Portfolio optimization for an exponential Ornstein-Uhlenbeck model with proportional transaction costs

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(joint work with Christoph Czichowsky, Philipp Deutsch and Hongzhong Zhang)
Outline of talk

- The exponential Ornstein-Uhlenbeck model with proportional transaction costs.

- Admissible self-financing trading strategies.

- Shadow price processes - definition and why we use them.

- Explicit construction of the shadow price process for the exponential OU model.

- Asymptotics for the no-trade region and the risky fraction when the transaction cost is small. Results extend the work of [GMS13], who deal with the Black-Scholes case, and show new phenomena.

- The verification argument, and links to excursion theory.

- Brief discussion on duality.
We work on some \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) and consider a financial market with one riskless bond with a constant price equal to 1 (i.e. zero interest rates) and a risky asset \(S_t\). We assume that \(S_t\) is given by the exponential of an Ornstein-Uhlenbeck process \(S_t = e^{X_t}\), where

\[
dX_t = \kappa (\bar{x} - X_t) dt + \sigma dW_t
\]

with \(\kappa, \bar{x}, \sigma > 0\). By Itô’s formula, \(S_t\) satisfies

\[
dS_t/S_t = [\kappa (\bar{x} - \log(S_t)) + \frac{\sigma^2}{2}] dt + \sigma dW_t =: \mu(S_t) dt + \sigma dW_t.
\]

We now model the bid-ask interval by \([(1 - \lambda)S_t, S_t]\) for some \(\lambda \in (0, 1)\). The investor pays \(S_t\) for each share bought, but only receives \((1 - \lambda)S_t\) for each share sold. Let \((\varphi_0^0, \varphi_t)\) denote our holding in the riskless and risky asset at time \(t\).

Investor wishes to maximize his expected long-term growth rate:

\[
\liminf_{T \to \infty} \frac{1}{T} \mathbb{E}[\log V_T(\varphi_0^0, \varphi_T)]
\]

where \(V_t((\varphi_0^0, \varphi_t)) = \varphi_0^0 + \varphi_t^+ (1 - \lambda)S_t - \varphi_t^- S_t\) is the liquidation value of the portfolio at time \(t\) (investor has log utility).
Admissible trading strategies

Assume the investor starts with \( x \) dollars in cash \( (x > 0) \). Then a pair of adapted processes \((\varphi_0^t, \varphi_t)\) is called an **admissible self-financing trading strategy** if both processes are predictable, have finite variation and:

(i) The **self-financing condition**:

\[
d\varphi_0^t = (1 - \lambda)S_t
d\varphi_t^\uparrow - S_t
d\varphi_t^\downarrow
\]  

for all \( 0 \leq t \leq T \). \( \varphi_t \) has F.V. so \( \varphi_t = \varphi_t^\uparrow - \varphi_t^\downarrow \), where \( \varphi_t^\uparrow, \varphi_t^\downarrow \) are two increasing processes.

(ii) The **solvency condition**: there exists an \( M > 0 \) such that the liquidation value

\[
V_t(\varphi_0^t, \varphi) = \varphi_0^t + \varphi_t^\uparrow(1 - \lambda)S_t - \varphi_t^\downarrow S_t \geq -M
\]

a.s., for all \( 0 \leq t \leq T \).

- The self-financing condition in (1) ensures that no funds are added or withdrawn to the portfolio, and (2) ensures that the investor cannot owe more than \( M \) dollars at any time.
Definition of a shadow price

Definition. A shadow price is a continuous semimartingale \( \tilde{S}_t \in [(1 - \lambda)S_t, S_t] \), such that the optimal trading strategy \((\varphi_t^0, \varphi_t)\) for a fictitious market with price process \(\tilde{S}_t\) and zero transaction costs exists, has finite variation and the number of stocks \(\varphi_t\) only increases when \(\tilde{S}_t = S_t\) and decreases when \(\tilde{S}_t = (1 - \lambda)S_t\).

Clearly any price process \(\tilde{S}_t\) with zero transaction costs which lies in \([(1 - \lambda)S_t, S_t]\) leads to more favourable terms of trade than the original market with transaction costs. But a shadow price process is a particularly unfavourable model, for which it’s optimal to only buy when \(\tilde{S}_t = S_t\), sell when \(\tilde{S}_t = (1 - \lambda)S_t\) + do nothing in between.
Proposition (Corollary 1.9 in Schachermayer et al.[GMS13]).
Let \( \tilde{S}_t \) be a shadow price process whose optimal trading strategy (for zero transaction costs) is given by \((\varphi^0_t, \varphi_t)\), with \( \varphi^0_t, \varphi_t \geq 0 \). Then under non-zero transaction costs, we have

\[
\sup_{(\psi^0, \psi)} \mathbb{E}[\log V_T((\psi^0, \psi))] \geq \mathbb{E}[\log V_T((\varphi^0, \varphi))]
\]

\[
\geq \mathbb{E}[\log V_T((\psi^0, \psi))] + \log(1 - \lambda)
\]

for any admissible \((\psi^0, \psi)\). Thus if we choose \( \lambda \) suff small so that \( |\log(1 - \lambda)| < \varepsilon \) and take the sup over all \((\psi^0, \psi)\), we see that \((\varphi^0, \varphi)\) is an \( \varepsilon \)-optimal trading strategy for the original problem.

Or take \( \liminf \) as \( T \to \infty \) + sup over all admissible strategies, we obtain

\[
\liminf_{T \to \infty} \frac{1}{T} \mathbb{E}[\log V_T((\varphi^0, \varphi))] = \sup_{(\psi^0, \psi)} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}[\log V_T((\psi^0, \psi))].
\]

Thus the optimal portfolio for the shadow price process is asymptotically optimal for the original problem under transaction costs, as \( \lambda \to 0 \) and/or as \( T \to \infty \).
First consider the case when $\lambda = 0$, i.e. zero transaction costs, and assume $S_t$ follows a general Itô process of the form $dS_t = S_t(\mu_t dt + \sigma_t dW_t)$ with zero interest rates.

For the frictionless case, we are looking to maximize:

$$E[\log V_T] = E[\log[x + \int_0^T \phi_t dS_t]]$$

$$= E[\log x + \int_0^T \frac{\phi_t S_t}{x + \int_0^t \phi_t dS_t} dS_t - \frac{1}{2} \int_0^T \frac{\phi_t^2 S_t^2 \sigma_t^2 dt}{(x + \int_0^t \phi_t dS_t)^2}]$$

$$= E[\log x + \int_0^T (\pi_t \mu_t - \frac{1}{2} \pi_t^2 \sigma_t^2 dt)].$$

Maximizing the integrand over all $\pi_t$, we obtain that $\hat{\pi}_t = \frac{\mu_t}{\sigma_t^2}$, which is known as the Merton fraction. For the Black-Scholes case, $\hat{\pi}_t = \hat{\pi} = \mu/\sigma^2$ is constant, but in general $\hat{\pi}_t$ has infinite variation and so will $\varphi_t$ (unlike the case $\lambda > 0$).

For the BS case, $dV_t = \hat{\pi} V_t dS_t / S_t = V_t(\hat{\pi} \mu dt + \hat{\pi} \sigma dW_t)$ is GBM, and so is $\phi_t = \hat{\pi} V_t / S_t$. 

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**Ansatz:** if $S$ increases from $a$ to $b$ without setting a new minimum in the meantime, then we guess that $\tilde{S}_t = g(S_t)$ for $0 \leq t \leq \tau_b$, for some $g \in C^2$ and target value $b = b(a, \lambda)$ to be determined. In general, $\tilde{S}_t = g(S_t; a_t, b(a_t, \lambda))$ where $a_t = \min_{0 \leq u \leq t} S_u$ up to time $\tau_b$. For $t \geq \tau_b$, we set $b_t = \max_{\tau_b \leq u \leq t} S_u$, and then $\tilde{S}_t = g(S_t; a(b_t, \lambda), b_t)$ until $S$ returns to the buy boundary (possibly along a new $g$ curve), and so on.
Explicit construction of the shadow price contd.

Assume that $\tilde{S}_0 = S_0 = a$ and $\tilde{S}_t = g(S_t)$ during an excursion from $S = a$ to $S = b$ (we postulate that no trading occurs until $S$ hits $b$, we will then show how to choose $b = b(a, \lambda)$).

- From the drawing we see that: $g(a) = a$ and $g(b) = (1 - \lambda)b$.
- **Smooth-pasting condition**: $g'(a) = 1$, $g'(b) = 1 - \lambda$ - this ensures that the volatility of $\tilde{S}_t$ vanishes on both boundaries (see below).
- $(1 - \lambda)s \leq g(s) \leq s$ for all $s \in [a, b]$.
- Applying Itô’s formula to the shadow price process, we obtain

\[
d\tilde{S}_t = dg(S_t) = g'(S_t)dS_t + \frac{1}{2}g''(S_t)\sigma^2 S_t^2 dt
\]

or

\[
dg(S_t)/g(S_t) = \hat{\mu}_t dt + \frac{1}{2}\hat{\sigma}_t dW_t,
\]

where

\[
\hat{\mu}_t = [g'(S_t)S_t \mu(S_t) + \frac{1}{2}g''(S_t)\sigma^2 S_t^2]/g(S_t), \quad \hat{\sigma}_t = g'(S_t)\sigma S_t / g(S_t).
\]

- But the optimal risky fraction for an investor who maximizes log-utility (with zero transaction costs) is given by

\[
\frac{\hat{\mu}(s)}{\hat{\sigma}(s)^2} = \frac{(g'(s)s \mu(s) + \frac{1}{2}g''(s)\sigma^2 s^2)/g(s)}{(g'(s)^2 \sigma^2 s^2)/g(s)^2} = \frac{\varphi g(s)}{\varphi^0 + \varphi g(s)}.
\]
ODE for the shadow price

- Multiplying the numerator and denominator of the right hand side of (3) by \( \frac{a}{\varphi^0 + \varphi_a} \) and setting \( \bar{\pi} = \frac{a\varphi}{\varphi^0 + \varphi_a} \) we have

\[
\frac{\bar{\pi} g(s)}{a \frac{\varphi^0}{\varphi^0 + \varphi_a} + \bar{\pi} g(s)} = \frac{\bar{\pi} g(s)}{a \frac{\varphi^0 + a\varphi - a\varphi}{\varphi^0 + \varphi_a} + \bar{\pi} g(s)} = \frac{\bar{\pi} g(s)}{a(1 - \bar{\pi}) + \bar{\pi} g(s)}.
\]

- Combining with (3) yields the following ODE for \( g \):

\[
\frac{1}{2} \sigma^2 s^2 g''(s) = \frac{g'(s)^2 \sigma^2 s^2}{g(s)} \frac{\bar{\pi} g(s)}{a(1 - \bar{\pi}) + \bar{\pi} g(s)} - g'(s) s \mu(s)
\]

\[
= g'(s)^2 \sigma^2 s^2 \frac{\bar{\pi}}{a(1 - \bar{\pi}) + \bar{\pi} g(s)} - g'(s) s \mu(s)
\]

which simplifies to

\[
g''(s) = \frac{2\bar{\pi} g'(s)^2}{a(1 - \bar{\pi}) + \bar{\pi} g(s)} - \frac{2g'(s) s \mu(s)}{\sigma^2 s^2}
\]

\[
= \frac{2\bar{\pi} g'(s)^2}{a(1 - \bar{\pi}) + \bar{\pi} g(s)} - \frac{2g'(s) \theta(s)}{s}
\]

where \( \theta(s) = \frac{\mu(s)}{\sigma^2} \).
Solution for the shadow price

- General solution to ODE in (4) with $g(a) = a, g'(a) = 1$:

\[
g(s) = g(s; a, \bar{\pi}) = a \frac{ah(a) + (1 - \bar{\pi})H(a, s)}{ah(a) - \bar{\pi}H(a, s)}
\]

where $H(a, s) = \int_a^s h(u)du$, $h(s) = \exp\left[\frac{\kappa}{\sigma^2} (\log(s) - \bar{x} - \frac{\sigma^2}{2\kappa})^2\right]$.

- Plugging this into $g(b) = (1 - \lambda)b, g'(b) = 1 - \lambda$ we obtain:

\[
\bar{\pi} = \frac{a(H(a, b) + \lambda bh(a) - bh(a) + ah(a))}{(a + \lambda b - b)H(a, b)}
\]

and $F(a, b, \lambda) = H(a, b)^2(\lambda - 1) + (a + b(\lambda - 1))^2h(a)h(b) = 0$.

- Solving the quadratic for $\lambda$ we find the physically meaningful solution is given by

\[
\lambda(a, b) = 1 - \frac{a}{b} - \frac{1}{2} \frac{H(a, b)^2}{b^2h(a)h(b)} - \frac{H(a, b)}{2b} \sqrt{\frac{H(a, b)^2}{b^2h(a)^2h(b)^2} + \frac{4a}{bh(a)h(b)}}.
\]

- For $\lambda$ fixed, eq has 2 solNs $b = b_{1/2}(a, \lambda)$ with $b_1 < a < b_2$. To choose the physically meaningful solution, it turns out there is a critical $a = a_0(\lambda)$ such that $b = b_2(a, \lambda)$ for $a \geq a_0$ and $b = b_1(a, \lambda)$ for $a < a_0$. 
Asymptotics

- Since $\lambda$ is smooth, we can expand it in a Taylor series around the point $b = a$:

$$\lambda(a, b) = \frac{\Gamma(a)}{6a^3}(b - a)^3 + O((b - a)^4), \quad (6)$$

where $\Gamma(s) = \theta(s)(1 - \theta(s)) - \theta'(s)s$.

- Inverting (6), for $a \notin \{a^*, b^*\}$, we obtain

$$b(a, \lambda) = a + a\left(\frac{6}{\Gamma(a)}\right)^{1/3}\lambda^{1/3} + O(\lambda^{2/3}).$$

- For the risky fraction $\bar{\pi}$, plugging this expansion into (5) we get:

$$\bar{\pi} = \theta(a) - \left(\frac{3}{4}\frac{\Gamma(a)^2}{\Gamma(a)}\right)^{1/3}\lambda^{1/3} + O(\lambda^{2/3}).$$

- For $a = a_0(\lambda)$ there are two $b$-values, and

$$b_{1,2}(a, \lambda) = a^* \pm a^*\sqrt{2}\left(\frac{3\sigma^4}{\kappa\sigma\sqrt{(4\kappa + \sigma^2)}}\right)^{1/4}\lambda^{1/4} + O(\sqrt{\lambda}),$$

$$\bar{\pi} = \theta(a) - \frac{\sqrt{\kappa}(4\kappa + \sigma^2)^{1/4}}{\sqrt{3\sigma^{3/2}}}\lambda^{1/2} + O(\sqrt{\lambda}),$$
The special value $a_0(\lambda)$ does show up for the Black-Scholes model where we do not see the $\lambda^{1/4}$ asymptotic behaviour.

In general $\tilde{S}_t = g(S_t; a_t, \bar{\pi}(a_t, b(a_t, \lambda), \lambda)$ for some continuous process $a_t$ with finite variation, which only increases/decreases when $\tilde{S}_t$ is on boundaries of the bid-ask cone.

If $S_0 = a$, then before $S$ hits $b(a, \lambda)$ or $a_0(\lambda)$, $a_t = \min_{0 \leq u \leq t} S_u$: in words, every time $S_t$ sets a new minimum, we need a new $a$-value for $g(.)$, but when $S_t$ makes an excursion away from its minimum process, $da_t = 0$, and $\tilde{S}_t$ just follows the $g$ curve from left to right.

For $t \geq \tau_b$, we set $b_t = \max_{\tau_b \leq u \leq t} S_u$, and then $\tilde{S}_t = g(S_t; a(b_t, \lambda), b_t)$ until $\tilde{S}$ returns to the buy boundary or $b_t$ hits the critical value $b_0(\lambda)$.

The $g$ curves change direction to the left of $a_0(\lambda)$ and to the right of $b_0(\lambda)$. At these critical values, there are two valid $g$ curves (not one).
We can show that the optimal number of shares $\varphi_t$ evolves as

$$ \frac{d\varphi_t}{\varphi_t} = -\frac{\Gamma(a_t; \lambda)}{\pi(a_t, b(a_t, \lambda), \lambda)} \frac{da_t}{a_t} $$

(7)

where $\Gamma(a; \lambda) = -a\pi'(a) + \pi(a)(1 - \pi(a))$, and here $\pi(a)$ is shorthand for $\pi(a, b(a, \lambda), \lambda)$.

Integrating (7) we get

$$ \log \frac{\varphi_t}{\varphi_0} = F(a_t) := -\int_{a_0}^{a_t} \frac{\Gamma(u; \lambda)}{\pi(u, b(u, \lambda), \lambda)u} du. $$

For $\tilde{S}_t$ to be a genuine shadow price, we have to verify that $d\varphi_t \geq 0$ when $S_t = \tilde{S}_t$, and $d\varphi_t \leq 0$ when $S_t = (1 - \lambda)\tilde{S}_t$ (this is the so-called verification argument).
Figure: Here we have plotted various shadow price curves $g(S; a, b)$ as a function of $s$, for $\lambda = .3$ and $\kappa = 3, \sigma = 1, \bar{x} = 1$, for which we find that $a_0(\lambda) = 1.3914, b_0(\lambda) = 5.31052$ and, and $a_1(\lambda) = 1.95159, a_2(\lambda) = 10.5602$ and $b_1(\lambda) = 0.699707, b_2(\lambda) = 3.78616$. 
Duality

Let \( \tilde{S}_t \in [(1 - \lambda)S_t, S_t] \) and \( Z^0_t \) denote the density process of an ELMM \( Q \) for \( \tilde{S}_t \), and let \( Z^1_t = Z^0_t \tilde{S}_t \).

Let \( V(y) = \sup_{x > 0}[U(x) - xy] \) denote the Fenchel-Legendre transform of \( U \), where \( U \) is a (concave) utility function.

Then (as before) trading the shadow price process is more favourable than trading the real risky asset, so for any admissible trading strategy \((\varphi^0, \varphi)\) we have

\[
\mathbb{E}[U(V_T(\varphi^0, \varphi))] \leq \mathbb{E}[U(x + \varphi_t \cdot d\tilde{S}_t)] \leq \mathbb{E}[V(yZ^0_T) + (x + \varphi_t \cdot \tilde{S}_t)yz^0_T)] \leq \mathbb{E}[V(yZ^0_T)] + xy
\]

because \( \mathbb{E}(Z^0_T) = 1 \) and \( \tilde{S}_t \) is a local martingale under \( Q \).

Taking sups and infs, we see that

\[
\sup_{(\varphi^0, \varphi)} \mathbb{E}[U(V_T(\varphi^0, \varphi))] \leq \inf_{(Z^0, Z^1, y)} \mathbb{E}[V(yZ^0_T)] + xy
\]

If the dual optimizers \((\hat{Z}^0, \hat{Z}^1, y)\) exist then we have equality, and \( U'[V_T(\varphi^0, \varphi)] = \hat{y}\hat{Z}^0_T \). For \( U(x) = \log x \), we have \( V(y) = -\log y - 1 \).
