The large-maturity smile for the Stein-Stein model

Martin Forde*
Dept. Mathematics
King’s College London,
4th April 2014

Abstract: We compute the large-maturity smile for the correlated Stein-Stein stochastic volatility model \(dS_t = S_tY_t dW_t^1, dY_t = \kappa(\theta - Y_t)dt + \sigma dW_t^2\), \(dW_t^1 dW_t^2 = \rho dt\), using the known closed-form solution for the characteristic function of the log stock price given in Schöbel&Zhu[SZ99]. The Stein-Stein model is not covered by the results in [FK13] and [JKRM13] because the volatility fails to satisfy the sublinear growth condition in [FK13] and is not an affine model.†

1. Introduction

The last few years have witnessed a number articles on large-time asymptotics for stochastic volatility models with/without a jump component. Using the Gärtner-Ellis theorem, [FJ11] compute the implied volatility smile for the popular Heston stochastic volatility model when \(\kappa > 0, \kappa > \rho \sigma\), in the large-time, large log-moneyness regime and [FJM10] compute the correction term using saddlepoint methods; the large-time smile is identical to the large-time smile for the Barndorff-Nielsen Normal Inverse Gaussian model, and [GJ11] show that the asymptotic smile can be computed in closed-form via the Gatheral SVI parameterization. [JM12] derive similar results for a displaced Heston model, and relax the aforementioned conditions on \(\kappa, \rho, \sigma\). [JKRM13] have extended the results in [FJ11] to a general class of affine stochastic volatility models (with jumps), which includes the Heston model with state-independent jumps, the Bates model with state-dependent jumps and the Barndorff-Nielsen-Shephard model.

[FP12] compute large-time asymptotics for the SABR model with \(\beta = 1, \rho \leq 0\) and \(\beta < 1, \rho = 0\); in particular for \(\beta = 1, \rho \leq 0\), they compute a closed-form expression for the asymptotic log stock price density and establish large-time asymptotics for the CEV model and the uncorrelated CEV-Heston model in the large-time, fixed-strike regime and a new large-time, large log-moneyness regime. [Forde11b] derives similar results for the modified SABR model in terms of the large-time asymptotic density of the Brownian exponential functional.

The long-term asymptotic behavior of the smile for exponential Lévy models and more general martingale models have been studied in [RT10], where it is proved that for fixed log-moneyness \(k\) and large maturity, the implied volatility converges to a constant value that does not depend on \(k\). This phenomenon is typically referred as the “smile-flattening” effect, which arises from the large deviation principle for i.i.d. random variables (see e.g. Cramér’s theorem in [DZ98]). For a general exponential Lévy model with mild conditions on the cumulant generating function, [GL11] derive an expansion of the form \(\sigma_t(x)^2 = \sigma_\infty^2 + a_1(x)/t + a_2(x)/t^2 + o((\log t)^2/t^3)\) as \(t \to \infty\) for the implied volatility \(\sigma_t(x)\) at log-moneyness \(x\) and maturity \(t\), where \(a_1(x)\) and \(a_2(x)\) are respectively affine and quadratic in \(x\).

In [Forde11], the author derives a large deviation principle for the log stock price under an uncorrelated stochastic volatility model driven by an Ornstein-Uhlenbeck process with a bounded volatility function. For this we use the fact that the occupation measure for the Ornstein-Uhlenbeck process satisfies an LDP with a good, convex lower semicontinuous rate function under the topology of weak convergence (and under the Prohorov metric), see section 7 in Donsker&Varadhan[DV76] (see also page 178 in Stroock[Str84] and [Pin85]), combined with the standard contraction principle and exponential tightness. In [FK13], we relax the assumptions of bounded volatility and zero correlation made in [Forde11]. The rate function for \(X_t/t\) now has the variational representation \(I(x) = \inf_{\mu \in P(\mathbb{R})} \frac{(x-M(\mu))^2}{2\nu(\mu)} + I_\alpha(\mu)\), for some linear functionals \(M, \nu\) which depend on the correlation \(\rho\). Using the LDP, we translate these results into large-time asymptotics for call options and implied volatility, and we extend the analysis to incorporate stochastic interest rates, by deriving a similar LDP for a three-factor model driven a CIR short rate process.

In this article, we look at the large-time behavior of the closed form expression for the characteristic function of the log stock price under the Stein-Stein model introduced in [SS91], which is derived in [SZ99]. Using the Gärtner-Ellis

---

*martin.forde@kcl.ac.uk
†We thank Rohini Kumar for insightful comments.
Theorem from large deviations theory, we compute a large-time large deviation principle for the log stock price. From this we can then characterize the large-time behavior of call option prices and implied volatility in the large-time, large log-moneyness regime. The Stein-Stein model reduces to a special case of the Heston model when the mean reversion level \( \theta = 0 \). We refer the reader to Deuschel et al.\[DFJV14\] for a discussion on tail asymptotics for the Stein-Stein model using Laplace’s method on Wiener space for a small-noise diffusion process and some simple scaling properties, and the earlier work on tail asymptotics for the zero correlation case in Gulisashvili&Stein\[GS10\].

2. Large deviation theory and the Gärtner-Ellis theorem

In this section, we recall some fundamental notions in large deviations theory (we refer the reader to Section 2.3 in \[DZ98\] and Section 2.2 in \[JM12\] for more details). A family of random variables \((Z_t)\) is said to satisfy the large deviation principle (LDP) as \( t \to \infty \) with good rate function \( I \) if for all \( B \in \mathcal{B}(\mathbb{R}) \) we have the following bounds

\[
- \inf_{x \in B^*} I(x) \leq \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{P}(Z_t \in B) \leq \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}(Z_t \in B) \leq - \inf_{x \in \bar{B}} I(x),
\]

where \( B^o (\bar{B}) \) denotes the interior (resp. closure) of \( B \).

We now assume that the cumulant generating function \( V_t(p) = \log \mathbb{E}(e^{p Z_t}) \) is finite on some neighborhood of zero and that the following limit exists as an extended real number

\[
V(p) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(e^{p Z_t}) \quad \forall p \in \mathbb{R}. \tag{2.1}
\]

Let \( \mathcal{D}_V = \{ p \in \mathbb{R} : |V(p)| < \infty \} \) and assume that \( \{0\} \in \mathcal{D}_V^c \). From Hölder’s inequality we can show that \( V_t \) is convex for all \( t > 0 \) and the limit \( V \) is also convex (see Lemma 2.3.9 in \[DZ98\]). Moreover \( V(0) = 0 \), thus (by convexity) we see that \( V(p) > -\infty \) for all \( p \in \mathbb{R} \). \( V : \mathbb{R} \to (-\infty, \infty) \) is called essentially smooth if \( V \) is differentiable in \( \mathcal{D}_V \) and satisfies \( \lim_{n \to \infty} |V'(p_n)| = \infty \) for every sequence \( (p_n) \) in \( \mathcal{D}_V^c \) which converges to a boundary point of \( \mathcal{D}_V^c \). A cgf \( V \) which satisfies this second property is called steep. The Fenchel-Legendre transform \( V^* \) of \( V \) is defined by the variational formula

\[
V^*(x) = \sup_{p \in \mathbb{R}} [px - V(p)]
\]

for all \( x \in \mathbb{R} \), with an effective domain \( \mathcal{D}_{V^*} = \{ x \in \mathbb{R} : V^*(x) < \infty \} \). In general \( V^* \) can be discontinuous and \( \mathcal{D}_{V^*} \) can be a strict subset of \( \mathbb{R} \) (see section 2.3 in \[DZ98\] for some simple examples).

We now state a simplified version of Gärtner-Ellis theorem (c.f. Theorem 2.3.6 in \[DZ98\]) which will needed in the next section.

**Theorem 2.1.** Let \((Z_t)_{t \geq 0}\) be a family of random variables for which \( V \) as defined in (2.1) satisfies \( \{0\} \in \mathcal{D}_V^c \). If \( V \) is essentially smooth and lower semicontinuous, then the LDP holds with good rate function \( V^* \).

3. The Stein-Stein model

From here on, we work on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \((\mathcal{F}_t)_{t \geq 0}\) throughout, supporting two independent Brownian motions and satisfying the usual conditions. We now recall the Stein-Stein stochastic volatility model for a log stock or forward price process \( X_t = \log S_t \):

\[
\begin{cases}
    dX_t = -\frac{1}{2} Y_t^2 \, dt + Y_t \, dW_t^1, \\
    dY_t = \kappa (\theta - Y_t) \, dt + \sigma dW_t^2
\end{cases} \tag{3.1}
\]

where \( \kappa, \sigma > 0, X_0 = x_0, Y_0 = y_0, \) and \( W^1, W^2 \) are Brownian motions such that \( dW_t^1 dW_t^2 = \rho dt, |\rho| < 1 \). The law of \( X_t - x_0 \) does not depend on \( x_0 \), so without loss of generality we set \( X_0 = 0 \).

We first verify the martingale property for \( S_t \).
Proposition 3.1. \((S_i)\) is a martingale.

Prove. Let \(0 < t_1 < t_2 < \infty\). We know that \(\sup_{t \geq 0} \mathbb{E}(e^{Y_0^2}) < \infty\) if \(c < \kappa/\sigma^2\), using that \(Y_t \sim N(e^{-\kappa t}y_0 + \theta(1 - e^{-\kappa t}), \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}))\). Now consider a uniform random variable \(U\) on \([t_1, t_2]\), independent of \(S\), and let \(\mathcal{F}_t^U = \sigma(Y_s; 0 \leq s \leq t)\) denote the filtration generated by the \(Y\) process. Then we have

\[
\mathbb{E}(e^{\frac{1}{2} J_{t_1}^t Y_s^2 dt}) = \mathbb{E}(e^{\frac{1}{2}(t_2-t_1)\mathbb{E}(Y_s^2 | \mathcal{F}_t^U)}) \\
\leq \mathbb{E}(\mathbb{E}(e^{\frac{1}{2}(t_2-t_1)Y_s^2} | \mathcal{F}_t^U)) \\
(\text{using the conditional Jensen’s inequality}) \\
= \mathbb{E}(\frac{1}{t_2-t_1} \int_{t_1}^{t_2} e^{\frac{1}{2}(t_2-t_1)Y_s^2} ds) < \infty \\
= \frac{1}{t_2-t_1} \int_{t_1}^{t_2} e^{\frac{1}{2}(t_2-t_1)\mathbb{E}(Y_s^2)} ds < \infty \\
(\text{by Fubini’s theorem})
\]

for \(\frac{1}{2}(t_2-t_1) \leq \kappa/\sigma^2\). By Corollary 5.14, p.199 in [KS91], we conclude that \(S_t = e^{-\frac{1}{2} \int_0^t Y_s^2 ds + \int_0^t Y_s dW_s}\) is a martingale. \(\Box\)

3.1. The large-time large deviation principle for the re-scaled log return

The following proposition establishes a large-time large deviation principle for the re-scaled log return for the Stein-Stein model:

Proposition 3.2. \(X_t/t\) satisfies a large-time LDP as \(t \to \infty\) with a good convex continuous rate function given by the Fenchel-Legendre transform

\[
I(x) = \sup_{p} \{px - V(p)\}
\]

where

\[
V(p) = V(p; \kappa, \theta, \sigma, \rho) = \begin{cases} \frac{1}{2} [\kappa - p\rho + \frac{(p-1)p\rho^2\kappa^2}{4\rho^2}] - \Gamma(p) & (p \in (p_-, p_+)) \\
+\infty & (p \notin (p_-, p_+)) \end{cases}
\]

\(\rho = \sqrt{1 - \rho^2}, \Gamma(p) = \sqrt{\kappa^2 - 2\rho\kappa\rho + p(1 - p\rho^2)}\) and \(p_\pm = \frac{\sigma^2 - 2\kappa\rho \pm \sqrt{4\kappa^2 - 4\kappa^2\rho^2 + \sigma^2}}{2\rho^2}\) are the roots of \(\Gamma(p)^2\). \(I\) is continuous and attains its minimum uniquely at \(x^* = V'(0) = -\frac{1}{2}(\theta^2 + \sigma^2/2\kappa)\).

Proof. From Eq 13 in [SZ99], we have the following closed-form expression for the characteristic function of the log return

\[
\phi_t(u) = \mathbb{E}(e^{iuX_t}) = e^{-\frac{1}{2}iu\rho(\sigma^2y^2 + \sigma^2) + \frac{1}{2}D(t, \hat{s}_1, \hat{s}_2) y^2 + B(t, \hat{s}_1, \hat{s}_2) y + C(t, \hat{s}_1, \hat{s}_2)}
\]

for \(u \in \mathbb{R}\), where

\[
D(t, T) = \frac{1}{\sigma^2} [\kappa - \gamma_1 \sinh(\gamma_1 t) + \gamma_2 \cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t)]
\]

\[
B(t, T) = \frac{1}{\sigma^2\gamma_1} \kappa \theta \gamma_1 - \gamma_2 \gamma_3 + \gamma_3 [\sinh(\gamma_1 t) + \gamma_2 \cosh(\gamma_1 t)]
\]

\[
C(t, T) = -\frac{1}{2} \log [\cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t)] + \frac{1}{2} \sigma^2 t + \frac{\kappa^2 \gamma_1^2 - \gamma_3^2}{2\sigma^2 \gamma_1} \left( \frac{\sinh(\gamma_1 t)}{\cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t)} - \gamma_1 t \right)
\]

\[
+ \frac{(\kappa \theta \gamma_1 - \gamma_2 \gamma_3) \gamma_3}{\sigma^2 \gamma_1^3} \left( \frac{\cosh(\gamma_1 t) - \gamma_1 t}{\cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t)} \right)
\]

where \(y = Y_0\) and \(\hat{s}_1 = \frac{1}{2} u^2 \rho^2 + \frac{1}{4} i u (1 - 2 \kappa \rho / \sigma), \hat{s}_2 = i u k \theta \rho^{-1}, \hat{s}_3 = \frac{1}{2} i u \kappa \rho^{-1}\). \(\phi_t(u)\) is regular in a neighborhood of the origin, so by Theorem 7.1.1 in Lukacs [Luk70], \(\phi_t(u)\) is also regular in the horizontal strip \(\{u \in \mathbb{C} : p_{-}(t) < u < p_{+}(t)\}\), where

\[
p_{+}(t) = \sup_{p \geq 1} \mathbb{E}(e^{p Y_t}) < \infty,
\]

\[
p_{-}(t) = \inf_{p \leq 0} \mathbb{E}(e^{p Y_t}) < \infty
\]
Note that $p_\pm(t)$ is not the same as $p_\pm$ as defined in the statement of the proposition, and we will show that $p_-(t) \leq p_-$ and $p_+ \leq p_+(t)$ (see discussion above (3.6)).

Looking at the expressions for $B, C, D$ on page 12 in [SZ99], we see that $\phi_t(u)$ has a pole at $u = -ip$ if and only if
\[
cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t) = 0.
\]
For $p \in (p_-, p_+)$ i.e. such that $\Gamma(p) > 0$, using that $\gamma_1 = \Gamma(p)$ and $\gamma_2 = -\Gamma(p)/(\kappa - \rho p\sigma)$, this equation is satisfied if $t = t^*(p) = \frac{1}{\gamma_1} \tanh^{-1}(\frac{1}{\gamma_2}) = \frac{1}{\Gamma(p)} \tanh^{-1}(\frac{\Gamma(p)}{\kappa - \rho p\sigma})$. But negative $t$-values are physically meaningless, so our preliminary analysis would indicate that
\[
T^*(p) = \begin{cases} \frac{1}{\gamma_1} \tanh^{-1}(\frac{1}{\gamma_2}) = \frac{1}{\Gamma(p)} \tanh^{-1}(\frac{\Gamma(p)}{\kappa - \rho p\sigma}) & (\kappa - \rho p\sigma < 0) \\
+\infty & (\kappa - \rho p\sigma \geq 0)
\end{cases}
\]
where $T^*(p) = \sup\{t : \mathbb{E}(e^{pX_t}) < \infty\}$ is the moment explosion time. We now first consider the case when $p > 1$. In this case, if $\rho \leq 0$ then $\kappa - \rho p\sigma > 0$ and for $p \in (p_-, p_+)$ we have that $\Gamma(p) > 0$, so $T^*(p) = +\infty$. Otherwise, if $\rho > 0$, then $\kappa - \rho p\sigma < 0$ if $p > p^*$ where $p^* = \kappa/\rho$. However
\[
p^* - p_+ = \frac{2\kappa - \rho \sigma - \rho \sqrt{4\kappa^2 - 4\kappa \rho \sigma + \sigma^2}}{2\rho \sigma^2},
\]
and using that
\[
(2\kappa - \rho \sigma)^2 - \rho^2(4\kappa^2 - 4\kappa \rho \sigma + \sigma^2) = 4\kappa \rho^2(\kappa - \rho \sigma) > 0
\]
we see that $p^* > p_+$, so it turns out that $T^*(p) = \infty$ for all $p \in (1, p_+)$. An almost identical calculation shows that $T^*(p) = \infty$ for all $p \in (p_-, 0)$. Moreover, for $p \in [0, 1]$, from Jensen’s inequality and the martingale property we have that $\mathbb{E}(S_t^p) \leq S_0^p < \infty$ for all $t$. Thus we have shown that $T^*(p) = \infty$ for all $p \in (p_-, p_+)$, so the mgf of $X_t$ is given by the analytic extension of $\phi$ to the imaginary axis for $p \in (p_-, p_+)$. The expression for $C(.)$ in [SZ99] is given by
\[
C(t, \hat{s}_1, \hat{s}_2, \hat{s}_3) = -\frac{1}{2} \log[cosh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t) + \frac{1}{2} \kappa t] + \frac{\kappa^2 \theta^2 - \gamma_1^2 - \gamma_2^2}{2^2 \gamma_1^2} \sinh(\gamma_1 t) + \gamma_2 \sinh(\gamma_1 t) - \frac{\Gamma(p)}{\Gamma(p)^2} - \Gamma(p) t \left[V(p) + \frac{1}{2} \rho \sigma \right] (t \to \infty).
\]
Letting $t \to \infty$ and using that $\cosh(\gamma_1 t) \to 1$ as $t \to \infty$, we also find that
\[
B(t, \hat{s}_1, \hat{s}_2, \hat{s}_3) \sim \frac{1}{\sigma^2 \gamma_1} \gamma_3 - \kappa \gamma_1 = O(1) \quad (t \to \infty),
\]
\[
D(t, \hat{s}_1, \hat{s}_2, \hat{s}_3) \sim \frac{1}{\sigma^2} (\kappa - \gamma_1) = O(1) \quad (t \to \infty),
\]
and thus constitute higher order terms as $t \to \infty$, which we can ignore at the order we are interested in. Thus we have
\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(e^{pX_t}) = V(p)
\]
for $p \in (p_-, p_+)$. This means that for $p \in (p_-, p_+)$ and $t < \infty$ fixed, we have $\mathbb{E}(e^{pX_t}) < \infty$, so
\[
p_-(t) \leq p_-, \quad p_+(t) \geq p_+.
\]
We now consider \( p \geq p_+ \). To this end we fix a \( q \in (1, p_+) \); then from the monotonicity of the \( L^p \) norm we have
\[
(\mathbb{E}(e^{qX_t}))^{1/q} \leq (\mathbb{E}(e^{pX_t}))^{1/p}.
\]
From this and (3.5) we obtain
\[
V(q) = \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{qX_t}) \leq \liminf_{t\to\infty} \frac{1}{t} \log (\mathbb{E}(e^{pX_t}))^{q/p}.
\]
But \( \forall K > 0 \), there exists a \( q(K) < p_+ \) such that \( V(q) \geq K \). Thus for \( t \) sufficiently large we have
\[
(e^{Kt})^{p/q(K)} \leq \mathbb{E}(e^{pX_t})
\]
or
\[
K \leq K - \frac{p}{q(K)} \leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{pX_t}).
\]
Thus letting \( K \to \infty \) we see that \( \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{pX_t}) = +\infty \). A similar analysis shows that \( \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{pX_t}) = +\infty \) for \( p \leq p_- \).

Differentiating \( V(p) \) we obtain
\[
V'(p) = \frac{(2p-1)\theta^2\kappa^2 + 2(1-p)p\theta^2\kappa^2}\Gamma'(p) - \frac{1}{2} (\rho\sigma + \Gamma'(p))
\]
and
\[
\Gamma'(p) = \frac{\sigma(-2\kappa\rho + (1 + 2p(-1 + \rho^2))\sigma)}{2\Gamma(p)}.
\]

Nothing that \( \Gamma(p_{\pm}) = 0 \), we see that \( V(p) \) and \( |V'(p)| \to +\infty \) as \( p \to p_{\pm} \). So \( V \) is essentially smooth, and \( V \) is lower semicontinuous. \( V_t(p) = \mathbb{E}(e^{pX_t}) \) satisfies Assumption 2.3.2 in [DZ98] as \( t \to \infty \), so by Lemma 2.3.9 in [DZ98] \( V \) is also convex, so from the Gärtner-Ellis Theorem (see Theorem 2.3.6 in [DZ98]) \( X_t/t \) satisfies the LDP with good convex rate function \( I(x) \).

We also have the upper bound
\[
I(x) \leq p_+ x \vee p_- x - V_{\min} < \infty
\]
where \( V_{\min} = \inf_{p \in [p_- \ldots p_+]} V(p) > -\infty \). But a convex function is continuous on the interior of its domain, so \( I \) is continuous. Finally, from elementary calculations we find that the unique minimum of \( I \) occurs at \( x^* = (I')^{-1}(0) = V'(0) \).

### 4. Call options and implied volatility

Let \( \mathbb{P}^*(A) = \frac{1}{S_0} \mathbb{E}(S_1 1_A) \) for \( A \in \mathcal{F}_t \) denote the Share measure (\( \mathbb{P}^* \) is a probability measure because \( S_t \) is a martingale by Proposition 3.1). From Girsanov’s theorem, it is easily shown that
\[
\begin{align*}
\{ d(-X_t) &= -\frac{1}{2} Y_t^2 dt - Y_t dW_t^1, \\
\quad dY_t &= [\kappa(\bar{\theta} - Y_t) + \rho\sigma Y_t] dt + \sigma dW_t^2 \\
&= \kappa(\bar{\theta} - Y_t) dt + \sigma dW_t^2,
\}
\tag{4.1}
\end{align*}
\]
where \( \bar{\kappa} = \kappa - \rho\sigma \), \( \bar{\theta} = \kappa\theta / (\kappa - \rho\sigma) \) and \( dW_t^1 dW_t^2 = \rho dt \) are independent \( \mathbb{P}^* \)-Brownian motions.

**Assumption 4.1.** From here on we further assume that \( \bar{\kappa} = \kappa - \rho\sigma > 0 \), which ensures that \( Y_t \) is mean-reverting under \( \mathbb{P}^* \).

From (4.1), we have the following trivial corollary of Proposition 3.2.
Fig 1. Here we have plotted $V(p)$ for $\kappa = 1.15, \theta = 0.1, \sigma = 0.2, \rho = -0.4$.

Fig 2. Here we have plotted the rate function $I(x)$ for the same parameter values as above.
Corollary 4.2. For \( \kappa > \rho \sigma \), \(-X_t/t\) satisfies the LDP under \( \mathbb{P}^* \) as \( t \to \infty \) with a good convex continuous rate function \( I_S(x) \) given by the Fenchel-Legendre transform of 

\[
V_S(p) = V(p; \tilde{\kappa}, \tilde{\theta}, \sigma, -\rho)
\]

and \( I_S \) is continuous and attains its minimum value uniquely at \(-x^+ = (V_S)'(0) = -\frac{1}{2}(\tilde{\theta}^2 + \sigma^2/2\tilde{\kappa})\).

By Corollary 4.2 and the continuity of the rate function \( I_S \), we obtain the following corollary, which will be used to characterize the large-time behaviour of call option prices.

Corollary 4.3.

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}^*(X_t > xt) = -I_S(x) \quad \text{for all} \quad x > x^+,
\]

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}^*(X_t < xt) = -I_S(x) \quad \text{for all} \quad x < x^+.
\]

Recall that the payoff of a European call option with strike \( K \) is \( \mathbb{E}(S_t - K)^+ \), and the payoff of a European put option with strike \( K \) is \( \mathbb{E}(K - S_t)^+ \).

Corollary 4.4. We have the following large-time asymptotic behaviour for put/call options in the large-time, large log-moneyness regime:

\[
- \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(S_t - S_0 e^{xt})^+ = I_S(x) \quad (x \geq x^+),
\]

\[
- \lim_{t \to \infty} \frac{1}{t} \log [\mathbb{E}(S_t - S_0 e^{xt})^+] = I_S(x) \quad (x^* \leq x \leq x^+),
\]

\[
- \lim_{t \to \infty} \frac{1}{t} \log(\mathbb{E}(S_0 e^{xt} - S_t)^+) = I_S(x) \quad (x \leq x^*).
\]

Proof. We first assume \( x > x^+ \), and recall that \( I_S(x) \) is non-decreasing for \( x > x^+ \). From Corollary 4.3, we know that for all \( \varepsilon > 0 \) there exists a \( t^* = t^*(\varepsilon) \) such that for all \( t > t^* \) we have

\[
\frac{1}{S_0} \mathbb{E}(S_t - S_0 e^{xt})^+ = \mathbb{P}^*(X_t > xt) - e^{xt} \mathbb{P}(X_t > xt) \leq \mathbb{P}^*(X_t > xt) \leq e^{-(I_S(x) - \varepsilon)/t}
\]

which gives the upper bound for the call price. For the lower bound we have

\[
\frac{1}{S_0} \mathbb{E}(S_t - S_0 e^{xt})^+ = \mathbb{E}^\mathbb{P}^*(1 - e^{xt} e^{-X_t})^+ = e^{xt} \mathbb{E}^\mathbb{P}^*(e^{-xt} - e^{-X_t})^+.
\]

Observe that for any \( \delta > 0 \),

\[
\mathbb{E}^\mathbb{P}^*(e^{-xt} - e^{-X_t})^+ \geq \mathbb{E}^\mathbb{P}^*[(e^{-xt} - e^{-X_t})^+ \mathbb{1}_{\{X_t < -xt - \delta\}}] \geq (e^{-xt} - e^{-xt-\delta}) \mathbb{P}^*(X_t < -xt - \delta).
\]

Combining this with (4.2) we have

\[
\frac{1}{S_0} \mathbb{E}(S_t - S_0 e^{xt})^+ \geq e^{xt}(e^{-xt} - e^{-xt-\delta}) \mathbb{P}^*(-X_t < -xt - \delta) = (1 - e^{-\delta}) \mathbb{P}^*(-X_t < -xt - \delta) = (1 - e^{-\delta}) \mathbb{P}^*(X_t/t > x + \delta/t) \geq (1 - e^{-\delta}) \mathbb{P}^*(X_t/t > x + \delta).
\]

Using Corollary 4.3 we get

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(S_t - S_0 e^{xt})^+ \geq I_S(x + \delta).
\]

This holds for all \( \delta > 0 \), so taking \( \lim_{\delta \to 0} \) and by the continuity of \( I_S(x) \) we obtain the first result that \( \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(S_t - S_0 e^{xt})^+ = I_S(x) \). The other cases follow similarly.
4.1. Implied volatility

Using the same proofs as in Corollary 1.7 and Corollary 2.17 in [FJ11] for the Heston model (or Theorem 14 in [JKRM13] for a general affine model), we have the following asymptotic behaviour in the large-time, large log-moneyness regime, where $\hat{\sigma}_t(x_t)$ is the implied volatility of a put/call option with strike $S_0e^{x_t}$ for the correlated Stein-Stein model:

$$\hat{\sigma}_\infty(x)^2 = \lim_{t \to \infty} \hat{\sigma}_t^2(x_t) = \begin{cases} 2(2I(x) - x - 2\sqrt{I(x)^2 - I(x)x}) & (x \notin [x^*, x_+]) \\ 2(2I(x) - x + 2\sqrt{I(x)^2 - I(x)x}) & (x \in (x^*, x_+) ) \end{cases}.$$
References


