouville Quantum Field Theory: LQFT is a 2d model of random surfaces, which (formally) we can view as a random metric in the context of quantum gravity, where we weight the classical free field action with an interaction term given by the exponential of a GMC and can be viewed as a toy model to understand in quantum gravity how the interaction with matter influences the geometry of space-time. LQFT is governed by the Liouville action, which for the simple case of a flat metric takes the form:

\[
S(\phi) = \frac{1}{4\pi} \int_D ((\nabla \phi)^2 + 4\pi \mu e^{\gamma \phi}) dx
\]

(3)

The \( \mu \) term here arises from the presence of a gravitational field and for classical gravity (i.e. ignoring the effects of quantum mechanics) computing the minima of the latter yields the Einstein field equations (\( S \) is known as the Einstein-Hilbert action). To make rigorous sense of \( S \), it is natural to interpret the \( (\nabla \phi)^2 \) term as the law of the Gaussian free field (GFF) with covariance function on a bounded domain \( D \) given by the Greens function \( G(x, y) \), which satisfies

\[
\Delta G(x, \cdot) = -2\pi \delta_x(\cdot)
\]

(4)

(see e.g. Proposition 1.11 in [Ber17]). To see this formally, note we can write

\[
e^{-\frac{1}{4\pi} \int_D ((\nabla \phi)^2 + 4\pi \mu e^{\gamma \phi}) dx} = e^{\int_D \phi \Delta \phi - 4\pi \mu e^{\gamma \phi} dx}
\]

2 see Lemma 3 in [BM03] to see why the critical \( q \) value is \( q^* \)
and from (4) we know that $-\frac{1}{2\epsilon^2} \Delta = G^{-1}$, where $G$ denotes the covariance operator $Gf = \int_D G(x,y)f(y)dy$, so we can further re-write the non-interaction term here as $e^{-\frac{1}{2} \int_D (\phi G^{-1} \phi)dx}$, which now looks like the familiar $x^T \Sigma^{-1} x$ in the exponent of the density of a standard Gaussian (see e.g. page 10 in [RV16] for related discussion).

Closer to the spirit of this article is [NR18], who consider a re-scaled modification of standard fractional Brownian motion, and formally show the re-scaled process tends weakly to an almost log-correlated Gaussian field (LGF) in the limit as $H \to 0$. Corollary 2.2 in [NR18] claims that their $\xi_H^*$ measure tends to a Gaussian multiplicative chaos as $H \to 0$, but they do not check whether the second condition in Theorem 25 in Shamov[Sha16] is satisfied. Similarly, their use of Proposition 2.5 in [RV16] to establish local multifractality is not justified, because they do not sandwich the covariance of their modified fBM between two covariances associated with the approximating Gaussian process for the standard MRW. See also [Hag19] for further interesting results which build on the ideas in [NR18].

Continuing in the same vein as [NR18], we consider a re-scaled Riemann-Liouville process $Z_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ in the $H \to 0$ limit. Using Lévy’s continuity theorem for tempered distributions, we show that $Z_t^H$ tends weakly to an almost log-correlated Gaussian field $Z$ (which we refer to as the Riemann-Liouville field) as $H \to 0$, which is a random tempered distribution, i.e. a random element of the dual of the Schwartz space $\mathcal{S}$. We later show that $Z$ is actually a Gaussian in the fractional Sobolev Hilbert space $H^{-\frac{1}{2},-\epsilon}$, and one can also show convergence of $(Z - Z_t^H)/H$ to some $\Gamma \in H^{-\frac{1}{2},-\epsilon}$, which we call the first order correction field. As a corollary, we show that $\xi_H^* (dt) = e^{\int_0^t (\frac{1}{2} \gamma^2 \text{Var}(Z_t^H)) dt}$ tends to a Gaussian multiplicative chaos (GMC) random measure $\xi^*$ for $\gamma \in (0, 1)$ as $H \to 0$. Unlike standard constructions of GMC, our approximating sequence $Z_t^H$ is not a martingale so we cannot appeal to the martingale convergence theorem. We later address the more difficult “$L^1$-regime” where $\gamma \in (0, \sqrt{2})$ using Strassen’s theorem and we show that our limiting GMC object using the RL process falls within the general setup of Shamov[Sha16] using randomized shifts on the Cameron-Martin space of the underlying Brownian motion. Our more involved analysis for $\gamma = 0, \sqrt{2}$ using Strassen’s theorem and [Sha16] also covers the simpler case $\gamma \in (0, 1)$ but we have included the simpler $L^2$ analysis for the latter for pedagogical reasons. We also construct a candidate GMC for the super-critical phase $\gamma > \sqrt{2}$, using an independent stable subordinator time-changed by our Riemann-Liouville GMC (similar to section 3 in [BJRV14]) to construct an atomic GMC with the correct (locally) multifractal exponent for $\gamma$-values greater than $\sqrt{2}$, which is closely related to the non-standard branch of gravity in conformal field theory.

These results have a natural application to the popular Rough Bergomi stochastic volatility model, since $\xi_H^*$ is the quadratic variation of the log stock price for this model and values of $H$ as low as .03 have been reported in empirical studies of this model (see e.g. Fukasawa et al.[FTW19]). Using our Riemann-Liouville GMC and Jacod’s stable convergence theorem, we prove the surprising result that the martingale component $X_t^H$ of the log stock price for the Rough Bergomi model tends weakly to $B_{\xi^*}(0,\epsilon)$ as $H \to 0$ where $B$ is a Brownian motion independent of everything else, which means the smile for the rBergomi model with $\rho \leq 0$ is symmetric in the $H \to 0$ limit for $\gamma \in (0, 1)$. More generally for any $\gamma > 0$, we derive a closed-form expression for the conditional third moment $\mathbb{E}(X_{t+h}^3 - X_t^3|\mathcal{F}_t)$ for $H > 0$, and we find that this expression decays exponentially fast or blows up exponentially fast depending on whether $\gamma$ is less than or greater than a critical $\gamma \approx 1.61711$. The expression for $\mathbb{E}(X_t^3)$ can then be used to calibrate a time-dependent correlation function $\rho(t)$ to the observed skew term structure, which satisfies an Abel integral equation (for which we establish existence and uniqueness) and we can also define a $H = 0$ model with non-zero skew for which $X_t/\sqrt{t}$ tends weakly to a non-Gaussian random variable $X_1$ with non-zero skewness as $t \to 0$.

2 The Riemann-Liouville process and its Gaussian multiplicative chaos as $H \to 0$

2.1 Weak convergence of the Riemann-Liouville process $Z^H$ to the Riemann-Liouville field $Z$

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t \geq 0}$ throughout, which satisfies the usual conditions. In this section we consider a re-scaled Riemann-Liouville process in the limit as $H \to 0$; To this end, let $(W_t)_{t \geq 0}$ denote a standard Brownian motion and consider the following family of re-scaled Riemann-Liouville processes:

$$Z_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$$

for $H \in (0, \frac{1}{2})$, for which

$$R_H(s,t) := \mathbb{E}(Z_s^H Z_t^H) = \int_0^{s \wedge t} (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} du$$

The integrand here is dominated by

$$h(u,s,t) = ((s-u)^{-\frac{1}{2}} \vee 1) \cdot ((t-u)^{-\frac{1}{2}} \vee 1)$$

which is integrable for $s < t$, so using the dominated convergence theorem, we find that

$$R_H(s,t) \to R(s,t) := \int_0^{s \wedge t} (s-u)^{-\frac{1}{2}} (t-u)^{-\frac{1}{2}} du$$
for $s \neq t$ as $H \to 0$ and $R_H(s,t) \to \infty$ for $s = t$. We note also that $R(0,0) = \int_0^\infty ds = \lim_{n \to \infty} \int_0^1 ds = 0$ (from the definition of Lebesgue integration) and we also note that $R_H(0,0) = 0$ so $\lim_{H \to 0} R_H(0,0) = R(0,0) = 0$. We can evaluate this integral to obtain

$$R(s,t) := 2 \tanh^{-1}(\sqrt{s}/\sqrt{t}) = \log \frac{1 + \sqrt{s}}{1 - \sqrt{s}/\sqrt{t}} = \frac{\sqrt{t} + \sqrt{s}}{\sqrt{t} - \sqrt{s}} = \frac{(\sqrt{t} + \sqrt{s})^2}{t-s} = \log \frac{1}{t-s} + g(s,t) \quad (7)$$

for $0 < s < t$, where

$$g(s,t) = 2 \log(\sqrt{s} + \sqrt{t}) \quad (8)$$

and note that $R(s,t) \geq 0$ for all $s,t \geq 0$. $\int_0^T \int_0^T R_H(s,t)dsdt \leq 2 \int_0^T \int_0^T ((s-u)^{-\frac{1}{2}} \vee 1) \cdot ((t-u)^{-\frac{1}{2}} \vee 1)dudsdt < \infty$, so from the dominated convergence theorem, we have

$$\lim_{H \to 0} \int_0^T \int_0^T \phi_1(s)\phi_2(t)R_H(s,t)dsdt = \int_0^T \int_0^T \phi_1(s)\phi_2(t)R(s,t)dsdt \quad (9)$$

for any $\phi_1, \phi_2 \in \mathcal{S}$, where $\mathcal{S}$ denotes the Schwartz space. Similarly, for any sequence $\phi_k \in \mathcal{S}$ with $\|\phi_k\|_{m,j} \to 0$ for all $m,j \in \mathbb{N}_0$ for any $n \in \mathbb{N}$ (i.e. under the Schwartz space semi-norm defined in Eq 1 in e.g. [BDW17])

$$\lim_{k \to \infty} \int_0^T \int_0^T \phi_k(s)\phi_k(t)R(s,t)dsdt = 0 \quad (10)$$

since $\mu(A) = \int_A R(s,t)dsdt$ is a bounded non-negative measure (since $\int_0^T \int_0^T R(s,t)dsdt = \int_0^T 2tdt = T < \infty$), and the convergence here implies in particular that $\phi_k$ tends to $\phi$ pointwise, so we can use the bounded convergence theorem. Thus if we define

$$L_{\mathcal{Z}^H}(f) = \mathbb{E}(e^{if(s)\mathcal{Z}^H_t}) = e^{-\frac{1}{2} \int_0^T f(s) \mathcal{Z}^H(s,t)dsdt}$$

$$\mathcal{L}(f) := e^{-\frac{1}{2} \int_0^T f(s) \mathcal{Z}(s,t)dsdt}$$

for $f \in \mathcal{S}$, then from (9) and (10) and Lévy’s continuity theorem for generalized random fields in the space of tempered distributions (see Theorem 2.3 and Corollary 2.4 in [BDW17]), we see that $L_{\mathcal{Z}^H}(f)$ tends to $\mathcal{L}_{\mathcal{Z}}(f)$ pointwise and $\mathcal{L}(. )$ is continuous at zero, then there exists a generalized random field $\mathcal{Z}$ (i.e. a random tempered distribution, see Appendix A for details) such that $L_{\mathcal{Z}} = L$ and $\mathcal{Z}^H$ tends to $\mathcal{Z}$ in distribution with respect to the strong and weak topology (see page 2 in [BDW17] for definition).

**Remark 2.1** Based on the right hand side of (7), we can say that $\mathcal{Z}$ is an almost log-correlated Gaussian field (LGF), see Appendix A for definitions and background on this.

### 2.1.1 Convergence of $\mathcal{Z}^H$ to $\mathcal{Z}$ in a fractional Sobolev space

**Proposition 2.1** $\mathcal{Z}^H$ converges in probability to some $\mathcal{Z}$ in the Sobolev space $H^{-\frac{1}{2}-\epsilon} (\mathbb{R}) \subset \mathcal{S}'$ (see [JSW18] for definition) with $\mathbb{E}(\|Z\|_{H^{-\frac{1}{2}-\epsilon}}^2) < \infty$ if and only if $\epsilon > 0$.

**Remark 2.2** For $s < \frac{1}{2}$, compactly supported Borel measures e.g. Dirac masses are included in $H^{-s}$ (see page 1 in [JSW18]).

**Proof.** For $0 < s \leq t < T$, $0 \leq R(s,t) \leq \log \frac{1}{|t-s|} + \bar{g}$. Then for $\epsilon > 0$

$$\mathbb{E}(\|Z\|_{H^{-\frac{1}{2}-\epsilon}}^2) = \mathbb{E}(\int_0^T (1+|k|^2)^{-\frac{1}{2}-\epsilon} |\mathcal{Z}_k|^2 dk)$$

$$= \mathbb{E}(\int_0^T (1+|k|^2)^{-\frac{1}{2}-\epsilon} \mathcal{Z}_k \overline{\mathcal{Z}_k} dk)$$

$$= \mathbb{E}(\int_0^T (1+|k|^2)^{-\frac{1}{2}-\epsilon} \int_0^T e^{ikt} \mathcal{Z}_t dt \int_0^T e^{-iks} \mathcal{Z}_s ds dk)$$

$$= \mathbb{E}(\int_0^T (1+|k|^2)^{-\frac{1}{2}-\epsilon} \int_0^T \int_0^T e^{ikt} \mathcal{Z}_t dt \int_0^T e^{-iks} \mathcal{Z}_s ds dt dk)$$

$$= \int_0^T (1+|k|^2)^{-\frac{1}{2}-\epsilon} \int_0^T \int_0^T e^{ikt} \mathcal{Z}_t ds dt dk$$
Using that $R \in L^1([0,T]^2)$, we see that $\int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}+\epsilon}\int_0^T \int_0^T E(Z_s Z_t) ds dt dk = \int_0^T \int_0^T R(s,t) ds dt \cdot \int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}+\epsilon} dk < \infty$ if $\epsilon > 0$, so by Fubini we have

$$E(\|Z\|^2_{H^{-\frac{1}{2}-\epsilon}}) = E\left(\int_0^T \int_0^T R(s,t) \int_{-\infty}^{\infty} e^{ik(t-s)} (1+|k|^2)^{-\frac{1}{2}+\epsilon} dk ds dt\right) \leq 2c_\epsilon \int_0^T \int_0^T R(s,t)(t-s)^\epsilon \text{BesselK}(c, t-s) ds dt$$

where we have used that the Fourier transform of $\hat{f}(k) := (1+|k|^2)^{-\frac{1}{2}+\epsilon}$ is $f(t) = c_\epsilon |t|^\epsilon \text{BesselK}(c, |t|)$ for some real constant $c_\epsilon$, and that $t^\epsilon \text{BesselK}(c, t)$ is bounded on $[0, T]$ if $\epsilon > 0$. For $\epsilon \leq 0$, the integrand in the triple integral in the first line is not absolutely integrable.

Moreover, using that

$$\chi(s,t,H,H_2) := E((Z_s^H - Z_t^H)(Z_s^H - Z_s^H)) = R_{H_2}(s,t) - E(Z_s^HZ_t^H) - E(Z_s^H) + R_H(s,t) \to 0$$

as $H, H_2 \to 0$ and that $|\chi(s,t,H,H_2)| \leq 4R(s,t) \vee R_\frac{1}{2}(s,t)$ (see also (6) above), we can use a similar argument to (11) and the dominated convergence theorem to show that

$$E(\|Z_{H_2}^H - Z_{H_1}^H\|^2_{H^{-\frac{1}{2}-\epsilon}}) \leq c_\epsilon \int_{[0,T]^2} \chi(s,t,H,H_2) ds dt \to 0$$

as $H, H_2 \to 0$, so $Z^H$ is a Cauchy sequence in the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P}; H^{-\frac{1}{2}-\epsilon})$ of $H^{-\frac{1}{2}-\epsilon}$-valued random variables $X$ with $E(\|X\|^2_{H^{-\frac{1}{2}-\epsilon}}) < \infty$, and thus converges in this space. Using that

$$\mathbb{P}(\|Z_{H_2}^H - Z^H\|^2_{H^{-\frac{1}{2}-\epsilon}} > \delta) \leq \frac{1}{\delta^2} E(\|Z_{H_2}^H - Z^H\|^2_{H^{-\frac{1}{2}-\epsilon}})$$

the claim is proved. \qed

**Remark 2.3** One can also show that $Z^H \to Z$ a.s. in $H^{-\frac{1}{2}-\epsilon}(\mathbb{R})$ on $[0, T]$ for $T \leq 1$, and $\frac{Z_i^H}{H_i}$ converges in probability to a Gaussian field $\Gamma \in H^{-\frac{1}{2}-\epsilon}(\mathbb{R})$ with covariance

$$R_1(s,t) = \int_0^\infty (s-u)^{-\frac{1}{2}}(t-u)^{-\frac{1}{2}} \log(s-u) \log(t-u) du$$

for $0 \leq s \leq t \leq T$, where $R_1(s,t) := \frac{1}{2} \Upsilon''(0) > 0$ where $\Upsilon(H) := E((Z_t - Z_1^H)(Z_s - Z_s^H))$, and $R_1(t,t) = \infty$.

### 2.2 Constructing a GMC measure from $Z^H$ as $H \to 0$

Using similar notation to [NR18], we now define the family of random measures:

$$\xi^H_{\gamma}(dt) := e^{\gamma Z_t^H - \frac{1}{2} \gamma^2 \text{Var}(Z_t^H)} dt.$$

**Theorem 2.2** Let $H_n \searrow 0$. Then for any $A \in \mathcal{B}([0,T])$ and $\gamma \in (0,1)$, $\xi^H_{\gamma}(A)$ tends to some non-negative random variable $\xi_{\gamma,A}$ in $L^2$ (and hence also converges in probability), $\xi_{\gamma, [0,T]}$ is a non-trivial random variable (i.e. has finite non-zero variance), and there exists a random measure $\xi_{\gamma}$ on $[0,T]$ such that $\xi_{\gamma}(A) = \xi_{\gamma,A}$ a.s. for all $A \in \mathcal{B}([0,T])$.

**Proof.** We wish to show that $E((\xi^H_{\gamma}(\cdot) - \xi^H_{\gamma}(\cdot))_2^2) \to 0$, i.e. that $\xi^H_{\gamma}(\cdot)$ is a Cauchy sequence in $L^2$. To this end, we first note that

$$E(\xi^H_{\gamma}([0,T])\xi^H_{\gamma}(\cdot)([0,T])) = E(\int_{[0,T]^2} e^{\frac{\gamma}{2}(Z_s^H + Z_t^H) - \frac{1}{2} \gamma^2 E((Z_t^H)^2)} e^{-\frac{1}{2} \gamma^2 E((Z_t^H)^2)} ds dt)$$

$$= E(\int_{[0,T]^2} e^{\frac{3}{2} \gamma^2 R_{H_n}(t,t) + \frac{1}{2} \gamma^2 R_{H_n}(s,s) + \gamma E(Z_t^H Z_s^H) - \frac{1}{2} \gamma^2 R_{H_n}(t,t) - \frac{1}{2} \gamma^2 R_{H_n}(s,s) ds dt$$

$$= E(\int_{[0,T]^2} e^{\gamma E(Z_t^H Z_s^H)} ds dt)$$

$^3$details available on request
The integrand here is bounded by \( e^{\frac{1}{2}h(u,s,t)}du \) (where \( h(u,s,t) \) is defined in (6)) and is integrable on \([0,T]^2\), and 
\[
E(Z_t^A Z_s^B) = \int_0^T (t-u)^{H_u - \frac{1}{2}} (s-u)^{H_u - \frac{1}{2}} du \to R(s,t) \text{ Lebesgue a.e. on } [0,T]^2 \text{ as } n,m \to 0,
\]
so from the dominated convergence theorem we see that
\[
E(\xi^H_\gamma([0,T])\xi^{H_m}_\gamma([0,T])) \to \int_{[0,T]^2} e^{2R(s,t)} ds dt \quad (n,m \to \infty)
\]
\[
= 2 \int_{[0,T]} \int_{[0,T]} e^{2R(s,t)} ds dt
\]
\[
= 2 \int_{[0,T]} \int_{[0,T]} \left( \frac{\sqrt{t} + \sqrt{u}}{\sqrt{t} - \sqrt{u}} \right)^{\gamma^2} ds dt
\]
\[
= 2 \int_{[0,T]} t \int_{[0,1]} \left( \frac{\sqrt{t} + \sqrt{ut}}{\sqrt{t} - \sqrt{ut}} \right)^{\gamma^2} du dt
\]
\[
= 2 \int_{[0,T]} t \int_{[0,1]} \left( \frac{1 + \sqrt{u}}{1 - \sqrt{u}} \right)^{\gamma^2} du dt
\]
\[
= 2 \int_{0}^{T} t \alpha_\gamma dt = a_\gamma T^2 < \infty
\]
(13)
for \( \gamma \in (0,1) \), where
\[
a_\gamma := \int_{[0,1]} \left( \frac{1 + \sqrt{u}}{1 - \sqrt{u}} \right)^{\gamma^2} du = \frac{2 \cdot 2F_1(2, -\gamma^2, 3 - \gamma^2, -1)}{(\gamma^2 - 1)(1 + \gamma)(\gamma^3 - 2)}
\]
(14)
where \( 2F_1(z) \) is the hypergeometric function, and using that \( 1 - \sqrt{u} \sim \frac{u}{2} (1 \sim u) \) as \( u \to 1 \), we can easily verify that \( a_\gamma \to \infty \) as \( \gamma \to 1 \).

We also know that \( E(\xi^H_\gamma([0,T])) = T \) for all \( n \) and we have already established \( L^2 \) convergence for \( \xi^H_\gamma(A) \) as \( n \to \infty \) which implies \( L^1 \) convergence, so \( E(\xi_\gamma([0,T])) = T \), which further implies that \( P(\xi_\gamma([0,T]) > 0) > 0 \) and
\[
E(\xi^2_\gamma([0,T])) = a_\gamma T^2
\]
so in particular \( \xi_\gamma \) is not multifractal at zero, since the power is 2 and not \( \xi(2) \). The \( L^2 \)-convergence also means that \( \xi^H_\gamma([0,T]) \to \xi_\gamma([0,T]) \) in \( L^q \) as \( H \to 0 \) for all \( q \in [1,2] \) which implies that
\[
\lim_{H \to 0} E(\xi^H_\gamma([0,T]))^q = E(\xi^q_\gamma([0,T])).
\]
(15)
Given that \( E(\xi_\gamma([0,T])) = T \) and \( \text{Var}(\xi_\gamma([0,T])) = \int_{[0,T]^2} e^{2R(s,t)} ds dt - T^2 > 0 \) since \( a_\gamma > 1 \) for \( \gamma \in (0,1) \), we see that \( \xi_\gamma([0,T]) \) is a non-trivial random variable.

For \( A, B \in \mathcal{B}([0,T]) \) disjoint, \( \xi^H_{\gamma,A \cup B} = \xi^H_{\gamma,A} + \xi^H_{\gamma,B} \) a.s. since \( \xi^H_\gamma \) is a measure, and we know that both sides tend to \( \xi_\gamma(A \cup B) \) and \( \xi_\gamma(A) + \xi_\gamma(B) \) in probability. By a standard result, if \( X_n \xrightarrow{P} X \) and \( X_n \xrightarrow{D} Y \), then \( X = Y \) a.s., hence
\[
\xi_\gamma(A \cup B) = \xi_\gamma(A) + \xi_\gamma(B)
\]
(16)
a.s. Moreover, we know that \( E(\xi_\gamma(A_n)) = \text{Leb}(A_n) \), so by Markov’s inequality \( \mathbb{P}(\xi_\gamma(A_n) > \delta) \leq \frac{\text{Leb}(A_n)}{\delta} \), so \( \xi_\gamma(A_n) \) tends to zero in probability. Similarly for any sequence \( A_n \downarrow \emptyset \) with \( A_n \in \mathcal{B}([0,T]) \), \( \xi_\gamma([0,T]) \) tends to zero in probability, and from (16), we know that \( \xi_\gamma(A_n) \) is decreasing, and hence also tends to some random variable \( Y \) a.s. (and hence also in probability). Thus by the same standard result discussed above, \( Y = 0 \) a.s. Thus by Theorem 9.1.XV in [DV07] (see also the end of Section 4 on page 18 in [RV10]), there exists a random measure \( \xi_\gamma \) on \([0,T]\) such that \( \xi_\gamma(A) = \xi_\gamma(A) \) a.s. for all \( A \in \mathcal{B}([0,T]) \).

**Remark 2.4** If we replace the definition of \( Z^H \) with the usual Riemann-Liouville process \( Z^H_t = \sqrt{2H} \int_0^t (t-s)^{H - \frac{1}{2}} ds \), then adapting the arguments above, we see that
\[
E(\left( \int_A e^{\gamma^2 Z^H_t - \frac{1}{2} \gamma^2 \text{Var}(Z^H_t)} dt \right)^2) = \int_A \int_A e^{2H R(s,t)} ds dt \to 1
\]
as \( H \to 0 \), for all \( A \in \mathcal{B}([0,T]) \). But we know that the first moment of this quantity is 1 as well (so its variance is zero), hence \( \int_A e^{\gamma^2 Z^H_t - \frac{1}{2} \gamma^2 \text{Var}(Z^H_t)} dt \to 1 \) in \( L^2 \).
2.3 Construction and properties of the usual Bacry-Delour-Muzy MRM via Gaussian white noise

Define \( \omega_l(t) \) as in Eq 7 in [BBM13] with \( \lambda = 1 \) and \( T = 1 \), and set \( \tilde{\omega}_l(t) := \omega_l(t) - E(\omega_l(t)) \), so

\[
\tilde{\omega}_l(t) = \int_{(u,s) \in \mathcal{A}_l(t)} dW(u,s)
\]

where \( dW(u,s) \) is 2-dimensional Gaussian white noise with variance \( s^{-2} duds \), and \( \mathcal{A}_l(t) \) is triangular region defined in Eq 11 [BM03] (for the special case when \( f(l) = f(q)(t) \) in their notation, see Eq 12 in [BM03]). Then

\[
K^T_l(s,t) := E(\tilde{\omega}_l(t) \tilde{\omega}_l(s)) = \begin{cases} \log \frac{T}{\tau} & l \leq \tau \leq T \\ \log \frac{T}{\tau} + 1 - \frac{\tau}{T} & \tau \leq l \\ 0 & \tau > T \end{cases}
\]

(17)

where \( \tau = |t-s| \), and one can easily verify that \( K^T_l(s,t) \leq \log \frac{T}{\tau} \) (see Eq 25 in [BM03]). From a picture, we also see that

\[
E(\tilde{\omega}_l(t) \tilde{\omega}_l(s)) = K_1(s,t)
\]

for \( l > l' \) (i.e. the answer does not depend on \( l' \)), and \( K^T_l(s,t) \sim \log \frac{T}{|t-s|} \) as \( l \to 0 \). We now define the measure

\[
M^T_l(dt) = e^{\gamma \tilde{\omega}_l(t)-\frac{1}{2} \gamma^2 \text{Var}(\tilde{\omega}_l(t))} dt
\]

and we use \( M^T_l(dt) \) as shorthand for \( M^T_{l,l}(dt) \). One can easily verify that \( M^T_l(A) \) is a backwards martingale with respect to the filtration \( \mathcal{F}_l := \sigma(W(A,B) : A \subseteq \mathbb{R}^+, B \subseteq [l,\infty]) \) (see e.g. subsection 5.1 in [BM03] and page 17 in [RV10]) and sup \( \|E(M^T_l(A)^q)\| < \infty \) (Lemma 3 i) in [BM03], so from the martingale convergence theorem, \( M^T_{l,l}(A) \) converges to \( M^T_l(A) \) in \( L^q \) for \( q \in (1,q^*) \), and from the reverse triangle inequality this implies that

\[
\lim_{l \to 0} E((M^T_{l,l}(A))^q) = E((M^T_l(A))^q)
\]

(18)

and \( M^T_l \) is perfectly multifractal, i.e.

\[
E(|M^T_l([0,t])|^q) = c_{q,T} t^{\xi(q)}
\]

(see e.g. Lemma 4 in [BM03]) for some finite constant \( c_{q,T} > 0 \), depending only on \( q \) and \( T \). For integer \( q \geq 1 \), we also note that

\[
E(M^T_l(A)^q) = \int_A \ldots \int_A e^{\gamma^2 \sum_{i \leq j \leq q} \log \frac{1}{|t_s-i|}} du_i \ldots du_q
\]

\[
= \int_A \ldots \int_A e^{\gamma^2 q(q-1) \log T + \sum_{i \leq j \leq q} \log \frac{1}{|t_s-i|}} du_i \ldots du_q
\]

so we see that

\[
c_{q,T} = c_{q,1} T^{\gamma^2 q(q-1)}
\]

(19)

where \( c_q = c_{q,1} \), and this also holds for non-integer \( q \) (see e.g. Theorem 3.16 in [Koz06]).

3 \( \xi_\gamma \) for the full sub-critical range \( \gamma \in (0,\sqrt{2}) \)

3.1 Weak convergence using Strassen’s theorem

Theorem 3.1 For \( \gamma \in (0,\sqrt{2}) \) and any interval \( I \subseteq [0,1] \), \( \xi^H_I \) tends weakly to some non-negative random variable \( \xi_\gamma(I) \) as \( H \to 0 \), and \( \xi_\gamma(I) \in L^q \) for all \( q \in [1,q^*) \).

Proof. Since \( R_H(s,t) \) is increasing for \( 0 < s, t < 1 \), Kahane’s inequality in Theorem 3.4 implies that the marginals of \( (\xi^H_I)_{H \in (0,\frac{1}{2})} \) increase in convex order as \( H \) tends to zero, and hence these marginals form a peacock. Then Strassen’s theorem (Theorem 8 in [Str65]) implies that there exists an (backwards) martingale \( \tilde{M}_H = \xi^H_I \) such that \( \tilde{\xi}^H_I \sim \xi_\gamma(I) \) for all \( H \in \left[\frac{1}{2},0\right) \).

From the upper bound part of the sandwich equation (31), we have the following inequality for \( 0 < s < t < 1 \):

\[
R_H(s,t) \leq K^\theta_{\gamma,I}(t,s)
\]

(20)
where \( \theta = 4 \cdot \sup(I) \) and \( K^T_I(s,t) \) is the covariance of the Bacry-Muzy model defined in (17), and \( l^*(H) \downarrow 0 \) as \( H \downarrow 0 \). Using Kahane’s inequality again, we see that

\[
\mathbb{E}[\xi(I)^q] = \mathbb{E}[\xi(I)^q] \leq \mathbb{E}[M^0_{l^*(H)}(I)^q]
\]

for \( k = 1..n \). Moreover, from Lemma 3 in [BM03] we know that

\[
\sup_{t>0} \mathbb{E}[M^0_{l^*(H)}(I)^q] < \infty
\]

for \( \gamma \in (0,\sqrt{2}) \) and \( q \in [1,q^*) \), so we have the uniform bound \( \sup_{H>0} \mathbb{E}[\xi(I)^q] < \infty \). Then (by Doob’s martingale convergence theorem for continuous martingales) \( \tilde{M}_H(I) \) tends to some random variable which we call \( \xi(I) \) a.s.\(^5\) and in \( L^q \) for all \( q \in (1,q^*) \) so clearly \( \tilde{M}_H \) also tends weakly to \( \xi(I) \). Thus (since \( \tilde{M}_H(I) \) mimics the law of \( \xi(I) \) \( \xi(I) \) also tends weakly to \( \xi(I) \) as \( H \to 0 \). Moreover, from the reverse triangle inequality, the aforementioned \( L^q \)-convergence of \( \tilde{M}_H(I) \) to \( M_H(I) \) implies that

\[
\mathbb{E}(\xi(I)^q) = \mathbb{E}(\tilde{M}_H(I)^q) \to \mathbb{E}(\xi(I)^q)
\]

as \( H \to 0 \), for \( q \in [1,q^*) \). Then for any \( f \in C_b \) with support on the negative line only, we know that \( \mathbb{E}(f(\xi(I))) = 0 \), so (from weak convergence) \( \mathbb{E}(f(\xi(I))) = 0 \), hence \( \xi(I) \geq 0 \).

### 3.2 Local multifractality of \( \xi \)

**Proposition 3.2** For \( \gamma \in (0,\sqrt{2}) \), \( \xi \) has the following locally multifractal behaviour away from zero:

\[
\lim_{\delta \to 0} \frac{\log \mathbb{E}(\xi([t,t+\delta])^q)}{\log \delta} = \zeta(q)
\]

for \( t \in (0,1) \) and \( q \in (0,q^*) \).

**Proof.** Since \( R_H(s,t) \) is increasing for \( 0 < s,t < 1 \), Kahane’s inequality in Theorem 3.4 implies that the marginals of \( (\xi_H(I))_{H \in (0,1)} \) increase in convex order as \( H \) tends to zero, and hence these marginals form a peacock. Then Strassen’s theorem (Theorem 8 in [Str65]) implies that there exists an (backwards) martingale \( \tilde{M}_H = \xi_H(I) \) on \( \mathbb{R}^n \) such that \( \xi_H(I) \sim \xi_H(I) \) for all \( H \in [\frac{1}{2},0) \). From the upper bound part of the sandwich equation (31), we have the following inequality for \( 0 < s < t < 1 \):

\[
R_H(s,t) \leq K^T_I(s,t)
\]

where \( \theta = 4 \cdot \sup(I) \) and \( K^T_I(s,t) \) is the covariance of the Bacry-Muzy model defined in (17), and \( l^*(H) \downarrow 0 \) as \( H \downarrow 0 \). Using Kahane’s inequality again, we see that

\[
\mathbb{E}[\xi(I)^q] = \mathbb{E}[\xi(I)^q] \leq \mathbb{E}[M^0_{l^*(H)}(I)^q]
\]

for \( k = 1..n \). Moreover, from Lemma 3 in [BM03] we know that

\[
\sup_{t>0} \mathbb{E}[M^0_{l^*(H)}(I)^q] < \infty
\]

for \( \gamma \in (0,\sqrt{2}) \) and \( q \in [1,q^*) \), so we have the uniform bound \( \sup_{H>0} \mathbb{E}[\xi(I)^q] < \infty \). Then (by Doob’s martingale convergence theorem for continuous martingales) \( \tilde{M}_H(I) \) tends to some random variable which we call \( \xi(I) \) a.s.\(^5\) and in \( L^q \) for all \( q \in (1,q^*) \) so clearly \( \tilde{M}_H \) also tends weakly to \( \xi(I) \). Thus (since \( \tilde{M}_H(I) \) mimics the law of \( \xi(I) \) \( \xi(I) \) also tends weakly to \( \xi(I) \) as \( H \to 0 \). Moreover, from the reverse triangle inequality, the aforementioned \( L^q \)-convergence of \( \tilde{M}_H(I) \) to \( M_H(I) \) implies that

\[
\mathbb{E}(\xi(I)^q) = \mathbb{E}(\tilde{M}_H(I)^q) \to \mathbb{E}(\xi(I)^q)
\]

as \( H \to 0 \), for \( q \in [1,q^*) \). Then for any \( f \in C_b \) with support on the negative line only, we know that \( \mathbb{E}(f(\xi(I))) = 0 \), so (from weak convergence) \( \mathbb{E}(f(\xi(I))) = 0 \), hence \( \xi(I) \geq 0 \).
3.3 Local multifractality of $\xi_\gamma$

**Proposition 3.3** For $\gamma \in (0, \sqrt{2})$, $\xi_\gamma$ has the following locally multifractal behaviour away from zero:

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}(\xi_\gamma([t, t+\delta])^q)}{\log \delta} = \zeta(q)$$

for $t \in (0, 1)$ and $q \in (0, q^*)$.

**Proof.** We first assume $q \in (1, q^*)$. The main result (which will also be applied elsewhere in this paper) is the following inequality:

$$K^{4\tau}_{l(H, \tau)}(k) \leq F_H(k) \leq R_H(s,t) \leq G_H(k) \leq K^{4\tau}_{l(H)}(k).$$

where $k = |t-s|$. We refer the reader to Appendix B for the definitions and basic properties of $G_H(k)$ and $F_H(k)$. For any $\tau \in [0,1]$, we now define

$$K^T_l(k) := K^T_l(\tau, \tau + k)$$

where $K^T_l$ is defined in (17), and choosing $l^* = l^*(H)$ such that $G_H(0) = \frac{(\tau + \delta)^{2H}}{2H} \leq \frac{1}{2H} = \log(\frac{4}{\tau})$, we find that

$$l^*(H) = 4 e^{-\frac{1}{2H}} \downarrow 0 \text{ as } H \to 0.$$

(C-4) implies that $G_H(k) \leq \log \frac{4}{k}$, and for $k \in [l^*, 4]$, $K^4_l(k) = \log \frac{4}{k}$, so in this case $G_H(k) \leq K^4_l(k)$. For $k \in (0, l^*)$, $K^4_l(k) = \log(\frac{4}{k}) + 1 - \frac{1}{2H} > \log \frac{4}{k}$, $G_H(0) > G_H(k)$, hence for both cases, we have established the upper bound

$$G_H(k) \leq K^4_l(k)$$

Note that this holds even if $\tau = 0$.

To obtain a lower bound we now assume $\tau \in (0, 1)$, and recall from Appendix B that $F_H(k) := R_H(\tau, \tau + k)$. The following facts will be used: $F_H(0) \downarrow -\infty$ as $k \downarrow 0$, $F_H(k) \uparrow F_0(k)$ uniformly on intervals not containing the origin (this is Dini's theorem), and that $F_H(k)$ is convex.

From the start of the lower bound part of Appendix B, we know that $F_0(k) > \log(\frac{4}{k})$, but we also know that $F_H(0) < \infty$ but clearly $\log(\frac{4}{k}) \to \infty$ as $k \downarrow 0$, so from the aforementioned uniform convergence, we see that for $H > 0$ sufficiently small there exists a $k^* = k^*(H, \tau) > 0$ such that

$$F_H(k^*) = \log \frac{4\tau}{k^*}$$

(29)

with

$$F_H(k) \geq \log \frac{4\tau}{k} \text{ for } k \in [k^*, 4\tau], \quad F_H(k) \leq \log \frac{4\tau}{k} \text{ for } k \leq k^*$$

(at least in a sufficiently small neighborhood of $k^*$). Now set $l_* = l_*(H, \tau)$ such that

$$|F_H(k^*)| = \frac{1}{l_*}.$$

Such an $l_*$ will be in $[\tau, \tau + \delta]$ for sufficiently small $H$. For such a choice we must have $l^* \geq k^*$, since

$$\frac{1}{k^*} = \left| \frac{d}{dk} \log \frac{4\tau}{k} \right|_{k=k^*} > |F_H'(k^*)|$$

where the final inequality follows from (30) and (29) and the convexity of $F_H(k)$ (see Figure 2). Then $K^4_{l_*(H)}(k) \leq F_H(k)$ for $k \in (0, l^*)$ and hence in $(0, k^*)$. We also see that $l_* \downarrow 0$ as $H \downarrow 0$, since $F_H(k^*(H)) \to -\infty$ as $H \to 0$. Thus we have shown that

$$G_H(k) = R_H(\tau + \delta - k, \tau + \delta) \leq K^4_{l_*(H)}(k)$$

and

$$K^{4\tau}_{l_*(H, \tau)}(k) \leq F_H(k) = R_H(\tau, \tau + k)$$

for $k \in [0, 4\tau]$. From Appendix B, we recall that

$$R_H(s, k+s) = \int_0^s (u(k+u))^{H-\frac{1}{2}} du$$

and if we restrict attention to $A_k := \{(s,t) : t-s = k, (s,t) \in [\tau, \tau + \delta]\}$ for $0 < \tau < \tau + \delta < 1$ with $k \in [0, \delta]$, then $R_H(s, t)$ is maximized at $s = \tau + \delta - k$ and minimized at $s = \tau$. Thus

$$K^{4\tau}_{l_*(H, \tau)}(k) \leq F_H(k) \leq R_H(s, t) \leq G_H(k) \leq K^4_{l_*(H)}(k)$$

(31)

for $(s, t) \in [\tau, \tau + \delta]^2$ where $k = |t-s|$.

We now recall the following fundamental result on GMC:
where subsection 3.1 which implies $\mathbb{E}([\gamma])$ for $\alpha$ and $\gamma > 0$ we see that $\mathbb{E}((\xi_\gamma)^q) \to \mathbb{E}((\gamma)^q)$ for $q \in (1, q^*)$ (see (26)), we see that

$$
\mathbb{E}((M^T_\gamma([\tau, \tau + \delta]))^q) \leq \mathbb{E}((\xi_\gamma([\tau, \tau + \delta]))^q) \leq \mathbb{E}((M^T_\gamma([\tau, \tau + \delta]))^q).
$$

Then using the multifractality property of $M^T_\gamma$ we see that:

$$
c_{q,4}(\delta^\gamma(q)) = c_{q,4}(4\tau)^{2^q(q-1)}\delta^\gamma(q) \leq \mathbb{E}((\xi_\gamma([\tau, \tau + \delta]))^q) \leq c_{q,4}\delta^\gamma(q) = c_{q,4}(4\tau)^{2^q(q-1)}\delta^\gamma(q).
$$

Taking the logarithm of the above inequality, dividing by $\log \delta$ and taking limits yields the local multifractality property for $\xi_\gamma$ (recall that we are assuming that $t > 0$ here).

For $q \in (0, 1)$, using that $|x^q - x^q| \leq |x - y|^q$ for $x, y > 0$, we see that for $[a, b] \subset [0, 1]$ we have

$$
|\mathbb{E}(\xi_\gamma([a, b])^q - \xi_\gamma^H([a, b])^q)| \leq \mathbb{E}((\xi_\gamma([a, b])^q - \xi_\gamma^H([a, b])^q)) \leq \mathbb{E}((\xi_\gamma([a, b]) - \xi_\gamma^H([a, b])^q)) \leq 0
$$

as $H \to 0$, where the final inequality follows from Jensen and the $L^q$-convergence of $\xi_\gamma^H([a, b])$ for $q \in (1, q^*)$ (see next subsection 3.1) which implies $L^1$ convergence), and similarly we find that $\mathbb{E}((M^T_\gamma([a, b])^q) \to \mathbb{E}((M^H_\gamma([a, b])^q)$ as $\epsilon \to 0$, where $M^\epsilon$ is defined as in Section 2.3. Then since $q \in (0, 1)$, the inequalities in Kahane’s inequality (33) are reversed:

$$
\mathbb{E}((M^T_\gamma([\tau, \tau + \delta]))^q) \leq \mathbb{E}((\xi_\gamma^H([\tau, \tau + \delta]))^q) \leq \mathbb{E}((M^T_\gamma([\tau, \tau + \delta]))^q)
$$

3.3.1 Hausdorff dimension of the support of $\xi_\gamma$

For $[a, b] \subset (0, T)$ we have

$$
\mathbb{E}(\int_{[0,T]^2} |I - s|^\alpha \xi_\gamma(ds)\xi_\gamma(dt)) \leq e^\bar{\gamma} \int_0^T \int_{|t-s|^\alpha} e^{\gammaK(s,t)}dsdt = \mathbb{E}(M_{\gamma+\alpha}([0,T])^2)
$$

which we know is finite if $\alpha + \gamma^2 < 1$. Thus (from Frostman’s lemma) the Hausdorff dimension of the support of $\xi_\gamma$ on $[0,T]$ is at least $1 - \gamma^2$ (see also Corollary 1.2 in [Aru17]).
abuse of notation) we denote this last integral as $\langle \gamma \rangle$. The third property above can be re-written as:

$$
\langle \gamma, Z^t \rangle = \langle W, Y^t \rangle = \int_0^t \frac{d}{ds} Y^t(s) dW_s.
$$

Proposition 3.5 For $\gamma \in (0, \sqrt{2})$ and any interval $I \subseteq [0, 1]$, $\xi_\gamma(I)$ has the same tail behaviour as $M_\gamma$ for all $q \in (1, q^*)$, i.e. $E(\xi_\gamma(I)^q) < \infty$ if $q \in (1, q^*)$, and $E(\xi_\gamma(I)^q) = \infty$ if $q > q^*$.

Proof. We first recall (33) for any interval $I = (\tau, \tau + \delta) \subset (0, 1)$, we have

$$
E((M_\gamma^{\tau, \tau + \delta}(I))^q) \leq E((\xi_\gamma^t(I))^q) \leq E((M_\gamma^{\tau, \tau + \delta}(I))^q).
$$

(34)

Thus letting $H \to 0$ and using the known $L^q$-convergence for $M_T^\gamma$ (see subsection 2.3) and $\xi_H^\gamma$ (see (26)), we obtain the result when $I$ is contained in $(0, 1)$. But clearly $\xi_\gamma([0, b]) \geq \xi_\gamma([a, b])$ for $a \in (0, b)$, so the critical moment for $\xi_\gamma([0, b])$ is clearly less than or equal to $q^*$. But we also know that the upper bound in (34) holds even when $I$ includes zero, so in fact $q = q^*$ for this case as well.

### 3.4 Existence of the GMC measure for $\gamma \in (0, \sqrt{2})$ using the Shamov approximation theorem

Recall that $\xi_\gamma^H(A) = \int_A e^{\gamma Z^H - \frac{1}{2} \gamma^2 \text{Var}(Z^H)} dt$ for $H > 0$ and we are now considering the full range $\gamma \in (0, \sqrt{2})$. Let $\mathcal{H} := H_0^2$ denote the Cameron-Martin space of $W$, and we make the following elementary observations:

1. $E(\xi_\gamma^t(dt)) = dt$.

2. $\xi_\gamma^t(dt)$ is measurable with respect to $W$, so we write $\xi_\gamma^t(dt) = \xi_\gamma^t(W, dt)$

3. For all $f \in \mathcal{H}$ we have:

$$
\xi_\gamma^t(W + f, dt) = e^{\gamma \int_0^t (t-s)^{H-\frac{1}{2}} f'(s) ds} \xi_\gamma^t(W, dt) \quad a.s.
$$

Using the nomenclature of Shamov[Sha16] (page 7), we now define a generalized $\mathcal{H}$-valued function $Y^H : \mathcal{H} \to L^0([0, 1])$ as

$$
Y^H(f) := \gamma \int_0^1 (t-s)^{H-\frac{1}{2}} f'(s) ds.
$$

We can re-write $Y^H(f)$ as $\langle Y^H(t), f \rangle$, where $Y^H(t) \in \mathcal{H}$ denotes the family of functions:

$$
Y^H(t) := \gamma \left( \frac{t^{H+\frac{1}{2}} - (t-s)^{H+\frac{1}{2}}}{H + \frac{1}{2}} \right) 1_{s<t} + \gamma \frac{t^{H+\frac{1}{2}}}{H + \frac{1}{2}} 1_{s\geq t}
$$

(note that $Y^H(t)$ also depends on $s$ and $Y^H(t) = 0$ at $s = 0$), and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathcal{H}$. Then the third property above can be re-written as:

$$
\xi_\gamma^t(W + f, dt) = e^{\langle Y^H(t), f \rangle} \xi_\gamma^t(W, dt)
$$

$Y^H(t) \notin \mathcal{H}$ for $H = 0$ as $\frac{d}{dt}Y^H(t)$ is not square integrable, however $\gamma \int_0^1 (t-s)^{-\frac{1}{2}} f'(s) ds$ still exists and thus (with mild abuse of notation) we denote this last integral as $\langle Y^0(t), f \rangle$.

With a similar abuse of notation we write:

$$
Z^H_t = \langle W, Y^H(t) \rangle = \int_0^1 \frac{d}{ds} Y^H(t) dW_s.
$$

Figure 2: Here we see simulations of $\xi_\gamma$ using the new Karhunen-Loève expansion in [FS19b] for (from left to right) $\gamma = 0.125, 0.25, 0.375$ and 0.5 with $n = 1000$ eigenfunctions, 1000 time points, $H = .0001$ and we have used Gauss-Legendre quadrature. For this range of $\gamma$-values, the first four raw sample moments are in very close agreement with the theoretical values for $H = 0$, see [FS19b] for tabulated values.
Of course $W \notin H$ a.s. but an “inner product” may be defined with elements of $\mathcal{H}$ by considering $W$ as a generalized random vector in $\mathcal{H}$ (see page 7 in [Sha16]):

$$W : f \to \langle W, f \rangle := \int_0^1 f'(s)dW_s.$$  

We say that $W$ is a standard Gaussian i.e. a continuous linear operator from $\mathcal{H}$ into $L^0(\mathbb{P})$ (in fact $L^2(\mathbb{P})$), where $\langle W, f \rangle$ is a centred Gaussian with variance $\langle f, f \rangle$, and $Y^H$ is a generalized $\mathcal{H}$-valued function on $[0,1]$ i.e. a continuous linear operator from $\mathcal{H}$ into $L^0(\text{Leb})$ (in fact $L^2(\text{Leb})$) noting that for $H > 0$ we have made the standard Riesz identification of $Y^H(t)$ with its dual.

It should be emphasized again that for $H > 0$, $Y^H(t)$ is in fact a member of $\mathcal{H}$ and so is referred to as a true $\mathcal{H}$-valued function. At $H = 0$, $Y^H$ is no longer “true” element of $\mathcal{H}$, but is still a generalized $\mathcal{H}$-valued function, since $Y^0$ still maps $\mathcal{H}$ to a measurable function.

This decomposition of $\gamma Z_t^H$ into the pair $(W, Y^H)$ is an instance of the Maurey-Nikishin Factorisation. With this new construction Shamov gives us an equivalent definition of the sub-critical Gaussian Multiplicative Chaos (GMC) over the pair $(W, Y^H)$ namely a random measure that satisfies the aforementioned three properties:

1. $\mathbb{E}(\xi^H(\dd t)) = \dd t$.

2. The measure $\xi^H$ is measurable with respect to $W$ so we can write $\xi^H(\dd t) = \xi^H(W, \dd t)$

3. For all $f \in \mathcal{H}$

$$\xi^H(W + f, \dd t) = e^{\langle Y^H(t), f \rangle} \xi^H(W, \dd t) \quad \text{a.s.}$$

As is shown in Shamov (Theorem 17), the existence of a sub-critical GMC $\xi^H$ over the Gaussian fields $(W, Y^H)$ with expectation $\text{Leb}(\dd t)$ is equivalent to the statement that $Y^H(t)$ is a randomized shift. A randomized shift is a generalized $\mathcal{H}$-valued function $Y(t)$ defined on $[0,1]$ such that if $t$ is sampled uniformly on $[0,1]$ independently from $W$ then the distribution of $W + Y(t)$ is absolutely continuous with respect to $W$. In the language of Shamov:

$$\text{Law}_{\mathbb{P}\otimes\text{Leb}}[W + Y(t)] \ll \text{Law}_\mathbb{P}[W].$$

We can easily verify this to be the case for our model when $H > 0$. Again this is due to the fact that $Y^H(t) \in \mathcal{H}$ and so the Cameron-Martin theorem yields the result.

The pair $(W, Y)$ form a generalized Gaussian field (see Definition 10 in [Sha16]) where this is proven to be equivalent to the more conventional definition via the Maurey-Nikishin factorization theorem. The question remains whether $Y^H(t)$ is a randomized shift and if so what is the relationship between $\xi^H$ and its approximating measures. This is answered (in general) by the Shamov[Sha16] Approximation Theorem (Theorem 25) which states that if we have a series of randomized shifts $Y_n$ with associated GMCs denoted $M_{Y_n}$ and kernels $K_{Y_n, Y_n}(t, s) := \langle Y_n(t), Y_n(s) \rangle$ satisfying:

- The family of random variables $\{M_{Y_n}\}$ is uniformly integrable.
- There exists a generalized $\mathcal{H}$-valued function that is the limit of $Y_n$ in the sense that

$$\forall f \in \mathcal{H} : \langle Y_n, f \rangle \overset{L^0(\text{Leb})}{\to} \langle Y, f \rangle$$

Then $Y$ is a randomized shift. If, furthermore

- The kernels $K_{Y_n, Y_n}$ converge to $K_Y, Y$ in $L^0(\mu \otimes \mu)$

then the sub-critical GMC $M_Y$ (associated to $Y$ with expectation $\mu$) is the limit of $M_{Y_n}$, in the sense that:

$$\forall f \in L^1(\mu) : \int f(t) M_{Y_n}(W, dt) \overset{L^1}{\to} \int f(t) M_Y(W, dt).$$

We address each of these points in turn:

- **Uniform Integrability.** As in the proof of multifractality, for each $H > 0$ we can bound the covariance of the RL process by the covariance of an approximate Bacry-Muzy multifractal random walk (see section 3.3). By Kahane’s inequality we can thus bound the $p$-th moment of our measure from above by the $p$-th moment of the Bacry-Muzy MRM which is shown in [BM03] to be uniformly bounded. Thus $\{M_{H_n}\}$ are uniformly integrable.
• **Convergence of the shifts.** The operator $Y^H$ is (up to an unimportant factor $\Gamma(H + \frac{1}{2})^{-1}$) the RL fractional integral $I^\alpha$ of order $\alpha = H + \frac{1}{2}$. As is proved in Samko et al.[SKM93] (Theorem 2.6), the RL integrals form a semigroup in $L^p(0, 1)$ for $p \geq 1$, which is continuous in the uniform topology for all $\alpha > 0$ and strongly for all $\alpha \geq 0$, which in our context implies that for all $f \in \mathcal{H}$:

$$\lim_{H \to 0} ||(Y^H, f) - (Y^0, f)||_{L^2} = \lim_{H \to 0} ||I^\alpha(f) - f||_{L^2} = 0.$$ 

Note that $f \in \mathcal{H}$ implies $f' \in L^2$, and convergence in $L^2$ implies convergence in measure.

• **Convergence of kernels.** These kernels are the same (up to a factor of $\gamma$) as the covariances $R_H(s, t)$ and $R(s, t)$. As discussed previously, away from the diagonal $\{s = t\}$, $R_H(s, t) \to R(s, t)$ pointwise and hence in measure.

Thus we have shown that

$$\int f(t)\xi^H_\gamma(dt) \xrightarrow{L^1} \int f(t)\xi_\gamma(dt)$$

for all $f \in L^1$.

**Remark 3.1** From section 6.8 in [Sha16], we know that $\xi_\gamma$ has no atoms.

### 3.4.1 Computing the adjoint of $Y^H$

As remarked earlier, $(Y^H, f, g) \in L^2[0, 1]$ for all $g \in \mathcal{H}$ and so, given $f' \in L^2[0, 1]$ the function $f'(Y^H, g) \in L^1[0, 1]$. This allows the definition of $\int_0^1 f'(t)Y^H(t)dt \in \mathcal{H}$:

$$\langle \int_0^1 f'(t)Y^H(t)dt, g \rangle := \int_0^1 \langle f'(t)Y^H(t), g \rangle dt.$$  

(35)

Acting on this element with the Brownian Motion:

$$\langle W, \int_0^1 f'(t)Y^H(t)dt \rangle = \int_0^1 f'(t)\langle W, Y^H(t) \rangle dt = \int_0^1 f'(t)Z^H_t dt.$$ 

This is the familiar action of $Z^H_t$ on $L^2$ test functions. If we define the operator $A : H \to L^2[0, 1]$:

$$Af' = \int_0^1 f'(t)Y^H(t)dt.$$ 

As described in Appendix A of Shamov, the generalized $\mathcal{H}$-valued function $Y$ corresponding to $Z^H$ is the adjoint of $A$ and we can easily see that:

$$\langle g, Af' \rangle_H = \langle g, \int_0^1 f'(t)Y^H(t)dt \rangle_H = \int_0^1 f'(t)\langle g, Y^H(t) \rangle_H dt = \langle f', Y^Hg \rangle_{L^2}.$$ 

By definition of the adjoint map we have $Y^H = (Y^H, \cdot)$.

### 3.4.2 $Y(t)$ as a generalized $\mathcal{H}$-valued function

Setting $H = 0$ in $Y^H(t)$ we see that

$$Y(t) := Y^0(t) = 2\gamma(t^{\frac{1}{2}} - (t - s)^{\frac{1}{2}})1_{s < t} + 2\gamma t^{\frac{1}{2}}1_{s \geq t}$$

and $Y'(t) := \gamma(t - s)^{-\frac{1}{2}}1_{s < t}$, so in particular $Y' \notin L^2([0, 1])$. Now let $\mathcal{H}$ denote the Cameron-Martin space of $W$ as above. Then $Y(t)$ is still a generalized $\mathcal{H}$-valued function, since for $h \in \mathcal{H}$ we know that

$$\langle Y(t), h \rangle := \int_0^t (t - s)^{-\frac{1}{2}}h'(s)ds \in L^2([0, 1]) \subset L^1([0, 1])$$

where the inclusion in $L^1([0, 1])$ here follows from the Samko et al.[SKM93] result used above. Then for all $f, g \in \mathcal{H}$, we have $(Y(t), g) \in L^2([0, 1])$ as already discussed, and $f' \in L^2([0, 1])$ by definition of $\mathcal{H}$. Then from Cauchy-Schwarz we see that

$$f'(t)(Y(t), g) \in L^1([0, 1]).$$

$f'(t)Y(t)$ is not in $\mathcal{H}$ for all $t$, but

$$\langle \int_0^1 f'(t)Y(t)dt, g \rangle := \int_0^1 f'(t)(Y(t), g)dt < \infty.$$
so we see that $\int_0^1 f'(t) Y(t) dt \in \mathcal{H}$. Since $\int_0^1 f'(t) Y(t) \in \mathcal{H}$, we can now form

$$\langle W, \int_0^1 f'(t) Y(t) dt \rangle.$$ 

Then by definition of a standard Gaussian

$$\mathbb{E}((\langle W, \int_0^1 f'(t) Y(t) dt \rangle)^2) = \int_0^1 (\int_0^1 f'(t) \frac{\partial}{\partial s} Y(s) dt)^2 ds$$

$$= \| \int_0^1 f'(t) Y(t) dt \|^2_{\mathcal{H}}$$

$$= \int_0^1 (\frac{\partial}{\partial s} \int_0^1 f'(t) Y(t) dt) (\frac{\partial}{\partial s} \int_0^1 f'(u) Y(u) du) ds$$

$$= \int_0^1 (\int_0^1 f'(t) \frac{\partial Y(t)}{\partial s} dt) (\frac{\partial}{\partial s} \int_0^1 f'(u) \frac{\partial Y(u)}{\partial s} du) ds$$

$$= \int_0^1 f'(t) f'(u) (\int_0^1 \frac{\partial}{\partial s} Y(t) \frac{\partial Y(u)}{\partial s} ds) du$$

$$= \int_0^1 f'(t) f'(u) (\int_0^1 (t-s)^{-\frac{1}{2}} 1_{s < t} (u-s)^{-\frac{1}{2}} 1_{s < u} ds) du$$

$$= \int_0^1 f'(t) f'(u) R(u, t) dt.$$ 

Then we see that $R(u, t) = \int_0^1 (t-s)^{-\frac{1}{2}} (u-s)^{-\frac{1}{2}} ds$ is the kernel $K(u, t) = \langle Y(t), Y(u) \rangle$ defined on page 14 in Shamov[Sha16]. Then by Theorem 26 in Shamov, there exists a sub-critical GMC over $(W, Y)$, and setting $Y_\epsilon(t) = \int_0^1 Y(t') \psi(t-t') dt'$ (for a suitable mollifier $\psi_\epsilon$), $Y_\epsilon$ tends weakly to $Y_\epsilon$ in probability; $M_\epsilon$ is the sequence of GMCs associated with the underlying sequence of Gaussian fields

$$X_\epsilon' = \int_{[0,1]} \psi_\epsilon(t-t') Z(t') dt' = " \int_{[0,1]} \psi_\epsilon \int_0^1 (t'-u)^{-\frac{1}{2}} dW_u dt' " = \int_{[0,1]} \int_0^1 \psi_\epsilon(t-t') (t'-u)^{-\frac{1}{2}} dt' dW_u$$

with covariance

$$\int_0^1 \psi_\epsilon(t-t') \psi_\epsilon(s-s') R(s', t') ds' dt'.$$

Thus $M_\epsilon$ and our Riemann-Liouville GMC $\xi_\gamma$ both satisfy the equation in Theorem 6 in [Sha16], and it is well known that Eq 7 in [Sha16] uniquely specifies the law of $M(X, dt)$, which we can prove using the notion of rooted measures and the disintegration theorem, see Corollary 18 in [Sha16] and also section 2.1 in [Aru17] for further discussion on this. Moreover, the measure $\xi_H$ for $H > 0$ is the same in this section and in section 4.

## 4 Supercritical Gaussian multiplicative chaos for the RL process as $H \to 0$ and local multifractality

Following Remark 5 in [BJRV14], we now construct an atomic GMC by considering a Radon measure $N_\epsilon$, whose law conditioned on $\xi_\gamma$ is a Poisson random measure on $\mathbb{R} \times \mathbb{R}_+$ with intensity $\frac{\xi_\gamma(dt) dx}{2^{1+\alpha}}$ for $\alpha \in (0, 1)$. Then $M(A) = \int_A \int_{\mathbb{R}_+} x N_\epsilon(dx, dt)$ satisfies

$$\mathbb{E}(e^{-uM(A)}) = \mathbb{E}(e^{-\int_A \int_{\mathbb{R}_+} (e^{-x-1}) \frac{dx}{2^{1+\alpha}}}) = \mathbb{E}(e^{-\frac{\Gamma(1-\alpha)}{1-\alpha} u^{\gamma+\alpha}}(A))$$

and in particular $A_t = M([0, t])$ conditioned on $\xi_\gamma$ is just an additive process, and $A_t$ has the same law (on path space) as $L_{\xi_\gamma([0,t])}$, where $L$ is a stable subordinator independent of $\xi_\gamma$, with $\mathbb{E}(e^{-uL_t}) = e^{-\frac{\Gamma(1-\alpha)}{1-\alpha} u^{\gamma+\alpha}}$.

Using the identity $x^\beta = \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty (1-e^{-xz}) \frac{dz}{z^{1-\beta}}$ twice (this is Eq 31 in [BJRV14]), we can then easily mimic the proof of Proposition 6 in [BJRV14] to show that

$$\mathbb{E}(\tilde{M}(A)^q) = c_q^\prime \mathbb{E}(\xi_\gamma(A)^{q/\alpha})$$

(36)

where $c_q^\prime = \frac{\Gamma(1-\alpha)\Gamma(1-\alpha)^{q/\alpha}}{\Gamma(1-q)\alpha^{q/\alpha}}$. Then setting $\gamma = 2$ so $\gamma > \sqrt{2}$, and $\alpha = \frac{1}{2} \gamma^2$, we find that $\zeta(q/\alpha) = \zeta(q)_{\gamma=\gamma}$ (see page 13 in [BJRV14]). Setting $A = [t, t + \delta]$ and taking the logarithm of (36), dividing by $\log \delta$ and taking the limit as $\delta \to 0$ and using the local multifractality property of $\xi_\gamma$, we see that

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}(\tilde{M}([t, t + \delta])^q)}{\log \delta} = \zeta(q)_{\gamma=\gamma}$$

for $q \in (0, \alpha = \frac{1}{2} \gamma^2 = \frac{2}{\sqrt{2}} = q^*_{\gamma=\gamma})$. Thus we see that $\tilde{M}$ is a natural candidate for a GMC with $\gamma$-value equal to $\gamma > \sqrt{2}$. 


5 Applications: the Rough Bergomi model in the $H \rightarrow 0$ limit

5.1 The skew flattening phenomenon

We now consider a driftless Rough Bergomi model for a stock price process $X_t^H$:

\[
\begin{align*}
\text{d}X_t^H &= \sqrt{V_t^H} \text{d}W_t, \\
V_t^H &= e^{\gamma Z_t^H - \frac{1}{2} \gamma^2 \text{Var}(Z_t^H)} \\
Z_t^H &= \int_0^t (s - t)^{H-\frac{1}{2}} (\rho \text{d}W_s + \text{d}B_t)
\end{align*}
\]

with $X_0^H = 0$, where $\gamma \in (0, 1)$, $|\rho| \leq 1$ and $W, W^\perp$ are independent Brownian motions.

**Theorem 5.1** $X^H$ tends to $B_{\xi_\gamma([0,t])}^\perp$ stably (and hence weakly) in law on any finite interval $[0, T]$, where $B^\perp$ is a Brownian motion independent of everything else.

**Remark 5.1** We call this the skew flattening phenomenon, so in particular $\tilde{X}_t$ (for a single fixed $t$) tends weakly to some weakly symmetric distribution $\mu$.

**Proof.** From Theorem 2.2, we know that $\langle X^H \rangle_t$ tends to a random variable $\xi_\gamma([0, t])$ in $L^2$ (and hence in probability), and $\langle X^H, W \rangle_t = \int_0^t \sqrt{V_t^H} \text{d}u$. But

\[
E((\sqrt{V_t^H})^2) = E(e^{\frac{1}{2} \gamma Z_t^H - \frac{1}{2} \gamma^2 \text{Var}(Z_t^H)}) = e^{-\frac{1}{2} \gamma^2 \text{Var}(Z_t^H)} \rightarrow 0
\]

as $H \rightarrow 0$, so (by Markov’s inequality) $P(\sqrt{V_t^H} > \delta) \leq \frac{1}{\delta^2} E(\sqrt{V_t^H}) \rightarrow 0$, so $\sqrt{V_t^H}$ tends to zero in probability, and hence

\[
G_t := \langle X^H, W \rangle_t \overset{P}{\rightarrow} 0.
\]

Moreover, for any bounded martingale $N$ orthogonal to $W$

\[
\langle X^H, N \rangle_t = 0
\]

Thus setting $Z_t = W_t$ and applying Theorem IX.7.3 in Jacod&Shiryaev[JS03] (see also Proposition II.7.5 and Definition II.7.8 in [JS03]), we can construct an extension $(\Omega, \mathcal{F}, (\mathcal{F}_t, \mathbb{P})$ of our original filtered probability space $\langle \Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P} \rangle$ and a continuous $\mathcal{F}$-progressive conditional PII martingale $X$ on this extension (see Definition 7.4 in chapter II in [JS03] for definition), such that $X^H$ converges stably (and hence weakly) to $X$ (see Definition 5.28 in chapter XIII in [JS03] for definition of stable convergence) for which

\[
\langle X \rangle_t = \xi_\gamma([0, t]) \\
\langle X, M \rangle_t = 0
\]

for all continuous (bounded) martingales $M$ with respect to the original filtration $\mathcal{F}_t$. From Proposition 7.5 and Definition 7.8 in Chapter 2 in [JS03], this means that

\[X_t = X'_t + \int_0^t u_s \text{d}W_s \]

where $X'$ is an $\mathcal{F}_t$-local martingale and $u$ is a predictable process on the original space $\langle \Omega, \mathcal{F}, \mathbb{P} \rangle$. One such $M$ is $M_t = W_t \wedge T_{b+b}$, where $T_b = \inf\{t : W_t = b\}$, so we have a pair of continuous local martingales $(M, X)$ with

\[
\langle X, M \rangle_t = \langle X, W \rangle_t = \int_0^t u_s \text{d}s = 0
\]

for $t \leq T_b \wedge T_{-b}$, so in fact $u_t \equiv 0$. Then applying F.Knight’s Theorem 3.4.13 in [KS91] with $M^{(1)} = X$ and $M^{(2)} = W$, if $T_t = \inf\{s \geq 0 : \langle X \rangle_s > t\}$, then $X_{T_t}$ is a Brownian motion independent of $W$. Hence $X$ has the same law as $B_{\xi_\gamma([0, t])}^\perp$ for any Brownian motion $B^\perp$ independent of $W$. \(\blacksquare\)

5.2 A closed-form expression for the skewness of $X_t^H$

In this subsection we compute an explicit expression for the skewness of $X_t^H$ (conditioned on its history), which (as a by-product) gives a more “hands-on” proof as to why the skew tends to zero as $H \rightarrow 0$, and also allows us to see how fast the skew decays and the $H$-value in $(0, \frac{1}{2})$ where the absolute value of the skew is maximized.
We first note that (trivially) $X^H$ has the same law as $X^H$ defined by

\[
\begin{align*}
\frac{dX^H_t}{dt} &= \sqrt{V_t^H}(\rho dB_t + \tilde{\rho}dW_t), \\
V_t^H &= e^{\gamma Z_t^H - \frac{1}{2} \gamma^2 \text{Var}(Z_t^H)} \\
Z_t^H &= \int_0^t (t-s)^{H-\frac{1}{2}} dB_t
\end{align*}
\]

(40)

where $B$ is independent of $W$, and this is the version of the model we use in this subsection. We henceforth use $\mathbb{E}_t(\cdot)$ as shorthand for the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_t^H)$, and we now replace the constant $\rho$ with a time-dependent $\rho(t)$, and replace our original $V_t^H$ process with

\[V_t^H = \xi_0(t)e^{\gamma Z_t^H - \frac{1}{2} \gamma^2 \text{Var}(Z_t^H)}\]

to incorporate a non-flat initial variance term structure.

**Proposition 5.2**

\[
\mathbb{E}_t((X^H_t - X^H_{t_0})^3) = 3\gamma \int_{t_0}^T \int_0^t \rho(s) \xi_0^2(s) \xi_0(t) e^{\frac{1}{2} \gamma^2 \text{Cov}_{t_0}(Z_s^H Z_t^H) - \frac{1}{2} \gamma^2 \text{Var}_{t_0}(Z_t^H)} (t-s)^{H-\frac{1}{2}} ds dt < \infty
\]

(41)

where $\xi_0(t) = \xi_0(t)e^{\gamma \int_0^t (t-s)^{H-\frac{1}{2}} dB_s - \frac{1}{2}(t-t_0)^{2H}}$. This simplifies to

\[
\mathbb{E}((X^H_t)^3) = 3\gamma V_0^3 \int_0^t \int_0^t \rho(s) e^{\frac{1}{2} \gamma^2 (R_H(s,t) - \frac{2H}{\gamma^2})} (t-s)^{H-\frac{1}{2}} ds dt < \infty
\]

(42)

if $t_0 = 0$ and $\xi_0(t) = V_0$ for all $t$ (i.e. flat initial variance term structure).

**Proof.** See Appendix C. \(\blacksquare\)

**Remark 5.2** Using that $R_H(s,t) \to R_{BM}(s,t)$ as $s,t \to 0$ (for $H > 0$ fixed), where $R_{BM}(s,t) = \frac{1}{2H} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right)$ is the covariance function of $\frac{1}{\sqrt{2H}}W^H$ where $W^H$ is a standard (one or two-sided) fractional Brownian motion, we find that the exponent in (42) behaves like $\frac{1}{16H}(s^{2H} + 2t^{2H} - 2(t-s)^{2H})$ for $s < t$ as $s,t \to 0$, and thus can effectively be ignored, so (for $\rho$ constant

\[
\mathbb{E}((X^H_t)^3) \sim \frac{4\rho \gamma V_0^3}{3 + 8H + 4H^2} T^{H+\frac{1}{2}}
\]

as $T \to 0$.

**Remark 5.3** We have tested (42) against Monte Carlo estimates for $\mathbb{E}((X^H_t)^3)$ and the results are in very close agreement. Note that $X^H$ is driftless so (40) is only a toy model at the moment, but we easily adapt Proposition 5.2 and the two remarks above to incorporate the additional $-\frac{1}{2} (X^H)_t$ drift term required to make $S_t = e^{X^H_t}$ a martingale. However, the relative contribution from this drift will disappear in the small-time limit, so we omit the tedious details, since rough stoc vol models are generally used (and considered more realistic) over small time horizons.

### 5.3 Convergence of the skew to zero

**Corollary 5.3** For $0 \leq t \leq T \leq 1$, $\mathbb{E}_t((X^H_t - X^H_{t_0})^3) \to 0$ a.s. as $H \to 0$.

**Proof.** See Appendix D. \(\blacksquare\)

### 5.4 Speed of convergence of the skew to zero

The following corollary quantifies the speed at which the skew goes to zero.

**Proposition 5.4** Let $\rho(.)$ be continuous and bounded away from zero with constant sign for $t$ sufficiently small. Then

\[
-\lim_{H \to 0} \log[\text{sgn}(\rho)\mathbb{E}((X^H_t)^3)] = \hat{r}(\gamma) = \begin{cases} 
\frac{1}{16} \gamma^2 & 0 \leq \gamma \leq 1, \\
\frac{1}{2} + \frac{1}{2} \log \gamma - \frac{3}{16} & \gamma \geq 1
\end{cases}
\]

(43)

**Proof.** The proof (by S.Gerhold) involves many intermediate lemmas, so we do not include it here, but is available in a supplementary note. \(\blacksquare\)

**Remark 4.4** Note that $\hat{r}(\gamma) \leq r(\gamma) \leq \frac{1}{16} \gamma^2$, where $r(\gamma) = \frac{1}{16} \gamma^2 1_{\gamma \leq 8} + \frac{1}{16}(1 + \log \frac{\gamma^2}{16}) 1_{\gamma \geq 8}$ and $\hat{r}(\gamma)$ is negative for $\gamma$ larger than the root at $\approx 1.61711$, which makes the integral explode as $H \to 0$ for such values of $\gamma$. 
could also solve for \((44)\) numerically using an Adams scheme. e.g. the first Theorem in [Atk74], we know that if \(g\) with strike \(\rho\) exceeds 1 for some \(t >\) then \((44)\) admits a unique solution of the form \(\rho(t) = O(1)\), and then choose \(V_0\) to ensure that \(|\rho(t)| \leq 1\). If the calibrated \(|\rho(t)|\) exceeds 1 for some \(t > 0\) for all choices of \(V_0\) this means either the model is wrong or we need a different \(H\)-value. We could also solve for \((44)\) numerically using an Adams scheme.

### 5.5 Calibrating a time-dependent correlation function to the skew term structure

From \((42)\) we see that

\[
\frac{\partial}{\partial T} \mathbb{E}((X_T^H)^3) = V_0 \frac{3}{2} \int_0^T \rho(s) e^{\frac{1}{2} s^2 \gamma^2 (R_{(s,T)} - \frac{2H}{T-0})} (T-s)^{H-\frac{1}{2}} ds.
\]

If \(g(T) = \frac{dT}{d\sigma} \mathbb{E}((X_T^H)^3)\) is known e.g. from the observed prices of call options in the market, this becomes a Volterra non-convolution integral equation of the first kind (also known as an Abel equation) for the unknown \(\rho(.)\). Then from e.g. the first Theorem in [Atk74], we know that if \(g(T) = T^\beta \tilde{g}(T)\) with \(\alpha + \beta > 0\) (where \(\alpha = \frac{1}{2} - H\) and \(\tilde{g} \in C^1[0,T]\), then \((44)\) admits a unique solution of the form \(\rho(t) = t^{\alpha+\beta-1} \tilde{f}(t)\) so some \(\tilde{f} \in C^1[0,T]\). For our application, we have to then choose \(\beta = H + \frac{1}{2}\) to ensure that \(\rho(t) = O(1)\), and then choose \(V_0\) to ensure that \(|\rho(t)| \leq 1\). If the calibrated \(|\rho(t)|\) exceeds 1 for some \(t > 0\) for all choices of \(V_0\) this means either the model is wrong or we need a different \(H\)-value. We could also solve for \((44)\) numerically using an Adams scheme.

### 5.6 A \(H = 0\) model - pros and cons

Returning to Section 4.1, we can circumvent the problem of vanishing skew, by considering a toy model of the form

\[
X_t = \sigma(W_t + \bar{\rho}B_{\xi_t,(0,t)]}^1)
\]

where \(\bar{\rho} = \sqrt{1-\rho^2}\), \(W\) and \(\xi_t,(0,t)]\) are defined as in Section 2.1 with \(\gamma \in (0,1)\), and \(B^1\) is a Brownian motion independent of \(W\). Then \(X\) is a self-similar process; more specifically \(X_t/\sqrt{t} \sim X_1\) for all \(t > 0\), and \(X_1\) has non-zero skewness for \(\alpha \neq 0\); more specifically

\[
\mathbb{E}((X_t^\gamma)^3) = 4\sigma^3 (1-\rho^2)^\gamma
\]

and \(\mathbb{E}(X_t^2) = \sigma^2\), and we can derive a similar (slightly more involved) expression for \(\mathbb{E}(X_t^4)\). We note that the skewness \((46)\) is minimized (resp. maximized) at \(\bar{\rho}^* = \pm \frac{1}{\sqrt{3}} \approx \pm 0.577\) (this does not imply that the density of \(X_t\) is symmetric when \(\rho = \pm 1\) even though the skewness is zero in this case). The \(\rho\) component achieves the goal of a \(H = 0\) model with non-zero skewness, and following a similar argument to Lemma 5 in [MT16] one can establish the following small-time behaviour for European put options in the Edgeworth Central limit theorem regime:

\[
\frac{1}{\sqrt{t}} \mathbb{E}(e^{i\sqrt{T}} - e^{X_t})^+ \sim e^{i\sqrt{T}} \mathbb{E}(e^{(x - X_t/\sqrt{t})}^+) \sim \mathbb{E}(e^{(x - X_t/\sqrt{t})}^+) \sim \mathbb{E}(e^{(x - X_1)^+})
\]

and \(\lim_{t \to 0} \hat{\sigma}_t(x,\sqrt{T}) = C_B(x,\cdot)^{-1}(C(x))\) for \(x > 0\), where \(\hat{\sigma}_t(x,t)\) denotes the implied volatility of a European call option with strike \(e^{i\sqrt{T}}\) maturity \(t\) and \(S_0 = 1\) \((C_B(x,\sigma)\) is the Bachelier model call price formula). Hence we see the full smile effect in the small-time FX options Edgeworth regime unlike the \(H > 0\) case discussed in e.g. [Fuk17], [EGR19], [FSV19], where the leading order term is just Black-Scholes, followed by a next order skew term, followed by an even higher order convexity term.

We can go from a toy model to a real model adding back the usual \(-\frac{1}{2} \langle X \rangle\) drift term for the log stock price \(X\) so \(S_t = e^{X_t}\) is a martingale, and in this case we lose self-similarity for \(X\) but \(X_t/\sqrt{t}\) still tends weakly to a non-Gaussian.
random variable, and in particular \( \lim_{t \to 0} E((X_t^3) = 4\sigma^3\rho \bar{\rho}^2 \gamma) \). This model overcomes two of the main drawbacks of the original Bacry et al.\[BDM01\] multifractal random walk (i.e. a Brownian motion evaluated at an independent MRW \( M_T^\gamma ([0, .]) \)), namely zero skewness and unrealistic small-time behaviour.

However, we suspect the property in (46) is not time-consistent, since if we define \( \eta^h_t := E((X_t^3)|F_t) \) for \( t > 0 \), then \( E((\eta^h_t)^2) = O(h^{-\gamma}) \) (and not \( O(1) \) as we would require for time consistency), so we do not pursue this model further in this article.

**Remark 5.5** It is also interesting to compare (46) against the small-time behaviour for a local vol model with \( \sigma(S) = \sigma_1 S < S_0 + \sigma_2 S \geq S_0 \), for which Pigato\[Pig19\] shows that the short-time implied volatility skew behaves like

\[
\sqrt{\frac{\pi}{2}} \frac{\sigma_+ - \sigma_-}{\sigma_+ - \sigma_- \sqrt{t}}
\]

as \( t \to 0 \), which attains the model-free upper bound in \[Fuk10\] in the limit as \( \sigma_- \to 0 \). Nevertheless, the extreme skew seen in \[Pig19\] will not hold as soon as \( S \) moves away from \( S_0 \), so this property is also not time-consistent.

**References**


\[\text{\textsuperscript{6}}\text{We can also replace the } \rho W_t \text{ component of } X \text{ with a second } \epsilon \text{Bergomi component with a non-zero } H \text{-value, and derive similar results.}\]


A Log-correlated Gaussian fields

The material in this Appendix is a brief summary of the material on log-correlated Gaussian fields in [DRSV17]. A log-correlated Gaussian field $X$ on $\mathbb{R}^d$ is a random distribution on $\mathbb{R}^d$ with a certain Gaussian structure: $X_t$ is not a random function on $\mathbb{R}^d$ since $X_t$ does not exist pointwise, but $X(f) = \langle X, f \rangle$ is well defined for any element $f$ of the Schwartz space $\mathcal{S} = \{ \phi \in C^\infty : p_{\alpha, \beta} = \| \phi \|^2_{\alpha, \beta} := \sup_{x} |x^\alpha D^\beta \phi| < \infty, \forall \alpha, \beta \}$ (the set of $f : \mathbb{R} \to \mathbb{R}$ whose derivatives of all orders exist and decay faster than any polynomial at infinity). $X$ is a Gaussian process indexed by elements of $\mathcal{S}$: $X$ is Gaussian here means that $\langle X, f_1 \rangle, \ldots, \langle X, f_n \rangle$ for $f_i \in \mathcal{S}$ has a multivariate Normal distribution, and $X$ has covariance

$$E(\langle X, \phi_1 \rangle \langle X, \phi_2 \rangle) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \log \frac{1}{|x-y|} \phi_1(y)\phi_2(x)dy \, dx$$

for all $\phi_1, \phi_2 \in \mathcal{S}$. $X$ is a (random) tempered distribution, i.e. an element of the dual space $\mathcal{S}'$ of $\mathcal{S}$ under the locally convex topology generated by the seminorms $p_{\alpha, \beta}$, which has a neighborhood base of convex sets the form $\{ f \in \mathcal{S} : p_{\alpha, \beta}(f - g) < \epsilon, \forall \alpha, \beta \}$: an element of $\mathcal{S}'$ is a continuous linear functional on $\mathcal{S}$ under this topology, i.e. $h$ is continuous iff $h(\phi_n) \to 0$ whenever $p_{\alpha, \beta}(\phi_n) \to 0$, and any $h \in \mathcal{S}'$ can be written as $h = \sum |\alpha + \beta| \leq k x^\alpha D^\beta u_{\alpha \beta}$ where $u_{\alpha \beta} \in C_b$ and differentiation is defined via integration by parts.
Figure 5: $R(s,t)$ is maximized at $s = \tau + \delta - k$, and minimized at $s = \tau$.

B Definition and properties of $F_H(k)$ and $G_H(k)$

We first note that

$$R_H(s,t) = \int_0^{s \wedge t} (s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}}du = \int_0^s u^{H-\frac{1}{2}}(t-s+u)^{H-\frac{1}{2}}du$$

for $0 \leq s \leq t$. Note that the integrand is non-negative. Going forward we will denote $k = t-s$. We restrict $R_H(s,t)$ to the line $\{t-s=k\}$ and the square $[\tau,\tau+\delta]^2$ with $\delta \in (0,1-\tau)$, i.e.

$$R_H(s,k+s) = \int_0^s (u(k+u))^{H-\frac{1}{2}}du.$$  

This expression is maximized at $s = \tau + \delta - k$ and minimized at $s = \tau$ for constant $k$ (see Figure 3), and this remains true if $\tau = 0$. In this case, the maximum is as before but the minimum is simply zero for all $H$ and all $k$. Define

$$G_H(k) := R_H(\tau + \delta - k, \tau + \delta)$$

$$F_H(k) := R_H(\tau, \tau + k).$$  

We now establish some basic properties of $G_H(k)$. By the above:

$$G_H(k) = \int_0^{\tau + \delta - k} (u(k+u))^{H-\frac{1}{2}}du.$$  

Taking the derivative with respect to $k$ and using the Leibniz rule:

$$G'_H(k) = -(\tau + \delta - k)^{H-\frac{1}{2}}(\tau + \delta)^{H-\frac{1}{2}} + (H - \frac{1}{2}) \int_0^{\tau + \delta - k} u^{H-\frac{1}{2}}(k+u)^{H-\frac{1}{2}}du < 0$$

so $G_H(k)$ is decreasing in $k$. The integral term in the previous equation explodes as $k \downarrow 0$:

$$\int_0^{\tau + \delta - k} u^{H-\frac{1}{2}}(k+u)^{H-\frac{1}{2}}du \geq \int_0^{\tau + \delta - k} (k+u)^{2H-2}du$$

$$= \frac{(\tau + \delta)^{2H-1}}{2H-1} - \frac{k^{2H-1}}{2H-1} \uparrow \infty.$$  

Hence $G_H(k)$ has a cusp at $k = 0$. Letting $H \to 0$ in (C-1), we see that

$$G_H(k) \uparrow \quad G_0(k) = \log \frac{1}{k} + 2\log(\sqrt{\tau + \delta - k} + \sqrt{\tau + \delta})$$

$$\quad \leq \qquad g(k) := \log \frac{1}{k} + \log(4(\tau + \delta))$$

with equality at $k = 0$ in the sense that both sides of the inequality are infinite.

Thus we have the following inequality:

$$G_H(k) \leq G_0(k) \leq g(k) \leq \log \frac{4}{k}.$$  

(C-4)
Proof of Corollary 5.3

For \( T \leq 1 \), using that \( R_H(t, s) \uparrow R(s, t) \) and \((t-s)^{-\frac{1}{2}} \uparrow (t-s)^{-\frac{1}{2}} \) we see that

\[
|\mathbb{E}_{t_0}((X_T^H - X_{t_0}^H)^3)| \leq 3|\rho| \gamma \int_0^T \int_0^T \xi_{t_0}^\gamma(s) |\xi_{t_0}(t)| e^{\frac{\gamma^2}{2} (R_{t_0}(s,t) - \bar{z}_T)(t-s)^{-\frac{1}{2}}} ds dt
\]

\[
\leq 3|\rho| \gamma \int_0^T \int_0^T \xi_{t_0}^\gamma(s) |\xi_{t_0}(t)| e^{\frac{\gamma^2}{2} (R(s,t) - \bar{z}_T)(t-s)^{-\frac{1}{2}}} ds dt
\]

\[
\leq 3\xi_{t_0}^*(s) \bar{\xi}_{t_0}(t) |\rho| \gamma \int_0^T \int_0^t e^{\frac{\gamma^2}{2} (1+\gamma^2) \log \frac{1}{s} + \frac{1}{2} \gamma^2 \bar{g} ds dt
\]

\[
\leq \text{const.} \cdot E(M_{\sqrt{(1+\gamma^2)^2}(0,T)})^2 < \infty
\]

for \( \gamma \in (0,1) \), where \( R_{t_0}(s,t) = \mathbb{E}_{t_0}(Z_tZ_{t_0}) = \int_0^t (s-u) - \frac{1}{2} (t-u) - \frac{1}{2} duds, \bar{g} = 2 \log(2\sqrt{2}), \bar{\xi}_t = \sup_{0 \leq s \leq t} \xi_s \). The result then follows from the dominated convergence theorem.