

# The QGARCH(1,1) model

(updated 20/07/2024). The QGARCH(1,1) model is a well known discrete-time model defined as

$$\begin{aligned} R_t &= \sqrt{V_t} Z_t \\ V_t &= \omega + \alpha R_{t-1}^2 + \beta V_{t-1} + \gamma R_{t-1} \end{aligned} \quad (1)$$

for  $t = 1, 2, \dots$  (e.g. days) where  $R_t = (S_t - S_{t-1})/S_{t-1}$  is the  $t$ 'th **stock price return** (note  $R_t \geq -1$  since  $S_t \geq 0$ ) and  $\omega, \alpha, \beta > 0$ , and  $Z_t$  is a sequence of i.i.d random variables with zero mean and variance  $\sigma^2$ , e.g.  $N(0, 1)$  or a student  $t$ -distribution with  $\nu$  degrees of freedom if we want fatter tails for which  $\sigma^2 = \frac{\nu}{\nu-2}$ , so we need  $\nu > 2$ . Since we can re-write the model as

$$V_t = V_{t-1} + (1 - \beta)(\bar{\omega} - V_{t-1}) + \alpha R_{t-1}^2 + \gamma R_{t-1} \quad (2)$$

where  $\bar{\omega} = \frac{\omega}{1-\beta}$ , we see that  $1 - \beta$  controls the **mean reversion** speed for  $V$ , and  $\bar{\omega}$  is level around which  $V$  mean reverts.  $\alpha$  controls the extent of **volatility clustering**, i.e. past large volatility giving rise to large future volatility and vice versa, and  $\gamma$  is a **skew term** which captures that squared volatility  $V_t$  tends to increase if  $R_{t-1} < 0$  since usually  $\gamma < 0$  as well so  $\gamma R_{t-1} > 0$  (the so-called **leverage** effect).  $\gamma < 0$  also allows the model to produce negatively skewed non-symmetric implied volatility smiles for European options which are seen in practice, particularly for Index and Equity options. The original Engle&Bollerslev **GARCH** model from 1986 has  $\gamma = 0$ , so the model above is sometimes known as the **asymmetric GARCH** model.

If we now instead say that  $V_{t+1}$  is  $V_t$ , then we can re-write the model in the **Euler-scheme** type form

$$\begin{aligned} S_t &= S_{t-1} + S_{t-1} \sqrt{V_{t-1}} Z_t \\ V_t &= V_{t-1} + (1 - \beta)(\bar{\omega} - V_{t-1}) + \alpha R_{t-1}^2 + \gamma R_{t-1} \\ &= V_{t-1} + (1 - \beta)(\bar{\omega} - V_{t-1}) + \alpha V_{t-1} Z_t^2 + \gamma \sqrt{V_{t-1}} Z_t \end{aligned} \quad (3)$$

then we see that  $(S_t, V_t)$  is **discrete-time Markov process**, since the distribution of  $S_t, V_t$  at time  $t - 1$  depends only on  $(S_{t-1}, V_{t-1})$  and does not require any further history of these two processes (note our original  $V_t$  is now  $V_{t-1}$  here).

Taking expectations in (1), we see that

$$\mathbb{E}(V_t) = \omega + \alpha \mathbb{E}(R_{t-1}^2) + \beta \mathbb{E}(V_{t-1}) + \gamma \mathbb{E}(R_{t-1}).$$

Using the **tower property** of conditional expectations, we can further re-write this as

$$\begin{aligned} \mathbb{E}(V_t) &= \omega + \alpha \mathbb{E}(\mathbb{E}(R_{t-1}^2) | V_{t-1}) + \beta \mathbb{E}(V_{t-1}) + \gamma \mathbb{E}(\mathbb{E}(R_{t-1}) | V_{t-1}) \\ &= \omega + \alpha \mathbb{E}(\sigma^2 V_{t-1}) + \beta \mathbb{E}(V_{t-1}) + 0 \end{aligned} \quad (4)$$

where we have also used that  $\mathbb{E}(R_{t-1}^2 | V_{t-1}) = \mathbb{E}(V_{t-1} Z_{t-1}^2 | V_{t-1}) = V_{t-1} \mathbb{E}(Z_{t-1}^2 | V_{t-1}) = V_{t-1} \mathbb{E}(Z_{t-1}^2) = V_{t-1} \sigma^2$ . For  $V_t$  to have a **stationary distribution**, i.e. for  $V_t$  to have the same distribution for all  $t$ , this clearly requires that  $\mathbb{E}(V_t) = \mathbb{E}(V_{t-1})$ , so we can further re-write (4) as

$$\mathbb{E}(V_t) = \omega + \alpha \sigma^2 \mathbb{E}(V_t) + \beta \mathbb{E}(V_t).$$

and

$$\mathbb{E}(R_t^2) = \mathbb{E}(\mathbb{E}(R_t^2 | V_t)) = \mathbb{E}(V_t).$$

Re-arranging, we see that

$$\mathbb{E}(V_t) = \frac{\omega}{1 - \alpha \sigma^2 - \beta}.$$

Since  $V_t$  cannot be negative, we must have that  $\alpha \sigma^2 + \beta < 1$ , which we call the **stationarity condition**. If  $V$  starts at time zero, then

$$\begin{aligned} \mathbb{E}(V_t) &= \frac{1}{1 - \alpha \sigma^2} (\omega + \beta \mathbb{E}(V_{t-1})) \\ \Rightarrow \mathbb{E}(V_t) - \bar{V} &= \frac{1}{1 - \alpha \sigma^2} (\omega + \beta \mathbb{E}(V_{t-1})) - \bar{V} = \frac{\beta}{1 - \alpha \sigma^2} (\mathbb{E}(V_{t-1}) - \bar{V}) \end{aligned}$$

i.e. a **linear recurrence** relation of the form  $r_t = \alpha r_{t-1}$ , with solution  $r_t = \mathbb{E}(V_t) - \bar{V} = \left(\frac{\beta}{1 - \alpha \sigma^2}\right)^t (V_0 - \bar{V})$ .

Moreover

$$V_t = \omega + \alpha R_{t-1}^2 + \beta V_{t-1} + \gamma R_{t-1} \geq \omega + \alpha R_{t-1}^2 + \gamma R_{t-1}$$

and (using basic calculus) the right-hand side is  $\geq 0$  for all  $R_{t-1}$  if  $\omega \geq \frac{\gamma^2}{4\alpha}$ . This is known as the **positivity condition**.

Let

$$\mathbb{E}(R_t^4) = \mathbb{E}(\mathbb{E}(R_t^4 | \mathcal{F}_{t-1})) = \mathbb{E}(V_t^2 \mathbb{E}(Z_t^4 | \mathcal{F}_{t-1})) = \mathbb{E}(Z_t^4) \mathbb{E}(V_t^2). \quad (5)$$

For  $\gamma = 0$  and  $\sigma = 1$ , we have

$$\begin{aligned} \mathbb{E}(V_t^2) &= (3 + K_\varepsilon) \mathbb{E}(V_t^2) \alpha^2 + 2 \mathbb{E}(R_{t-1}^2 V_{t-1}) \alpha \beta + \mathbb{E}(V_t^2) \beta^2 + 2 \mathbb{E}(V_t) \alpha \omega + 2 \mathbb{E}(V_t) \beta \omega + \omega^2 \\ &= (\dots) + 2 \alpha \beta \mathbb{E}(V_{t-1} \mathbb{E}_{t-2}(R_{t-1}^2)) \\ &= (\dots) + 2 \alpha \beta \mathbb{E}(V_t^2) \end{aligned}$$

Re-arranging the final expression, we see that

$$\mathbb{E}(V_t^2) = \frac{\omega(2 \mathbb{E}(V_t)(\beta + \alpha) + \omega)}{1 - ((3 + K_\varepsilon) \alpha^2 + \beta^2 + 2 \alpha \beta)}.$$

if the denominator is positive.

## Maximum likelihood estimates for the QGARCH parameters

The joint density of  $R_1, \dots, R_n$  can be easily expressed as a product of conditional densities of the returns:

$$L = p(R_1) p(R_2 | R_1) p(R_3 | R_1, R_2) \dots = p(R_1) p(R_2 | V_2) \dots p(R_n | V_n) = \prod_{j=1}^n f\left(\frac{R_j}{\sqrt{V_j}}\right) \frac{1}{\sqrt{V_j}} = p(R_1) p(R_2 | V_2) \dots p(R_n | V_n)$$

where  $f$  is the density of each  $Z_t$  in (1). This is true because

$$\mathbb{P}(R_j \leq x | V_j) = \mathbb{P}(Z_j \leq \frac{x}{\sqrt{V_j}} | V_j) = F\left(\frac{x}{\sqrt{V_j}}\right)$$

where  $F$  is the distribution function of  $Z_t$ . Using *observed* values for  $R_1, \dots, R_n$ , and given parameter values for the model, the values of  $Z_j = \frac{R_j}{\sqrt{V_j}}$  are known as the **residuals** and  $L$  is the likelihood function of  $R_1, \dots, R_n$ . We can then maximize  $L$  over all admissible parameter combinations to compute MLEs for the model parameters  $\omega, \alpha, \beta, \gamma$ , and the parameter(s) for the distribution of each  $Z_t$  (this is conceptually similar to Part 2).

## Goodness-of-fit tests for the residuals

If e.g. we assume  $Z_t \sim N(0, 1)$ , we can then perform standard normality tests like **Kolmogorov Smirnov**, **Shapiro-Wilk**, **Jarque-Bera** or **Andersen-Darling** to test whether the  $Z_t$  values are indeed i.i.d. Normals. Otherwise, if we use a different distribution for  $Z_t$  (e.g. a **t-distribution** with  $\nu$  degrees of freedom which will give the returns fatter tails), we have to transform these back  $Z$  values to Normal RVs before applying these normality tests, using inverse cdfs.

## Estimating $V_0$ from the stock price history

If we assume  $\gamma = 0$  for simplicity, then iterating the definition of  $V_t$  we see that

$$\begin{aligned} V_t &= \omega + \beta V_{t-1} + \alpha R_{t-1}^2 \\ &= \omega + \beta(\omega + \beta V_{t-2} + \alpha R_{t-2}^2) + \alpha R_{t-1}^2 \\ &= \omega + \beta(\omega + \beta(\omega + \beta V_{t-3} + \alpha R_{t-3}^2) + \alpha R_{t-2}^2) + \alpha R_{t-1}^2 \\ &= \omega(1 + \beta + \beta^2 + \dots) + \frac{\alpha}{\beta} \sum_{\tau=1}^{\infty} \beta^\tau R_{t-\tau}^2 = \bar{\omega} + \frac{\alpha}{\beta} \sum_{\tau=1}^{\infty} e^{-b\tau} R_{t-\tau}^2 \end{aligned} \quad (6)$$

where  $b$  is defined by  $\beta = e^{-b}$  and  $\bar{\omega}$  is defined above, and note the first term on the right-hand side is the mean reversion level from above. So we see that the effect of past returns on volatility decays exponentially, and re-doing this computation with  $\gamma \neq 0$ , we find that the last line just changes to

$$V_t = \frac{\omega}{1 - \beta} + \frac{\alpha}{\beta} \sum_{\tau=1}^{\infty} e^{-b\tau} R_{t-\tau}^2 + \frac{\gamma}{\beta} \sum_{\tau=1}^{\infty} e^{-b\tau} R_{t-\tau}.$$

In particular, we also see that

$$V_0 = \bar{\omega} + \frac{\alpha}{\beta} \sum_{\tau=1}^{\infty} e^{-b\tau} R_{-\tau}^2 + \frac{\gamma}{\beta} \sum_{\tau=1}^{\infty} e^{-b\tau} R_{-\tau}$$

so we can estimate  $V_0$  by truncating this sum in practice rather than fitting  $V_0$  as an additional free parameter for the MLE maximization computation described above, since  $V_0$  is already fixed by the history of the returns.

## Stochastic volatility as the diffusive limit of QGARCH

Consider the following variant of the model above:

$$\begin{aligned} S_t &= S_{t-\Delta t} + S_{t-\Delta t} \sqrt{V_{t-\Delta t}} Z_t \\ V_t &= V_{t-\Delta t} + \kappa \theta \Delta t + \frac{\eta}{\sqrt{\Delta t}} (R_t^2 - V_{t-\Delta t} \Delta t) - \kappa V_{t-\Delta t} \Delta t + \gamma R_t \\ &= V_{t-\Delta t} + \kappa (\theta - V_{t-\Delta t}) \Delta t + \frac{\eta}{\sqrt{\Delta t}} V_{t-\Delta t} (Z_t^2 - \Delta t) + \gamma \sqrt{V_{t-\Delta t}} Z_t \\ &= V_{t-\Delta t} + \bar{\kappa} (\bar{\theta} - V_{t-\Delta t}) \Delta t + \frac{\eta}{\sqrt{\Delta t}} R_t^2 + \gamma R_t \end{aligned}$$

for some  $\bar{\kappa}$ ,  $\bar{\theta}$ , with  $Z_1, Z_2, \dots$  i.i.d. as above and  $V_{t-1}$  here is our old  $V_t$ , and now assume  $\text{Var}(Z_t) = \Delta t$  and  $\eta = O(1)$ , and impose that  $\nu > 4$  so  $\mathbb{E}(Z_t^4) < \infty$ , and from the final line we see that  $V_t$  is still of the QGARCH(1,1) form in (3). Then as  $\Delta t \rightarrow 0$ , the model tends to the mean-reverting **Markov stochastic volatility** model:

$$\begin{aligned} dS_t &= S_t \sqrt{V_t} dW_t \\ dV_t &= \kappa (\theta - V_t) dt + 2\eta V_t dB_t + \gamma \sqrt{V_t} dW_t \end{aligned} \quad (7)$$

where  $W$  and  $B$  are standard independent Brownian motions, so we see that the specific form of the distribution of the  $Z_t$ 's does not show up in the  $\Delta t \rightarrow 0$  limit and the independent Brownian motion  $B$  appears almost by magic. When  $\eta$  is larger, the implied volatility smile will be more  $U$ -shaped as a function of strike  $K$ , and will be symmetric as a function of  $x = \log \frac{K}{S_0}$  if  $\gamma = 0$ . If  $\nu$  is smaller, the smile may just be monotonically decreasing as a function of  $K$  over relevant strike ranges.

The limiting model in (7) is hybrid of the well known **Hull-White** and **Heston** models (the well known Heston model has a  $\sqrt{V_t}$  term in it). To see why this is true, we first note that

$$\frac{1}{\sqrt{\Delta t}} \sum_{i=1}^{[nt]} (Z_i^2 - \Delta t) = \sqrt{n} \sum_{i=1}^{[nt]} (\Delta t \tilde{Z}_i^2 - \Delta t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} (\tilde{Z}_i^2 - 1) \quad (8)$$

where  $\tilde{Z}_i = Z_t / \sqrt{\Delta t} \sim N(0, 1)$ , and that  $\text{Var}(\tilde{Z}_i^2 - 1) = \mathbb{E}((\tilde{Z}_i^2 - 1)^2) = 3 - 2 + 1 = 2$ .

We now recall **Donsker's theorem**. Let  $X_i$  be a sequence of i.i.d. random variables with  $\mathbb{E}(X_i) = 0$  and  $\text{Var}(X_i) = 1$ , and let  $S_n = \sum_{i=1}^n X_i$ . Now consider the **random function**:

$$W_t^n = \frac{S_{[nt]}}{\sqrt{n}} \quad (t \in [0, 1])$$

where  $[nt]$  denotes the largest integer less than or equal to  $nt$ . Then by the **Central Limit Theorem**,  $W_1^n = \frac{S_n}{\sqrt{n}}$  tends to an  $N(0, 1)$  random variable as  $n \rightarrow \infty$ . More precisely,  $\lim_{n \rightarrow \infty} \mathbb{E}(F(W_1^n)) = \mathbb{E}(F(Z))$  for any bounded continuous function  $F$  (this is known as **weak convergence**). Donsker's theorem, states that the random function  $W_t^n$  tends weakly to a random function which is a Brownian motion as  $n \rightarrow \infty$ . This shows that we can numerically approximate Brownian motion using  $X_i$ 's with any distribution with finite variance. Thus (8) falls exactly under the framework of Donsker's theorem, aside from  $\tilde{Z}_i^2 - 1$  having a variance of 2 not 1, which is why there is a **factor of 2** in (7).

## Changing from $\mathbb{P}$ to $\mathbb{Q}$ measure

If the  $Z_t$ 's have a non-zero density under  $\mathbb{P}$ , then the  $Z_t$ 's can have any non-zero density under  $\mathbb{Q}$  (does not have to be equal to the original density), so long as  $\mathbb{E}^{\mathbb{Q}}(Z_t) = 0$ , then  $S$  will still be a martingale under  $\mathbb{Q}$ , which is equivalent to  $\mathbb{P}$  since both densities are non-zero by assumption.

## Intraday dynamics consistent with the QGARCH model

The  $t$ -distribution is infinitely divisible which means a random variable  $Z$  with this distribution can be written as a sum of  $n$  i.i.d random variables  $Z_i^n$ , for any  $n$ . The characteristic function  $\mathbb{E}(e^{iuZ_i^n})$  of  $Z_i^n$  is then  $\phi(u)^{1/n}$  where  $\phi(u) = \mathbb{E}(e^{iuZ})$ . This gives us a way to extend the model from modelling daily returns to intraday returns with  $n$  i.i.d residuals per day, keeping  $V$  constant within any given day.

### Bayesian analysis

If we set  $X = (R_1, \dots, R_n)$  and  $\theta = (\alpha, \beta, \gamma, \nu)$ , then from Bayes formula, we know that

$$p(\theta|X) = \frac{p(X|\theta)p(\theta)}{p(X)}$$

where the  $p$ 's refer to densities or conditional densities here.  $p(X)$  does not depend on  $\theta$ , and if assume a uniform prior  $p(\theta) = \text{const.}$  for  $\theta$  on some finite hypercube in  $\mathbb{R}^4$  (and zero elsewhere), then

$$p(\theta|X) = \text{const.} \times p(X|\theta)$$

so the conditional density of  $\theta$  given  $X$  is proportional to the likelihood function  $p(X|\theta)$ , and by integrating in the other 3 parameters we can compute e.g. the marginal density of  $\alpha$ ,  $\beta$ ,  $\gamma$  or  $\nu$  given  $X$ . This is easier if e.g. we fix  $\gamma = 0$  and fix  $1 - \beta$  to its lower bound, so we only have two free parameters.

### Power kernel model

We can modify the model as follows:

$$\begin{aligned} R_t &= \sqrt{V_t} Z_t \\ V_t &= \omega + c \sum_{\tau=1}^{\infty} \tau^{-\alpha} R_{t-\tau}^2 + \gamma \sum_{\tau=1}^{\infty} \tau^{-\alpha_2} R_{t-\tau} \end{aligned}$$

for  $\alpha, \alpha_2 > 2$  (add mean reversion?) which corresponds to **power decay**, and again we have to take care to ensure positivity and stationarity. In this case, using the same tower law argument as above

$$\mathbb{E}(V_t) = \omega + c \sum_{\tau=1}^{\infty} \tau^{-\alpha} \mathbb{E}(R_{t-\tau}^2) = \omega + c \sum_{\tau=1}^{\infty} \tau^{-\alpha} \mathbb{E}(\mathbb{E}(R_{t-\tau}^2 | V_{t-\tau})) = \omega + c\sigma^2 \sum_{\tau=1}^{\infty} \tau^{-\alpha} \mathbb{E}(V_{t-\tau}).$$

If  $V$  is stationary, then

$$\mathbb{E}(V_t) = \omega + c\sigma^2 \sum_{\tau=1}^{\infty} \tau^{-\alpha} \mathbb{E}(V_t) = \omega + c\sigma^2 \mathbb{E}(V_t) \zeta(\alpha)$$

which we can re-arrange as  $\mathbb{E}(V_t) = \frac{\omega}{1 - c\sigma^2 \zeta(\alpha)}$ , where  $\zeta(\alpha) = \sum_{n=1}^{\infty} n^{-\alpha}$  denotes the **zeta function**, so clearly a necessary condition for stationarity is that  $c\sigma^2 \zeta(\alpha) < 1$ .

If  $\alpha = \alpha_2$ , then can re-write as

$$V_t = \sum_{\tau=1}^{\infty} \tau^{-\alpha} (\bar{\omega} + cR_{t-\tau}^2 + \gamma R_{t-\tau})$$

where  $\bar{\omega} = \frac{\omega}{\zeta(\alpha)}$ , so we have essentially the same **positivity condition** as before  $\bar{\omega} \geq \frac{\gamma^2}{4c}$ . This is a discrete-time version of the **rough Heston model**.

### Quadratic Rough Heston-type model

We can also generalize to a quadratic rough Heston-type model:

$$V_t = \omega + c \sum_{\tau=1}^{\infty} \tau^{-\alpha} R_{t-\tau}^2 + b \left( \sum_{\tau=1}^{\infty} \tau^{-\alpha} R_{t-\tau} - a \right)^2 + \gamma \sum_{\tau=1}^{\infty} \tau^{-\alpha} R_{t-\tau}.$$

Then again assuming stationarity, we now see that

$$\begin{aligned}
\mathbb{E}(V_t) &= \omega + c\sigma^2\zeta(\alpha)\mathbb{E}(V_t) + b\mathbb{E}\left(\sum_{\tau=1}^{\infty} \tau^{-\alpha} R_{t-\tau} - a\right)^2 \\
&= \omega + c\sigma^2\zeta(\alpha)\mathbb{E}(V_t) + b\mathbb{E}\left(\sum_{\tau=1}^{\infty} \tau^{-\alpha} R_{t-\tau}\right)^2 + a^2 \\
&= \omega + c\sigma^2\zeta(\alpha)\mathbb{E}(V_t) + b(\zeta(2\alpha)\mathbb{E}(V_t) + a^2)
\end{aligned}$$

using that  $\mathbb{E}(R_i R_j) = \mathbb{E}(R_i \mathbb{E}(R_j | R_i, V_j)) = 0$  for  $i < j$ , so the stationarity condition now reads as  $c\sigma^2\zeta(\alpha) + b(\zeta(2\alpha) < 1$ .

## Numerical results

Below we compute MLEs and apply the Kolmogorov Smirnov, Shapiro-Wilk and Jarque-Bera normality tests on the (transformed) residuals implied by the MLEs for the model in (1) using daily prices, with a 1yr/3yr/1yr test window (the initial 1yr window is used to compute the  $V_0$  for the middle window from the initial 1yr history of returns; the middle 3yr period is used for in-sample (i/s) testing, and final year used for out-of-sample testing, all three periods are consecutive with no gaps/overlap), ending 11/08/2023. Although the fits are very good, the sample variance of the MLEs using synthetic paths with the fitted parameters are much higher than we would ideally like.

MLEs/ <i>p</i> -vals	$\alpha$	$\beta$	$\gamma$	$\nu$	KS i/s	SW i/s	JB i/s	KS o/s	SW o/s	JB o/s
EUR/USD	0.0293	0.962	-5.405e-05	8.684	0.835	0.870	0.706	0.912	0.714	0.643
GBP/USD	0.0303	0.932	-0.000252	6.192	0.966	0.836	0.712	0.119	0.224	0.279
USD/JPY	0.0830	0.875	-0.000299	5.9611	0.292	0.476	0.352	0.0603	0.0907	0.229
AMZN	0.03482	0.9420	-0.000505	5.008	0.401	0.811	0.951	0.560	0.607	0.570
BRK-B	0.103	0.868	-0.00103	8.929	0.168	0.921	0.950	0.611	0.676	0.984
INTC	0.0280	0.943	-5.940e-05	3.914	0.375	0.0634	0.0404	0.229	0.262	0.236
AZN	0.0496	0.904	-0.000897	4.153	0.247	0.587	0.428	0.103	0.206	0.195
N225	0.0982	0.856	-0.00129	6.271	0.281	0.443	0.349	0.0713	0.236	0.354
HSI	0.06222	0.898	-0.000834	5.108	0.491	0.226	0.358	0.530	0.121	0.161

To fix SPX historical prices well, we need a skewed  $t$ -distribution for the residuals