Large-time option pricing using the Donsker-Varadhan LDP - correlated stochastic volatility with stochastic interest rates and jumps

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Abstract: We establish a large-time large deviation principle (LDP) for a general mean-reverting stochastic volatility model with non-zero correlation and sublinear growth for the volatility coefficient, using the Donsker-Varadhan [DV83] LDP for the occupation measure of a Feller process under mild ergodicity conditions. We verify that these conditions are satisfied when the process driving the volatility is an Ornstein-Uhlenbeck (OU) process with a perturbed (sublinear) drift. We then translate these results into large-time asymptotics for call options and implied volatility and we verify our results numerically using Monte Carlo simulation. Finally we extend our analysis to include a CIR short rate process and an independent driving Lévy process.

1. Introduction

The last few years has seen the emergence of a number of articles on large-time asymptotics for stochastic volatility models, with and without jumps. Using the Gärtner-Ellis theorem from large deviations theory, [FJ11] compute the asymptotic (i.e. leading order) implied volatility smile for the well known Heston model in the so-called large-time, large log-moneyness regime, under a mild restriction on the model parameters, and the rate function is computed numerically as a Fenchel-Legendre transform which is just a one-dimensional root-finding exercise. [GJ11] show that the asymptotic smile can actually be computed in closed-form via the SVI parameterization and [FJM10] compute the correction term to this smile using saddlepoint methods; [Forde14] derives a similar result for the Stein-Stein model and [JM12] derive a similar result for a displaced Heston model (and relax the aforementioned condition on the parameters). [JKRM12] extended the results in [FJ11] to a general class of affine stochastic volatility models (with jumps), which includes the Heston, Bates and the Barndorff-Nielsen-Shephard model, and under mild assumptions, they show that the limiting smile necessarily corresponds to the smile generated by an exponential Lévy model. More recently, [FZ15] compute large-time asymptotics for a fractional local-stochastic volatility model and large-time asymptotics for European and barrier options under conventional and fractional exponential Lévy models, using the deAcosta LDP for a Lévy process on path space.

[Forde11] derives a large deviation principle for the log stock price under an uncorrelated stochastic volatility model driven by an Ornstein-Uhlenbeck process with a bounded volatility function. For this we use the fact that the occupation measure of the Ornstein-Uhlenbeck process satisfies an LDP with a good, convex lower semicontinuous rate function under the topology of weak convergence (and under the Prokhorov metric), see section 7 in Donsker & Varadhan [DV76] (see also page 178 in Stroock [Str84] and Proposition 1.3 in [KM05]), combined with the standard contraction principle and exponential tightness. The large-time regime is also closely related to the small-time, fast mean reverting regime considered in Feng, Fouque & Kumar [FFK12] for...
a more general stochastic volatility model. The problem then falls into the class of homogenization and averaging problems for nonlinear HJB type equations, where the fast volatility variable lives on a non-compact set.

1.1. Outline of article

In this article, we consider a stochastic volatility model for a log stock price process \( X_t \) of the form

\[
\begin{align*}
    dX_t &= -\frac{1}{2}\sigma(Y_t)^2dt + \sigma(Y_t)(\sqrt{1-\rho^2}dW^1_t + \rho dW^2_t), \\
    dY_t &= (-\alpha Y_t + g(Y_t))dt + dW^2_t
\end{align*}
\]

where \( W^1, W^2 \) are independent Brownian motions. We first relax the assumptions that \( \sigma \) is bounded and \( g \) are zero that are imposed in [Forde11]. This requires an auxiliary result, namely that the variance of a probability measure on the real line can be bounded in terms the Donsker-Varadhan rate function of the measure. Using this property, we then establish an LDP for \( (X_t/t) \) using the trivial joint LDP for the two independent variables \((W^1_t/t, \mu_t)\) (where \( \mu_t \) is the occupation measure of \( Y \)), combined with the extended contraction principle for non-continuous functionals given in Theorem 4.2.23 in [DZ98]. This is the same theorem which can be used to prove the Freidlin-Wentzell small-noise LDP from Schilder’s theorem, despite the lack of continuity of the Itô map in the sup norm topology (see e.g. proof of Theorem 5.6.7 in [DZ98]), and is also used in rough paths theory to prove the small-noise LDP for a rough differential equation driven by fractional Brownian motion (c.f. section 15.7 and Proposition 19.14 in [FV10]). The rate function for \( X_t/t \) in this article has the variational representation

\[
I_{X_t} = \inf_{\mu \in \mathcal{P}(\mathbb{R})} \left\{ \int_{\mathbb{R}} \left[ \frac{(x-M(\mu))^2}{2\nu(\mu)} + I_Y(\mu) \right] d\mu \right\}
\]

which can be used to prove the principal eigenvalue \( \lambda_1 \) of an associated Sturm-Liouville equation.

In section 5, we translate these results into large-time asymptotics for call options and implied volatility; this requires computing the corresponding LDP for the log stock price under the so-called Share measure \( \mathbb{P}^s \) associated with using the stock price process as the numéraire, and in section 6, we compute \( I(x) \) numerically, using the Ritz method from the theory of calculus of variations. The Ritz method is described at length in Gelfand&Fomin [GF00] - we choose an \( n \)-dimensional subspace of the space of admissible functions, in this case the Hilbert space \( L^2(\mu_{\infty}) \), and then minimize the objective function \( \frac{(x-M(\mu))^2}{2\nu(\mu)} + I_Y(\mu) \) by minimizing over the subspace for the \( n \) Fourier coefficients.

In section 7, we enrich the general model with an additional independent CIR short rate process \( r_t \) and an independent driving Lévy process \( Z_t \). It is well known that stochastic interest rates make a significant difference to the price of European options at large maturities, but to our knowledge this effect has never been properly quantified using asymptotics; specifically, we show that the log stock price now satisfies the LDP with rate function

\[
I_t(x) = \inf_{a,y,z:a+z=x} \left[ I(y) + I_{\text{CIR}}(a) + V_{\text{F}}(z) \right]
\]

where \( I_{\text{CIR}}(a) = \kappa^2(a-\theta)^2/(2\sigma^2) \) is the rate function for \( \frac{1}{t} \int_0^t r_s ds \), and \( V_{\text{F}}(x) \) is the rate function for \( Z_t/t \).

2. The Donsker-Varadhan large deviation principle

Let \( \Omega \) denote the space of real-valued functions \( \omega(.) \) on \(-\infty < t < \infty\) with discontinuities of the first kind, normalized to be right continuous, and with convergence induced by the Skorokhod topology on bounded intervals. Let \((Y_t, \mathbb{P}_y)\) be a Markov process on \( \Omega \) with invariant distribution \( \mu_{\infty}(dy) \) such that the mapping \( y \mapsto \mathbb{P}_y \) is weakly continuous (which implies the Feller property for the process \( Y \)). Let \( p(t, x, dy) \) denote the transition probability for \( Y_t \), \( P_t \) denote the semigroup associated with \( Y \), and let \( L \) denote the infinitesimal generator of \( P_t \) and \( D = D(L) \subset C_b(\mathbb{R}) \) its domain. For each \( t > 0 \) and \( A \in \mathcal{B}(\mathbb{R}) \), let

\[
\mu_t(A) = \frac{1}{t} \int_0^t 1_A(Y_s) ds
\]

denote the occupation time distribution of \( Y \), i.e. the proportion of time that \( Y \) spends in the set \( A \). For each \( t > 0 \) and \( \omega \), \( \mu_t(.) \) is a probability measure on \( \mathbb{R} \). Let \( \mathcal{P}(\mathbb{R}) \) denote the space of probability measures on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\). Then from [DV76] (or page 178 in Stroock [Str84] or Pinsky [Pin85]), under suitable recurrence
and transitivity conditions (see next subsection for details), $\mu(\cdot)$ satisfies the LDP as $t \to \infty$ in the topology of weak convergence, with a convex, lower semicontinuous rate function $I_Y : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ given by

$$I_Y(\mu) = -\inf_{u \in \mathcal{D}^+} \int_{-\infty}^\infty \frac{L u}{u} \, d\mu$$

(2.1)

for each $\mu \in \mathcal{P}(\mathbb{R})$, where $\mathcal{D}^+$ is the set of $u$ in the domain $\mathcal{D}$ of $L$ with $u \geq \varepsilon > 0$ for some $\varepsilon > 0$. More precisely, if we define a probability measure $Q_{t,y}$ on $\mathcal{P}(\mathbb{R})$ by $Q_{t,y} = \mathbb{P}_y \circ \mu^{-1}$, then for any closed set $C \subset \mathcal{P}(\mathbb{R})$ (weak topology) and for any open set on $G \subset \mathcal{P}(\mathbb{R})$ we have

$$\limsup_{t \to \infty} \frac{1}{t} \log Q_{t,y}(C) \leq -\inf_{\mu \in C} I_Y(\mu),$$

$$\liminf_{t \to \infty} \frac{1}{t} \log Q_{t,y}(G) \geq -\inf_{\mu \in G} I_Y(\mu).$$

(2.2)

$I_Y(\cdot)$ is known as the $I$-function for the process $Y$.

**Remark 2.1.** By the ergodic theorem, $Q_{t,y} \xrightarrow{w} \delta_{\mu_{\infty}}$ as $t \to \infty$, and it is well known that $I_Y(\mu) = 0$ if and only if $\mu = \mu_{\infty}$ (see e.g. the proof of Corollary 1.5 in [Pin85]).

### 2.1. Sufficient conditions for the LDP upper and lower bounds

In [Var84] (page 34) and [DV76], [DV83] it is shown that the following five conditions imply the LDP upper bound in (2.2):

There exists a sequence $u_n$ of functions in $\mathcal{D}(L)$ with the five properties:

1. $u_n(y) \geq c > 0$ for $y$ and $n$.
2. For all compact sets $K \subset \mathbb{R}$, there exists a constant $C_K$ such that $\sup_{y \in K} \sup_n u_n(y) \leq C_K$.
3. $V_n(y) := -(Lu_n/u_n)(y) \geq -C$ for all $n$ and $y$.
4. There exists a function $V(y)$ such that for all $y \in \mathbb{R}$, $\lim_{n \to \infty} V_n(y) = V(y)$.
5. The set $\{ y : V(y) \leq \ell \}$ is compact for all $\ell < \infty$.

Moreover, the following two conditions imply the LDP lower bound:

There exists a density function for $p(1, x, dy)$ with respect to a reference measure $\alpha$ on $\mathbb{R}$ such that

- $p(1, x, dy) = p(1, x, y)\alpha(dy)$.
- $p(1, x, \cdot)$ as a mapping from $\mathbb{R} \to L^1(\alpha)$ is continuous.

These two conditions are given in [Var84], [DV83], where the LDP is proved as a corollary of a more general LDP on path space in terms of the entropy function (see Theorem 13.1.31 in [Var84]). These two conditions simplify the more cumbersome conditions for the LDP lower bound given on page 393 in [DV76].

### 2.2. Examples: the OU process, and the perturbed OU process

- For the Ornstein-Uhlenbeck process

  $$dY_t = -\alpha Y_t dt + dW_t$$

  the conditions 1-5 in subsection 2.1 are satisfied with $u_n(y) = \cosh(n\theta(y/n))$, if $\theta(y) = y$ for $0 \leq y \leq 1$ and $\theta, \theta', \theta''$ are uniformly bounded on $\mathbb{R}$ and $\theta$ is odd (see sections 7 in [DV76] and [DV83]), and in this case $V(y) = -L\alpha(y) = -\frac{1}{2} + y\alpha \tanh y$, which tends to $|y|$ as $y \to \pm \infty$. In the next bullet point, we will show that the two lower bounds are also satisfied, as a special case of a more general perturbed OU process. Thus $\mu_\nu$ satisfies the LDP with rate $I_\nu(\mu)$ as in (2.1) as $t \to \infty$. The OU process has a unique invariant distribution given by $\mu_{\infty}(y) = (\frac{\pi}{\alpha})^\frac{1}{2} e^{-\alpha y^2}$ i.e. $N(0, \frac{1}{2\alpha})$. If $\mu$ is absolutely continuous
with respect to \( \mu_\infty \), then (because the OU process is symmetric, i.e. its generator is self adjoint with respect to \( \mu_\infty(y) \)) we can simplify the rate function \( I_Y \) to

\[
I_Y(\mu) = \frac{1}{2} \int_{-\infty}^{\infty} \psi'(y)^2 \mu_\infty(dy)
\]

(2.3)

if \( \psi' \in L^2(\mu_\infty) \), where \( \phi = \frac{d\mu}{d\mu_\infty} \) is the Radon-Nikodým derivative and \( \psi = \sqrt{\phi} \) (see p.179 under Exercise (8.28) in Stroock [Str84]). If \( \mu \) is not absolutely continuous with respect to \( \mu_\infty \), then \( I_Y(\mu) = \infty \). The representation in (2.3) will be used for the numerics in section 6 using the Ritz method.

- For a perturbed OU process of the form

\[
dY_t = (-\alpha Y_t + g(Y_t)) dt + dW_t
\]

(2.4)

where \( g \) is \( C^3 \) with sublinear growth at \( \pm \infty \) and continuous bounded derivatives of all orders up and including 3, the \( -\alpha y \) drift term swamps the \( g(y) \) term as \( |y| \to \infty \) and the five conditions 1-5 for the LDP upper bound are still satisfied with the same \( u_n(y) \) as above. This includes the case when e.g. \( g(y) = \alpha \theta \) for a constant \( \theta \) which is the mean-reversion level for \( Y \).

**Lemma 2.1.** The perturbed OU process in (2.4) satisfies the two lower bound conditions I and II above.

*Proof.* See Appendix B.

We also note that the process \( Y \) in (2.4) has a unique invariant distribution given by

\[
\mu_\infty(y) = \frac{e^{-\alpha y^2} \int_{-\infty}^{y} e^{-\alpha u^2} g(u) du}{\int_{-\infty}^{\infty} e^{-\alpha u^2} \int_{-\infty}^{y} e^{-\alpha v^2} g(v) dv du}.
\]

(2.5)

**2.3. The Prokhorov metric on \( \mathcal{P}(\mathbb{R}) \) and goodness of the rate function \( I_Y(\mu) \)**

We can also topologize \( \mathcal{P}(\mathbb{R}) \) with the Prokhorov metric, defined as

\[
d(\mu, \mu_1) = \inf\{\delta > 0 : \mu(C) \leq \mu_1(C^\delta) + \delta \text{ for all closed } C \in \mathcal{B}(\mathbb{R})\}
\]

for \( \mu, \mu_1 \in \mathcal{P}(\mathbb{R}) \), where \( C^\delta \) is the \( \delta \)-neighborhood of \( C^1 \) (see page 96 in Ethier&Kurtz[EK86]). Under this metric, \( \mathcal{P}(\mathbb{R}) \) is a metric space (note also that \( d(\mu, \mu_1) \leq 1 \) for all \( \mu, \mu_1 \)). Moreover, \( \mathbb{R} \) is separable, so convergence of measures in the Prokhorov metric is equivalent to weak convergence of measures (see Theorem 3.1 part a) and part b) in [EK86] for details), so the Donsker-Varadhan LDP for \( \mu_t \) also holds in the topology induced by the metric \( d \).

**Remark 2.2.** By Lemma 7.1 (see also page 461) in [DV76] \( \mu_\infty \) is exponentially tight in the weak topology (and thus also in the Prokhorov topology), and thus (by Lemma 1.2.18 in [DZ98]) \( I_Y(.) \) is a good rate function.

**2.4. The tail behaviour of probability measures inside the level sets of \( I_Y \)**

The following lemma is the main observation on which the article is based, which characterizes the tail behaviour of the measures inside a level set of \( I_Y \).

**Lemma 2.2.** Consider the perturbed OU process in (2.4). Then for \( \mu \in \mathcal{P}(\mathbb{R}) \) we have the following bound for the second moment of \( \mu \) in terms of \( I_Y(\mu) \):

\[
\int_{-\infty}^{\infty} y^2 \mu(dy) \leq K_2(\alpha) I_Y(\mu) + K_3(\alpha)
\]

for some constants \( K_2(\alpha) > 0 \) and \( K_3(\alpha) \).

\(^1\text{The set of all points which are of distance } \leq \delta \text{ from } C.\)
Proof. The infinitesimal generator $\mathcal{L}$ of $Y$ coincides with the differential operator $\mathcal{L} = (-\alpha y + g(y))\partial_y + \frac{1}{2}\partial_{yy}^2$ on $C_0^2(\mathbb{R})$. Define a function $\psi$ such that

$$\psi(y) := \begin{cases} 
y & (0 \leq y \leq 1) \
\frac{y}{2} & (y \geq 2) \end{cases}$$

and $\psi$ is an odd function. Consequently, $\psi, \psi', \psi''$ are uniformly bounded, $\psi' \geq 0$ and $\psi(u)/u > 0$ and is uniformly bounded when $u \neq 0$. Let $u_n(y) = e^{\frac{c}{2}[\psi(\frac{y}{n})]^2}$ with $c \in \left(0, \alpha \wedge \left(\frac{\alpha y}{\sup_u \sup_{\psi(u)} \psi'(u)}\right)\right)$. Then

$$-Lu_n(y) = -\left[(-\alpha y + g(y))(cy\psi'(\frac{y}{n}))\psi'(\frac{y}{n}) + \frac{1}{2} \left(c^2n^2\psi^2(\frac{y}{n})\psi'(\frac{y}{n})^2 + c\psi(\frac{y}{n})\psi''(\frac{y}{n})\right)\right]_{\{\frac{y}{n}\leq 2\}} u_n(y)$$

$$= -\left[(-\alpha y + yg(y))(cy\psi'(\frac{y}{n}))\psi'(\frac{y}{n}) + \frac{1}{2} \left(c^2\psi^2(\frac{y}{n})\psi'(\frac{y}{n})^2 + c\psi(\frac{y}{n})\psi''(\frac{y}{n})\right)\right] u_n(y)$$

$$-\frac{Lu_n}{u_n}(y) = c\psi(\frac{y}{n})\psi'(\frac{y}{n})/[cy - yg(y)] - \frac{cy^2}{2}\psi(\frac{y}{n})\psi'(\frac{y}{n}) - \frac{1}{2}[c\psi(\frac{y}{n})\psi''(\frac{y}{n})]$$

$$= I + II.$$ 

Observe that the second term $II$ is uniformly bounded as $\psi, \psi', \psi''$ are uniformly bounded. For the first term $I$, note that $\frac{cy}{n}\psi'(u) > 0$ and is uniformly bounded for $u \neq 0$, hence $-\frac{cy}{2}\psi(u)\psi'(u) > -\frac{cy}{2}$ if $c\psi'(u)/\psi(u) < \alpha$. Moreover, since $g(y)$ has sublinear growth, there exists a constant $c_1 > 0$ such that $y^2 - yg(y) > c_1$ for all $y$. Hence $-\frac{Lu_n}{u_n}(y)$ is uniformly bounded from below. Since $\psi(y/n) = y/n$ and $\psi'(y/n) = 1$ for $|y| \leq n$, it is trivial to check that $-\frac{Lu_n}{u_n}(y) \to (\alpha - \frac{cy}{2})y^2 - cyg(y) - \frac{cy}{2}$ pointwise as $n \to \infty$ and $Lu_n \in C_0$ because $Lu_n(y) = 0$ for $y$ sufficiently large, so $u_n \in D^+$. From this we obtain

$$I_Y(\mu) = \sup_{u \in D^+} \int_{-\infty}^{\infty} \frac{Lu}{u} \, d\mu \geq \int_{-\infty}^{\infty} \frac{Lu_n}{u_n} \, d\mu.$$ 

Taking the liminfs of both sides as $n \to \infty$ and using Fatou’s lemma we obtain

$$I_Y(\mu) \geq \liminf_{n \to \infty} \int_{-\infty}^{\infty} \frac{Lu_n}{u_n} \, d\mu \geq \int_{-\infty}^{\infty} \left(\alpha - \frac{cy}{2}\right)y^2 - cyg(y) = \frac{1}{2}c_2 \mu(dy) - K_1(\alpha) - \frac{cy}{2}c_2$$

Since $yg(y)$ is subquadratic, we can find a positive constant $K_1(\alpha)$ such that $(\alpha - c/2)y^2 - cyg(y) = (\alpha - c/2)y^2 + (\alpha - c/2)y^2 - cyg(y) \geq (\alpha - c/2)y^2 - K_1(\alpha)$. Thus we have that $I_Y(\mu) \geq (\alpha - c/2)\int_{-\infty}^{\infty} y^2 \mu(dy) - K_1(\alpha) - \frac{cy}{2}c_2$, and the result follows by re-arranging. \[\square\]

3. The stochastic volatility model

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ unless otherwise stated, with a filtration $(\mathcal{F}_t)_{t \geq 0}$ and satisfying the usual conditions. We consider the following stochastic volatility model for a log stock price process $X_t = \log S_t$ driven by a perturbed Ornstein-Uhlenbeck process $Y$:

$$\begin{cases} 
dX_t = -\frac{1}{2}\sigma(Y_t)^2dt + \sigma(Y_t)(\rho dW_t + \bar{\rho}dW^1_t), \\
dY_t = (-\alpha Y_t + g(Y_t))dt + dW^2_t, \end{cases} \tag{3.1}$$

where $\alpha > 0, X_0 = x_0, Y_0 = y_0, W^1, W^2$ are two independent standard Brownian motions, $\rho \in (-1, 1), \bar{\rho} = \sqrt{1 - \rho^2}$ and we make the following assumptions on $\sigma$ and $g$ throughout:

**Assumption 3.1.** $\sigma : \mathbb{R} \mapsto (0, \infty)$ and $g : \mathbb{R} \mapsto \mathbb{R}$ are both continuous and satisfy the sublinear growth conditions

$$\sigma(y) \vee g(y) \leq K_1(1 + |y|^p) \tag{3.2}$$

for some $K_1 > 0, p \in (0, 1)$.

\[\footnote{The same condition appears in Feng, Fouque, and Kumar [FFK12].}\]
Assumption 3.2. $g$ has continuous bounded derivatives of all orders order up and including 3, and if $\rho \neq 0$ then $\sigma$ is differentiable and $|\sigma'(y)|$ is bounded.

Remark 3.3. Note that for the seemingly more general model:

$$
\begin{align*}
\begin{cases}
    dX_t & = -\frac{1}{2}f(V_t)^2 dt + f(V_t)(\rho dW_t^2 + \bar{\rho} dW_t^1), \\
    dV_t & = [\alpha(m - V_t) + \tilde{h}(V_t)] dt + \beta dW_t^2
\end{cases}
\end{align*}
$$

for $\alpha, \beta > 0$ and $\tilde{h}$ satisfying the same conditions as $\sigma$ and $g$ above, if we set $Y_t = \frac{1}{\beta}(V_t - m)$ and $\sigma(y) = f(\beta y + m)$, $g(y) = \tilde{h}(\beta y + m)$, then we are transformed back to a model of the model in (3.1), so there is no loss of generality in our assumption of zero mean reversion level and vol-of-vol (i.e. diffusion coefficient) equal to 1 in the $Y$ process in (3.1).

We also set $S_0 = 1$ throughout (i.e. $x_0 = 0$) without loss of generality, because $X_t - x_0$ is independent of $x_0$ as the SDEs have no dependence on $x$.

### 3.1. The integrated variance

Now let $F : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}^+$ denote the linear functional defined by

$$
F(\mu) = \int_{-\infty}^{\infty} \sigma^2(y) \mu(dy).
$$

Note that $F$ may not be continuous in the weak topology because $\sigma^2$ may not be bounded. Define

$$
F(\mu_t) = \int_{-\infty}^{\infty} \sigma^2(y) \mu_t(dy) = \frac{1}{t} \int_0^t \sigma^2(Y_s) ds
$$

where $\mu_t(dy)$ is the occupation measure of $Y$; then we see that $F(\mu_t)$ is the time-average of the instantaneous variance for $Y$. We also define

$$
\bar{\sigma}^2 = \int_{-\infty}^{\infty} \sigma^2(y) \mu_\infty(dy)
$$

where $\mu_\infty$ is defined in (2.5).

### 4. Large-time asymptotics for the stochastic volatility model

#### 4.1. The main result: the LDP for the log stock price as $t \to \infty$

We now state the first main result, which is a large deviation principle for the re-scaled log stock price $(X_t/t)$ as $t \to \infty$.

**Theorem 4.1.** Consider the process $X$ defined in (3.1) and let $b(y) = \sigma(y)(\alpha y - g(y)) - \frac{1}{2}\sigma'(y), G(\mu) = \int_{-\infty}^{\infty} b(y) \mu(dy)$. Then under Assumptions 3.1 and 3.2, $X_t/t$ satisfies the LDP as $t \to \infty$ with a good rate function given by

$$
I(x) = \inf_{\mu \in \mathcal{P}(\mathbb{R})} \left[ \frac{(x - M(\mu))^2}{2\nu(\mu)} + I_Y(\mu) \right]
$$

where $M(\mu) = -\frac{1}{2}F(\mu) + \rho G(\mu)$, $\nu(\mu) = \bar{\rho}^2 F(\mu)$ and $I_Y(\mu)$ is the rate function for the occupation measure of $Y$ defined in (2.1).
Proof. Integrating (3.1), we see that
\[
X_t = -\frac{1}{2} \int_0^t \sigma(Y_s)^2 ds + \int_0^t \sigma(Y_s) (\rho dW_s^2 + \tilde{\rho} dW_s^1).
\]
If we let \( \chi(y) = \int_{y_0}^y \sigma(u) du \), then
\[
d\chi(Y_t) = \sigma(Y_t) dY_t + \frac{1}{2} \sigma'(Y_t) d(Y_t) = \sigma(Y_t) ((-\alpha Y_t + g(Y_t)) dt + dW_t^2) + \frac{1}{2} \sigma'(Y_t) dt
\]
which we can integrate and re-arrange as follows
\[
\int_0^t \sigma(Y_s) dW_s^2 = \chi(Y_t) + \int_0^t [\sigma(Y_s) (\alpha Y_s - g(Y_s)) - \frac{1}{2} \sigma'(Y_s)] ds = \chi(Y_t) + \int_0^t b(Y_s) ds.
\]
Now let \( Z_t = W_t^1/t \) and \( \tilde{X}_t = X_t/t \). Conditioning on \( (Y_s; 0 \leq s \leq t) \) we obtain
\[
\tilde{X}_t \overset{d}{=} -\frac{1}{2} F(\mu_t) + \rho G(\mu_t) + \tilde{\rho} W_t^1(\mu_t) = -\frac{1}{2} F(\mu_t) + \rho G(\mu_t) + \frac{1}{t} \chi(Y_t) + \frac{\tilde{\rho} \sqrt{F(\mu_t)}}{t} W_t^1(\mu_t)
\]
where \( M(\mu) = -\frac{1}{2} F(\mu) + \rho G(\mu) \) and \( \nu(\mu) = \tilde{\rho}^2 F(\mu) \). From the Gärtner-Ellis theorem we know that \( Z_t \) satisfies a large time LDP with good rate function \( \frac{1}{2} z^2 \), we also know that \( \mu_t \) satisfies a large time LDP with good rate function \( I_\tau(\mu) \). Moreover, \( Z_t \) and \( \mu_t \) are independent, so we have
\[
\mathcal{I}(z, \mu) = \lim_{\delta \to 0} \lim_{t \to \infty} \frac{1}{t} \log P(Z_t \in B_\delta(z), \mu_t \in B_\delta(\mu)) = \lim_{\delta \to 0} \lim_{t \to \infty} \frac{1}{t} \log P(|Z_t| > \delta z) + P(\mu_t \not\in B_\delta(\mu))
\]
Thus \((Z_t, \mu_t)\) satisfies the weak LDP with rate \( \mathcal{I}(z, \mu) = \frac{1}{2} z^2 + I_\tau(\mu) \). Since \( \mu_t \) is exponentially tight (by Remark 2.2), for any \( c > 0 \), there exists a compact set \( K_c \in \mathcal{P}(\mathbb{R}) \) such that \( \limsup_{t \to \infty} \frac{1}{t} \log P(\mu_t \not\in K_c) \leq -c \). Thus for any \( c > 0 \), there exists a compact set \([-\sqrt{2c}, \sqrt{2c}] \times K_c \subset \mathbb{R} \times \mathcal{P}(\mathbb{R}) \) such that
\[
\limsup_{t \to \infty} \frac{1}{t} \log P(|Z_t| \notin [-\sqrt{2c}, \sqrt{2c}] \times K_c) \leq \limsup_{t \to \infty} \frac{1}{t} \log P(|Z_t| > \sqrt{2c}) + P(\mu_t \not\in K_c)
\]
\[
\leq \max \left\{ \limsup_{t \to \infty} \frac{1}{t} \log P(|Z_t| > \sqrt{2c}), \limsup_{t \to \infty} \frac{1}{t} \log P(\mu_t \not\in K_c) \right\}
\]
\[
\leq -c.
\]
Thus \((Z_t, \mu_t)\) is exponentially tight, so \((Z_t, \mu_t)\) satisfies the full LDP and (by Lemma 1.2.18b in [DZ98]) the rate function \( \mathcal{I}(z, \mu) \) is good. From (4.2) we have
\[
\tilde{X}_t \overset{d}{=} \tilde{X}_t := \varphi(z_t, \mu_t) + \frac{\tilde{\rho}}{t} \chi(Y_t)
\]
where \( \varphi : \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R} \) is given by \( \varphi(z, \mu) = M(\mu) + \sqrt{\nu(\mu)} \cdot z \). Similarly, define
\[
\tilde{X}_t^m = \varphi^m(Z_t, \mu_t),
\]
where \( \varphi^m(z) = M^m(\mu) + \sqrt{\nu^m(\mu)} \cdot z \), where we have truncated the integrands in \( M(\mu) \) and \( \nu(\mu) \) to get
\[
M^m(\mu) = \int \left[ (-\frac{1}{2} \sigma^2(y) + \rho \sigma^2(m))1_{|y| \leq m} + (-\frac{1}{2} \sigma^2(m) + \rho \sigma^2(m))1_{|y| > m} + (-\frac{1}{2} \sigma^2(-m) + \rho \sigma^2(-m))1_{|y| < -m} \right] \mu(dy)
\]
and
\[
\nu^m(\mu) = \tilde{\rho}^2 \int \left[ \sigma^2(y)1_{|y| \leq m} + \sigma^2(m)1_{y > m} + \sigma^2(-m)1_{y < -m} \right] \mu(dy).
\]
Since the integrands are bounded and continuous functions of \( \mathbb{R} \), \( M^m(\mu) \) and \( \nu^m(\mu) \) are continuous functionals of \( \mu \) under the weak topology. Using the Hölder continuity of the square root function: \( |\sqrt{x} - \sqrt{y}| \leq |x - y| \), we have
\[
|\tilde{X}_t - \tilde{X}_t^m| \leq |M(\mu_t) - M^m(\mu_t)| + \sqrt{\nu(\mu_t) - \nu^m(\mu_t)} |Z_t| + \frac{\tilde{\rho}}{t} \chi(Y_t).
\]
Thus by Theorem 4.2.23 in [DZ98], \( \hat{X}_t \) satisfies the LDP with good rate function

\[
I(x) = \inf_{(z, \mu) : M(\mu) + \sqrt{\nu(\mu)} z = x} \left[ \frac{1}{2} z^2 + I_Y(\mu) \right].
\]  

(4.6)

Thus the proof proceeds as in [DZ98], where \( \hat{X}_t \) satisfies the LDP with good rate function.

\[
\nu(\mu) = \rho^2 \int_{-\infty}^\infty \sigma^2(y) \mu(dy) > 0
\]

because \( \sigma^2 \) is strictly positive. Thus we can re-write the right hand side of (4.6) as

\[
\inf_{\mu \in \mathbb{P}(\mathbb{R})} \left[ \frac{1}{2} z^2 + I_Y(\mu) \right].
\]

(4.6)
4.2. Properties of the rate function $I(x)$

The following two corollaries establish some basic properties of $I(x)$:

**Corollary 4.2.** The infimum of $I(x)$ in (4.1) is attained uniquely at

$$x_{\text{min}} = M(\mu_\infty) = \frac{1}{2}\sigma^2$$

where $M(.)$ is defined as in Theorem 4.1 and $\sigma$ is defined in (3.5).

**Proof.** Let $I(x, \mu) = \frac{(x_{\text{min}}-M(\mu))^2}{2\sigma^2(\mu)} + I_Y(\mu)$. Then, by (4.1), $I(x) = \inf_{\mu \in P(\mathbb{R})} I(x, \mu)$. Setting $\mu = \mu_\infty$ we have $I(x_{\text{min}}, \mu_\infty) = \frac{(x_{\text{min}}-M(\mu_\infty))^2}{2\sigma^2(\mu_\infty)} + I_Y(\mu_\infty) = 0$. Therefore

$$0 \leq I(x_{\text{min}}) = \inf_{\mu \in P(\mathbb{R})} I(x_{\text{min}}, \mu) \leq I(x_{\text{min}}, \mu_\infty) = 0$$

so $I(x_{\text{min}}) = 0$.

We show that $x_{\text{min}}$ is the unique minimum by contradiction. Suppose there exists an $x \neq x_{\text{min}}$ such that $\lim_{n \to \infty} I(x, \mu_n) = 0$ for some sequence $(\mu_n)$ with $\mu_n \in P(\mathbb{R})$. If $I(x, \mu_n) \to 0$ as $n \to \infty$ then $I_Y(\mu_n) \to 0$ and $M(\mu_n) \to x$ as $n \to \infty$. We first show that $(\mu_n)$ is a tight sequence. For any $k > 0$,

$$k^2\mu_n[{-k,k}]^c \leq \int_{[-k,k]^c} y^2 \mu_n(dy) \leq K_2(\alpha)I_Y(\mu_n) + K_3(\alpha)$$

where we have used Lemma 2.2 for the last inequality. Since $I_Y(\mu_n) \to 0$ as $n \to \infty$, we can find a $C < \infty$ such that $\sup_n I_Y(\mu_n) \leq C$. Hence $k^2\mu_n[-k,k]^c \leq K_2(\alpha)C + K_3(\alpha)$ for all $n$. Thus, given $\epsilon > 0$, we can choose $k$ large enough such that

$$\sup_n \mu_n[-k,k]^c \leq \frac{K_2(\alpha)C + K_3(\alpha)}{k^2} < \epsilon ,$$

so $(\mu_n)$ is tight as required.

Hence $(\mu_n)$ has a convergent subsequence. Without loss of generality we denote the convergent subsequence by $(\mu_n)$ and let $\mu$ denote the limit point. Then $I_Y(\mu) = 0$ by lower semicontinuity of $I_Y$ (i.e. $I_Y(\mu) \leq \liminf_{\mu_n \to \mu} I_Y(\mu_n) = 0$) and by uniqueness of minimizer of $I_Y$ we obtain $\mu = \mu_\infty$. We will next show that $M(\mu_n) \to M(\mu_\infty) = x_{\text{min}}$ which gives the contradiction.

Let $m > 0$. Then

$$|M(\mu_n) - M(\mu_\infty)| \leq |M_m(\mu_n) - M_m(\mu_\infty)| + |M(\mu_n) - M_m(\mu_n)| + |M(\mu_\infty) - M_m(\mu_\infty)|$$

$$\leq |M_m(\mu_n) - M_m(\mu_\infty)| + c(m)(C_1 I_Y(\mu_n) + C_2) + c(m)(C_1 I_Y(\mu_\infty) + C_2)$$

(where we have applied Lemma A.1, and $C_1, C_2$ are constants and $c(m) = 1/m^2 + 1/m^{2-q}$ for some $q \in (0, 2)$)

$$= |M_m(\mu_n) - M_m(\mu_\infty)| + c(m)(C_1 I_Y(\mu_n) + C_2) + c(m)C_2.$$

Taking $n \to \infty$, we see that

$$\lim_{n \to \infty} |M(\mu_n) - M(\mu_\infty)| \leq 0 + 2c(m)C_2$$

(4.8)

because $M_m$ is a continuous functional. Since this holds for any arbitrary $m > 0$, taking $m \to \infty$ and noting that $c(m) \to 0$ as $m \to \infty$, we get $M(\mu_n) \to M(\mu_\infty) = x_{\text{min}}$.

Finally, using the definition of $M(\cdot)$ in Theorem 4.1, we find that $M(\mu_\infty) = -\frac{1}{2}\sigma^2 + \bar{\rho} \bar{b}$ where $\bar{b} = \int_{-\infty}^{\infty} b(y)\mu_\infty(y)dy$. Recall that $b(y)$ is defined in Theorem 4.1 as $b(y) = (ay + g(y))\sigma(y) - \frac{1}{2}\sigma'(y)$. Then we
have
\[
\int b(y)\mu_\infty(dy) = \text{const.} \times \left[ \frac{1}{\alpha}(\sigma(y) - g(y)) \right] e^{\frac{1}{\alpha} \int_{\sigma(y)}^{\sigma(u)} d\mu(y)} - \frac{1}{\alpha} \frac{\sigma'(y)}{\sigma(y)} dy - \int_{-\infty}^{\infty} \frac{1}{\alpha^2} \sigma''(y) e^{\frac{1}{\alpha} \int_{\sigma(y)}^{\sigma(u)} d\mu(y)} dy
\]
\[
= \text{const.} \times \left[ \frac{1}{\alpha}(\sigma(y) - g(y)) \right] e^{\frac{1}{\alpha} \int_{\sigma(y)}^{\sigma(u)} d\mu(y)} - \frac{1}{\alpha} \frac{\sigma'(y)}{\sigma(y)} dy - \int_{-\infty}^{\infty} \frac{1}{\alpha^2} \sigma''(y) e^{\frac{1}{\alpha} \int_{\sigma(y)}^{\sigma(u)} d\mu(y)} (-2\sigma(y) + 2g(y)) dy
\]
where we have integrated by parts in the second expression of the last line. Thus we see that \(x_{\text{min}} = -\frac{1}{2} \sigma^2\).

**Corollary 4.3.** \(I(x)\) in (4.1) is continuous.

**Proof.** Let \(I(x, \mu)\) be as defined in Corollary 4.2. Then, \(I(x, \mu)\) is upper semicontinuous in \(x\) for \(\mu\) fixed, and \(I(x) = \inf_\mu I(x, \mu)\). The pointwise supremum of a family of LSC functions is LSC (see e.g. Lemma 2.41 on page 43 in [AB06]), hence the pointwise infimum of a family of USC functions is USC, so \(I(x)\) is USC. But \(I(x)\) is also a rate function, hence \(I\) is also LSC.

### 4.3. The case \(x = 0\) with \(\rho = 0\) - the Rayleigh-Ritz formula

**Corollary 4.4.** For \(x = 0\), \(\rho = 0\), \(I(0)\) reduces to
\[
I(0) = \lambda_1 = \inf_{\mu \in \Pi(\mathbb{R})} \left\{ \frac{1}{2} F(\mu) + I_Y(\mu) \right\} = \inf_{\psi \in L^2(\mu_\infty) : \|\psi\|_2 = 1} \int_{-\infty}^{\infty} \left[ \frac{1}{8} \sigma^2(y) \psi(y)^2 + \frac{1}{2} \psi'(y)^2 \right] \mu_\infty(y) dy
\]
(4.9)

**Proof.** The first equality in (4.9) just follows by setting \(x = 0\) in (4.1) and simplifying. The second equality just follows by re-writing \(\mu\) in terms of \(\psi\).

**Remark 4.5.** (4.9) is the classical Rayleigh-Ritz formula for the lowest eigenvalue \(\lambda_1\) for the Sturm-Liouville problem \((-\alpha y + g(y))u' + \frac{1}{2} u'' - \frac{1}{2} \sigma^2(y) u = -\lambda_1 u\) (see page 2 in [DV75II] for more details).

### 4.4. A general vol-of-vol coefficient

For a more general model of the form
\[
\begin{cases}
    dX_t = -\frac{1}{\hat{\beta}} \sigma(V_t)^2 dt + \sigma(V_t)(\rho dW_t^2 + \tilde{\rho} dW_t^1), \\
    dV_t = (-\alpha V_t + g(V_t)) dt + \beta(V_t)dW_t^2
\end{cases}
\]
for \(g, \sigma\) satisfying the same conditions as before, \(\beta \in C^1\) with bounded first derivative (so \(\beta\) is Lipschitz), \(0 < \beta \leq \beta(v) \leq \beta < \infty\), \(\beta(v) \to \beta_\infty\) as \(|v| \to \infty\) and \(\frac{1}{\beta(v)} - \frac{1}{\beta_\infty} = O(1/1 + |v|^\gamma)\) for some \(\gamma > 0\), then making the transformation \(Y_t = U(V_t)\), where \(U(v) = \int_{0}^{v} \frac{dz}{\beta(z)}\), we find that
\[
\begin{align*}
    dY_t &= U'(V_t) dV_t + \frac{1}{2} U''(V_t) d(V_t)^2 = U'(V_t)(-\alpha V_t + g(V_t)) dt + \beta(V_t)dW_t^2 + \frac{1}{2} U''(V_t) \beta(V_t)^2 dt \\
    &= \frac{1}{\beta(V_t)} \left[ (-\alpha V_t + g(V_t)) dt + dW_t^2 - \frac{1}{2} \beta'(V_t) dt \right] \\
    &= \left[ -\alpha Y_t + \left( \frac{\alpha V_t}{\beta(V_t)} \right) - \frac{1}{2} \beta'(V_t) \right] dt + dW_t^2 \\
    &= \left[ -\alpha Y_t + (\alpha Y_t - \frac{\alpha}{\beta(U^{-1}(Y_t))} U^{-1}(Y_t)) + \left( \frac{g(U^{-1}(Y_t))}{\beta(U^{-1}(Y_t))} \right) dt + dW_t^2 \\
    &= \left[ -\alpha Y_t + (\alpha Y_t - \frac{\alpha}{\beta(U^{-1}(Y_t))} U^{-1}(Y_t)) + \left( \frac{g(U^{-1}(Y_t))}{\beta(U^{-1}(Y_t))} \right) dt + dW_t^2.
\end{align*}
\]
We need to show that the terms $\alpha Y_t - \frac{1}{\beta(U^{-1}(Y_t))}U^{-1}(Y_t)$ and $\frac{g(U^{-1}(Y_t))}{\beta(U^{-1}(Y_t))} - \frac{1}{2}\beta'(U^{-1}(Y_t))$ satisfy the sublinear growth condition in Assumption 3.1. Henceforth, “sublinear growth” will mean that equation (3.2) is satisfied.

We first look at the term $\frac{g(U^{-1}(Y_t))}{\beta(U^{-1}(Y_t))} - \frac{1}{2}\beta'(U^{-1}(Y_t))$. Since $1/\beta(\cdot)$ and $\beta'(\cdot)$ are bounded functions, it is sufficient to show that $g(U^{-1}(Y_t))$ has sub linear growth in $Y_t$. By the definition of $Y$ and bounds on $\beta(\cdot)$ we get $V_t/\beta \leq V_t \leq V_t/\beta$ which then gives us the inequality $\beta Y_t \leq V_t = U^{-1}(Y_t) \leq \beta Y_t$. Since $g$ has sublinear growth and $V_t$ grows linearly with $t$, we get that $g(U^{-1}(Y_t))$ is a sub linear function of $Y_t$.

We next show that $\left| y - \frac{U^{-1}(y)}{\beta(U^{-1}(y))} \right| \leq \text{constant} \times (1 + |y|)^\delta$ for some $\delta \in (0,1)$. By definition of $Y$ and properties of $\beta(\cdot)$ we get

$$y = U(v) = \int_0^v \frac{1}{\beta(z)} dz = \frac{v}{\beta_\infty} + \int_0^v \left( \frac{1}{\beta(z)} \right) dz = \frac{v}{\beta_\infty} + O(1 + |v|^{1-\gamma})$$

and

$$\frac{v}{\beta(v)} = \frac{v}{\beta_\infty} + O(1 + |v|^{1-\gamma}).$$

Putting this together we get

$$y - \frac{U^{-1}(y)}{\beta(U^{-1}(y))} = U(v) - \frac{v}{\beta(v)} = O(1 + |v|^{1-\gamma}).$$

Since $V_t$ grows linearly with $Y_t$, we get $|y - \frac{U^{-1}(y)}{\beta(U^{-1}(y))}| = O(1 + |y|^{1-\gamma})$. So $|y - \frac{U^{-1}(y)}{\beta(U^{-1}(y))}| \leq \text{constant}$ if $\gamma > 1$ and $|y - \frac{U^{-1}(y)}{\beta(U^{-1}(y))}| \leq \text{constant} \times (1 + |y|^{1-\gamma})$ if $\gamma \in (0,1)$. Thus

\[
\begin{align*}
\{ dX_t & = -\frac{1}{2}\sigma(Y_t)^2 dt + \sigma(Y_t)(\rho dW_t^1 + \bar{\rho} dW_t^2), \\
\{ dY_t & = (-\alpha Y_t + \bar{g}(Y_t)) dt + dW_t^2
\end{align*}
\]

for some $\sigma, \bar{g}$ which satisfy Assumptions 3.1 and 3.2, so we are back to a model of the form in (3.1) and thus the main result in Theorem 4.1 still holds.

If we want to impose less stringent conditions on $\beta$ we would have to manually verify the upper bound conditions 1-5 and the lower bound conditions A,B in subsection 2.1.

5. Call options and implied volatility

We now verify the martingale property for $S_t = e^{X_t}$. This will be used to define the Share measure $\mathbb{P}^*$ below.

**Proposition 5.1.** $(S_t)_{0 \leq t \leq \infty}$ defined in (3.1) is a martingale.

**Proof.** See Appendix D. □

We consider the family of probability measures $\mathbb{P}^*_T(A) := \frac{1}{S_0} \mathbb{E}(S_T 1_A)$ defined for each $T > 0$, for $A \in \mathcal{F}_T$ and $t \leq T$. ($\mathbb{P}^*_T$ is a probability measure on $\mathcal{F}_T$ because $(S_t)_{0 \leq t \leq T}$ is a martingale by Proposition 5.1). From Girsanov’s theorem, we have that

\[
\begin{align*}
\{ dX_t & = \frac{1}{2}\sigma(Y_t)^2 dt + \sigma(Y_t)(\rho dW_t^1 + \bar{\rho} dW_t^2), \\
\{ dY_t & = (-\alpha Y_t + \bar{g}(Y_t)) dt + dW_t^2
\end{align*}
\]

where $W_t^1, W_t^2$ are independent $\mathbb{P}^*_T$-Brownian motions. Let $\mathbb{P}^*$ be a probability measure under which $(X, Y)$ satisfies (5.1) for all $t > 0$ with $X_0 = 0$ and $Y_0 = y_0$.

**Proposition 5.2.** $X_t/t$ satisfies the LDP under $\mathbb{P}^*$ as $t \to \infty$ with a good rate function given by

$$I^*(x) = \inf_{\mu \in \mathbb{P}(\mathbb{R})} \left\{ \frac{(x - M^*(\mu))^2}{2\nu(\mu)} + I_Y(\mu) \right\}$$

where $M^*(\mu) = \frac{1}{2} F(\mu) + \rho G^*(\mu)$, where $G^*(\mu) = \int_0^\infty \left[ (\alpha y - g(y) - \rho \sigma(y)) \sigma(y) - \frac{1}{2} \sigma'(y) \right] \mu(dy)$, and the minimum of $I^*(x)$ is attained uniquely at $x^*_{\text{min}} = M^*(\mu^*_\infty)$; where $\mu^*_\infty$ is the invariant distribution of the $Y$ process under $\mathbb{P}^*$. 

Proof. If we let \( \hat{g}(y) = g(y) + \rho \sigma(y) \), then \( \hat{g} \) also has sublinear growth, and the proof then just follows by an almost identical argument to the proofs of Theorem 4.1 and Corollary 4.2.

Corollary 5.3. The unique minimizers \( x_{min} \) and \( x^*_{min} \) of the rate functions \( I \) and \( I^* \) respectively (defined in Corollary 4.2 and Proposition 5.2 respectively), satisfy the inequality \( x^*_{min} > x_{min} \).

Proof. Recall the formula of the invariant density for the perturbed OU process given in (2.5). Then \( \mu^*_\infty(y) = \frac{e^{-\alpha y^2} e^{\alpha y \int_0^\infty \hat{g}(u) du}}{\int_{-\infty}^\infty e^{-\alpha y^2} e^{\alpha y \int_0^\infty \hat{g}(u) du} du} \), where \( \hat{g}(y) = g(y) + \rho \sigma(y) \) and \( \mu_\infty(y) = \frac{e^{-\alpha y^2} e^{\alpha y \int_0^\infty \hat{g}(u) du}}{\int_{-\infty}^\infty e^{-\alpha y^2} e^{\alpha y \int_0^\infty \hat{g}(u) du} du} \). Observe that

\[
G^*(\mu^*_\infty) = \text{const} \int_{-\infty}^{\infty} \left[ (\alpha y - \hat{g}(y)) \sigma(y) - \frac{1}{2} \sigma'(y) \right] e^{-\alpha y^2} e^{2 \int_0^y \hat{g}(u) du} dy = \text{const} \int_{-\infty}^{\infty} -\frac{1}{2} \frac{d}{dy} \left( \sigma(y) e^{-\alpha y^2} e^{2 \int_0^y \hat{g}(u) du} \right) dy = 0.
\]

Similarly, \( G(\mu_\infty) = 0 \). Thus

\[
x_{min} = -\frac{1}{2} F(\mu_\infty) + \rho G(\mu_\infty) = -\frac{1}{2} F(\mu_\infty) + \rho G^*(\mu^*_\infty) = \frac{1}{2} F(\mu^*_\infty) + \rho G^*(\mu^*_\infty) = x^*_{min}.
\]

By Proposition 5.2, i.e. the LDP for \( (X_t/t) \) under \( \mathbb{P}^* \), and the continuity of the rate function \( I^* \), we obtain the following corollary, which will be used to characterize the large-time behaviour of call option prices.

Corollary 5.4. For the model in (3.1), we have the following large-time behaviour for digital call options

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}^*(X_t > xt) = -\Lambda^*(x) \quad (x > x^*_{min}),
\]

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}^*(X_t < xt) = -\Lambda^*(x) \quad (x < x^*_{min}),
\]

where

\[
\Lambda^*(x) = \begin{cases} 
\inf_{y > x} I^*(y), & \text{if } x \geq x^*_{min}, \\
\inf_{y < x} I^*(y), & \text{if } x \leq x^*_{min}.
\end{cases}
\]

Remark 5.5. From the definition of \( \Lambda^* \), we see that \( \Lambda^* \) is non-increasing for \( x < x^*_{min} \) and non-decreasing for \( x > x^*_{min} \), and (by the continuity of \( I^*(x) \), which can be proved by a similar argument to Corollary 4.3) \( \Lambda^* \) is continuous.

Recall that the payoff of a European call option of strike \( K \) is \( \mathbb{E}(S_t - K)^+ \), and the payoff of a European put option with strike \( K \) is \( \mathbb{E}(K - S_t)^+ \).

Corollary 5.6. For the model in (3.1), we have the following large-time asymptotic behaviour for put/call options in the large-time, large-log-moneyness regime:

\[
- \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(S_t - S_0 e^{xt})^+ = \Lambda^*(x) \quad (x \geq x^*_{min}),
\]

\[
- \lim_{t \to \infty} \frac{1}{t} \log [\mathbb{E}(S_t - S_0 e^{xt})^+] = \Lambda^*(x) \quad (x_{min} \leq x \leq x^*_{min}),
\]

\[
- \lim_{t \to \infty} \frac{1}{t} \log (\mathbb{E}(S_0 e^{xt} - S_t)^+) = \Lambda^*(x) \quad (x \leq x_{min}),
\]

(5.2)

Proof. This is now a standard result, see e.g. Corollary 2.4 in [FJ11].
6. Numerical results

We now consider the following three-factor model for the log stock price \( X_t \) under \( \mathbb{P} \), which incorporates stochastic volatility and a stochastic short rate driven by a CIR square root process:

\[
\begin{align*}
    dX_t &= (r_t - \frac{1}{2}\sigma(Y_t)^2)dt + \sigma(Y_t)(\rho dW_t^2 + \tilde{\rho}dW_t^1) + dZ_t, \\
    dY_t &= (-\alpha Y_t + g(Y_t))dt + dW_t^2, \\
    dr_t &= \kappa_r(\theta_r - r_t)dt + \sigma_r \sqrt{r_t}dW_t^3
\end{align*}
\]

(7.1)

where \( W^1, W^2, W^3 \) are independent Brownian motions, \( x_0, \kappa, \theta, \sigma > 0, |\rho| < 1 \) and \( 2\kappa, \theta > \sigma^2 \), and \( Z_t \) is a Lévy process independent of \( W^1, W^2, W^3 \) such that \( e^{Z_t} \) is a martingale, with cumulant generating function (cgf) \( V_f(p) \) so that \( \mathbb{E}(e^{rZ_t}) = e^{V_f(r)p} \), and \( g, \sigma \) satisfy Assumptions 3.1 and 3.2.
Assume that $V'_J(p) > 0$ and $V_J(p)$ is essentially smooth on some interval $(p_-, p_+)$ (i.e. $|V'_J(p)| \to \infty$ as $p \nearrow p_+$ and $p \searrow p_-)$ with $p_- < 0 < 1 < p_+$. If we let $x_+ = V'_J(0)$ and $V_J(x) = \sup_{p}[px - V_J(p)]$ denote the Legendre transform of $V_J$, then by the Gärtner-Ellis theorem, $Z_t/t$ satisfies the LDP with rate function $V_J(x)$, and $x_-$ is the unique minimum of $V_J(x)$ where $V_J(x_-) = 0$ (see [FFJ11] for details).

We will need the following result:

**Lemma 7.1.** For the model in (7.1), $\Gamma_t = \frac{1}{t} \int_0^t r_s ds$ satisfies the LDP as $t \to \infty$ with good rate function given by the Fenchel-Legendre transform of $V_{\text{CIR}}$:

$$I_{\text{CIR}}(a) = \sup_{a > 0} \{pa - V_{\text{CIR}}(p)\} = \frac{\kappa^2 r^2(a - \theta_r)^2}{2\alpha \sigma^2},$$

where

$$V_{\text{CIR}}(p) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(e^{p \int_0^t r_s ds}) = \begin{cases} \frac{\kappa^2 \theta_r^2}{2\alpha \sigma^2}[\kappa_r - \sqrt{\kappa_r^2 - 2\sigma^2 p}], & \text{for } p \in (-\infty, p_+] \\ \infty, & \text{for } p \notin (-\infty, p_+], \end{cases} \tag{7.2}$$

and $p_+ = \frac{\kappa^2}{2\sigma^2}$. $I_{\text{CIR}}$ clearly attains its minimum value of zero at $a = \theta_r$.

**Proof.** Just follows from the known closed-form expression for the moment generating function of $\Gamma_t$ given in e.g. section 3 in [CGMY03] and the Gärtner-Ellis theorem from large deviations theory, using a similar argument to Theorem 2.1 in Forde&Jacquier [FJ11].

From the contraction principle, we now have:

**Corollary 7.1.** $X_t/t$ satisfies the LDP as $t \to \infty$ with rate function $I_r(x) = \inf_{a,g,z:a+y+z=x} [I(y) + I_{\text{CIR}}(a) + V_J(z)] = \inf_{a,y} [I(y) + I_{\text{CIR}}(a) + V_J(x - a - y)],$ where $I(x)$ is defined as in Theorem 4.1.

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**Fig 1.** Here we have plotted the right half of the (symmetric) asymptotic implied volatility $\sigma_\infty(x)$ for the Ornstein-Uhlenbeck model with $\rho = 0$, $\alpha = 1$ and $\sigma(y) = \sqrt{\log(1 + e^{y})}$ (solid blue line) using the Ritz method with the NMinimize command in Mathematica and $n = 7$, and the values obtained using Monte Carlo simulation for $t = 75$ years (grey diagonal crosses) and $t = 30$ years (black crosses). For the Monte Carlo, we use 1,000,000 simulations and 1000 time steps and we use the usual conditioning trick for $\rho = 0$ by simulating the integrated variance $\int_0^t \sigma(Y_s)^2 ds$ and then plugging this into the Black-Scholes formula. In this case $x^*_{\min} = 0.376131$ and $x_{\min} = -x^*_{\min}$. Note that $\sigma(y) \sim \sqrt{y}$ as $y \to \infty$ and thus satisfies the sublinear growth condition.
Remark 7.2. For the model in (7.1), if there is no Lévy process component, by conditioning on $\Gamma_t = \frac{1}{t} \int_0^t r_s ds$, we can prove the following asymptotic behaviour for the price of a digital call option in the large-time, large log-moneyness regime:

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(e^{-\int_0^t r_s ds} 1_{X_t > x_t}) = - \inf_{y > x} I_r(y)$$

where $I_r(x) = \inf_{a \in \mathbb{R}^+} [a + I(x-a) + I_{CIR}(a)]$.

Remark 7.3. We can adapt this result to compute e.g. large-time asymptotics for European call options.

References

Appendix A: Linear functionals of the occupation measure

**Lemma A.1.** Consider a linear functional $\Lambda : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ defined by $\Lambda(\mu) = \int_\mathbb{R} \lambda(y)\mu(dy)$, where $\lambda$ satisfies the growth condition

$$|\lambda(y)| \leq A(1 + |y|^q)$$

for $q \in (0, 2)$, $A > 0$. Then

$$|\Lambda(\mu) - \Lambda^m(\mu)| \leq 2A\left(\frac{1}{m^2} + \frac{1}{m^{2-q}}\right)(K_2(\alpha)I_Y(\mu) + K_3(\alpha))$$

where $\Lambda^m(\mu) = \int \left[\lambda(y)1_{|y| \leq m} + \lambda(m)1_{|y| > m} + \lambda(-m)1_{|y| < -m}\right] \mu(dy)$ and $K_2(\alpha) > 0$ and $K_3(\alpha)$ are the constants introduced in Lemma 2.2.

**Proof.** For $I_Y(\mu) \leq c$, using the growth condition on $\lambda$ we obtain

$$|\Lambda(\mu) - \Lambda^m(\mu)| = \int_{|y| > m} \left[(\lambda(y) - \lambda(m))1_{|y| > m} + (\lambda(y) - \lambda(-m))1_{|y| < -m}\right] \mu(dy)$$

$$\leq \int_{|y| > m} \left[|\lambda(y)| + |\lambda(m)|1_{|y| > m} + (|\lambda(y)| + |\lambda(-m)|)1_{|y| < -m}\right] \mu(dy)$$

$$\leq 4\int_{|y| > m} A(1 + |y|^q)\mu(dy)$$

$$\leq 4A\left(\frac{1}{m^2} + \frac{1}{m^{2-q}}\right)\int_{-\infty}^{\infty} y^2\mu(dy)$$

$$\leq 4A\left(\frac{1}{m^2} + \frac{1}{m^{2-q}}\right)(K_2(\alpha)I_Y(\mu) + K_3(\alpha))$$

where we have used Lemma 2.2 in the final line.

**Lemma A.2.** $\sigma^2(y)$ satisfies the subquadratic growth condition

$$\sigma^2(y) \leq A_1(1 + |y|^{2p})$$

where $A_1 = 3K_1^2$; thus $F$ (as defined in (3.5)) satisfies the conditions of Lemma A.1 with $\lambda(y) = \sigma^2(y)$, $A = A_1$ and $q = 2p \in (0, 2)$. 


Proof. From the sublinear growth condition $\sigma(y) \leq K_1(1 + |y|^p)$, we see that
\[ \sigma(y)^2 \leq A^2(1 + |y|^p)^2 = A^2(1 + 2|y|^p + |y|^{2p}) \leq 3A^2(1 + |y|^{2p}), \]
where the final inequality just comes from the inequality $|y|^p \leq 1 + |y|^{2p}$. \qed

Lemma A.3. $b$ satisfies the growth condition
\[ |b(y)| \leq A_2(1 + |y|^{1+p}) \quad (A-3) \]
for some $A_2 > 0$; hence the functional $G$ defined in Theorem 4.1 satisfies the conditions in Lemma A.1 with $\lambda(y) = b(y)$, $A = A_2$ and $q = 1 + p \in (0, 2)$.

Proof. Using the sublinear growth condition (3.2) and the boundedness of $\sigma'$ we see that
\[ |b(y)| \leq \alpha|y|K_1(1 + |y|^p) + \frac{1}{2}||\sigma'|| = \alpha K_1 |y| + \alpha K_1 |y|^{1+p} + \frac{1}{2}||\sigma'|| \]
\[ \leq \alpha K_1 (1 + |y|^{1+p}) + \alpha K_1 |y|^{1+p} + \frac{1}{2}||\sigma'|| \]
\[ \leq A_2(1 + |y|^{1+p}) \]
for some $A_2 > 0$. \qed

Lemma A.4. Let $m(y) = -\frac{1}{2}\sigma^2(y) + \rho b(y)$. Then $m$ satisfies the growth condition
\[ |m(y)| \leq A_3(1 + |y|^{1+p}) \]
for some $A_3 > 0$; thus $M$ satisfies the conditions in Lemma A.1 with $\lambda(y) = m(y)$, $A = A_3$ and $q = 1 + p \in (0, 2)$.

Proof. Using (A-2) and (A-3)
\[ m(y) = | -\frac{1}{2}\sigma^2(y) + \rho b(y) | \leq \frac{1}{2}A_1(1 + |y|^{2p}) + \rho A_2(1 + |y|^{1+p}) \leq A_3(1 + |y|^{1+p}) \]
for some $A_3 > 0$. \qed

Appendix B: Proof of Lemma 2.1

To verify the lower bound conditions I and II, we have to show that $p(1, x, dy)$ admits a density $p(1, x, y)$ and that
\[ \lim_{x_2 \to x_1} \int_0^\infty |p(1, x_2, y) - p(1, x_1, y)| \, dy = 0. \quad (B-1) \]

For the rest of the proof we assume that $Y_0 = x$. Let $\tilde{G}(y) = \int_x^y g(u) \, du$; then $\tilde{G}$ has sub-quadratic growth and recall that $|g'|$ is bounded by assumption. Let $h(y) := -\frac{a}{2}y^2 + \tilde{G}(y)$. Then the perturbed OU process $Y$ in (2.4) satisfies $dY_t = h'(Y_t) \, dt + dW_t$ and
\[ dh(Y_t) = h'(Y_t)(h'(Y_t) \, dt + dW_t) + \frac{1}{2}h''(Y_t) \, dt. \quad (B-2) \]

We now define a measure $\mathbb{Q}$ such that
\[ \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} := e^{-\frac{1}{2} \int_0^t h''(Y_s)^2 \, ds - \int_0^t h'(Y_s) \, dW_s} = e^{h(x) - h(Y_t) + \frac{1}{2} \int_0^t \tilde{g}(Y_s) \, ds} \quad (B-3) \]
where $\tilde{g}(y) = h''(y) + (h'(y))^2 = (-\alpha + g'(y)) + (-\alpha y + g(y))^2$ and we have used (B-2) to remove the stochastic integral term in (B-3). To check that the right hand side in (B-3) is a $\mathbb{P}$-martingale, we first define an intermediate change of measure $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} := M_1(t)$, where
\[ M_1(t) = e^{-\frac{1}{2} \int_0^t \tilde{g}(Y_s) \, ds - \int_0^t g(Y_s) \, dW_s}. \quad (B-4) \]
Then $M_1$ is a $\mathbb{P}$-martingale since the Novikov condition is satisfied following the same argument as Appendix D, and

$$dY_t = -\alpha Y_t dt + d\bar{W}_t$$

where $\bar{W}_t = W_t - \int_0^t g(Y_s) ds$ is a Brownian motion under $\mathbb{P}^{OU}$, i.e. $Y$ is an (unperturbed) OU process under $\mathbb{P}^{OU}$. Now define $\frac{d\mathbb{Q}}{d\mathbb{P}^{OU}}|_{\mathcal{F}_t} := M_2(t)$, where

$$M_2(t) = e^{-\frac{1}{2} \int_0^t (\alpha^2 Y_s^2) ds} - \int_0^t \alpha Y_s d\bar{W}_s.$$ 

If $M_2(t)$ is a $\mathbb{P}^{OU}$-martingale, then we can go straight from $\mathbb{P}$ to $\mathbb{Q}$ and define as in (B-3):

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} := e^{-\frac{1}{2} \int_0^t h'(Y_s)^2 ds} - \int_0^t h'(Y_s) d\bar{W}^{(2)}_s = M_1(t) M_2(t)$$

and $M_1 M_2$ will be a $\mathbb{P}$-martingale. To verify the Novikov condition,

$$\mathbb{E}^\mathbb{P}^{OU} \left[ e^{\frac{1}{2} \int_0^t (\alpha^2 Y_s^2) ds} \right] = \mathbb{E}^\mathbb{Q} \left[ e^{\frac{1}{2} \int_0^t h'(Y_s)^2 ds} \right] \leq \frac{1}{\epsilon} \int_s^{t+\epsilon} \mathbb{E}^\mathbb{P}^{OU} \left[ e^{\frac{1}{2} \int_0^t (\alpha^2 Y_s^2) ds} \right] ds$$

by Jensen’s inequality. Under $\mathbb{P}^{OU}$, $Y_u \sim N(y_0 e^{-\alpha u}, \frac{1-e^{-2\alpha u}}{2\alpha})$, so taking $\epsilon$ small enough (say $\epsilon = \frac{1}{4\alpha}$), we get $\mathbb{E}^\mathbb{Q} \left[ e^{\frac{1}{2} \int_0^t (\alpha^2 Y_s^2) ds} \right] < \infty$, for any $s > 0$. Thus by Corollary 5.5.14 on page 199 in [KS91], $M_2$ is a $\mathbb{P}^{OU}$-martingale.

By Girsanov’s theorem, $Y$ is standard Brownian motion under $\mathbb{Q}$ and for any $f \in C_b(\mathbb{R})$,

$$\int_{-\infty}^{\infty} f(y) p(t, x, dy) = \mathbb{E}^\mathbb{P} \left[ f(Y_t) \right] = \mathbb{E}^\mathbb{Q} \left[ f(Y_t) e^{h(Y_t)-h(x)-\frac{1}{2} \int_0^t \hat{g}(Y_s) ds} \right]$$

$$= \int_{-\infty}^{\infty} f(y) e^{h(y)-h(x)} \mathbb{Q} \left[ \left. e^{\frac{1}{2} \int_0^t \hat{g}(Y_s) ds} \right| Y_t = y \right] \gamma(t, x, y) dy$$

$$= \int_{-\infty}^{\infty} f(y) e^{h(y)-h(x)} \phi(t, x, y) \gamma(t, x, y) dy$$

(see also Eqs 6-8 in [Rog85]), where $\gamma(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$, $h(y) = \int_x^y (-\alpha u + g(u)) du = -\frac{\alpha}{2} y^2 + \bar{G}(y) + \frac{\alpha}{2} x^2 - \bar{G}(x)$ and $\bar{G}(y) = \int_x^y g(u) du$ and

$$\phi(t, x, y) = \mathbb{P}^{x,y}(e^{-\frac{1}{2} \int_0^t \hat{g}(Y_s) ds})$$

where $\mathbb{P}^{x,y}$ is a probability measure under which $Y$ is a Brownian bridge with $Y_0 = x$ and $Y_t = y$. Thus $Y$ has a transition density given by

$$p(t, x, y) = \gamma(t, x, y) e^{h(y)-h(x)} \phi(t, x, y).$$

$\bar{G}(y)$ is sub-quadratic so we can choose a constant $c > 0$ such that $\bar{G}(y) \leq c + \frac{\alpha y^2}{2}$. Then we see that

$$e^{h(Y_t)-h(x)} = e^{-\frac{\alpha y^2}{2} + \bar{G}(Y_t) + \frac{\alpha x^2}{2} - \bar{G}(x)} \leq e^{\frac{\alpha y^2}{2} - \bar{G}(x)} e^{-\frac{\alpha y^2}{2}},$$

and

$$\phi(t, x, y) = \mathbb{P}^{x,y}(e^{-\frac{1}{2} \int_0^t \hat{g}(Y_s) ds})$$

$$= \mathbb{P}^{x,y}(e^{\int_0^t (-\frac{1}{2} (-\alpha Y_s + g(Y_s))^2 - \frac{1}{2} (-\alpha + g(Y_s))^2) ds})$$

$$\leq e^{\frac{\alpha + \frac{\alpha y^2}{2}}{2} t}$$

Thus we have

$$p(1, x, y) \leq \frac{1}{\sqrt{2\pi}} e^{\frac{\alpha y^2}{2} - \bar{G}(x)} e^{\frac{-\alpha y^2}{2}} e^{\frac{-\alpha y^2}{2}} = C_1 e^{-\frac{\alpha y^2}{2} + C_2}$$

for some constants $C_1, C_2$ which are independent of $y$. Thus $\sup_{x \in K} p(1, x, y) \leq c_1 e^{-cy^2}$ for any compact set $K \subset \mathbb{R}$. From the main theorem in [Rog85] we also know that $p(t, x, y)$ is continuous in $x$. Hence we can apply the dominated convergence theorem to establish (B-1).
Appendix C: Proof of Lemma 4.1

Using the sublinear growth condition on $\sigma$ we have

$$|\chi(y)| = \left| \int_{y_0}^{y} \sigma(u)du \right| \leq \int_{y_0}^{y} K_1(1+|u|^p)du \leq K_1|y-y_0| + K_1 \int_{y_0}^{y} |u|^pdu \leq K_1|y-y_0| + \frac{K_1}{1+p}(|y|^{1+p} + |y_0|^{1+p}) .$$

Thus $\limsup_{|y| \to \infty} \frac{|\chi(y)|}{|y|^r} \leq \frac{K_1}{1+p}$ which implies that

$$\lim \inf_{|y| \to \infty} \frac{|\chi^{-1}(y)|}{|y|^r} \geq \tilde{K} \quad (C-1)$$

for some $\tilde{K} > 0$, where $r = \frac{1}{p+1} \in (\frac{1}{2}, 1)$ (note that $\chi^{-1}(.)$ is well defined because $\chi'(y) = \sigma(y) > 0$). Then from the analysis in the previous Appendix, we have

$$\mathbb{P}(\chi(Y_t) > tu) = \mathbb{E} \mathbb{Q}[e^{\int_{y_0}^{Y_t} \sigma(Y_s)ds - \frac{1}{2} \int_{y_0}^{Y_t} \sigma^2(Y_s)ds}1_{Y_t > \chi^{-1}(tu)}] \leq \mathbb{E} \mathbb{Q}[e^{\int_{y_0}^{Y_t} \sigma(Y_s)ds - \frac{1}{2} \int_{y_0}^{Y_t} \sigma^2(Y_s)ds}1_{Y_t > \chi^{-1}(tu)}] \leq c_1 e^{-c_2 t^{2r}}$$

for $t$ sufficiently large, for some constants $c_1, c_2 > 0$, where we have used (C-1) in the penultimate line and that $Y_t \sim N(y_0, t)$ under $\mathbb{Q}$.

Appendix D: Proof of Proposition 5.1

To show that $S_t = e^{-\frac{1}{2} \int_{0}^{t} \sigma^2(Y_s)ds + \int_{0}^{t} \sigma(Y_s)(\delta dW^1_s + \rho dW^2_s)}$ is a martingale, by Corollary 5.13, p.199 in [KS91], it is sufficient to check the Novikov condition:

$$\mathbb{E}(e^{\frac{1}{2} \int_{0}^{t} \sigma^2(Y_s)ds}) < \infty; \quad 0 \leq t < \infty .$$

Fix $0 < t < \infty$. Define $u_n$ as in the proof of Lemma 2.2. Then, as in the proof of Lemma 2.2, $-\frac{L_{un}}{un}(y) \to c_0 y^2 - c_1 yg(y) - c_2$ pointwise as $n \to \infty$, where $c_0 > 0$. Thus by Fatou’s lemma we have

$$\int_{-\infty}^{\infty} (c_0 y^2 - c_1 yg(y) - c_2)\mu(dy) \leq \liminf_{n \to \infty} \int_{-\infty}^{\infty} -\frac{L_{un}}{un}(y)\mu(dy) \quad a.s. $$

and

$$\mathbb{E}[e^{\int_{-\infty}^{\infty} (c_0 y^2 - c_1 yg(y) - c_2)\mu(dy)}] \leq \mathbb{E}(e^{\liminf_{n \to \infty} \int_{-\infty}^{\infty} -\frac{L_{un}}{un}(y)\mu(dy)}) = \mathbb{E}(\liminf_{n \to \infty} e^{\int_{-\infty}^{\infty} -\frac{L_{un}}{un}(y)\mu(dy)}) \leq \liminf_{n \to \infty} \mathbb{E}[e^{\int_{-\infty}^{\infty} -\frac{L_{un}}{un}(y)\mu(dy)}].$$

As in the proof of Lemma 2.2, using the sublinear growth of $g$, we can find a constant $C_1$ such that $c_0 y^2 - c_1 yg(y) - c_2 \geq \frac{c_0}{2} y^2 - C_1$. From this we see that

$$\mathbb{E}[e^{-C_1 \int_{0}^{t} e^{\frac{1}{2} \int_{0}^{s} \sigma^2(Y_u)du}ds}] \leq \liminf_{n \to \infty} \mathbb{E}[e^{\int_{-\infty}^{\infty} -\frac{L_{un}}{un}(y)\mu(dy)}]. \quad (D-1)$$

The righthand side in (D-1) can be bounded as

$$\mathbb{E}[e^{-\int_{0}^{t} \frac{L_{un}}{un}(Y_u)ds}] \leq e^{log u_n(Y_t) \mathbb{E}[e^{log u_n(Y_t) - log u_n(Y_0) - \int_{0}^{t} \frac{L_{un}}{un}(Y_u)du}]} \leq u_n(Y_0), \quad (D-2)$$

where the inequality follows because $log u_n(y) = \frac{c_0}{2} [\sigma(y)^2] \geq 0$, and the last equality follows because $M_t = e^{log u_n(Y_t) - \int_{0}^{t} \frac{L_{un}}{un}(Y_u)du}$ is a local martingale with $M_0 = 1$. Applying this to (D-1) and using the definition of $u_n(y)$ we get

$$\mathbb{E}[e^{-C_1 \int_{0}^{t} e^{\frac{1}{2} \int_{0}^{s} \sigma^2(Y_u)du}ds}] \leq e^{\frac{c_0}{2} Y_t^2} < \infty. \quad (D-3)$$
From Assumption 3.1 we know that $\sigma^2(y)$ has sub-quadratic growth and hence there exists a constant $C_2$ such that $\frac{1}{2}\sigma^2(y) \leq \frac{C_2}{2}y^2 + C_2$. Therefore

$$E[e^{\frac{1}{2} \int_0^t \sigma^2(Y_s) ds}] \leq E[e^{C_2t} e^{\frac{C_0}{2} \int_0^t Y_s^2 ds}] \leq e^{\frac{C_0}{2}y_0^2 + C_2t + C_1t} < \infty$$

from (D-3).