

# THE SMALL-TIME SMILE AND TERM STRUCTURE OF IMPLIED VOLATILITY UNDER THE HESTON MODEL

MARTIN FORDE, ANTOINE JACQUIER, AND ROGER LEE

ABSTRACT. We characterise the asymptotic smile and term structure of implied volatility in the Heston model at small maturities. Using saddlepoint methods we derive a small-maturity expansion formula for call option prices, which we then transform into a closed-form expansion (including the leading-order and correction terms) for implied volatility. This refined expansion reveals the relationship between the small-expiry smile and all Heston parameters (including the pair in the volatility drift coefficient), sharpening the leading-order result of [Forde, Jacquier, ‘Small-time asymptotics for implied volatility under the Heston model’, *IJTAF*, 12(6): 861-876, 2009] which found the relationship between the zero-expiry smile and the diffusion coefficients.

## 1. INTRODUCTION

Stochastic models are used extensively by traders and quantitative analysts in order to price and hedge financial products. Once a model has been chosen for its realistic features, one has to calibrate it. This calibration must be robust and stable and should not be too computer intensive. This latter constraint often rules out global optimisation algorithms which are very slow despite their accuracy. For this reason closed-form asymptotic approximations have grown rapidly in the past few years. They have proved to be very efficient (i) to provide some information about the behaviour of option prices in some extreme regions such as small or large strikes or maturities (where standard numerical schemes lose their accuracy), (ii) to improve calibration efficiency. Indeed one can first perform an instantaneous calibration on the closed-form and then use this result as a starting point to calibrate the whole model. In practice calibration is often performed using the implied volatility—i.e. the volatility parameter to be used in the Black-Scholes formula in order to match the observed market price—rather than option prices.

For these reasons, there has recently been an explosion of literature on small-time asymptotics for stochastic volatility and exponential Lévy models (see [3], [7], [8], [9], [10], [14], [15], [16], [24] and [22]). All these articles characterise the behaviour of the Black-Scholes implied volatility for European options in the small-maturity limit. Varadhan ([28], [29]) and Freidlin & Wentzell [13] initiated the study of large deviations for strong solutions of stochastic differential equations, and showed that on a logarithmic scale the small-time behaviour of such a diffusion process can be characterised in terms of a distance function on a Riemannian manifold, whose metric is equal to the inverse of the diffusion coefficient. Higher-order expansions in powers of the small time parameter have extended these seminal works. Molchanov [21] provided a rigorous probabilistic proof of this heat kernel expansion at leading order, and Bellaïche [2] improved this expansion for non-compact manifolds under mild technical conditions.

---

*Date:* July 25, 2012.

*1991 Mathematics Subject Classification.* AMS 91G20, 65C50.

*Key words and phrases.* Heston model, implied volatility, asymptotics, saddlepoint expansion, calibration.

In mathematical finance, Henry-Labordère [16] was the first to introduce heat kernel methods to study asymptotics of the implied volatility, both for local and for stochastic volatility models, and initiated a stream of research in this area. For one-dimensional local volatility models Gatheral et al. [14] provided a rigorous proof of the small-time expansion—up to second-order in the maturity—for the transition density and for the implied volatility. On the analytic side, Berestycki et al. [3] showed that for a stochastic volatility model with coefficients satisfying certain growth conditions, the small-maturity implied volatility is given by a distance function obtained as the unique viscosity solution to a non-linear eikonal first-order Hamilton-Jacobi PDE. Paulot [24] derived a small-time expansion for call options under a general local-stochastic volatility model (including the SABR model) by applying the Laplace method to integrate the heat kernel over the range of the volatility variable. It is interesting to note that the small-maturity at-the-money implied volatility has a qualitatively different behaviour than the rest of the smile and cannot be dealt with in the same way.

This heat kernel asymptotic approach however does not apply to the Heston model (2.1) where the variance process follows a square-root diffusion, since the associated Riemannian manifold is not complete (see [16, Chapter 6] for more details about this phenomenon). Using the affine properties of the Heston model, Forde and Jacquier [9] developed a large deviations approach to obtain the small-time behaviour of the implied volatility (the large-maturity case is treated in [11] by analogous arguments, and we refer the interested reader to [25] for a detailed review). In this paper we refine this analysis by providing the first-order correction of the small-maturity expansion for the implied volatility in this model. The methodology in use here—similar to the one used in [12] for the large-maturity case—is based on saddlepoint expansions in the complex plane and the properties of holomorphic functions. We first derive an asymptotic expansion for European call options, which we then translate into implied volatility asymptotics. Namely for a European option written on the underlying  $(S_t)_{t \geq 0}$  with strike  $S_0 e^x$ , we prove that the expansion

$$\sigma_t^2(x) = \sigma_0^2(x) + a(x)t + o(t)$$

holds for any non-zero real number  $x$  for the implied volatility  $\sigma_t$  as the maturity  $t$  tends to zero (Theorem 4.2). The correction term  $a(x)$  is important since it takes into account the drift terms in the SDEs (2.1) for the Heston model. The genuine limit  $\sigma_0(x)$  (also derived in [9]) fails to capture these drifts because large deviations theory is only sharp on a logarithmic scale.

The paper is organised as follows: we recall the Heston model and the main ingredients that will be needed in Section 2. Section 3 presents the main results of the paper, namely the small-maturity asymptotic expansion for European call option prices, both for the Heston and for the Black-Scholes model. The lengthy proof of the main theorem is deferred to Section 6. In Section 4 we translate these expansions into implied volatility asymptotics and provide numerical evidence of the accuracy of our formulae. Finally we propose a calibration methodology based on these closed-form approximations in Section 5.

## 2. MODEL AND NOTATIONS

We work on a probability space  $(\Omega, \mathcal{F}, P)$  with a filtration  $(\mathcal{F}_t)_{t \geq 0}$  supporting two Brownian motions and satisfying the usual conditions. Let  $(S_t)_{t \geq 0}$  be a stock price process and we denote its logarithm by  $X_t := \log(S_t)$ . Without loss of generality we shall assume that interest rates and dividends are null—otherwise we can just model the dynamics of the forward price directly instead of the stock price. In the Heston model the

process  $(X_t)_{t \geq 0}$  satisfies the following system of SDEs:

$$(2.1) \quad \begin{aligned} dX_t &= -\frac{1}{2}Y_t dt + \sqrt{Y_t} dW_t, & X_0 &= x_0 \in \mathbb{R}, \\ dY_t &= \kappa(\theta - Y_t) dt + \sigma\sqrt{Y_t} dZ_t, & Y_0 &= y_0 > 0, \\ d\langle W, Z \rangle_t &= \rho dt \end{aligned}$$

where  $\kappa > 0$ ,  $\theta > 0$ ,  $\sigma > 0$ ,  $|\rho| < 1$  and  $W$  and  $Z$  are two standard Brownian motions. We further assume as in [19] that the inequality  $\kappa > \rho\sigma$  holds. This will be needed in Section 6.2 to characterise the effective domain of the moment generating function; without this condition, the variance process fails to be mean-reverting under the Share measure. This assumption is however not restrictive in practice since the correlation is usually either negative (in equity markets) or close to zero (implied volatility smiles are almost symmetric in foreign exchange markets). The  $Y$  process is a Feller diffusion and the coefficients of the stochastic differential equation satisfy the Yamada-Watanabe condition [17, Section 5.2.C, Proposition 2.13] so it admits a unique strong solution. The  $X$  process is a stochastic integral of the  $Y$  process and thus is well defined. Let us now define the two real numbers  $p_- < 0$  and  $p_+ > 0$  by

$$(2.2) \quad p_- := \frac{2}{\sigma\bar{\rho}} \arctan\left(\frac{\bar{\rho}}{\rho}\right) \mathbf{1}_{\{\rho < 0\}} - \frac{\pi}{\sigma} \mathbf{1}_{\{\rho = 0\}} + \frac{2}{\sigma\bar{\rho}} \left( \arctan\left(\frac{\bar{\rho}}{\rho}\right) - \pi \right) \mathbf{1}_{\{\rho > 0\}},$$

$$p_+ := \frac{2}{\sigma\bar{\rho}} \left( \arctan\left(\frac{\bar{\rho}}{\rho}\right) + \pi \right) \mathbf{1}_{\{\rho < 0\}} + \frac{\pi}{\sigma} \mathbf{1}_{\{\rho = 0\}} + \frac{2}{\sigma\bar{\rho}} \arctan\left(\frac{\bar{\rho}}{\rho}\right) \mathbf{1}_{\{\rho > 0\}},$$

where  $\bar{\rho} := \sqrt{1 - \rho^2}$ . We further define the function  $\Lambda : (p_-, p_+) \rightarrow \mathbb{R}$  by

$$(2.3) \quad \Lambda(p) := \frac{y_0 p}{\sigma(\bar{\rho} \cot(\sigma\bar{\rho}p/2) - \rho)}, \quad \text{for all } p \in (p_-, p_+).$$

It can easily be checked that  $\Lambda$  is a real strictly convex function on the interior of its domain, and that  $\Lambda(p) > 0$  for all  $p \in (p_-, p_+) \setminus \{0\}$ , and  $\Lambda(0) = 0$ .

**Remark 2.1.** In [9] the authors proved that the function  $\Lambda$  corresponds to the limiting cumulant generating function of the Heston model, i.e.

$$\Lambda(p) = \lim_{t \rightarrow 0} t \log \mathbb{E} \left( e^{p(X_t - x_0)/t} \right), \quad \text{for all } p \in (p_-, p_+).$$

Let us finally define the Fenchel-Legendre transform  $\Lambda^* : \mathbb{R} \rightarrow \mathbb{R}$  of the function  $\Lambda$  by

$$(2.4) \quad \Lambda^*(x) := \sup_{p \in (p_-, p_+)} \{px - \Lambda(p)\}, \quad \text{for all } x \in \mathbb{R}.$$

In [9] the authors proved that the small-maturity limit of the implied volatility is fully characterised by this dual transform. We shall of course recover this result while proving higher-order expansions for the short-maturity implied volatility. The function  $\Lambda^*$  does not admit a closed-form representation, but a straightforward analysis shows that the equality  $\Lambda^*(x) = p^*(x)x - \Lambda(p^*(x))$  holds for any real number  $x$  where  $p^*(x) \in (p_-, p_+)$  is defined as the unique solution to the equation  $\Lambda'(p^*(x)) = x$ . By strict convexity and the fact that  $|\Lambda'(p)|$  tends to infinity whenever  $p$  tends to  $p_-$  or  $p_+$ , this number is always uniquely well defined. As proved in [9], the identities  $p^*(x) > 0$  for  $x > 0$ ,  $p^*(x) < 0$  for  $x < 0$  and  $p^*(0) = 0$  always hold. We also have  $\Lambda^*(x) > 0$  and  $\Lambda''(p^*(x)) > 0$  for all  $x \neq 0$ , and  $\Lambda^*(0) = 0$ .

In the Black-Scholes model, the share price process  $(S_t)_{t \geq 0}$  satisfies the SDE  $dS_t = \Sigma S_t dW_t$ , with  $S_0 > 0$  and where the volatility  $\Sigma$  is a strictly positive real number. We shall use the notation  $C_{\text{BS}}(x, t, \Sigma)$  to represent

the Black-Scholes price of a European call option written on the share price  $S$  with strike  $S_0 e^x$ , maturity  $t$  and volatility  $\Sigma$ . In the rest of the paper,  $\Re$  and  $\Im$  will respectively denote the real and the imaginary parts of a complex number, and  $\mathcal{N}$  the cumulative distribution function of the standard Gaussian distribution.

### 3. SMALL-TIME BEHAVIOUR OF EUROPEAN CALL OPTIONS

In this section we derive small-time expansions for European call option prices in the Heston model (Theorem 3.1) and in the Black-Scholes model (Proposition 3.4). We postpone the proof of Theorem 3.1 to Section 6.

**Theorem 3.1.** *In the Heston model (2.1) the asymptotic behaviour for European call options*

$$\frac{\mathbb{E}(e^{X_t} - S_0 e^x)_+}{S_0} = (1 - e^x)_+ + \exp\left(-\frac{\Lambda^*(x)}{t}\right) \left(\frac{A(x)}{\sqrt{2\pi}} t^{3/2} + \mathcal{O}(t^{5/2})\right),$$

holds for any  $x \in \mathbb{R} \setminus \{0\}$  as the maturity  $t$  tends to zero, where

$$A(x) := \frac{e^x U(p^*(x))}{p^*(x)^2 \sqrt{\Lambda''(p^*(x))}},$$

$$U(p) := \exp\left(\frac{\kappa\theta}{\sigma^2} \left( (\mathrm{i}\rho\sigma - d_0) \mathrm{i}p - 2 \log\left(\frac{1 - g_0 e^{-\mathrm{i}d_0 p}}{1 - g_0}\right) \right)\right) \\ \exp\left(\frac{y_0 e^{-\mathrm{i}d_0 p}}{(1 - g_0 e^{-\mathrm{i}d_0 p}) \sigma^2} \left( (\mathrm{i}\rho\sigma - d_0) \mathrm{i}p d_1 - (\kappa - d_1) (1 - e^{\mathrm{i}d_0 p}) + \frac{(\mathrm{i}\rho\sigma - d_0) (1 - e^{-\mathrm{i}d_0 p}) (g_1 - \mathrm{i}d_1 g_0 p)}{1 - g_0 e^{-\mathrm{i}d_0 p}} \right)\right),$$

where the functions  $\Lambda$  and  $\Lambda^*$  are defined in (2.3) and in (2.4), and

$$(3.1) \quad d_0 := \sigma \bar{\rho}, \quad d_1 := \frac{2\kappa\rho - \sigma}{2\bar{\rho}} \mathrm{i}, \quad g_0 := \frac{\mathrm{i}\rho - \bar{\rho}}{\mathrm{i}\rho + \bar{\rho}}, \quad g_1 := \frac{(2\kappa - \rho\sigma)}{\sigma \bar{\rho} (\mathrm{i}\rho + \bar{\rho})^2}.$$

**Remark 3.2.** Although this is not obvious from its representation, the function  $U$  maps the interval  $(p_-, p_+)$  to the real line. The reason for this is explained in Remark 6.2. Since  $p^*(x)$  takes values in  $(p_-, p_+) \setminus \{0\}$ , and  $\Lambda''(p^*(x)) > 0$  for all  $x \in \mathbb{R}^*$ , then  $A(x)$  is a well-defined real number.

**Remark 3.3.** Note that the theorem does not deal with the at-the-money case  $x = 0$ . We refer the interested reader to Remark 6.10 for further details about this case.

Since we shall eventually be interested in deriving small-time asymptotics for the implied volatility, we need an analogous result to Theorem 3.1 for the Black-Scholes model. This is the purpose of the following proposition, the proof of which can be found in Appendix 7.1.

**Proposition 3.4.** *Let  $\Sigma > 0$ ,  $c \in \mathbb{R}$ , define  $\sigma_t := \sqrt{\Sigma^2 + ct}$  for  $t > 0$  and assume that  $t \in (0, \Sigma^2/|c|)$  if  $c < 0$ . Then the following behaviour holds as the maturity  $t$  tends to zero,*

$$\frac{C_{\text{BS}}(x, t, \sigma_t)}{S_0} = \begin{cases} (1 - e^x)_+ + \exp\left(-\frac{x^2}{2\Sigma^2 t} + \frac{x}{2} + \frac{cx^2}{2\Sigma^4}\right) \left(\frac{\Sigma^3}{x^2 \sqrt{2\pi}} t^{3/2} + \mathcal{O}(t^{5/2})\right), & \text{if } x \neq 0, \\ \frac{\Sigma}{\sqrt{2\pi}} t^{1/2} + \frac{1}{2\Sigma \sqrt{2\pi}} \left(\frac{c}{\Sigma} - \frac{\Sigma^4}{12}\right) t^{3/2} + \mathcal{O}(t^{5/2}), & \text{if } x = 0. \end{cases}$$

The behaviour of European call options in the standard Black-Scholes model follows immediately.

**Corollary 3.5.** *In the standard Black-Scholes model (i.e.  $c = 0$ ) we have  $\sigma_t = \Sigma$  for all  $t > 0$  and*

$$\frac{C_{\text{BS}}(x, t, \Sigma)}{S_0} = \begin{cases} (1 - e^x)_+ + \exp\left(-\frac{x^2}{2\Sigma^2 t}\right) \left(\frac{A_{\text{BS}}(x, \Sigma)}{\sqrt{2\pi}} t^{3/2} + \mathcal{O}(t^{5/2})\right), & \text{if } x \neq 0, \\ \frac{\Sigma}{\sqrt{2\pi}} t^{1/2} - \frac{\Sigma^3}{24\sqrt{2\pi}} t^{3/2} + \mathcal{O}(t^{5/2}), & \text{if } x = 0, \end{cases}$$

where

$$(3.2) \quad A_{\text{BS}}(x, \Sigma) := \frac{\Sigma^3}{x^2} \exp\left(\frac{x}{2}\right), \quad \text{for any } x \neq 0.$$

#### 4. SMALL-TIME BEHAVIOUR OF IMPLIED VOLATILITY

In this section we translate the call option asymptotics stated above into small-time expansions for the implied volatility. For any  $t > 0$  and  $x \in \mathbb{R}^*$ ,  $\sigma_t(x)$  shall denote the implied volatility of a European call option with maturity  $t$  and strike  $S_0 e^x$ . We first define the two functions  $\sigma_0 : \mathbb{R} \rightarrow \mathbb{R}$  and  $a : \mathbb{R} \rightarrow \mathbb{R}$  by

$$(4.1) \quad \sigma_0(x) := \frac{|x|}{\sqrt{2\Lambda^*(x)}} \quad \text{and} \quad a(x) := \frac{2\sigma_0^4(x)}{x^2} \log\left(\frac{A(x)}{A_{\text{BS}}(x, \sigma_0(x))}\right), \quad \text{for all } x \in \mathbb{R} \setminus \{0\},$$

where the functions  $\Lambda^*$ ,  $A$  and  $A_{\text{BS}}$  are respectively defined in (2.4), in Theorem 3.1 and in (3.2). These two functions clearly exist for all  $x \in \mathbb{R} \setminus \{0\}$ , and Corollary 4.3 below ensures that  $\sigma_0(0)$  and  $a(0)$  are well defined by continuity.

**Remark 4.1.** In [9], the authors proved that the function  $\sigma_0$  is the genuine limit of the implied volatility smile as the maturity tends to zero and that it is continuous at the origin. Although the two functions  $A$  and  $A_{\text{BS}}$  are not continuous at the origin, the function  $a$  is, as shown below in Corollary 4.3.

The following theorem is the core of this section and gives the out-of-the-money implied volatility expansion for small maturities. We defer its proof to Section 7.2.

**Theorem 4.2.** *The asymptotic expansion  $\sigma_t^2(x) = \sigma_0^2(x) + a(x)t + o(t)$  holds for all  $x \in \mathbb{R} \setminus \{0\}$ .*

The following corollary—proved in Section 7.3—is a direct consequence of this theorem and provides information on the behaviour of the short-time implied volatility near the money in terms of the model parameters. It also highlights the moneyness dependence of the coefficients  $\sigma_0$  and  $a$  near the money.

**Corollary 4.3.** *The following expansions for the functions  $\sigma_0$  and  $a$  hold when  $x$  is close to zero,*

$$\begin{aligned} \sigma_0(x) &= \sqrt{y_0} \left(1 + \frac{\rho\sigma x}{4y_0} + \frac{1}{24} \left(1 - \frac{5\rho^2}{2}\right) \frac{\sigma^2 x^2}{y_0^2}\right) + \mathcal{O}(x^3), \\ a(x) &= -\frac{\sigma^2}{12} \left(1 - \frac{\rho^2}{4}\right) + \frac{y_0 \rho \sigma}{4} + \frac{\kappa}{2} (\theta - y_0) + \frac{\rho\sigma}{24y_0} (\sigma^2 \rho^2 - 2\kappa(\theta + y_0) + y_0 \rho \sigma) x \\ &\quad + \frac{176\sigma^2 - 480\kappa\theta - 712\rho^2\sigma^2 + 521\rho^4\sigma^2 + 40y_0\rho^3\sigma + 1040\kappa\theta\rho^2 - 80y_0\kappa\rho^2}{7680} \frac{\sigma^2 x^2}{y_0^2} + \mathcal{O}(x^3). \end{aligned}$$

These approximations for the functions  $\sigma_0$  and  $a$  are symmetric in the log-moneyness when the correlation parameter  $\rho$  is null. This is consistent with the fact that uncorrelated stochastic volatility models generate symmetric smiles (see [26]). In the at-the-money case, we have the following corollary:

**Corollary 4.4.** *If there exists some  $\varepsilon > 0$  such that the map  $(t, x) \mapsto \sigma_t^2(x)$  is of class  $C^{1,1}$  on  $[0, \varepsilon) \times (-\varepsilon, \varepsilon)$ , then  $\sigma_t^2(0) = \sigma_0^2(0) + a(0)t + o(t)$ .*

*Proof.* Define the function  $f$  by  $f(t, x) := \sigma_t^2(x)$ . Applying Taylor's theorem twice we obtain

$$f(t, 0) - f(0, 0) = t \frac{\partial f}{\partial t}(0, 0) + o(t) = t \lim_{x \rightarrow 0} \frac{\partial f}{\partial t}(0, x) + o(t) = \lim_{x \rightarrow 0} a(x)t + o(t) = a(0)t + o(t).$$

□

The following corollary explains why the correction term  $a(x)$  is important.

**Corollary 4.5.** *The following small-time approximation for call options holds as  $t$  tends to zero:*

$$\mathbb{E} (e^{X_t} - S_0 e^x)_+ \sim C_{\text{BS}} \left( x, t, \sqrt{\sigma_0^2(x) + a(x)t} \right), \quad \text{for all } x > 0,$$

where  $\sim$  denotes asymptotic equivalence:  $f(t) \sim g(t)$  means  $f(t)/g(t)$  converges to one as  $t$  tends to zero.

When  $x < 0$ , it is clear from Theorem 3.1 and Proposition 3.4 that the leading orders  $(1 - e^x)_+ = (1 - e^x)$  dominate, so that  $\mathbb{E} (e^{X_t} - S_0 e^x)_+ \sim C_{\text{BS}}(x, t, \sigma_0(x))$  holds. When  $x > 0$ , the leading orders  $(1 - e^x)_+$  are zero, and the corollary follows from Theorem 3.1, Proposition 3.4 and (4.1):

$$\lim_{t \rightarrow 0} \frac{\mathbb{E} (e^{X_t} - S_0 e^x)_+}{C_{\text{BS}} \left( x, t, \sqrt{\sigma_0(x)^2 + a(x)t} \right)} = \frac{A(x)x^2}{\sigma_0^3(x)} \exp \left( -\frac{x}{2} - \frac{a(x)x^2}{2\sigma_0^4(x)} \right) = 1.$$

The necessity of the correction term  $a(x)$  when  $x > 0$  can be understood by the following limit ratio:

$$\lim_{t \rightarrow 0} \frac{C_{\text{BS}} \left( x, t, \sqrt{\sigma_0(x)^2 + a(x)t} \right)}{C_{\text{BS}}(x, t, \sigma_0(x))} = \exp \left( \frac{a(x)x^2}{2\sigma_0^4(x)} \right),$$

which is clearly not independent of the correction term—and in particular is not equal to 1 unless  $a(x) = 0$ .

We now test the accuracy of the small-time expansion for the implied volatility derived in Theorem 4.2.

**Example 4.6.** Let us consider the following set of parameters:  $\kappa = 1.15$ ,  $\sigma = 0.2$ ,  $\theta = y_0 = 0.04$  and  $\rho = -0.4$ . In Figure 1, we plot the smiles computed numerically<sup>1</sup> as well as the zeroth and the first order approximations (i.e.  $\sigma_0(x)$  and  $\sqrt{\sigma_0^2(x) + a(x)t}$  as given in (4.1)) for the three maturities  $t = 0.1$  year,  $t = 0.25$  year and  $t = 0.5$  year. We observe that our approximation and the generated data are very close for the maturities  $t = 0.1$  and  $t = 0.25$ , and are still close (within 0.18 percentage points uniformly in the displayed strikes) even at  $t = 0.5$ , an expiry approaching moderate size. The correction term  $a(x)$  is essentially the smile-flattening effect which is a stylised feature of implied volatility surfaces observed in the market. It is interesting to note that the error between our refined expansion and the true value of the Heston smile is almost constant over all strikes. We also plot in Figure 2 the correction term  $a(x)$  given by the formula in (4.1).

## 5. CALIBRATION METHODOLOGY

Based on the asymptotic expansion derived in Theorem 4.2 above, we find a calibration formula that generates parameter estimates, which can then serve as starting points to be input into a standard numerical optimiser. For the sake of clarity in this section we introduce the notation  $\alpha := \kappa\theta$ . Given implied variances for five contracts, our objective is to find explicit formulae to calibrate the five Heston parameters  $(v, \sigma, \rho, \alpha, \kappa)$  to the five implied variances. For some configurations of contracts, one cannot expect to solve this problem. For example, a set of contracts with the same expiry would be uninformative in regard to the term structure

<sup>1</sup> The implementation is based on a refined quadrature scheme for inverse Fourier transforms using the Zeliade Quant Framework by Zeliade Systems, where the second author was working.

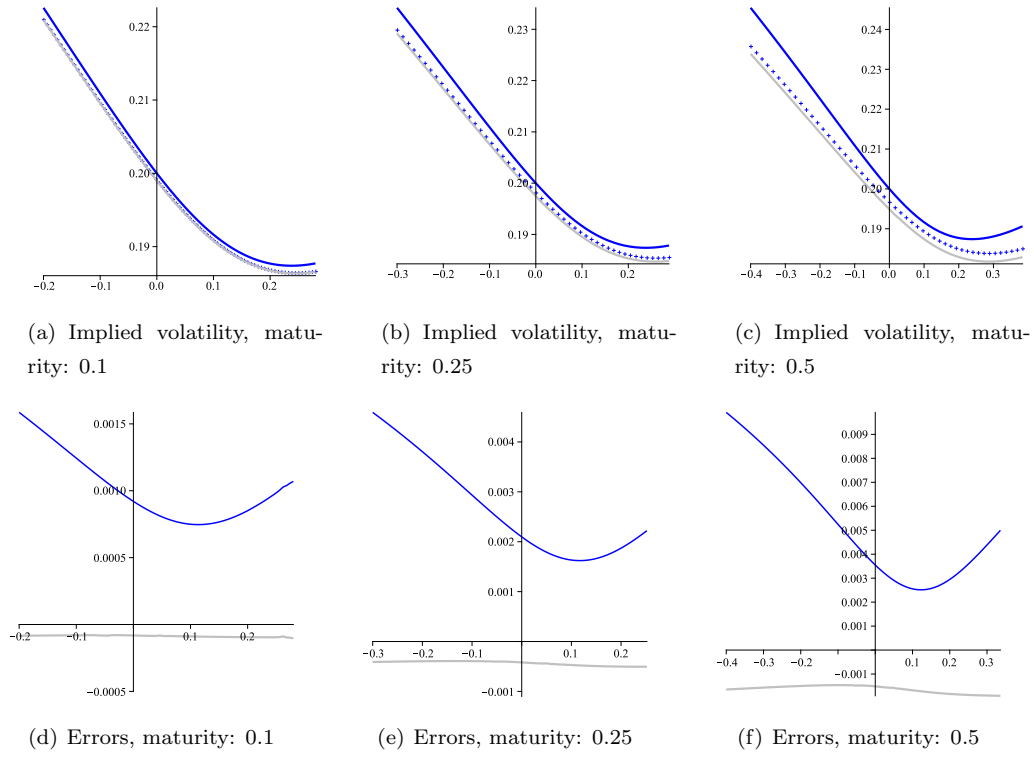


FIGURE 1. The three graphs on top represent the implied volatilities for different maturities. The three graphs below represent the corresponding errors (difference between our formula and the true implied volatility) for the same maturities. The solid blue line corresponds to the leading-order smile  $\sigma_0(x)$ , the solid grey one is the refined formula  $\sqrt{\sigma_0^2(x) + a(x)t}$ , and the blue crosses account for the true Heston implied volatilities. The horizontal axis represents the log-moneyness  $x$ .

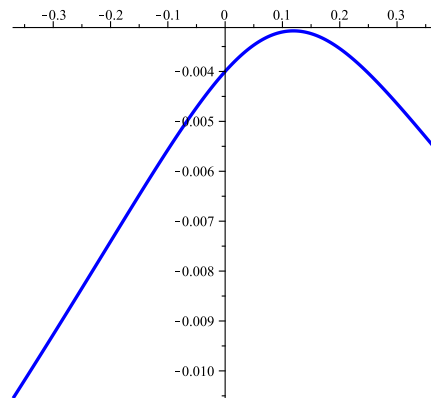


FIGURE 2. Plot of the correction term  $a(x)$  defined in (4.1).

of implied variance, hence uninformative in regard to the mean-reversion parameter  $\kappa$ . We therefore choose configurations which involve multiple expiries. In such a setting our maturity-dependent formula plays a crucial role in capturing the maturity effects needed to calibrate the full set of Heston parameters. Recall that Theorem 4.2 and Corollary 4.3 imply  $\sigma_t^2(x) = H(x, t) + \mathcal{O}(x^3) + o(t)$ , as  $x$  and  $t$  tend to zero, where

$$(5.1) \quad \begin{aligned} H(x, t) &:= H(x, t; y_0, \rho, \sigma, \alpha, \kappa) := y_0 \left( 1 + \frac{\rho \sigma x}{2 y_0} + \left( 1 - \frac{7}{4} \rho^2 \right) \frac{\sigma^2 x^2}{12 y_0^2} \right) \\ &+ \left( \left( \frac{\rho \sigma y_0}{2} - \frac{\sigma^2}{6} \left( 1 - \frac{1}{4} \rho^2 \right) + \alpha - \kappa y_0 \right) + \frac{\rho \sigma}{12 y_0} (\sigma^2 (1 - \rho^2) + \rho \sigma y_0 - 2\alpha - 2\kappa y_0) x \right) \frac{t}{2} \\ &+ \frac{\sigma^2}{7680 y_0^2} \left( (176 - 712 \rho^2 + 521 \rho^4) \sigma^2 + 40 \sigma \rho^3 y_0 + 80 (13 \rho^2 - 6) \alpha - 80 \kappa \rho^2 y_0 \right) x^2 t. \end{aligned}$$

Let us consider a ‘‘skew’’  $\mathbf{V} : \mathbb{K} \rightarrow \mathbb{R}$ , where each point in the configuration  $\mathbb{K} \subset \mathbb{R} \times [0, \infty]$  represents a (log moneyness, expiration), and where  $\mathbf{V}$  represents the squared implied volatility. We shall say that  $(v, \rho, \sigma, \alpha, \kappa)$  calibrates  $H$  to the skew  $\mathbf{V} : \mathbb{K} \rightarrow \mathbb{R}$  if the equality  $H(x, t; y_0, \rho, \sigma, \alpha, \kappa) = \mathbf{V}(x, t)$  holds for all  $(x, t) \in \mathbb{K}$ . This definition demands exact fitting of  $H$  to the given volatility skew at all points in  $\mathbb{K}$ . Consider now the configuration  $\mathbb{K} = \{(0, 0), (x_0, t_1), (-x_0, t_1), (x_0, t_2), (-x_0, t_2)\}$  where  $0 < t_1 < t_2$  and  $x_0 > 0$ . Given  $\mathbf{V} : \mathbb{K} \rightarrow \mathbb{R}$ , define

$$\begin{aligned} \mathbf{V}_0 &:= \mathbf{V}(0, 0), \quad \mathbf{S} := \frac{\mathbf{V}_+ - \mathbf{V}_-}{2x_0}, \quad \mathbf{C} := \frac{\mathbf{V}_+ - 2\mathbf{V}_0 + \mathbf{V}_-}{2x_0^2}, \\ \mathbf{V}_\pm &:= \frac{t_2}{t_2 - t_1} \mathbf{V}(\pm x_0, t_1) - \frac{t_1}{t_2 - t_1} \mathbf{V}(\pm x_0, t_2). \end{aligned}$$

**Theorem 5.1.** *The parameter choices  $(\tilde{y}_0, \tilde{\rho}, \tilde{\sigma}, \tilde{\alpha}, \tilde{\kappa})$  calibrate  $H$  to the skew  $\mathbf{V} : \mathbb{K} \rightarrow \mathbb{R}$ , where*

$$(5.2) \quad \begin{pmatrix} \tilde{y}_0 \\ \tilde{\sigma} \\ \tilde{\rho} \end{pmatrix} := \begin{pmatrix} \mathbf{V}_0 \\ \sqrt{7\mathbf{S}^2 + 12\mathbf{V}_0\mathbf{C}} \\ 2\mathbf{S}/\sqrt{7\mathbf{S}^2 + 12\mathbf{V}_0\mathbf{C}} \end{pmatrix},$$

$$(5.3) \quad \begin{pmatrix} \tilde{\alpha} \\ \tilde{\kappa} \end{pmatrix} := \mathbf{M}^{-1}(\mathbf{q} - \mathbf{r}),$$

with

$$\mathbf{q} := \frac{1}{2t_1} \begin{pmatrix} \mathbf{V}(x_0, t_1) - \mathbf{V}(-x_0, t_1) - \mathbf{V}_+ + \mathbf{V}_- \\ \mathbf{V}(x_0, t_1) + \mathbf{V}(-x_0, t_1) - \mathbf{V}_+ - \mathbf{V}_- \end{pmatrix},$$

and

$$\mathbf{r} := \begin{pmatrix} \frac{\tilde{\rho}\tilde{\sigma}^3(1 - \tilde{\rho}^2) + \tilde{y}_0\tilde{\rho}^2\tilde{\sigma}^2}{24\tilde{y}_0} x_0 \\ \frac{12\tilde{y}_0\tilde{\rho}\tilde{\sigma} + (\tilde{\rho}^2 - 4)\tilde{\sigma}^2}{48} + \frac{(176 - 712\tilde{\rho}^2 + 521\tilde{\rho}^4)\tilde{\sigma}^4 + 40\tilde{y}_0\tilde{\rho}^3\tilde{\sigma}^3}{7680\tilde{y}_0^2} x_0^2 \end{pmatrix},$$

and

$$\mathbf{M} := \begin{pmatrix} -\frac{\tilde{\rho}\tilde{\sigma}}{12\tilde{y}_0} x_0 & -\frac{\tilde{\rho}\tilde{\sigma}}{12} x_0 \\ \frac{1}{2} + \frac{(13\tilde{\rho}^2 - 6)\tilde{\sigma}^2}{96\tilde{y}_0^2} x_0^2 & -\frac{\tilde{y}_0}{2} - \frac{\tilde{\rho}^2\tilde{\sigma}^2}{96\tilde{y}_0} x_0^2 \end{pmatrix},$$

provided that  $\tilde{y}_0 \neq 0$ ,  $\tilde{\rho} \neq 0$  and  $\tilde{\rho}^2 \neq \frac{3}{7}(1 - 16\tilde{y}_0^2/(x_0^2\tilde{\sigma}^2))$ .



This theorem follows immediately by substituting (5.2), (5.3), and each  $(x, t) \in \mathbb{K}$ , into (5.1). The following provides a simple numerical example based on the implied volatility smiles in Figure 1.

**Example 5.2.** Suppose that the true Heston parameters are the ones that generate Figure 1. Let  $t_1 := 0.1$ ,  $t_2 := 0.25$ , and  $x_0 := 0.1$ . Then the squared implied volatilities are

$$\mathbb{V}(0, 0) = 0.04, \mathbb{V}(x_0, t_1) = 0.03644, \mathbb{V}(-x_0, t_1) = 0.04395, \mathbb{V}(x_0, t_2) = 0.03610, \mathbb{V}(-x_0, t_2) = 0.04325.$$

Given these data points, the explicit calibration formulae in Theorem 5.1 produce the estimates

$$\tilde{\theta} := \tilde{\alpha}/\tilde{\kappa} = 0.04105, \quad \tilde{\kappa} = 1.104, \quad \tilde{\rho} = -0.4069, \quad \tilde{\sigma} = 0.1907$$

of the true parameters  $(\theta, \kappa, \rho, \sigma) = (0.04, 1.15, -0.4, 0.2)$ .

## 6. PROOF OF THE CALL PRICE EXPANSION (THEOREM 3.1)

We split the proof of Theorem 3.1 into several parts, from Section 6.1 to Section 6.3 below. From [20], we know that European call option prices can be written as an inverse Fourier transform (6.1) along some horizontal contour in the complex plane. We then rescale this integral (subsection 6.2) and move the horizontal contour of integration so it passes through the saddlepoint (see Definition 6.6) of the small-time approximation of the integrand (Lemma 6.1). In Proposition 6.9 we prove an asymptotic expansion of the integral in (6.2). Lemma 6.4 is a technical lemma needed to justify this saddlepoint expansion. In the whole proof—according to the statement of Theorem 3.1—we assume that  $x \neq 0$ .

**6.1. The Fourier inversion formula for call options.** For each non-negative real number  $t$ , define the sets  $A_{t,X} \subset \mathbb{R}$  and  $\Lambda_{t,X} \subset \mathbb{C}$  by

$$A_{t,X} := \{\nu \in \mathbb{R} : \mathbb{E}(\exp(\nu(X_t - x_0))) < \infty\}, \quad \Lambda_{t,X} := \{z \in \mathbb{C} : -\Im(z) \in A_{t,X}\},$$

and the characteristic function  $\phi_t : \mathbb{C} \rightarrow \mathbb{C}$  of the logarithmic return  $(X_t - x_0)_{t \geq 0}$  by

$$\phi_t(z) := \mathbb{E}\left(e^{iz(X_t - x_0)}\right), \quad \text{for all } z \in \Lambda_{t,X}.$$

By [20, Theorem 5.1], we know that for any  $\alpha \in \mathbb{R}$  such that  $\alpha + 1 \in A_{t,X}$  and  $\alpha \neq 0$ , we have the following Fourier inversion formula for the price of a call option

$$(6.1) \quad \begin{aligned} \frac{\mathbb{E}(e^{X_t} - S_0 e^x)_+}{S_0} &= \phi_t(-\mathbf{i}) \mathbf{1}_{\{-1 < \alpha < 0\}} + \left(\phi_t(-\mathbf{i}) - e^x \phi_t(0)\right) \mathbf{1}_{\{\alpha < -1\}} + \left(\phi_t(-\mathbf{i}) - \frac{e^x}{2} \phi_t(0)\right) \mathbf{1}_{\{\alpha = -1\}} \\ &+ \frac{1}{\pi} \int_{0 - \mathbf{i}\alpha}^{+\infty - \mathbf{i}\alpha} \Re\left(e^{-izx} \frac{\phi_t(z - \mathbf{i})}{\mathbf{i}z - z^2}\right) dz. \end{aligned}$$

The first three terms on the right-hand side are complex residues that arise when we cross the pole of  $(\mathbf{i}z - z^2)^{-1}$  at  $z = 0$  and at  $z = \mathbf{i}$ . Setting  $u = \mathbf{i} - z$  and using the fact that the process  $(e^{X_t})_{t \geq 0}$  is a true martingale (see [19, Theorem 2.5]) we obtain

$$(6.2) \quad \begin{aligned} \frac{\mathbb{E}(e^{X_t} - S_0 e^x)_+}{S_0} &= \mathbf{1}_{\{-1 < \alpha < 0\}} + (1 - e^x) \mathbf{1}_{\{\alpha < -1\}} + \left(1 - \frac{e^x}{2}\right) \mathbf{1}_{\{\alpha = -1\}} \\ &+ \frac{e^x}{\pi} \int_{0 + \mathbf{i}(\alpha + 1)}^{+\infty + \mathbf{i}(\alpha + 1)} \Re\left(e^{\mathbf{i}ux} \frac{\phi_t(-u)}{\mathbf{i}u - u^2}\right) du. \end{aligned}$$

**6.2. Rescaling the variable of integration.** In this subsection, we first rescale the integrand above in order to perform an asymptotic expansion, and we then deform the contour of integration along a line in the complex plane that passes through the point  $\mathbf{i}p^*(x)$ , where  $p^*(x)$  is defined on Page 3. The reason for such a choice will be made clear in Lemma 6.7. As proved in [9], since  $p^*(x) \in (p_-, p_+)$  for any real number  $x$  (see Page 3), then the change of variable  $u =: k/t$  together with  $\alpha + 1 = p^*(x)/t$  in (6.2) is valid, and hence the equality

$$(6.3) \quad \frac{1}{S_0} \mathbb{E} (e^{X_t} - S_0 e^x)_+ = (1 - e^x) \mathbf{1}_{\{x < 0\}} + \frac{e^x}{\pi} \Re \left( \int_{0 + \mathbf{i}p^*(x)}^{+\infty + \mathbf{i}p^*(x)} \exp \left( \frac{\mathbf{i}xk}{t} \right) \frac{\phi_t(-k/t)}{\mathbf{i}k/t - k^2/t^2} \frac{dk}{t} \right)$$

holds for  $t$  sufficiently small. Indeed, from (6.2), the set  $\{-1 < \alpha < 0\}$  is equivalent to the set  $\{0 < p^*(x) < t\}$ , the set  $\{\alpha < -1\}$  is equal to the set  $\{p^*(x) < 0\}$ , and the set  $\{\alpha = -1\}$  corresponds to the set  $\{p^*(x) = 0\}$ . Since we consider  $t$  sufficiently small and  $x \neq 0$ , only the second set remains, and the properties of the function  $p^*$  on Page 3 imply (6.3). The first residue term corresponds to the intrinsic value of the call option. For  $k \neq 0$ , we have  $t^{-1} (\mathbf{i}k/t - k^2/t^2)^{-1} = -(t/k^2) (1 + \mathcal{O}(t/k)) = -(t/k^2) (1 + \mathcal{O}(t))$  uniformly in  $k$ , therefore we can rewrite (6.3) as

$$\frac{\mathbb{E} (e^{X_t} - S_0 e^x)_+}{S_0} = (1 - e^x) \mathbf{1}_{\{x < 0\}} - t \frac{e^x}{\pi} \Re \left( \int_{\mathbf{i}p^*(x)}^{+\infty + \mathbf{i}p^*(x)} \exp \left( \frac{\mathbf{i}xk}{t} \right) \frac{\phi_t(-k/t)}{k^2} (1 + \mathcal{O}(t)) dk \right).$$

Let us now define, for each  $p \in \mathbb{R}$  the explosion time  $t^*(p) := \sup\{t > 0, \mathbb{E} (e^{p(X_t - x_0)}) < \infty\}$ . In [9], using [19] (which applies since  $\kappa > \rho\sigma$ ), the authors proved that for any  $p \in (p_-, p_+)$  the explosion time  $t^*(p/t)$  is strictly larger than  $t$  for  $t$  sufficiently small, so that  $p^*(x)/t \in A_{t,X}$  for  $t$  sufficiently small. We also note that

$$\Re \left( e^{\mathbf{i}kx/t} \frac{\phi_t(-k/t)}{k^2} \right) = \Re \left( \frac{\exp(\mathbf{i}kx/t)}{k^2} \mathbb{E} \left( e^{-\mathbf{i}k(X_t - x_0)/t} \right) \right) = \mathbb{E} \left( \Re \left( \frac{\exp(\mathbf{i}kx/t)}{k^2} e^{-\mathbf{i}k(X_t - x_0)/t} \right) \right),$$

and we can easily show that this expression is an even function of  $\Re(k)$  and an odd function of  $\Im(k)$ . We can thus rewrite the normalised call price as

$$(6.4) \quad \frac{\mathbb{E} (e^{X_t} - S_0 e^x)_+}{S_0} = (1 - e^x) \mathbf{1}_{\{x < 0\}} - t \frac{e^x}{2\pi} \Re \left( \int_{\zeta_x} e^{\mathbf{i}xk/t} \phi_t(-k/t) \left( \frac{1}{k^2} + \mathcal{O}(t) \right) dk \right),$$

where we define the contour  $\zeta_x$  for every real number  $x \neq 0$  by

$$(6.5) \quad \zeta_x : (-\infty, \infty) \rightarrow \mathbb{C} \text{ such that } \zeta_x(u) := u + \mathbf{i}p^*(x).$$

Let us also define the set  $\mathcal{Z} \in \mathbb{C}$  by

$$(6.6) \quad \mathcal{Z} := \{k \in \mathbb{C} : \Im(k) \in (p_-, p_+)\},$$

where  $p_-$  and  $p_+$  are defined in (2.2).

**6.3. Saddlepoint expansion of the integral in (6.4).** The proof of Theorem 3.1 relies on an asymptotic expansion of the integral in (6.4), stated in Proposition 6.9 below. Before this though, we need to introduce several tools and prove some preliminary results. We start with the following lemma which characterises the asymptotic behaviour of the rescaled characteristic function when the maturity  $t$  is small.

**Lemma 6.1.** *Let  $k := k_r + \mathbf{i}k_i \in \mathbb{C}$ . For any fixed  $k_i$  such that  $-k_i \in (p_-, p_+) \setminus \{0\}$ , the expansion*

$$\phi_t \left( -\frac{k}{t} \right) = U(-\mathbf{i}k) \exp \left( \frac{\Lambda(-\mathbf{i}k)}{t} \right) (1 + \mathcal{O}(t)), \quad \text{holds uniformly in } k_r \text{ as } t \text{ tends to zero,}$$

where the function  $\Lambda$  is defined in (2.3) and the function  $U$  in Theorem 3.1.

**Remark 6.2.** Let  $p \in (p_-, p_+)$ . From [9] we know that for all  $t > 0$ ,  $\phi_t(ip/t)$  is a well-defined real number. The proof of Lemma 6.1 can be carried out analogously step by step for such a  $\phi_t(ip/t)$ , which implies that the number  $U(p)$  in Theorem 3.1 is real as well.

*Proof.* Albrecher et al. [1] derived the following closed-form representation for the characteristic function  $\phi_t$  under the Heston model:

$$(6.7) \quad \phi_t(k) := \mathbb{E} \left( e^{ik(X_t - x_0)} \right) = \exp \left( C(k, t) + y_0 D(k, t) \right),$$

where

$$\begin{aligned} C(k, t) &:= \frac{\kappa\theta}{\sigma^2} \left( (\kappa - i\rho\sigma k - d(k)) t - 2 \log \left( \frac{1 - g(k)e^{-d(k)t}}{1 - g(k)} \right) \right), \\ D(k, t) &:= \frac{\kappa - i\rho\sigma k - d(k)}{\sigma^2} \frac{1 - e^{-d(k)t}}{1 - g(k)e^{-d(k)t}}, \\ g(k) &:= \frac{\kappa - i\rho\sigma k - d(k)}{\kappa - i\rho\sigma k + d(k)}, \quad \text{and} \quad d(k) := \left( (\kappa - i\rho\sigma k)^2 + \sigma^2 k(i + k) \right)^{1/2}, \end{aligned}$$

and we take the principal branch for the complex logarithm. Lee [20] proved that the function  $\phi_t$  could be analytically extended in the complex plane inside a strip of regularity. From the above definitions we have the following asymptotic behaviour for  $d(-k/t)$  and  $g(-k/t)$  as the ratio  $t/k$  tends to zero:

$$\begin{aligned} d\left(-\frac{k}{t}\right) &= t^{-1} \left( \sigma^2 \bar{\rho}^2 k^2 + (2\kappa\rho - \sigma) i\sigma k t + \kappa^2 t^2 \right)^{1/2} = \frac{k}{t} d_0 + d_1 + \mathcal{O}(t), \\ g\left(-\frac{k}{t}\right) &= \frac{\kappa t + i\rho\sigma k - \left( \sigma^2 \bar{\rho}^2 k^2 + (2\kappa\rho - \sigma) i\sigma k t + \kappa^2 t^2 \right)^{1/2}}{\kappa t - i\rho\sigma k + \left( \sigma^2 \bar{\rho}^2 k^2 + (2\kappa\rho - \sigma) i\sigma k t + \kappa^2 t^2 \right)^{1/2}} = g_0 + \frac{t}{k} g_1 + \mathcal{O}(t^2), \end{aligned}$$

where we define the following quantities (similar to (3.1)):

$$d_0 := \sigma \bar{\rho} c_k, \quad d_1 := \frac{2\kappa\rho - \sigma}{2\bar{\rho}} i c_k, \quad g_0 := \frac{i\rho - \bar{\rho} c_k}{i\rho + \bar{\rho} c_k}, \quad g_1 := \frac{(2\kappa - \rho\sigma) c_k}{\sigma \bar{\rho} (i\rho + \bar{\rho} c_k)^2}, \quad c_k := \text{csgn}(k),$$

and where the sign function  $\text{csgn}$  for complex numbers is defined by  $\text{csgn}(k) = 1$  if  $\Re(k) > 0$ , or if  $\Re(k) = 0$  and  $\Im(k) > 0$ , and  $-1$  otherwise. Since  $|k| \geq |\Im(k)| > 0$  we have

$$\begin{aligned} D\left(-\frac{k}{t}, t\right) &= \frac{1}{\sigma^2} \left( \kappa + i\rho\sigma \frac{k}{t} - d_0 \frac{k}{t} - d_1 + \mathcal{O}(t) \right) \frac{1 - \exp(-d_0 k - d_1 t + \mathcal{O}(t^2))}{1 - g(k) \exp(-d_0 k - d_1 t + \mathcal{O}(t^2))} \\ &= \frac{(i\rho\sigma - d_0) k}{\sigma^2 t} \frac{1 - e^{-d_0 k}}{1 - g_0 e^{-d_0 k}} + \frac{1}{(1 - g_0 e^{-d_0 k}) \sigma^2} \left( e^{-d_0 k} (i\rho\sigma - d_0) d_1 k + (\kappa - d_1) (1 - e^{-d_0 k}) \right) \\ &\quad + \frac{1 - e^{-d_0 k}}{(1 - g_0 e^{-d_0 k})^2 \sigma^2} (i\rho\sigma - d_0) (g_1 - g_0 k d_1) e^{-d_0 k} + \mathcal{O}(t). \end{aligned}$$

Similarly we have

$$\begin{aligned} C\left(-\frac{k}{t}, t\right) &= \frac{\kappa\theta}{\sigma^2} \left( \kappa t + i\rho\sigma k - d\left(-\frac{k}{t}\right) t - 2 \log \left( \frac{1 - g(-k/t) \exp(-d(-k/t)t)}{1 - g(-k/t)} \right) \right) \\ &= \frac{\kappa\theta}{\sigma^2} \left( (i\rho\sigma - d_0) k - 2 \log \left( \frac{1 - g_0 \exp(-d_0 k)}{1 - g_0} \right) \right) + \mathcal{O}(t). \end{aligned}$$

We know that for all  $t > 0$ , the characteristic function in the Heston model is even in the argument  $d$  (see [20, Page 31]). Therefore the limiting behaviour of  $\phi_t(-k/t)$  does not depend on the sign function  $c_k$ , and the lemma follows from the definition of the functions  $U$  and  $\Lambda$  in Theorem 3.1 and in (2.3).  $\square$

**Remark 6.3.** Note that the assumption  $|\rho| < 1$  on Page 3 implies that  $1 - g_0$  is not null, and therefore the expansion for  $C(-k/t, t)$  is well defined. Concerning the denominator in  $D(-k/t)$ , simple calculations show that  $1 - g_0 \exp(-d_0 k) = 0$  if and only if  $k = -\log\left(\frac{(i\rho + \bar{\rho})}{(i\rho - \bar{\rho})}\right) / (\sigma\bar{\rho})$ , which is a purely imaginary number. Tedious but straightforward computations further reveal that this value precisely corresponds—depending on the sign of  $\rho$ —to the boundary points  $p_-$  and  $p_+$ , which are excluded. Therefore the expansion for  $D(-k/t)$  is also well defined.

We now state and prove the following lemma which we shall need later in the proof of Proposition 6.9.

**Lemma 6.4.** *Let  $k := k_r + ik_i \in \mathcal{Z}$ . The map  $k_r \mapsto \Re(-ikx - \Lambda(-ik))$  has a unique minimum at zero for any  $k_i \in (p_-, p_+)$ .*

*Proof.* Fix  $k_i \in (p_-, p_+)$ . It is then clear that  $\Re(-ikx - \Lambda(-ik)) = k_i x - \Re(\Lambda(k_i - ik_r))$ . Hence the lemma is equivalent to proving that the function  $k_r \mapsto \Re(\Lambda(k_i - ik_r))$  has a unique maximum at zero. Using the double angle formulae for trigonometric functions we have

$$\Re(\Lambda(p + iq)) = \Re\left(\frac{(p + iq)y_0}{\sigma(\bar{\rho}\cot(\frac{1}{2}\sigma(p + iq)\bar{\rho}) - \rho)}\right) = \frac{y_0 M(q)}{\sigma N(q)},$$

where the functions  $M$  and  $N$  are defined by

$$\begin{aligned} M(q) &:= p\left(\rho \cos(p\bar{\rho}\sigma) + \bar{\rho} \sin(p\bar{\rho}\sigma)\right) - p\rho \cosh(q\bar{\rho}\sigma) - q\bar{\rho} \sinh(q\bar{\rho}\sigma), \\ N(q) &:= \cosh(q\bar{\rho}\sigma) + (1 - 2\rho^2) \cos(p\bar{\rho}\sigma) - 2\rho\bar{\rho} \sin(p\bar{\rho}\sigma). \end{aligned}$$

Note that  $N(0) > 0$ , and  $N'(q) > 0$  for  $q > 0$ . We need to show that  $M(q)/N(q) < M(0)/N(0)$  for  $q \neq 0$ . By the symmetry  $q \mapsto -q$ , we can take  $q > 0$  and by the symmetry  $(p, \rho) \mapsto (-p, -\rho)$ , we can take  $p \geq 0$ . It hence suffices to show that for  $q > 0$ ,

$$(6.8) \quad \frac{M'(q)}{N'(q)} < \frac{M(0)}{N(0)},$$

since the following inequality will therefore hold for all  $q > 0$ :

$$\begin{aligned} \frac{M(q)}{N(q)} &= \frac{1}{N(q)} \left( M(0) + \int_0^q M'(u) du \right) < \frac{1}{N(q)} \left( M(0) + \frac{M(0)}{N(0)} \int_0^q N'(u) du \right) \\ &= \frac{1}{N(q)} \left( M(0) + \frac{M(0)}{N(0)} (N(q) - N(0)) \right) = \frac{M(0)}{N(0)}. \end{aligned}$$

Since  $M'(q)/N'(q) = -1/\sigma - p\rho - q\bar{\rho} \coth(q\bar{\rho}\sigma) < -p\rho - 2/\sigma$ , the inequality (6.8) will be satisfied as soon as

$$(6.9) \quad -\frac{2}{\sigma} < p\rho + \frac{M(0)}{N(0)}.$$

If  $\rho \geq 0$  then (6.9) holds since  $M(0)/N(0) = \sigma\Lambda(p)/y_0$  and because  $\Lambda(p) \geq 0$  as outlined in Section 2. Otherwise, for  $\rho < 0$ , note that the derivative of the right-hand side of (6.9) with respect to  $\rho$  is equal to a positive factor multiplied by

$$(2 - p\rho\sigma)\bar{\rho} - 2\rho^2\bar{\rho} \cos(p\bar{\rho}\sigma) - \rho\bar{\rho}^2 \sin(p\bar{\rho}\sigma) \geq \bar{\rho}(2 - p\rho\sigma - 2\rho^2 + p\rho\sigma) > 0,$$

since  $|\bar{\rho}^2 \sin \alpha| \leq \alpha$  for all  $\alpha > 0$ . So it suffices to verify (6.9) in the limit as  $\rho$  tends to  $-1$ . Since

$$\lim_{\rho \searrow -1} \left( p\rho + \frac{M(0)}{N(0)} \right) = -\frac{2p\sigma}{\sigma(2 + p\sigma)} > -\frac{2}{\sigma},$$

then (6.9) is satisfied and the lemma follows.  $\square$

**Remark 6.5.** Using the functions  $M$  and  $N$  defined in the proof of Lemma 6.4 above, it is easy to see that the ratio  $M(q)/N(q)$  tends to  $-\infty$  as  $|q|$  tends to infinity. This implies that for any  $k \in \mathcal{Z}$ ,  $\Re(-ikx - \Lambda(-ik))$  tends to infinity as  $|k|$  tends to infinity.

In order to prove an asymptotic expansion of the integral in (6.4), we first need to choose the optimal contour  $\zeta_x$  defined in (6.5) along which to integrate. By “optimal”, we mean that the contour  $\zeta_x$  passes through the saddlepoint (see Definition 6.6 and Lemma 6.7 below) of the integrand in (6.4).

**Definition 6.6.** (see [4]) Let  $F : \mathcal{Z} \rightarrow \mathbb{C}$  be an analytic complex function on an open set  $\mathcal{Z}$ . A point  $z_0 \in \mathcal{Z}$  such that the complex derivative  $dF/dz$  vanishes is called a saddlepoint.

Let us now define the function  $F : \mathcal{Z} \rightarrow \mathbb{C}$  by

$$(6.10) \quad F(k) := -ikx - \Lambda(-ik), \quad \text{for all } k \in \mathcal{Z},$$

where the function  $\Lambda$  is defined in (2.3) and the set  $\mathcal{Z}$  in (6.6). In the Heston case we are considering, this saddlepoint can be expressed in closed-form as follows.

**Lemma 6.7.** For any real number  $x$  the function  $F$  defined in (6.10) has a saddlepoint at  $k^*(x) = ip^*(x)$ , where  $p^*(x)$  is the real number defined on Page 3.

**Remark 6.8.** By the characterisation of the Fenchel-Legendre transform  $\Lambda^*$  of  $\Lambda$  in (2.4) and the remarks following this definition, the equality  $F(k^*(x)) = \Lambda^*(x)$  holds for any real number  $x$ .

*Proof.* Let  $x \in \mathbb{R}$ . By definition of  $p^*(x)$  on Page 3, we know that the equation  $\Lambda'(p) = x$  has a unique solution  $p^*(x) \in (p_-, p_+)$ . Therefore  $F'(k^*(x)) = -ix + i\Lambda'(p^*(x)) = 0$  and the lemma follows.  $\square$

We now have all the ingredients to prove the following proposition which, combined with (6.4), finishes the proof of Theorem 3.1.

**Proposition 6.9.** For any real number  $x \neq 0$  the following equality holds as  $t$  tends to zero,

$$-\frac{e^{xt}}{2\pi} \Re \left( \int_{\zeta_x} e^{ikx/t} \phi_t \left( -\frac{k}{t} \right) \left( \frac{1}{k^2} + \mathcal{O}(t) \right) dk \right) = \exp \left( -\frac{\Lambda^*(x)}{t} \right) \frac{A(x)t^{3/2}}{\sqrt{2\pi}} (1 + \mathcal{O}(t)),$$

where the functions  $A$  and  $\Lambda^*$  are given in Theorem 3.1 and in (2.4), and the contour  $\zeta_x$  in (6.2).

*Proof.* Since  $F(k) = -ikx - \Lambda(-ik)$  by (6.10), Lemma 6.1 applied on the contour  $\zeta_x$  implies

$$(6.11) \quad \Re \left( \int_{\zeta_x} e^{ikx/t} \phi_t \left( -\frac{k}{t} \right) \left( \frac{1}{k^2} + \mathcal{O}(t) \right) dk \right) = \Re \left( \int_{\zeta_x} U(-ik) e^{-F(k)/t} (1 + \mathcal{O}(t)) \left( \frac{1}{k^2} + \mathcal{O}(t) \right) dk \right).$$

The functions  $F$  and  $u$  are both analytic along  $\zeta_x$ . Lemma 6.4 implies the inequality  $\Re(F(k)) > \Re(F(ip^*(x)))$  for all  $k \in \zeta_x \setminus \{ip^*(x)\}$ , and  $\Re(F(k))$  tends to infinity as  $|k|$  tends to infinity by Remark 6.5 and Definition 6.6. We further know from [9, Proof of Theorem 1.1] that the quantity  $F''(ip^*(x))$  is not null. Therefore the Laplace expansion in [23, Chapter 4, Theorem 7.1] leads to the following expression:

$$\begin{aligned} \frac{e^{xt}}{2\pi} \Re \left( \int_{\zeta_x} U(-ik) \exp \left( -\frac{F(k)}{t} \right) \frac{dk}{k^2} \right) &= \frac{e^x}{\sqrt{\pi}} \exp \left( -\frac{F(k^*(x))}{t} \right) \frac{U(-ik^*(x)) t^{3/2}}{k^*(x)^2 \sqrt{2F''(k^*(x))}} (1 + \mathcal{O}(t)) \\ &= -\frac{e^x}{\sqrt{\pi}} \exp \left( -\frac{\Lambda^*(x)}{t} \right) \frac{U(p^*(x)) t^{3/2}}{p^*(x)^2 \sqrt{2\Lambda''(p^*(x))}} (1 + \mathcal{O}(t)) \\ &= -\exp \left( -\frac{\Lambda^*(x)}{t} \right) \frac{A(x) t^{3/2}}{\sqrt{2\pi}} (1 + \mathcal{O}(t)), \end{aligned}$$

as  $t$  tends to zero, where the equalities  $F(k^*(x)) = \Lambda^*(x)$  and  $F''(k^*(x)) = \Lambda''(p^*(x))$  follow from the definition of the function  $F$  in (6.10) and the properties of  $p^*(x)$  on Page 3. The  $\mathcal{O}(t)$  terms in (6.11) constitute higher order terms which we can neglect at the order we are interested in.  $\square$

**Remark 6.10.** The methodology for the general case  $x \neq 0$  above does not hold in the at-the-money case  $x = 0$  since the horizontal contour of integration passes through the saddlepoint  $k^*(0) = ip^*(0) = 0$  of the integrand, and the ratio  $U(-ik)/k^2$  (appearing in the proof of Proposition 6.9) is not analytic at the origin, so the Laplace expansion [23, Theorem 7.1, chapter 4] does not apply any more. Note however that Corollary 4.4 provides a small-maturity expansion of the at-the-money implied volatility.

**Acknowledgements.** The work of M. Forde was supported by an SFI grant for the Edgeworth Centre for Financial Mathematics. A. Jacquier acknowledges financial support from MATHEON. A former version of this paper circulated under the title "Small-time asymptotics for implied volatility under the Heston model: Part 2". The authors would like to thank the anonymous referees and the associate editor for useful comments.

#### REFERENCES

- [1] H. Albrecher, P. Mayer, W. Schoutens and J. Tistaert. The Little Heston Trap. *Wilmott Magazine*, 83-92, January 2007.
- [2] C. Bellaïche. Comportement asymptotique de  $p_t(x; y)$ . *Astérisque*, 84/85: 151-187, 1981.
- [3] H. Berestycki, J. Busca and I. Florent. Computing the Implied Volatility in Stochastic Volatility models. *Comm. Pure App. Math.*, 57: 1352-1373, 2004.
- [4] N. Bleistein and R.A. Handelsman. Asymptotic expansions of integrals. Holt, Rinehart and Winston (New York), 1975.
- [5] P. Carr and D. Madan. Saddlepoint methods for option pricing. *The Journal of Computational Finance*, 13: 49-61, 2009.
- [6] V. Durrleman. From implied to spot volatilities. *PhD dissertation*, Princeton University, 2004.
- [7] J. Feng, J.P. Fouque and R. Kumar. Small-time asymptotics for fast mean-reverting stochastic volatility models. Forthcoming in *Annals of Applied Probability*, 2011.
- [8] J. Feng, M. Forde and J.P. Fouque. Short maturity asymptotics for a fast mean-reverting Heston stochastic volatility model. *SIAM Journal on Financial Mathematics*, 1: 126-141, 2010.
- [9] M. Forde and A. Jacquier. Small-time asymptotics for implied volatility under the Heston model. *IJTAF*, 12(6): 861-876, 2009.
- [10] M. Forde and A. Jacquier. Small-time asymptotics for an uncorrelated local-stochastic volatility model. *Applied Mathematical Finance*, 18(6): 517-535, 2011.
- [11] M. Forde and A. Jacquier. The large-maturity smile for the Heston model. *Finance and Stochastics*, 15(4): 755-780, 2011.
- [12] M. Forde, A. Jacquier and A. Mijatović. Asymptotic formulae for implied volatility in the Heston model. *Proceedings of the Royal Society A*, 466(2124): 3593-3620, 2010.
- [13] M.I. Freidlin and A. Wentzell. Random perturbations of dynamical systems. Springer-Verlag, New York, 1984.
- [14] J. Gatheral, E. Hsu, P. Laurence, C. Ouyang and T-H. Wang. Asymptotics of implied volatility in local volatility models. Forthcoming in *Mathematical Finance*, 2012.
- [15] P. Hagan, D. Kumar, A.S. Lesniewski and D.E. Woodward, Managing Smile Risk. *Wilmott Magazine*, 84-108, September 2002.
- [16] P. Henry-Labordère. Analysis, Geometry and Modeling in Finance: Advanced Methods in Option Pricing. Chapman & Hall/CRC, Financial Mathematics Series, 2008.
- [17] I. Karatzas and S. Shreve. Brownian motion and Stochastic Calculus. Springer-Verlag, 1991.
- [18] S. Karlin and H.M. Taylor. A second course in stochastic processes. Academic Press, 1981.
- [19] M. Keller-Ressel. Moment Explosions and Long-Term Behavior of Affine Stochastic Volatility Models. *Mathematical Finance*, 21(1): 73-98, 2011.
- [20] R.W. Lee. Option Pricing by Transform Methods: Extensions, Unification, and Error Control. *Journal of Computational Finance*, 7(3): 51-86, 2004.
- [21] S. Molchanov. Diffusion processes and Riemannian geometry. *Russian Mathematical Surveys*, 30(1): 1-63, 1975.
- [22] J. Muhle-Karbe and M. Nutz. Small-Time Asymptotics of Option Prices and First Absolute Moments. *Journal of Applied Probability*, 48(4): 1003-1020, 2011.

- [23] F.W. Olver. Asymptotics and Special Functions. Academic Press, 1974.
- [24] L. Paulot. Asymptotic implied volatility at the second order with application to the SABR model. *Preprint*, SSRN, 2009.
- [25] H. Pham. Large deviations in Mathematical Finance. Lecture notes, Third SMAI European Summer School in Financial Mathematics, Paris, [felix.proba.jussieu.fr/pageperso/pham/Lecture-LD-Smai.pdf](http://felix.proba.jussieu.fr/pageperso/pham/Lecture-LD-Smai.pdf), 2010.
- [26] E. Renault and N. Touzi. Option hedging and implied volatilities in a stochastic volatility model. *Mathematical Finance*, 6(3): 279-302, 1996.
- [27] E.M. Stein and R. Sharkarchi. Complex Analysis. Princeton University Press, 2003.
- [28] S.R.S. Varadhan. On the behavior of the fundamental solution of the heat equation with variable coefficients. *Comm. Pure App. Math.*, 20: 431-455, 1967.
- [29] S.R.S. Varadhan. Diffusion processes in a small time interval. *Comm. Pure App. Math.*, 20: 659-685, 1967.
- [30] Zeliade Systems. Heston 2010. *White paper*, [www.zeliade.com/whitepapers/zwp-0004.pdf](http://www.zeliade.com/whitepapers/zwp-0004.pdf), 2011.

## 7. APPENDIX

**7.1. Proof of Proposition 3.4.** Define  $d_{\pm} := (-x \pm \sigma_t^2 t/2) / (\sigma_t \sqrt{t})$ . We first consider the case  $x > 0$ . Note that  $d_{\pm}$  tends to  $-\infty$  as  $t$  tends to 0. Substituting the asymptotic series

$$1 - \mathcal{N}(z) = (2\pi)^{-1/2} e^{-z^2/2} (z^{-1} - z^{-3} + \mathcal{O}(z^{-5})), \quad \text{as } z \text{ tends to infinity,}$$

into the Black-Scholes call option formula with implied volatility  $\sigma_t$ , we obtain

$$\begin{aligned} \mathbb{E}(e^{X_t} - S_0 e^x)_+ &= S_0 \mathcal{N}(d_+) - S_0 e^x \mathcal{N}(d_-) = S_0 (1 - \mathcal{N}(-d_+) - e^x + e^x \mathcal{N}(-d_-)) \\ (7.1) \quad &= \frac{S_0}{\sqrt{2\pi}} \exp\left(-\frac{d_+^2}{2}\right) \left(-\frac{1}{d_+} + \frac{1}{d_-} + \frac{1}{d_+^3} - \frac{1}{d_-^3} + \mathcal{O}(d_+^{-5})\right), \end{aligned}$$

as  $t$  tends to zero, where we used the fact that  $-d_-^2/2 + x = -d_+^2/2$ . Since the expansion

$$\begin{aligned} \exp\left(-\frac{d_+^2}{2}\right) &= \exp\left(-\frac{x^2}{2\Sigma^2 t} + \frac{x(\Sigma^4 + cx)}{2\Sigma^4} - \frac{\Sigma^8 + 4x^2 c^2 t}{\Sigma^6} \frac{t}{8} + \mathcal{O}(t^2)\right) \\ &= \left(1 - \frac{\Sigma^8 + 4x^2 c^2 t}{\Sigma^6} \frac{t}{8} + \mathcal{O}(t^2)\right) \exp\left(-\frac{x^2}{2\Sigma^2 t} + \frac{x(\Sigma^4 + cx)}{2\Sigma^4}\right) \end{aligned}$$

holds as well as  $d_-^{-1} - d_+^{-1} + d_+^{-3} - d_-^{-3} = x^{-2} \Sigma^3 t^{3/2} + \mathcal{O}(t^{5/2})$ , we can then plug this expression into (7.1) and the desired result follows. The proof of the case  $x < 0$  is analogous.

When  $x = 0$ , note that  $d_{\pm}$  converges to zero as  $t$  tends to zero, so that we use the asymptotic series  $\mathcal{N}(z) = 1/2 + (2\pi)^{-1/2} (z - \frac{1}{6}z^3 + \mathcal{O}(z^5))$  and the proposition follows from

$$\begin{aligned} \mathbb{E}(e^{X_t} - S_0 e^x)_+ &= S_0 \mathcal{N}(d_+) - S_0 \mathcal{N}(d_-) = \frac{S_0}{\sqrt{2\pi}} \left( (d_+ - d_-) - \frac{1}{6} (d_+^3 - d_-^3) + \mathcal{O}(d_+^5) \right) \\ &= \frac{S_0}{\sqrt{2\pi}} \left( \Sigma t^{1/2} + \frac{12c - \Sigma^4}{24\Sigma} t^{3/2} + \mathcal{O}(t^{5/2}) \right). \end{aligned}$$

**7.2. Proof of Theorem 4.2.** Let us first assume  $x > 0$ . Note first that the equalities (4.1) follow from equating the leading order and the correction terms for the Heston and the Black-Scholes models as

$$(7.2) \quad \Lambda^*(x) = \Lambda_{\text{BS}}^*(x) = \frac{x^2}{2\sigma_0^2(x)}, \quad \text{and} \quad A(x) = A_{\text{BS}}(x, \sigma_0(x)) \exp\left(\frac{a(x)x^2}{2\sigma_0^4(x)}\right).$$

We now have to make this argument rigorous, since we do not know that  $\sigma_t$  admits an expansion of the form stated in the theorem. However Theorem 3.1 and (7.2) imply that for all  $\varepsilon > 0$ , there exists  $t^*(\varepsilon)$  such that

$$\frac{\mathbb{E}(e^{X_t} - S_0 e^x)_+}{S_0} \leq \frac{A(x)}{\sqrt{2\pi}} t^{3/2} \exp\left(-\frac{\Lambda^*(x)}{t}\right) e^{\varepsilon} = \frac{A_{\text{BS}}(x, \sigma_0(x))}{\sqrt{2\pi}} \exp\left(\frac{a(x)x^2}{2\sigma_0^4(x)}\right) t^{3/2} \exp\left(-\frac{x^2}{2\sigma_0^2(x)t}\right) e^{\varepsilon}$$

holds for all  $t < t^*(\varepsilon)$ . The function  $a$  is continuous on  $(0, \infty)$ . Therefore for any  $\delta > 0$  sufficiently small we can choose  $\varepsilon' > 0$  such that the equality

$$\exp\left(\frac{1}{2} \frac{a(x) x^2}{\sigma_0^4(x)}\right) e^\varepsilon = \exp\left(\frac{1}{2} \frac{(a(x) + \delta) x^2}{\sigma_0^4(x)}\right) e^{-\varepsilon'},$$

holds and hence Theorem 3.1 implies the inequalities

$$\begin{aligned} \mathbb{E}(e^{X_t} - S_0 e^x)_+ &\leq \frac{A_{\text{BS}}(x, \sigma_0(x))}{\sqrt{2\pi}} \exp\left(\frac{(a(x) + \delta) x^2}{2\sigma_0^4(x)}\right) t^{3/2} \exp\left(-\frac{x^2}{2\sigma_0^2(x)t}\right) e^{-\varepsilon'} \\ &\leq C_{\text{BS}}\left(x, t, \sqrt{\sigma_0^2(x) + (a(x) + \delta)t}\right) \end{aligned}$$

for  $t$  sufficiently small. Since the Black-Scholes formula is a strictly increasing function of the volatility then the upper bound  $\sigma_t^2(x) \leq \sigma_0^2(x) + a(x)t + \delta t$  holds for all  $x > 0$ . We proceed similarly for the lower bound and for the case  $x < 0$ , and the theorem follows.

**7.3. Proof of Corollary 4.3.** We wish to compute a series expansion for  $p^*(x)$  defined on page 3 as the unique solution to  $x = \Lambda'(p^*(x))$  when  $x$  is close to zero (recall that  $p^*(0) = 0$ ). We substitute an ansatz power series expansion for  $p^*(x)$  and then recursively determine the coefficients such that the power series of the composition  $\Lambda'(p^*(x))$  equals  $x$ . We only indicate below the important auxiliary quantities:

$$\Lambda(p) = \frac{y_0}{2} p^2 + \frac{\rho\sigma y_0}{4} p^3 + \frac{\sigma^2 y_0}{24} (1 + 2\rho^2) p^4 + \frac{\rho\sigma^3 y_0}{48} (2 + \rho^2) p^5 + \mathcal{O}(p^6),$$

$$\begin{aligned} U(p) &= 1 - \frac{y_0}{2} p + \left(\kappa(\theta - y_0) - \sigma\rho y_0 + \frac{y_0^2}{2}\right) \frac{p^2}{4} \\ &\quad + \left(\frac{\kappa\rho\sigma}{3}(\theta - 2y_0) - \frac{\sigma^2 y_0}{3}(1 + \rho^2) - \frac{y_0}{2}(\kappa(\theta - y_0) - \rho\sigma y_0) - \frac{y_0^3}{12}\right) \frac{p^3}{4} + \mathcal{O}(p^4), \end{aligned}$$

$$\Lambda^*(x) = \frac{x^2}{2y_0} - \frac{\rho\sigma}{4y_0^2} x^3 + \frac{\sigma^2}{96y_0^3} (19\rho^2 - 4) x^4 + \mathcal{O}(x^5),$$

$$\sigma_0(x) = \sqrt{y_0} + \frac{\rho\sigma x}{4\sqrt{y_0}} + \left(1 - \frac{5}{2}\rho^2\right) \frac{\sigma^2 x^2}{24y_0^{3/2}} + \mathcal{O}(x^3),$$

$$A(x) = \frac{y_0^{3/2}}{x^2} + \frac{\sqrt{y_0}}{4x} (3\rho\sigma + 2y_0) - \frac{1}{96\sqrt{y_0}} \left(11\rho^2\sigma^2 - 8\sigma^2 - 48\rho\sigma y_0 - 24\kappa\theta + 24\kappa y_0 - 12y_0^2\right) + \mathcal{O}(x).$$

From these formulae we deduce the expansion

$$\frac{A(x)}{A_{\text{BS}}(x, \sigma_0(x))} = 1 + \frac{\sigma^2(\rho^2 - 4) + 12\rho\sigma y_0 + 24\kappa(\theta - y_0)}{96y_0^2} x^2 - \frac{\rho\sigma x^3}{96y_0^3} \left(10\rho\sigma y_0 + 3\sigma^2(\rho^2 - 2) - 20\kappa y_0 + 28\kappa\theta\right) + \mathcal{O}(x^4),$$

which implies the expression for  $a(x) = x^{-2} 2\sigma_0^4(x) \log(A(x)/A_{\text{BS}}(x, \sigma_0(x)))$  in Corollary 4.3.

DEPARTMENT OF MATHEMATICS, KING'S COLLEGE LONDON  
E-mail address: martin.forde@kcl.ac.uk

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON  
E-mail address: ajacquier@imperial.ac.uk

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO  
E-mail address: rogerlee@math.uchicago.edu