Applications of large deviations in finance and mathematical physics.

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Motivation and definition of the large deviation principle (LDP).

Examples - Brownian motion, Cramér’s theorem, Lévy processes, Sanov’s theorem.

The Brownian sheet.

Saddlepoint methods; the Feynman path integral.

The Donsker-Varadhan LDP for the occupation measure of the Ornstein-Uhlenbeck process $dY_t = -\theta Y_t dt + dW_t$ for $\theta > 0$.

Applications to stochastic volatility models - the Ornstein-Uhlenbeck and CEV-Heston models.

Large deviations for the maximum likelihood estimator of $\theta$.

Application to SPDEs - Freidlin-Wentzell theory for the stochastic heat equation.
The Large deviation principle (LDP): motivation

- Suppose we have sequence of random variables \((X_n)\) such that \(X_n\) is concentrated around \(x_0\) as \(n \to \infty\), and for sets \(A\) away from \(x_0\), \(\mathbb{P}(X_n \in A)\) tends to zero exponentially rapidly in \(n\):

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) = -I(A)
\]

i.e. \(\forall \delta > 0, \quad e^{-n(I(A)+\delta)} \leq \mathbb{P}(X_n \in A) \leq e^{-n(I(A)-\delta)}\)

for \(n = n(\delta)\) sufficiently large, and some rate function \(I \geq 0\).

- Example: for standard Brownian motion \((W_t)\), \(W_t \to 0\) a.s. as \(t \to 0\) and (by SLLN) \(\frac{W_t}{t} \to 0\) a.s. as \(t \to \infty\), but

\[
\lim_{t \to 0} t \log \mathbb{P}(W_t > x) = -\frac{1}{2}x^2, \\
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}\left(\frac{W_t}{t} > x\right) = -\frac{1}{2}x^2
\]

for \(x > 0\).
**Definition.** A sequence of random variables \((X_n)\) in a topological space \(S\) satisfies the LDP with non-negative lower semicontinuous rate function \(I\) if we have the following exponential upper/lower bounds for \(A \in \mathcal{B}(S)\):

\[
- \inf_{x \in \overline{A}} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) \leq - \inf_{x \in \bar{A}} I(x).
\]

**Definition.** \(X_n\) is said to satisfy the weak LDP if

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in B_\delta) = -I(x).
\]
Cramér’s theorem. Let \((X_i)\) be an i.i.d. sequence of random variables with finite mean \(E(X_1) < \infty\) and cumulant generating function

\[
V(p) = \log E(e^{pX_1}).
\]

Then \(\tilde{S}_n = \frac{1}{n} \sum_{i=1}^n X_i\) satisfies the LDP with rate function equal to the Fenchel-Legendre transform \(V^*(x) = \sup_{p \in \mathbb{R}} \{px - V(p)\}\). For Brownian motion, \(V(p) = \frac{1}{2} p^2\), \(V^*(x) = \frac{1}{2} x^2\).

A Lévy process \((X_t)\) has i.i.d. increments, so \((\frac{X_t}{t})\) satisfies an LDP as \(t \to \infty\) with rate function \(V^*(x)\).
Sketch proof of Cramér’s theorem

- Cramér upper bound proved using a simple Chebychev argument:

\[ \mathbb{P}(\bar{S}_n \geq x) = \mathbb{E}(1_{\{s_n \geq nx\}}) \leq \mathbb{E}(e^{-\theta nx} e^{\theta S_n}) = e^{-n\theta x} e^{nV(\theta)}. \]

We then tighten the bound by taking the inf over \( \theta \) on the right hand side:

\[ \mathbb{P}(\bar{S}_n \geq x) \leq e^{-n \sup_\theta [\theta x - V(\theta)]} = e^{-nV^*(x)}. \]

- Lower bound is obtained by changing to a different measure \( \mathbb{P}_{\theta^*}(x) \) under which \( \{\bar{S}_n \geq x\} \) is no longer a rare, large deviation event.
Let \((X_i)\) be a sequence of \(n\) i.i.d. random variables in \(\mathbb{R}\) with common probability measure \(\mu\). The sample distribution:

\[
L^n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}
\]

is a random probability measure (a.k.a. the empirical measure).

Let \(P_n = \mu^n \circ (L^n)^{-1}\) denote the distribution of \(L^n\), where \(\mu^n\) is the product measure. \(P_n\) is a probability measure on \((\mathcal{P}(\mathbb{R}), \mathcal{B}(\mathcal{P}(\mathbb{R}))\) - i.e. \(P_n \in \mathcal{P}(\mathcal{P}(\mathbb{R}))\). By SLLN, we can show that \(P_n \overset{w}{\to} \delta_{\mu}\).

**Theorem** (Sanov). \((L^n)\) satisfies an LDP in the topology of weak convergence\(^1\) as \(n \to \infty\) with rate function given by the infinite dimensional counterpart of \(V^*(x)\):

\[
R(\nu | \mu) = \sup_{p \in B(\mathbb{R})} \left[ \int p d\nu - \log \int e^p d\mu \right],
\]

where \(B(\mathbb{R})\) is the space of bounded, measurable functions on \(\mathbb{R}\). (see [Var10]), so \(\mathbb{P}(L^n \in A) \approx e^{-t \inf_{\nu \in A} R(\nu | \mu)}\) for \(\mu \notin A\).

\(^1A \subseteq \mathcal{P}(\mathbb{R})\) is closed iff for any \((\mu_n) \in A\) with \(\mu_n \overset{w}{\to} \mu \in \mathcal{P}(\mathbb{R})\), we have \(\mu \in A\).
By solving the variational problem on the previous slide, we can show that the rate function simplifies to

\[ R(\nu|\mu) = \begin{cases} 
\int_{-\infty}^{\infty} (\log \frac{d\nu}{d\mu}) d\nu = \int_{-\infty}^{\infty} \frac{d\nu}{d\mu} (\log \frac{d\nu}{d\mu}) d\mu & \text{if } (\nu \ll \mu), \\
\infty & \text{otherwise}
\end{cases} \]
Let \((Z_t)\) be the Brownian sheet, i.e. the centred Gaussian process on \([0, 1]^2\) with zero mean and covariance structure:

\[ \mathbb{E}(Z_t Z_s) = (s_1 \wedge t_1)(s_2 \wedge t_2) \]

where \(t = (t_1, t_2), s = (s_1, s_2)\). \(Z\) is a “two-parameter” Brownian motion.

Then \(\sqrt{\epsilon} Z\) satisfies the LDP on \(C_0([0, 1]^2)\) with rate function

\[
I(f) = \begin{cases} 
\frac{1}{2} \int_{[0,1]^2} \left( \frac{\partial^2 f}{\partial s \partial t} \right)^2 \, dsdt & \text{if } \frac{\partial^2 f}{\partial s \partial t} \in L^2, \\
+\infty & \text{otherwise}.
\end{cases}
\]
Saddlepoint approximations

The LDP gives crude exponential bounds. For a Lévy process \((X_t)\) with density \(p_t(x)\), we can sharpen these bounds using saddlepoint methods, proved using contour integration:

▶ **Large-time estimate**

\[
p_t(xt) \sim \frac{e^{-t(p^* x - V(p^*))}}{\sqrt{2\pi t V''(p^*)}} = \frac{e^{-tV^*(x)}}{\sqrt{2\pi t V''(p^*)}} \quad (t \to \infty)
\]

(see F-López, Forde&Jacquier[FLFJ11]).

▶ **Tail estimate**

\[
p_t(x) \sim \frac{e^{-p^*(\frac{x}{t})x + tV(p^*(\frac{x}{t}))}}{\sqrt{2\pi t V''(p^*(\frac{x}{t}))}} \quad (x \to \infty).
\]

(see F-López, Forde[FLF11]). \(p^* = p^*(x)\) is the unique solution to the saddlepoint equation \(V'(p^*) = x\).

▶ Similar saddlepoint estimates can be obtained for the well known **Heston** stochastic volatility model for large-time[FJ09],[FJM10], small-time[FJL10] and tail regimes (see Friz et al.[FGGS10]).
Consider the Feynman path integral for a wavefunction $\psi(x, t)$:

$$
\psi(x, t) = (2\pi i)^{-\frac{n}{2}} \int_{\gamma : \gamma_t = x} e^{\frac{i}{\hbar} \left[ \frac{1}{2} m \int_0^t \dot{\gamma}^2 d\tau - \int_0^t V(\gamma_\tau) d\tau \right]} \psi(\gamma_0, 0) \mathcal{D}\gamma
$$

$$
= (2\pi i)^{-\frac{n}{2}} \int_{\mathcal{H}} e^{-\frac{i}{\hbar} \frac{m}{2} \int_0^t \dot{\gamma}^2 d\tau} d\mu(\gamma)
$$

for $x \in \mathbb{R}^n$, with $\psi(x, 0) = e^{\frac{i}{\hbar} f(y)} \chi(y)$.

The first line is the formal expression for the path integral which we define rigorously via the Fresnel integral in the second line over $\mathcal{H} = \{ \gamma \in C[0, t] : \dot{\gamma} \in L^2[0, t], \gamma_t = x \}$ for $V, \psi(., 0) \in \mathcal{F}(\mathcal{H})$.

$f(\gamma) = e^{-\frac{i}{\hbar} \int_0^t V(\gamma_\tau) d\tau} \psi(\gamma_0, 0)$ is the Fourier transform:

$$
f(\gamma) = \int_{\mathcal{H}} e^{(\gamma, \gamma_1)} d\mu(\gamma_1)
$$

of $\mu \in \mathcal{M}(\mathcal{H})$ with B.V., where $(\gamma, \gamma_1) = \frac{m}{\hbar} \int_0^1 \dot{\gamma} \dot{\gamma}_2 d\tau$, [AHM08]$^2$.

$\psi(x, t)$ satisfies the Schrödinger eq: $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V(x)\psi$.

$^2$Feynman integral can also be defined via analytic continuation of Wiener measure.
Letting $\hbar \to 0$ - the semi-classical expansion

- The integrand $e^{i\frac{\hbar}{\hbar}S_t} = e^{i\frac{\hbar}{\hbar} \left[ \frac{1}{2} m \int_0^t \dot{\gamma}^2 d\tau - \int_0^t V(\gamma_{\tau}) d\tau \right]}$ is an infinite-dimensional oscillatory integral. If we let $\hbar \to 0$, we tend towards classical everyday Newtonian mechanics and the integral becomes highly oscillatory, so we expect the main contribution to come from the classical path $\gamma^*$ which make $S_t$ stationary (in analogy with the finite-dimensional method of stationary phase).

- From this we can compute the semi-classical expansion:

$$\psi(x, t) \sim (2\pi i)^{-\frac{n}{2}} \frac{1}{\sqrt{\det(...)}} e^{i\frac{\hbar}{\hbar} \left[ \frac{1}{2} m \int_0^t (\dot{\gamma}^*)^2 d\tau - \int_0^t V(\gamma_{\tau}^*) d\tau \right]} \chi(y)$$

as $\hbar \to 0$ (see [AHM08]).

- The stationary path $\gamma^*$ is just the classical path $m\ddot{\gamma} = -\nabla V$ followed by a particle moving under the potential $V(x)$, which goes from $y$ to $x$ in time $t$ with initial momentum $f'(y)$ (IF there is a unique non-degenerate stationary path $\gamma^*$ with this property).
The Donsker-Varadhan LDP for the occupation measure of the Ornstein Uhlenbeck process

Let $dY_t = -\theta Y_t dt + dW_t$ be an OU process for $\theta > 0$. Let

$$\mu_t(A) = \frac{1}{t} \int_0^t 1_A(Y_s) ds$$

denote the proportion of time that $Y$ spends in $A$, for $A \in \mathcal{B}$. For each $t > 0$ and $\omega$, $\mu_t(\omega,.) \in \mathcal{P}$. Then from [DV76] (or [Str84]) $\mu_t(.)$ satisfies the LDP as $t \to \infty$ in the topology of weak convergence, with a good\(^3\), convex, lower semicontinuous rate function given by:

$$I_B(\mu) = -\inf_{u \in D^+} \int_{-\infty}^{\infty} \frac{Lu}{u} d\mu$$

where $L = -\theta y \frac{d}{dy} + \frac{1}{2} \frac{d^2}{dy^2}$ is the infinitesimal generator for $Y$ and $D^+$ is the set of $u$ in the domain $D$ of $L$ with $u > \epsilon$ for some $\epsilon > 0$.

\(^3\)good means that the level set $\{x : I(x) \leq \alpha\}$ is compact.
We can simplify \( I_B \) to the following:

\[
I_B(\mu) = \frac{1}{2} \int_{-\infty}^{\infty} |\partial_y \sqrt{(\frac{d\mu}{d\mu_\infty})(y)}|^2 \mu_\infty(dy)
\]

for \( \mu \ll \mu_\infty \), where \( \mu_\infty(y) = \left(\frac{\theta}{\pi}\right)^{\frac{1}{2}} e^{-\theta y^2} \) is the unique stationary distribution for \( Y \), i.e. \( N(0, 1/(2\theta)) \). \( \frac{d\mu}{d\mu_\infty} \) is the Radon-Nikodým derivative. If \( \mu \) is not absolutely cts wrt \( \mu_\infty \), then \( I_B(\mu) = \infty \).

\( \mathcal{P}(\mathbb{R}) \) can be made into a (non-compact) metric space using the Prokhorov metric.

\( I_B(\mu) \) clearly attains its minimum value of zero at \( \mu = \mu_\infty \), and we can show that \( \mu_\infty \) is the unique minimizer of \( I_B(\mu) \).
Consider a stochastic volatility model for a log stock price process $X_t = \log S_t$:

\[
\begin{aligned}
    dX_t &= -\frac{1}{2} \sigma^2(Y_t) dt + \sigma(Y_t) dW_t^1, \\
    dY_t &= -\theta Y_t dt + dW_t^2
\end{aligned}
\]

for $\theta > 0$, where $f(y) = \sigma^2(y)$ is a continuous non-decreasing function with $0 < f_{\min} \leq f(y) \leq f_{\max}$ and $d\langle W_1, W_2 \rangle = 0$ with $x_0 = 0$.

The distribution of $X_t$, conditional on $A_t = \frac{1}{t} \int_0^t \sigma^2(Y_s) ds$, is $N(-\frac{1}{2} A_t t, A_t t)$. 

Applications of large deviations in finance and mathematical physics.
Using the contraction principle

Let $F(\mu) = \int_{-\infty}^{\infty} f(y) \mu(dy)$ for $\mu \in \mathcal{P}(\mathbb{R})$. Then we can re-write $A_t$ as

$$A_t = F(\mu_t) = \int_{-\infty}^{\infty} f(y) \mu_t(dy) = \frac{1}{t} \int_{0}^{t} f(Y_s)ds.$$  

$F : \mathcal{P}(\mathbb{R}) \mapsto [f_{\min}, f_{\max}]$ is a bounded, continuous functional $^4$, because if $\mu_n \xrightarrow{w} \mu$ then $\int f(y) \mu_n(dy) \rightarrow \int f(y) \mu(dy)$, because $f \in C_b$.

Thus, by the contraction principle from large deviations theory, $A_t$ also satisfies the LDP, with rate function

$$I_f(a) = \inf_{\mu \in \mathcal{P}(\mathbb{R}) : F(\mu) = a} I_B(\mu), \quad a \in [f_{\min}, f_{\max}]. \quad (2)$$

$^4$in the topology of weak convergence.
**Proposition** [Forde11a]. \((X_t/t, A_t)\) satisfies a LDP on \(\mathbb{R} \times [f_{\text{min}}, f_{\text{max}}]\) as \(t \to \infty\) with rate function

\[
I(x, a) = aV^*(\frac{x}{a}) + l_f(a)
\]

where \(V^*(x) = \frac{1}{2}(x + \frac{1}{2})^2\).

**Sketch proof.** Let \(Z_t = X_t/t\). We first note that \((Z_t, A_t) \overset{d}{=} (\frac{1}{t} W_{tA_t} - \frac{1}{2} A_t, A_t)\). Conditioning on \(A_t\), formally we have

\[
P(|Z_t - x| < \delta, |A_t - a| < \delta) \approx \cdots \times e^{-aV^*(\frac{x}{a})/t} e^{-l_f(a)/t}
\]

as \(t \to \infty\), where \(aV^*(\frac{x}{a})\) is the rate function of \(W_{ta} - \frac{1}{2} a\), for a fixed. This argument can be made rigorous.
**Corollary** [Forde11a]. \((X_t/t)\) satisfies the LDP as \(t \to \infty\) with a good rate function given by

\[
I(x) = \inf_{a \in [f_{\min}, f_{\max}]} \left\{ \frac{(x + \frac{1}{2}a)^2}{2a} + I_f(a) \right\} \leq \frac{(x + \frac{1}{2}\bar{\sigma}^2)^2}{2\bar{\sigma}^2} \tag{3}
\]

**Proof** The LDP with a good rate function just follows from the contraction principle.

This can be applied to price call options with value \(\mathbb{E}(e^{X_t} - K)^+\).

We can relax the assumption that \(\sigma\) is bounded to a sublinear growth condition \(\sigma(y) \leq A(1 + |y|^p), \ A > 0, \ p \in (0, 1);\) in this case we take the infimum over all \(a \in (0, \infty)\) in (3) (see [Forde11b]).

The LDP can be also be extended to a Lévy process or a CEV process evaluated the OU time-change \(\int_0^t f(Y_s)ds\).
For the case of sublinear growth, the following lemma is the key observation:

**Lemma.** If $I_B(\mu) \leq \alpha$ and $k \in (0, 1)$, we have

$$
\int_{-\infty}^{\infty} y^2 \mu(dy) \leq \frac{\alpha + k}{2k(1-k)}.
$$

**Proof.** If we consider the test function $u = e^{ky^2}$ in $I_B(\mu) = -\inf_{u \in D^+} \int_{-\infty}^{\infty} \frac{Lu}{u} d\mu$, then $-\frac{Lu}{u}(y) = k[2(1-k)y^2 - 1]$. From this we obtain

$$
\alpha \geq I_B(\mu) = -\inf_{u \in D^+} \int_{-\infty}^{\infty} \frac{Lu}{u} d\mu = \sup_{u \in D^+} -\int_{-\infty}^{\infty} \frac{Lu}{u} d\mu
$$

$$
\geq \int_{-\infty}^{\infty} k[2(1-k)y^2 - 1] \mu(dy)
$$

$$
= 2k(1-k) \int_{-\infty}^{\infty} y^2 \mu(dy) - k.
$$
The CEV model

- The CEV model is defined by the SDE
  \[ dS_t = \delta S_t^\beta \, dW_t \]  
  (4)
  with \( \beta \in (0, 1) \), \( \delta > 0 \) and \( S = 0 \) absorbing so \((S_t)\) is a martingale.

- The transition density is
  \[ p(t, S_0, S) = \frac{S^{-2\bar{\beta}} - \frac{3}{2} S_0^{\frac{1}{2}}}{\delta^2 |\bar{\beta}| t} \exp\left( - \frac{S_0^{-2\bar{\beta}} + S^{-2\bar{\beta}}}{2\delta^2 \bar{\beta}^2 t} \right) I_{\nu}\left( \frac{S_0^{-\bar{\beta}} S^{-\bar{\beta}}}{2\delta^2 \bar{\beta}^2 t} \right) \]  
  (S > 0),
  where \( \bar{\beta} = \beta - 1 \), \( \nu = \frac{1}{2|\bar{\beta}|} \), and \( I_{\nu}(\cdot) \) is the modified Bessel function of the first kind (see [DavLin01]).

- **Proposition.** Let \( \gamma = 1/|\bar{\beta}|. \) Then using (5) we can show that \((S_t/t^\gamma)\) satisfies the LDP on \([0, \infty)\) as \( t \to \infty \) with continuous rate function
  \[ I_{\text{CEV}}(K) = \frac{K^2|\bar{\beta}|}{2\delta^2 \bar{\beta}^2} \]  
  \((K \geq 0)\).
Combining the CEV model with a CIR time-change, we can define the uncorrelated \textit{CEV-Heston model}, governed by the following SDEs

\[
\begin{align*}
\frac{dS_t}{S_t} &= \beta \sqrt{Y_t} dW^1_t, \\
\frac{dY_t}{Y_t} &= \kappa(\theta - Y_t) dt + \sigma \sqrt{Y_t} dW^2_t
\end{align*}
\]

with \(dW^1_t dW^2_t = 0\), \(Y_0 = y_0 > 0\).

Conditioning on \(\int_0^t Y_s ds\), we can write \(S_t = X \int_0^t Y_s ds\), where \(X\) is now just the standard CEV process \(dX_t = \delta X_t^\beta \, dW_t\) with \(\delta = 1\).

By a similar argument to that used for the OU model, we have: \textit{Proposition}. \((S_t/t^\gamma)\) satisfies the LDP on \([0, \infty)\) as \(t \to \infty\) with a good rate function given by

\[
l_{\text{CEVH}}(K) = \inf_{a \in (0, \infty)} [a l_{\text{CEV}}(\frac{K}{a^\gamma}) + l_{\text{CIR}}(a)] \leq \theta l_{\text{CEV}}(\frac{K}{\theta^\gamma}) \quad (K \geq 0),
\]

where \(l_{\text{CIR}}(a)\) is the rate function of \(A_t = \frac{1}{t} \int_0^t Y_s ds\), and the infimum of \(l\) is attained uniquely at \(K = 0\), where \(l(K) = 0\).
We can show that call options have the same large-time behaviour

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(S_t - Kt^\gamma)^+ = I_{CEVH}(K).$$

For the large-time, fixed-strike regime, we can show that

$$S_0 - \mathbb{E}(S_t - K)^+ = cK (\theta t)^{-\frac{\gamma}{2}} (1 + o(1)) \quad (t \to \infty)$$

where

$$c = \frac{1}{\Gamma\left(1 + \frac{\gamma}{2}\right)} \left[ \frac{1}{2} \left( \frac{S_0^{-2\beta}}{\delta^2 \beta^2} \right) \right]^{\frac{\gamma}{2}}.$$
The large-maturity smile for the CEV-Heston model

Figure: Here we have plotted the implied volatility for the CEV-Heston model in the large-time, large-strike regime for $t = 30$ years using Corollary 7.1 in Gao&Lee[GL11], with $\delta = 1, \beta = .7, S_0 = 1$ and $\kappa = 1.15, \theta = .04, \sigma = 0.2$. Working in the large-time, large-strike parameterizaton allows us to see the slope and the convexity effect.
The Maximum likelihood estimator of $\theta$ for the OU process

Let $\theta_0$ denote the true value of $\theta$.

Let $\mathbb{P}^T_{\theta}$ be the measure induced on $(C[0, T], \mathcal{B}(C[0, T]))$ by the solution of $dY_t = -\theta Y_t dt + dW_t$. Then, from Girsanov’s theorem, we have the likelihood ratio

$$L(\theta) = \frac{d\mathbb{P}^T_{\theta}}{d\mathbb{P}^T_0} = e^{-\int_0^T \theta Y_t dY_t - \frac{1}{2} \int_0^T \theta^2 Y_t^2 dt}$$

(note that $\mathbb{P}^T_0$ is just the Wiener measure).

Taking the log of $L(\theta)$, differentiating wrt $\theta$ and setting to zero, we obtain the classical maximum likelihood estimator for $\theta$:

$$\hat{\theta}_T = -\frac{\int_0^T Y_t dY_t}{\int_0^T Y_t^2 dt},$$

(see [Kut04]) and $\hat{\theta}_T$ is a consistent estimator of $\theta_0$ (i.e. $\hat{\theta}_T \to \theta_0$ in probability as $T \to \infty$).
It can be shown (see [FLP99]) that $\hat{\theta}_T$ satisfies the LDP with good rate function

$$J(\theta) = \begin{cases} \frac{1}{4\theta}(\theta - \theta_0)^2 & (\theta \geq \frac{1}{3}\theta_0), \\ -2\theta + \theta_0 & (\theta < \frac{1}{3}\theta_0). \end{cases}$$

Figure: Here we have plotted $J(\theta)$ for $\theta = .04$. 

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Applications of large deviations in finance and mathematical physics.
Consider the stochastic heat equation with small-noise:

\[ \partial_t u^\epsilon_{t,x} = \frac{1}{2} \partial_{xx} u^\epsilon_{t,x} + \sqrt{\epsilon} \dot{W}_{t,x} \]  

(7)
on $[0, T] \times [0, 1]$, with Dirichlet boundary condition $u^\epsilon_{0,x} \in C^{2\alpha}$, $0 \leq \alpha < \frac{1}{4}$ and $u^\epsilon_{t,0} = u^\epsilon_{t,1} = 1$. $\dot{W}$ is space-time white noise, which is a Gaussian random set function such that $W_A \sim N(0, \text{Leb}(A))$ for $A \in \mathcal{B}([0, T] \times [0, 1])$ and $\mathbb{E}(W_A W_B) = \text{Leb}(A \cap B)$.

$W_{t,x} := W_{[0,t] \times [0,x]}$ is the previously defined Brownian sheet.

We can give a rigorous meaning to (7) by writing the solution in the integrated form

\[ u^\epsilon_{t,x} = \int_0^1 G_t(x, y) u_0(y) dy + \sqrt{\epsilon} \int_0^t \int_0^1 G_{t-s}(x, y) W(ds, dy) \]

where the stochastic integral on the right is defined in a similar way to the classical Itô integral, and $G_t(x, y)$ is the usual Green kernel for the non-stochastic heat eq $\partial_t u = \frac{1}{2} \partial_{xx} u$ with the same Dirichlet boundary condition (see Pardoux[Par93]).
The skeleton of $h = h(t, x)$ in the Cameron-Martin space for $W$ is given by

$$Z^h_{t, x} = \int_0^1 G_t(x, y)u_0(y)dy + \sqrt{\epsilon} \int_0^t \int_0^1 G_{t-s}(x, y) \frac{\partial^2 h}{\partial t \partial x}(s, y)dsdy.$$  

By a generalized contraction principle, $u^\epsilon$ satisfies the LDP on $\chi = C^{x, 0}([0, T] \times [0, 1])$ with rate function

$$S(f) = \begin{cases} 
\inf \{I(h) : Z^h = f\}, & f \in \text{Im}(Z) \\
+\infty & (\text{otherwise})
\end{cases}$$ (8)

(see [CM07]), where $I(h) = \frac{1}{2} \int_{[0,1]^2} \left( \frac{\partial^2 h}{\partial s \partial t} \right)^2 dsdt$ is the previously defined rate function for the Brownian sheet.

We can also compute a small-noise LDP for the alternative way of approaching SPDEs as a Hilbert-space valued SDE driven by a Hilbert-spaced valued Brownian motion (see [DPZ92]).
References


