THE LARGE-MATURITY SMILE FOR THE HESTON MODEL

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Abstract. Using the Gärtner-Ellis theorem from large deviations theory, we characterize the leading-order behaviour of call option prices under the Heston model, in a new regime where the maturity is large and the log-moneyness is also proportional to the maturity. Using this result, we then derive the implied volatility in the large-time limit in the new regime, and we find that the large-time smile mimics the large-time smile for the Barndorff-Nielsen Normal Inverse Gaussian model. This makes precise the sense in which the Heston model tends to an exponential Lévy process for large times. We find that the implied volatility smile does not flatten out as the maturity increases, but rather it spreads out, and the large-time, large-moneyness regime is needed to capture this effect. As a special case, we provide a rigorous proof of the well known result by Lewis [40] for the implied volatility in the usual large-time, fixed-strike regime, at leading order. We find that there are two critical strike values where there is a qualitative change of behaviour for the call option price, and we use a limiting argument to compute the asymptotic implied volatility in these two cases.

1. Introduction

In contrast to the small-time case, there is a comparatively small literature on large-time asymptotics for stochastic volatility models. Lewis [40] approximated the characteristic function of the logarithmic return for the Heston model at large maturities by the leading order term of its eigenfunction expansion. Using Laplace's method for contour integrals along a horizontal line going through the saddle point of the eigenvalue function on the imaginary axis (see the sequel paper by Forde, Jacquier & Mijatović [24] and Theorem 7.1, chapter 4 of Olver [43] for more details), Lewis derived an asymptotic formula for the price of a call option in the large-time, fixed-strike regime, and developed a similar asymptotic result for the Black-Scholes model.

By equating the asymptotic call option formulae under the Heston model and the Black-Scholes model at zeroth order and $O(1/t)$, Jacquier [32, 33] computed the corresponding asymptotic formula for implied volatility. The leading order term in the expansion is a fixed number, independent of the log-moneyness, which is equal to eight times the principal eigenvalue of the Sturm-Liouville operator associated with the instantaneous variance process. The correction (or "skew") term is affine in x/t, where t represents the maturity and x is the log-moneyness. From this we see that this approach captures the skew effect, but not the smile effect; we have to go to higher order to see the x^2/t^2 term. The arguments of Lewis and Jacquier were not rigourous as they did not explicitly compute the contour of steepest descent, or justify the truncation of the eigenfunction expansion with tail estimates.

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Keller-Ressel [37] provides a general treatise on the long-term properties of a general affine stochastic volatility model (in the sense of Duffie, Filipovic & Schachermayer [12]). These models can incorporate an auxiliary finite or infinite jump process in addition to the diffusion component, and many properties of these models can be analyzed by using differential equations of the generalized Riccati type. In [37], the author derives conditions for the price process to be a martingale and conservative (i.e. without explosions or killing), and conditions for the existence of an invariant distribution of the stochastic variance process, and characterizes this distribution in terms of its cumulant generating function. Under mild conditions, the stochastic variance process will converge in law to the invariant distribution. Keller-Ressel also provides explicit expressions for the explosion time at which a given moment of the terminal stock price becomes infinite and establishes results for the long-term properties of the price process, showing that as time tends to infinity, its rescaled cumulant generating function (cgf) tends to the cgf of an infinitely divisible random variable (which has a Normal Inverse Gaussian distribution in the case of the Heston model).

The theory of large deviations provides a natural approach to the approximation of the exponentially small probabilities associated with the behaviour of a diffusion process over a very small time interval, or a very large time interval when the process has ergodic behaviour. In recent years there has been an explosion of literature on small-time asymptotics for stochastic volatility models (see Berestycki et al. [5], [6], Forde & Jacquier [21, 22], Hagan et al. [29], Henry-Labordère [30], Laurence [38] and Robertson [44]). All these articles characterize the behaviour of the Black-Scholes implied volatility for European options in the small maturity limit, and they are essentially applications and/or higher order corrections to the seminal work of Varadhan [47, 48] and the Freidlin-Wentzell theory of large deviations for stochastic differential equations [25], the main point being that the small-time behaviour of a diffusion process can be characterized in terms of an energy/distance function on a Riemmanian manifold, whose metric is induced from the inverse of the diffusion coefficient.

The Heston model is the canonical example of an affine stochastic volatility model, and is widely used because the characteristic function of the log stock price can be computed in closed-form by solving Riccati equations, so call options prices are easily calculated using Fourier inversion techniques (see Lee [39] et al.). The critical moment $p^* = \inf\{p \geq 1 : \mathbb{E}\left(S_t^p\right) = \infty\} \in (1, \infty)$ (see Andersen & Piterbarg [2] and Lions & Musiela [41]), so the right tail of the density of S_t is fatter than the standard Black-Scholes model (for which $p^* = \infty$), thinner than the popular SABR model with $\beta = 1$, $\rho = 0$ (which has $p^* = 1$), and comparable to a SABR model with $\beta = 1$, $\rho < 0$, where $p^* \in (1, \infty)$ (see Jourdain [35]).

We can characterize the limiting behaviour of the implied volatility in the small-time limit using the Gärtner-Ellis Theorem from large deviations theory (see Forde & Jacquier [21]), or we can go further and compute a small-time expansion for call options and implied volatility using Laplace's method for contour integrals (see Forde, Jacquier & Lee [23]). As the maturity of a European option goes larger, the smile for the Heston model flattens much quicker than the smile for a time-homogenous local volatility model (e.g. the CEV model), and tends to a single number in the large-time limit, which is related to the principal eigenvalue for a certain Sturm-Liouville operator (see Lewis [40]).

In this article, we establish a large-time large deviation principle for the log return divided by the time-to-maturity. We accomplish this by applying the Gärtner-Ellis theorem to the exponential affine closed-form solution for the log stock price moment generating function (Theorem 2.1). As a corollary, we derive an asymptotic formula for the price of a call option under the Heston model in the large-maturity limit, when the log-moneyness is also proportional to the maturity (Corollary 2.4). To prove Corollary 2.4, we first prove a corresponding large-time large deviation principle under the Share measure, and then use the trick on page 4 in Carr & Madan [8] expressing the call option price as the probability of the log stock price minus an independent Exponential random variable exceeding the strike price under the Share measure. We derive a similar result for the Black-Scholes model (Corollary 2.4).

We derive the corresponding asymptotic formula for the implied volatility (Corollary 2.14) which shows (contrary to popular belief) that the smile effect does not disappear at large maturities, but rather it just spreads out. Moreover, we find that the asymptotic implied volatility smile is the same as that for the Normal Inverse Gaussian model, because both models have the same limiting behaviour for the cgf of the log stock price. Gatheral & Jacquier [28] have also recently proved that under a suitable change of variables and Condition (2.2), this asymptotic volatility smile is algebraically equal to the Stochastic Volatility Inspired (SVI) parameterisation proposed by Gatheral in [27], thus confirming Gatheral's conjecture about the large time form of the Heston smile. Friz, Gerhold, Gulisashvili & Sturm [26] have recently derived a *large-strike* expansion of the form $\sigma_{BS}(k,T)^2T = (\beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} ...)^2$ for the implied volatility $\sigma_{BS}(k,T)$ as a function of the log-moneyness k under the Heston model, where all constants are explicitly known in terms of p^*. Their analysis is based entirely on affine principles; they do not require the explicit form of the Heston cgf, but rather they extract all the necessary information on the Mellin transform by analysing the corresponding Riccati equations near the critical point, using higher order Euler estimates. They also noted that the SVI parameterisation is not compatible with their implied volatility expansion at finite maturities; this anomaly is resolved by noting that the β_2 term in the Friz et al. expansion is $O(T^{-1/2})$ as $T \to \infty$. Finally, they proved that the initial level of the instantaneous variance and its long run level do not affect the tail asymptotics.

Forde, Jacquier & Mijatović [24] took a less probabilistic approach, and used Laplace's method for contour integrals as detailed in Olver [43] to derive asymptotic formulae for the price of a call option and the implied volatility under the Heston model, in the same large-time, large-strike regime considered in this article. The correction term for implied volatility takes account of the initial level of the instantaneous volatility process and provides a sharp estimate for call option prices in the large-time, large-strike limit. For the usual large-time, *fixed-strike* regime, [24] give a rigorous proof of the expansion proposed by Jacquier [32]. The large-strike expansion in [26] and the large-time, large-strike expansion in [24] suggest that the SVI parameterisation should be modified at finite maturities. For the affine class of models, Keller-Ressel [37] makes the statement that "the marginal distributions of the price process approach those of an exponential-Lévy process". For the Heston model, this statement can be disproved using saddlepoint methods (see [24]), essentially because it takes account of the rate function in the exponent but not the pre-factor in front of the exponent for the mgf.

The leading order term for the implied volatility in the large-time, large-strike regime is independent of the initial level of the instantaneous variance process, due to the ergodic property of this process. The variance process for the Heston model is a Cox-Ingersoll-Ross (CIR) diffusion, and it is well known that the invariant distribution for this process is a Gamma distribution, with mean equal to the mean reversion level of the process. Mathematically, it is more natural to work in this regime, because we retain more information about the full implied volatility smile at large maturities, rather than characterizing its behaviour with just one or two numbers as in the previous works of Lewis and Jacquier. We show how to prove Lewis' result rigourously at leading order in Corollary 2.17.

The leading order large-time asymptotics in this article are closely related to the work of Feng, Forde & Fouque [18]. Using an extension of the Gärtner-Ellis theorem with Gamma convergence, they establish a large deviation principle for the log stock price in a small-time, fast mean-reverting regime, and derive corresponding results for call option prices and implied volatility. In this article, the drift term for the log stock price shows up in the leading order asymptotics, but in [18], the only effect of the drift is the lack of lower semicontinuity of the limiting logarithmic moment generating function, which is why Gamma convergence is needed because the Gärtner-Ellis theorem does not apply.

Our work is also related to a recent paper by Tehranchi [46]. Tehranchi merely imposes that the stock price process is a non-negative local martingale under a locally equivalent measure, and that the stock price tends to zero almost surely as time tends to infinity. The latter condition is equivalent to the reasonable economic assumption that the call price tends to the initial stock price as time goes large (the Heston model satisfies these conditions). Under these weak assumptions, Tehranchi derives an expression for the asymptotic implied variance in the fixed-strike, large-time regime, which depends only on the expectation of the minimum of the stock price and the strike price. Theorem 4.7 in Tehranchi [46] essentially proves and corrects a missing term in Equation 3.8 in chapter 6 of Lewis [40] (which was also heuristically corrected by Jacquier [32]) for the large-time implied volatility in the Heston model in the fixed-strike regime. See also Forde [19] and Jacquier [34] for a summary of large deviations results for small and large time under the Heston model.

The asymptotics in this paper can be traced back to the seminal work of Donsker & Varadhan [13, 14, 15, 16], who showed that for an ergodic diffusion process with reasonable coefficients, the occupation measure satisfies a large-time large deviation principle, and the rate function has a variational representation as the infimum of a certain functional over the space of probability measures. Intuitively, this rate function measures the exponentially small probability of the realized occupation measure deviating from the stationary (invariant) measure for the process. They also showed that the large-time asymptotics can be characterized in terms of the principal eigenvalue for the associated second-order elliptic operator for the process plus a potential term, and they derive a variational representation for this eigenvalue which generalizes the classical Rayleigh-Ritz formula for a self-adjoint operator. If the occupation measure for the instantaneous volatility process for a stochastic volatility model satisfies a large deviation principle, then the integrated variance (which is just a linear functional of the occupation measure) also satisfies the occupation measure (by the contraction principle), and the rate function will govern the large-time behaviour of digital and variance call options on logarithmic scale.

2. Main results

We work on a model $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ supporting two Brownian motions, and satisfying the usual conditions. All expectations are taken under $\mathbb{P}$ unless otherwise stated, and we assume that the prevailing risk-free interest rate is zero. All the results below are very easily amended in the case of constant interest rates, where it is more convenient to deal directly with the forward price rather than the stock price.

Let $(S_t)_{t\geq 0}$ denote a stock price process with a strictly positive non-random initial value S_0, and assume that $X_t = \log S_t$ is governed by the Heston stochastic volatility model, defined by the following stochastic differential equations

$$
\begin{aligned}
dX_t &= -\tfrac{1}{2} Y_t dt + \sqrt{Y_t} dW_t^1, \\
dY_t &= \kappa(\theta - Y_t) dt + \sigma \sqrt{Y_t} dW_t^2, \\
d\left\langle W^1, W^2 \right\rangle_t &= \rho dt,
\end{aligned}
\tag{2.1}
$$

with κ, θ, $\sigma > 0$, $|\rho| < 1$, $X_0 = x_0 \in \mathbb{R}$, $Y_0 = y_0 > 0$ and $2\kappa\theta > \sigma^2$, which ensures that zero is an unattainable boundary for the process Y. The process Y is a Cox-Ingersoll-Ross (CIR) diffusion (also known as the square root process), and the stochastic differential equation for the CIR diffusion satisfies the Yamada-Watanabe condition (see page 291, Proposition 2.13 in Karatzas & Shreve [36]), so it admits a unique strong solution. The process X can be expressed as a stochastic integral of Y, so it is also well defined.

2.1. A large-time large deviation principle.

In this section we derive the following result which describes the large-time asymptotic behaviour of the moment generating function and the distribution function of the log stock price under the Heston model, in the regime where the maturity is large, and the log-moneyness is also proportional to the maturity.

Theorem 2.1. *Assume that*

$$
\kappa > \rho\sigma, \tag{2.2}
$$

then the process $\frac{1}{t}(X_t - x_0)$ *satisfies a large deviation principle as* t *tends to infinity, with rate function* $V^*(x) = V^*(x; \kappa, \theta, \sigma, \rho)$ *equal to the Legendre transform of*

$$
\begin{aligned}
V(p) &= \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left(e^{p(X_t - x_0)}\right) \\
&= \begin{cases} \frac{\kappa\theta}{\sigma^2}\left(\kappa - \sigma\rho p - \sqrt{(\kappa - \sigma\rho p)^2 - \sigma^2 p(p-1)}\right), & \text{for } p \in [p_-, p_+] \\ \infty, & \text{for } p \notin [p_-, p_+], \end{cases}
\end{aligned}
\tag{2.3}
$$

where

$$
p_\pm = \frac{\sigma - 2\kappa\rho \pm \eta}{2(1-\rho^2)\sigma}, \quad \text{and} \quad \eta = \sqrt{\sigma^2 + 4\kappa^2 - 4\rho\sigma\kappa}.
$$

V^* is continuous, attains its minimum value at $x^* = V'(0) = -\frac{1}{2}\theta < 0$ and we note that $V(0) = V(1) = 0$, $p_- < 0$, $p_+ > 1$, and for all $a < b$, we have

$$-\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\frac{X_t - x_0}{t}\in(a,b)\right) = \inf_{x\in(a,b)} V^*(x)$$

$$= \begin{cases} 0, & \text{for } a \le x^* \le b,\\ V^*(b), & \text{for } a \le b \le x^*,\\ V^*(a), & \text{for } x^* \le a \le b.\end{cases} \tag{2.4}$$

Remark 2.2. By Lemma 2.3 in Andersen & Piterbarg [2], we know that $\kappa - \rho\sigma$ is the mean reversion level of the Y process under the so-called Share measure $\mathbb{P}^*$, defined in Section 3.2.1. We postpone the discussion on the physical significance of this measure to the proof of Theorem 2.1 . If $\kappa < \rho\sigma$, the Y process is not mean reverting anymore under $\mathbb{P}^*$, and thus does not have the required ergodic behaviour under $\mathbb{P}^*$, and we have that the process Y_t grows exponentially as $t \to \infty$. We can see this using the representation of the CIR process as a time-changed Bessel process, see Lemma 2.1 in Atlan [3] for instance. Note that Condition (2.2) also appears in section 6.1 of Keller-Ressel [37]. In Equity markets, the correlation ρ is almost always negative, so this condition is not overly restrictive.

Remark 2.3. If we set $\alpha = \frac{\sqrt{\sigma^2+4\kappa^2-4\kappa\rho\sigma}}{2\sigma\bar\rho^2}$, $\beta = \frac{2\kappa\rho-\sigma}{2\sigma\bar\rho^2}$, $\mu = -\frac{\kappa\theta\rho}{\sigma}$, $\delta = \frac{\kappa\theta\bar\rho}{\sigma}$, $\bar\rho = \sqrt{1-\rho^2}$, then, by completing the square, we find that V is the cumulant generating function of a Normal inverse Gaussian distribution $NIG(\alpha,\beta,\mu,\delta)$, which reads

$$V(p) = \delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+p)^2}\right) + \mu p,$$

for $\alpha, \delta > 0, \mu \in \mathbb{R}$ and $|\beta| < \alpha$. Note that we can rewrite α as $\alpha = \frac{\sqrt{\beta^2+4\kappa^2\bar\rho^2}}{2\sigma\bar\rho^2}$, so that $|\beta| < \alpha$ (see also Remark 2.16). This does not imply that the transition density for $\frac{1}{t}(X_t - x_0)$ tends to that of a Normal inverse Gaussian distribution, and this can be shown to be false because for the transition density we also have to take account of the leading eigenfunction for the Sturm-Liouville operator associated with the Heston model (see Lewis [40] and Forde, Jacquier & Mijatović [24] and also Remark 2.16).

For a standard one-dimensional Brownian motion $(W_t)_{t\ge0}$, the moment generating function is given by $\mathbb{E}\left(e^{pW_t}\right) = \exp\left(\frac{1}{2}p^2 t\right)$, and $V(p) = \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left(e^{p(X_t-x_0)}\right) = \frac{1}{2}p^2$. Hence Equation (2.4) still holds, but in this case the rate function $V^*(x)$ is given by the Legendre transform of V, which easily reads $V^*(x) = \frac{1}{2}x^2$. We can also extend Theorem 2.1 to a Heston model with jumps, which has an additional independent driving Lévy process (mean corrected so the stock price process remains a martingale). In this case, the limit $V(p)$ is easily computed in terms of the characteristic function for the Lévy process (see Jacquier [34] and Keller-Ressel [37] for details).

The convergence of the rescaled cumulant generating function to $V(p)$ in Equation (2.3) is proved in Section 2 as the first step of the proof of the theorem. For the Heston model (and more general affine stochastic volatility models), an alternative proof is possible using Theorem 3.4 in Keller-Ressel [37] [1], although our approach makes clear the physical meaning of the condition $\kappa - \rho\sigma > 0$.

2.2. Pricing Digital and European options in the large-time limit.

[1] We thank an anonymous referee for pointing out this alternative proof.

2.2.1. *The Heston model.* In this section, we prove the following useful corollary of Theorem 2.1 which is a rare event estimate for pricing European put/call options in the large-time, large-strike regime.

Corollary 2.4. *For the Heston model defined in Equation* (2.1) *under Condition* (2.2)*, we have the following large-time asymptotic behaviour for put/call options*

$$\begin{aligned}
-\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left(S_t - S_0e^{xt}\right)^+ &= V^*(x) - x, && \text{for } x \geq \frac{1}{2}\bar{\theta},\\
-\lim_{t\to\infty}\frac{1}{t}\log(S_0 - \mathbb{E}\left(S_t - S_0e^{xt}\right)^+) &= V^*(x) - x, && \text{for } -\frac{1}{2}\theta \leq x \leq \frac{1}{2}\bar{\theta},\\
-\lim_{t\to\infty}\frac{1}{t}\log(\mathbb{E}\left(S_0e^{xt} - S_t\right)^+) &= V^*(x) - x, && \text{for } x \leq -\frac{1}{2}\theta,
\end{aligned}$$

where $\bar{\theta} = \frac{\kappa\theta}{\kappa - \rho\sigma}$.

Remark 2.5. If we set $V_S^*(-x) = V^*(x) - x$, then $V_S^*(x)$ is the rate function associated with the family of random variables $-\frac{1}{t}(X_t - x_0)$ under the Share measure $\mathbb{P}^*$ (see Corollary 3.1 and Section 3.2.1). The value $x_S^* = \frac{1}{2}\bar{\theta}$ is the turning point of $V_S^*(-x)$, where $V_S^*(-\frac{1}{2}\bar{\theta}) = {V_S^*}'(-\frac{1}{2}\bar{\theta}) = 0$.

Remark 2.6. We note that the "effective" rate function $V^*(x) - x$ for call options is different to the rate function $V^*(x)$ for digital call options, in contrast to the small-time regime discussed in Forde & Jacquier [21], where both rate functions are the same.

We also have the following corollaries of Theorem 2.1 and Corollary 2.4 for the usual large-time, fixed-moneyness regime.

Corollary 2.7. *We have the following large-time asymptotic behaviour for digital call options of strike* $K = S_0e^x > 0$ *on* S_t*,*

$$\begin{aligned}
-\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}(S_t > K) &= -\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}(X_t - x_0 > x)\\
&= V^*(0) = \frac{\kappa\theta}{2\sigma^2(1-\rho^2)}(-2\kappa + \rho\sigma + \eta) > 0.
\end{aligned}$$

Remark 2.8. Note that $V^*(0)$ is minus the minimum value of V.

Corollary 2.9. *We have the following large-time asymptotic behaviour for European call options of strike* $K > 0$ *on* S_t

$$-\lim_{t\to\infty}\frac{1}{t}\log(S_0 - \mathbb{E}(S_t - K)^+) = V_S^*(0) = V^*(0) > 0.$$

Remark 2.10. Note that the limits in Corollaries 2.7 and 2.9 are the same and independent of the fixed strike K i.e. we do not see the smile effect in the fixed-strike regime.

2.2.2. *The Black-Scholes model.* In this section, we consider the standard Black-Scholes model with zero interest rates for a log stock price process $(X_t)_{t\geq 0}$ with volatility $\sigma > 0$, which satisfies $dX_t = -\frac{1}{2}\sigma^2 dt + \sigma dW_t$, with $X_0 = x_0 \in \mathbb{R}$. We obtain similar results to Theorem 2.1 and Corollary 2.4.

Theorem 2.11. *For the Black-Scholes model, the process* $\frac{1}{t}(X_t - x_0)$ *satisfies a large deviation principle as* t *tends to infinity, with rate function* $V_{\mathrm{BS}}^*(x,\sigma)$ *equal to the Legendre transform of*

$$V_{\mathrm{BS}}(p,\sigma) = \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left(e^{p(X_t - x_0)}\right) = \frac{1}{2}\sigma^2(p^2 - p). \tag{2.5}$$

The function $x \mapsto V^*_{\mathrm{BS}}(x,\sigma)$ is continuous, attains its minimum value at $x^* = \partial_p V_{\mathrm{BS}}(p,\sigma)|_{p=0} = -\frac{1}{2}\sigma^2 < 0$, and $V_{\mathrm{BS}}(0,\sigma) = V_{\mathrm{BS}}(1,\sigma) = 0$. Furthermore, for all $a < b$, we have

$$-\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\frac{X_t - x_0}{t} \in (a,b)\right) = \inf_{x\in(a,b)} V^*_{\mathrm{BS}}(x,\sigma) = \begin{cases} 0, & \text{for } a \le x^* \le b, \\ V^*_{\mathrm{BS}}(b,\sigma), & \text{for } a \le b \le x^*, \\ V^*_{\mathrm{BS}}(a,\sigma), & \text{for } x^* \le a \le b. \end{cases}$$

Corollary 2.12. *For the Black-Scholes model, we have the following asymptotic behaviour for call and put options*

$$\begin{aligned} -\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}(S_t - S_0e^{xt})^+ &= V^*_{\mathrm{BS}}(x,\sigma) - x, && \text{for } x \ge \frac{1}{2}\sigma^2, \\ -\lim_{t\to\infty}\frac{1}{t}\log(S_0 - \mathbb{E}(S_t - S_0e^{xt})^+) &= V^*_{\mathrm{BS}}(x,\sigma) - x, && \text{for } -\frac{1}{2}\sigma^2 \le x \le \frac{1}{2}\sigma^2, \\ -\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}(S_0e^{xt} - S_t)^+ &= V^*_{\mathrm{BS}}(x,\sigma) - x, && \text{for } x \le -\frac{1}{2}\sigma^2, \end{aligned}$$

where

$$V^*_{\mathrm{BS}}(x,\sigma) = \frac{(x + \frac{1}{2}\sigma^2)^2}{2\sigma^2}.$$

Remark 2.13. Corollary 2.12 is very important in the proof of Corollary 2.14, where we characterize the implied volatility under the Heston model in the large-time, large-strike regime.

2.3. **Large-time behaviour of implied volatility.** We can also compute the asymptotic implied volatility in the large-time limit. Let $\sigma_t(x)$ denote the Black-Scholes implied volatility of a European call option with strike price $K = S_0e^{xt}$ under the Heston model. In Section 2.3 we prove the following result:

Corollary 2.14. *Under Condition* (2.2)*, we have the following large-time asymptotic behaviour for $\sigma_t(x)$*

$$\begin{aligned} \sigma^2_\infty(x) &= \lim_{t\to\infty}\sigma^2_t(x) \\ &= \begin{cases} 2\left(2V^*(x) - x - 2\sqrt{V^*(x)^2 - V^*(x)x}\right), & \text{for } x \in \mathbb{R}\setminus\left[-\frac{1}{2}\theta, \frac{1}{2}\bar{\theta}\right], \\ 2\left(2V^*(x) - x + 2\sqrt{V^*(x)^2 - V^*(x)x}\right), & \text{for } x \in \left(-\frac{1}{2}\theta, \frac{1}{2}\bar{\theta}\right). \end{cases} \end{aligned} \tag{2.6}$$

Remark 2.15. The main practical use of Corollary 2.14 is to approximate the Heston call price $\mathbb{E}(S_t - S_0e^{xt})^+$ by $C^{\mathrm{BS}}(S_0, S_0e^{xt}, t, \sigma_\infty(x))$ in some sense (for large t), where $C^{\mathrm{BS}}(S_0,K,t,\sigma)$ is the Black-Scholes call option formula with initial stock price S_0, strike price K, maturity t and volatility σ. It can also be used for implied volatility smile extrapolations at large maturities, where Monte Carlo and PDE methods break down. However, it is important to note that Corollary 2.14 does not imply that

$$\mathbb{E}(S_t - Ke^{xt})^+ \to C^{\mathrm{BS}}(S_0, S_0e^{xt}, t, \sigma_\infty(x)), \qquad \text{as } t\to\infty.$$

This is because the exponential bounds associated with the large deviation principle are too crude to be able to say anything sharper. However, if we calculate the next term in the expansion for the implied volatility as $\sigma_t(x)^2 = \sigma^2_\infty(x) + \frac{1}{t}a(x) + O\left(1/t^2\right)$, then it is true that

$$\mathbb{E}(S_t - Ke^{xt})^+ \to C^{\mathrm{BS}}\left(S_0, Ke^{xt}, t, \sqrt{\sigma^2_\infty(x) + \frac{a(x)}{t}}\right), \qquad \text{as } t\to\infty,$$

(see the sequel paper by Forde, Jacquier & Mijatović [24] and Forde [20], where the correction term $a(x)$ is computed using Laplace's method for contour integrals).

Remark 2.16. The proof of Corollary 2.14 is a immediate consequence of the limiting behaviour $V(p) = \lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}(e^{p(X_t-x_0)})$, and nothing else. This means that the Heston model has the same rate function $V^*(x)$ and the same asymptotic implied volatility $\sigma_\infty(x)$ as the Barndorff-Nielsen Normal inverse Gaussian model where $X_t - x_0 \sim NIG(\alpha, \beta, \mu t, \delta t)$ and $\mathbb{E}(e^{p(X_t-x_0)}) = e^{V(p)t}$ for all $t > 0$, if $\alpha, \beta, \mu, \delta$ are chosen as in Remark 2.3 (see Benaim & Friz [4], Cont & Tankov [9], Keller-Ressel [37] et al. for further details).

Gatheral & Jacquier [28] recently proved that under a suitable change of variables and Condition (2.2), this asymptotic volatility smile was algebraically equal to the Stochastic Volatility Inspired (SVI) parameterisation proposed by Gatheral in [27], thus confirming the conjecture about the large time form of the Heston smile.

2.3.1. *The cases $x = \frac{1}{2}\bar{\theta}$, $x = -\frac{1}{2}\theta$ and $x = 0$.* Using the fact that $V_S^*(-\frac{1}{2}\bar{\theta}) = 0$ and $V^*(-\frac{1}{2}\theta) = 0$, we find that

$$\begin{aligned} \lim_{x\to\frac{1}{2}\bar{\theta}} \sigma_\infty^2(x) &= \bar{\theta}, \\ \lim_{x\to-\frac{1}{2}\theta} \sigma_\infty^2(x) &= \theta. \end{aligned}$$

For the at-the-money case $x = 0$, we have

$$\sigma_\infty^2(0) = 8V_S^*(0) = 8V^*(0) = \frac{4\kappa\theta}{\sigma^2(1-\rho^2)}(-2\kappa + \rho\sigma + \eta), \tag{2.7}$$

which agrees with the expressions in Equations 3.9 and 4.3 in chapter 6 of Lewis [40] at leading order.

2.4. **The large-time, fixed-strike regime.** We can compute the leading order asymptotic implied volatility in the large-time, fixed-strike regime given in Equations 3.9 and 4.3 in Chapter 6 of Lewis [40].

Corollary 2.17. *Let $K > 0$. For the standard Heston model in Theorem 2.1, we have the following asymptotic behaviour for the implied volatility $\hat{\sigma}_t(x)$ of a European call option on $S_t = e^{X_t}$, with strike $K = S_0 e^x$ as t tends to infinity,*

$$\lim_{t\to\infty} \hat{\sigma}_t(x)^2 = 8V_S^*(0) = 8V^*(0) = \frac{4\kappa\theta}{\sigma^2(1-\rho^2)}(-2\kappa + \rho\sigma + \eta). \tag{2.8}$$

Remark 2.18. Note that the answer is the same as for the at-the-money case $x = 0$ in Equation (2.7). See Theorem 4.6 in Tehranchi [46] for a similar result.

2.5. **Numerics.** In this section, we provide some visual explanations of the results presented above: Figure 1 shows the domain of existence and essential smoothness of the limiting moment generating function V as well as the behaviour of the function V_{BS}^*, which serves as a tool in the proof of Corollary 2.14. It is interesting to compare this figure with the asymptotic implied volatility smile in Figure 3 and the effective rate function $V_S^*(-x) = V^*(x) - x$ in Figure 2: as ρ grows from a negative value to a positive one, the asymmetry of V translates into that of the smile and (inversely) of V^*. The intuition behind this result is that, when considering the positive axis for instance, a large value of $V(p)$ means high values of the stock price process (almost surely), hence larger values for call options, which implies the asymmetry of the smile. We also zoom in on Figure 2 not to lead to the wrong impression that the minimum of V_S^* is independent of the correlation ρ.

Note that from Corollary 2.14, the asymptotic implied volatility smile under the Heston model does not depend on the initial value y_0 of the variance process. The form of this large time smile was conjectured

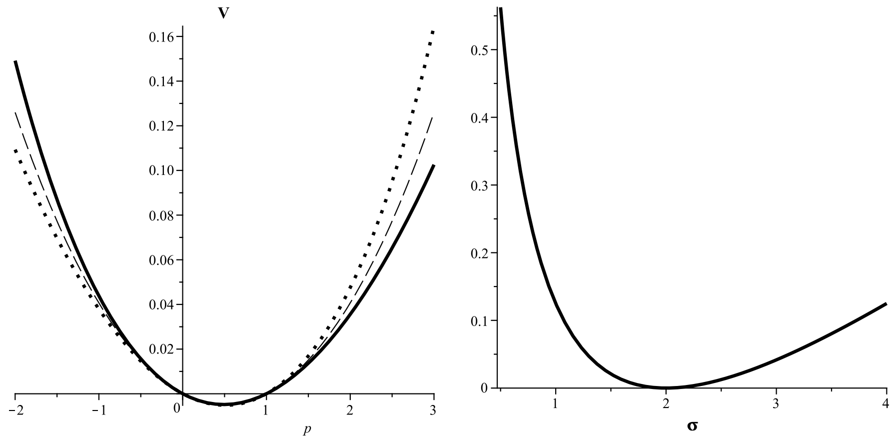

FIGURE 1. The left figure is the graph of the function V defined in (2.3) for $\rho = -.4, 0, 0.4$ (solid, dashed and dotted lines respectively). We can immediately identify the points $(0,0)$ and $(1,0)$ where $V'(0) = -\frac{1}{2}\theta$ and $V'(1) = \frac{1}{2}\bar{\theta}$. On the right, we plot the function $\sigma \mapsto V^*_{\mathrm{BS}}(-1,\sigma)$ given in Corollary 2.12. This graph is essentially here as a visual aid for the proof of Corollary 2.14 in Section 3.6.

by Gatheral in [27], who called it the Stochastic Volatility Inspired parameterisation, and it was recently proved in [28] that it is precisely the limit of the Heston smile as the maturity goes to infinity, which is indeet independent of y_0. In all the graphs below, we take $\kappa = 1.15$, $\theta = 0.04$, and $\sigma = 0.2$ and $\rho = -0.4, 0$ and 0.4.

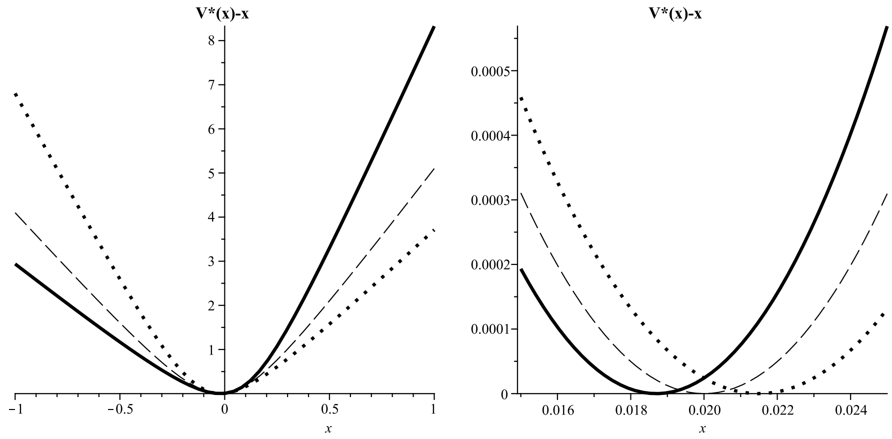

FIGURE 2. We plot here the function $x \mapsto V^*(x) - x$ for $\rho = -0.4, 0, 0.4$, which achieves its minimum value of zero at $x = \frac{1}{2}\bar{\theta} = 0.0187$ for $\rho = -0.4$, 0.02 for $\rho = 0$ and 0.0215 for $\rho = 0.4$. The right graph is a zoomed-in version which shows the dependence of the minimum on ρ.

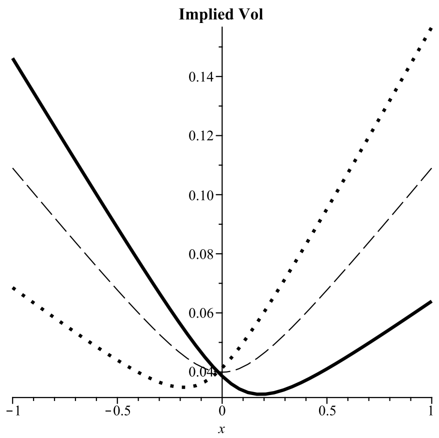

FIGURE 3. This graph represents the asymptotic implied volatility smile $\sigma_\infty(x)$ for $\rho = -0.4, 0, 0.4$ given in Corollary 2.14. $\rho = 0$ leads to a symmetric smile (dashed) and the sign of ρ indicates the asymmetry of the smile.

3. PROOF OF THE MAIN RESULTS

We first prove Theorem 2.1. It is trivial to prove the form of $V(p)$ when it exists; the more delicate issue is proving that $V(p)$ is infinite outside (p_-, p_+). For this, we recall the two conditions that appear in Theorem 3.1 in Hurd & Kuznetsov [31] and Proposition 3.1 in Andersen & Piterbarg [2]. We show that for our purposes, one of the conditions implies the other. In section 2.2, we prove Corollary 2.4 in a series of steps. We first introduce the Share measure $\mathbb{P}^*$. It turns out that the effective rate function for call options is equal to the rate function for the log stock price divided by the maturity under $\mathbb{P}^*$ and not $\mathbb{P}$, in contrast to small-time large deviations theory where both rate functions are the same.

In Lemma 3.2, we establish a simple relationship between the rate functions under $\mathbb{P}$ and $\mathbb{P}^*$ which simplifies subsequent calculations. In the final proof of Corollary 2.4, we exploit an interesting identity derived in Carr & Madan [8], namely that the price of a call option under $\mathbb{P}$ is equal to the price of a digital call option on the log stock price minus an independent Exponential variable, under the measure $\mathbb{P}^*$. Corollaries 2.7 and 2.9 then follow almost immediately from the smoothness properties of the Legendre transforms under $\mathbb{P}$ and $\mathbb{P}^*$. The Black-Scholes versions of these results are then proved using similar arguments. This enables us to derive the implied volatility asymptotic behaviour, namely Corollary 2.14, for which we use Theorem 2.12 to sandwich the Heston call option price between two Black-Scholes call prices whose volatilities are very close to each other, and from this we obtain bounds for the implied volatility itself using the monotonicity of the Black-Scholes call price as a function of the volatility. Finally, Corollary 2.17 follows from a simple combination of Corollaries 2.7 and 3.3.

3.1. **Proof of Theorem 2.1.** The process $S_t = \exp(X_t)$ is a true martingale (see Proposition 2.5 in Andersen & Piterbarg [2] and Proposition 2.10 in Buehler [7]) and, by Lemma 2.3 in [2], we have

$$\mathbb{E}\left(e^{p(X_t-x_0)}\right) = \mathbb{E}^{\mathbb{Q}_p}\left(e^{\frac{1}{2}p(p-1)\int_0^t Y_s ds}\right), \tag{3.1}$$

with

$$\begin{aligned} dY_t &= \kappa(\theta - Y_t)dt + \rho\sigma p Y_t dt + \sigma\sqrt{Y_t}d\tilde{W}_t^2 \\ &= \tilde{\kappa}(\tilde{\theta} - Y_t)dt + \sigma\sqrt{Y_t}d\tilde{W}_t^2, \end{aligned}$$

where $\tilde{W}^2$ is a $\mathbb{Q}_p$-Brownian motion, $\tilde{\kappa} = \kappa - \rho\sigma p$, $\tilde{\theta} = \frac{\kappa\theta}{\kappa - \rho\sigma p}$ and $Y_0 = y_0$. Note that we use Equation (3.1) because we have not seen a rigourous (existence and uniqueness) proof of the characteristic function for $X_t - x_0$ using Riccati equations, and Hurd & Kuznetsov [31] is the only article we are aware of that provides a rigorous probabilistic proof of the moment generating function of $\int_0^t Y_s ds$, using Girsanov's theorem and the well known non-central chi square transition density for the CIR process, (see also Dufresne [11]). This proof circumvents the need for Riccati equations, and then having to use analytic continuation to go from the characteristic function of $X_t - x_0$ to the mgf of $X_t - x_0$ inside the strip of analyticity, which is contrary to the probabilistic spirit of the article. Equation (3.1) is also used in Andersen & Piterbarg [2], Feng et al. [18], Lewis [40] and Lions & Musiela [41]. Also, although the statement of Lemma 2.3 in [2] is limited to the case of $p \geq 1$, the proof is not limited to that case, allowing $p \in \mathbb{R}$. From Theorem 3.1 in Hurd & Kuznetsov [31], if the following two conditions are satisfied

$$\tilde{\kappa} > 0, \tag{3.2}$$

$$\omega \leq \frac{\tilde{\kappa}^2}{2\sigma^2}, \tag{3.3}$$

then

$$\mathbb{E}^{\mathbb{Q}_p}\left(e^{\omega\int_0^t Y_s ds}\right) < \infty.$$

for all $t > 0$ (ω here corresponds to d_1 in [31]). Condition (3.2) ensures that Y has mean-reverting behaviour under $\mathbb{Q}^p$, and Condition (3.3) ensures that the square root in the expression v_1 in Theorem 3.1 in [31] is real, which is needed to define a real Girsanov change of measure in the proof. Intuitively, from Condition (3.3), we see that the mean reversion effect and the volatility compete with each other - higher values of $\tilde{\kappa}$ thin the tails of $\int_0^t Y_s ds$, but higher values of σ have the opposite effect. To use (3.1), we set $\omega = \frac{1}{2}p(p-1)$ and the conditions read

$$\kappa > \rho\sigma p, \tag{3.4}$$

$$(\kappa - \rho\sigma p)^2 \geq p(p-1)\sigma^2. \tag{3.5}$$

Both these inequalities also appear in Proposition 3.1 in [2]; together they ensure that the moment explosion time $T^*(p)$ is infinite for $X_t - x_0$ under the original measure $\mathbb{P}$. Note that this is a special case of the general affine time-homogenous Markov process used in Keller-Ressel [37], and this solution can also be obtained using ordinary differential equations of the Riccati type. Under these conditions, we have the exponential affine closed-form solution

$$\mathbb{E}^{\mathbb{Q}_p}\left(e^{\frac{1}{2}p(p-1)\int_0^t Y_s ds}\right) = e^{m(t)-n(t)y_0} < \infty, \tag{3.6}$$

where

$$\begin{aligned} m(t) &= \frac{2\kappa\theta}{\sigma^2}\log\left(\frac{\bar{b}e^{\frac{bt}{2}}}{\bar{b}\cosh(\frac{1}{2}\bar{b}t) + b\sinh(\frac{1}{2}\bar{b}t)}\right), \\ n(t) &= -p(p-1)\frac{\sinh(\frac{1}{2}\bar{b}t)}{\bar{b}\cosh(\frac{1}{2}\bar{b}t) + b\sinh(\frac{1}{2}\bar{b}t)}, \\ b &= \kappa - \rho\sigma p, \\ \bar{b} &= \sqrt{(\kappa-\rho\sigma p)^2 - \sigma^2 p(p-1)}. \end{aligned}$$

Equation (3.5) is equivalent to $p_- \leq p \leq p_+$, and recall that $p_\pm$ is given by

$$p_\pm = \frac{\sigma - 2\kappa\rho \pm \sqrt{\sigma^2 + 4\kappa^2 - 4\rho\sigma\kappa}}{2(1-\rho^2)\sigma} = \frac{\sigma - 2\kappa\rho \pm \sqrt{(\sigma-2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)}}{2(1-\rho^2)\sigma}, \tag{3.7}$$

so we see that $p_- < 0$. From Appendix B, we also know that $p_+ > 1$. For $0 \leq \rho < 1$, if $\kappa > \rho\sigma p_+$ then $\kappa > \rho\sigma p$ for $p \in [p_-, p_+]$. But p_+ depends on κ itself, so we have

$$\begin{aligned} \kappa > \rho\sigma p_+ &\Leftrightarrow 2\kappa(1-\rho^2) > \rho\sigma - 2\kappa\rho^2 + \rho\sqrt{(\sigma-2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)} \\ &\Leftrightarrow 2\kappa - \rho\sigma > \rho\sqrt{(\sigma-2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)} \\ &\Leftrightarrow (2\kappa-\rho\sigma)^2 > \rho^2(\sigma-2\kappa\rho)^2 + 4\kappa^2\rho^2(1-\rho^2) \\ &\Leftrightarrow \kappa - \rho\sigma > 0, \end{aligned} \tag{3.8}$$

which is true by assumption. Thus (3.4) is implied by (3.5). For $\rho \leq 0$, choosing $\kappa > \rho\sigma p_-$ ensures that Condition (3.4) is satisfied for all $p \in [p_-, p_+]$. But p_- also depends on κ, and we see that (3.8) is clearly true for all $\rho < 0$.

Analyzing Equation (3.6) for t large and $p \in [p_-, p_+]$, we find that

$$\begin{aligned} \frac{m(t)}{t} &\to \frac{\kappa\theta}{\sigma^2}(b - \bar{b}), \\ \frac{n(t)}{t} &\to 0, \end{aligned}$$

as t tends to infinity, so the $m(t)$ term dominates the $n(t)$ term (in contrast to the small-time regime where the $n(t)$ term dominates, see Forde & Jacquier [21]).

We have shown that (3.5) implies (3.4). Taking the contrapositive, if (3.4) does not hold, then (3.5) does not hold and $p \notin [0,1]$ (recall that $p_- < 0$ and $p_+ > 1$); thus, by Proposition 3.1 in [2], $\mathbb{E}(e^{p(X_t - x_0)}) = \infty$ for t sufficiently large, and (2.3) holds. We also note that $V(0) = V(1) = 0$, and

$$V(p_\pm) = \frac{\kappa\theta(2\kappa - \rho\sigma \mp \rho\eta)}{2\sigma^2(1-\rho^2)} < \infty,$$

so $V : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous, but is not continuous at $p = p_\pm$. Differentiating $V(p)$ we obtain

$$\text{(3.9)} \qquad V'(p) = \frac{\kappa\theta}{\sigma^2}\left(-\rho\sigma + \frac{1}{2}\frac{2\rho\kappa - \sigma + 2p\sigma(1-\rho^2)}{\sqrt{(\kappa - \rho\sigma p)^2 + \sigma^2(p^2 - p)}}\right),$$

$$\text{(3.10)} \qquad V''(p) = \frac{\kappa\theta\eta^2}{4((\kappa - \rho\sigma p)^2 + \sigma^2(p - p^2))^{\frac{3}{2}}},$$

for any $p \in (p_-, p_+)$. From this, we see that $|V'(p)| \to \infty$ as $p \nearrow p_+$ or $p \searrow p_-$, and $V^{''}(p) > 0$ for $p \in (p_-, p_+)$. Thus V is convex, essentially smooth (see Appendix A for a definition) and lower semicontinuous, so we can appeal to the Gärtner-Ellis theorem recalled in Appendix A to establish that the process $\frac{1}{t}(X_t - x_0)$ satisfies a large deviation principle with rate function V^* equal to the Fenchel-Legendre transform of V. By the essential smoothness property of V, the equation

$$\frac{\partial}{\partial p}(px - V(p))|_{p^*} = 0$$

has a solution $p^* = p^*(x)$ in (p_-, p_+), which equivalently solves $x = V'(p^*(x))$. This equation is solvable in closed-form and we obtain

$$\text{(3.11)} \qquad p^*_\pm(x) = \frac{\sigma - 2\kappa\rho \pm (\kappa\theta\rho + x\sigma)\eta}{2(1-\rho^2)\sigma\sqrt{x^2\sigma^2 + 2x\kappa\theta\rho\sigma + \kappa^2\theta^2}}.$$

Note that the square root appearing in the denominator of (3.11) is well defined for all $x \in \mathbb{R}$. The square root in (3.9) has to be positive, so setting $V'(p) = x$ and rearranging (3.7), we have the sign condition

$$\text{(3.12)} \qquad \frac{2\rho\kappa - \sigma + 2p\sigma(1-\rho^2)}{x\sigma + \rho\kappa\theta} > 0,$$

and substituting $p = p^*_\pm(x)$, the left hand-side of (3.12) becomes

$$\pm\frac{\eta}{\sqrt{x^2\sigma^2 + \theta^2\sigma^2 + 2\kappa\theta\rho x\sigma}},$$

so we see that $p^*_+(x)$ is the only valid root. By a direct calculation, we find that

$$p^*_+(0) = \frac{1}{2}\frac{\sigma - 2\rho\kappa - |\rho|\eta}{\sigma(1-\rho^2)}.$$

The unique minimum x^* of V^* occurs at $x^* = (V^{*'})^{-1}(0) = V'(0) = -\frac{1}{2}\theta$, and $V^*(-\frac{1}{2}\theta) = 0$. For $p \in (p_-, p_+)$, $V'(p)$ is negative when $p < p^*_+(0)$ and positive when $p > p^*_+(0)$. V is C^2, essentially smooth (see Definition A.4 in Appendix A) and strictly convex inside (p_-, p_+), so using standard calculus, we can easily show that $x = V'(p)$ for $p \in (p_-, p_+)$ if and only if $p = V^{*'}(x)$ (see the proof of Lemma 2.5 in Feng, Forde & Fouque [18] for a very similar analysis, and Theorem 26.5 on page 258 in Rockafellar [45]). Thus for $p > 0$, we have

$$\begin{aligned} x &= V'(p) > V'(0) = -\frac{1}{2}\theta, \\ p^* &= V^{*'}(x) > 0. \end{aligned}$$

and the Fenchel-Legendre transform reduces to the Legendre transform. Consequently, V^* is strictly increasing when $x > -\frac{1}{2}\theta$. Similarly, we can show that V^* is strictly decreasing when $x < -\frac{1}{2}\theta$, and (2.4) follows from Lemma A.3 in Appendix A.

3.2. **Proof of Corollary 2.4.** The proof of Corollary 2.4 requires some intermediate notions and results.

3.2.1. *The Share measure.* We can rewrite the SDEs for the Heston model (2.1) as follows

$$\begin{aligned} dX_t &= -\frac{1}{2}Y_t dt + \sqrt{Y_t}\left(\rho dB_t^2 + \sqrt{1-\rho^2}dB_t^1\right), \\ dY_t &= \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t}dB_t^2, \end{aligned}$$

where B^1 and B^2 are independent Brownian motions under $\mathbb{P}$. As the process $(S_t)_{t\geq 0}$ is a martingale (see [2] and [7]), by Girsanov's theorem, we can define the so-called Share measure $\mathbb{P}^*$ as

$$\mathbb{P}^*(A) = \mathbb{E}^{\mathbb{P}}\left(\frac{S_t}{S_0}1_A\right) = \mathbb{E}^{\mathbb{P}}\left(e^{-\frac{1}{2}\int_0^t Y_s ds + \int_0^t \rho\sqrt{Y_s}dB_s^2 + \int_0^t \sqrt{1-\rho^2}\sqrt{Y_s}dB_s^1}1_A\right),$$

for every $A \in \mathcal{F}_t$. $\mathbb{P}^*$ is the measure associated with using $S_t = e^{X_t}$ as the numéraire. Setting $B_t^{*1} = B_t^1 - \int_0^t \sqrt{1-\rho^2}\sqrt{Y_s}ds$, $B_t^{*2} = B_t^2 - \int_0^t \rho\sqrt{Y_s}ds$, we have

$$\begin{aligned} dX_t &= \frac{1}{2}Y_t dt + \sqrt{Y_t}\left(\rho dB_t^{*2} + \sqrt{1-\rho^2}dB_t^{*1}\right), \\ dY_t &= \kappa(\theta - Y_t)dt + \rho\sigma Y_t dt + \sigma\sqrt{Y_t}dB_t^{*2} \\ &= \bar\kappa(\bar\theta - Y_t)dt + \sigma\sqrt{Y_t}dB_t^{*2}, \end{aligned}$$

where $\bar\kappa = \kappa - \rho\sigma$, $\bar\theta$ is defined in Corollary 2.4, and B^{*1} and B^{*2} are two independent $\mathbb{P}^*$ Brownian motions. From this we see that

$$\begin{aligned} d(-X_t) &= -\frac{1}{2}Y_t dt + \sqrt{Y_t}dZ_t^{*1}, \\ dY_t &= \bar\kappa(\bar\theta - Y_t)dt + \sigma\sqrt{Y_t}dZ_t^{*2}, \end{aligned} \tag{3.13}$$

where Z^{*1} and Z^{*2} are two correlated Brownian motions under $\mathbb{P}^*$ with $d\left\langle Z^{*1}, Z^{*2}\right\rangle_t = -\rho dt$ (note the minus sign).

3.2.2. *A large-time large deviation principle under the Share measure.* Working under the Share measure $\mathbb{P}^*$, we have the following corollary of Theorem 2.1.

Corollary 3.1. *Under $\mathbb{P}^*$, $-\frac{1}{t}(X_t - x_0)$ satisfies a large deviation principle as t tends to infinity, with rate function V_S^* equal to the Legendre transform of*

$$\begin{aligned} V_S(p) &= V(p;\bar\kappa,\bar\theta,\sigma,-\rho) = \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{P}^*}\left(e^{-p(X_t-x_0)}\right) \\ &= \begin{cases} \frac{\bar\kappa\bar\theta}{\sigma^2}\left(\bar\kappa + \sigma\rho p - \sqrt{(\bar\kappa+\sigma\rho p)^2 - \sigma^2 p(p-1)}\right), & \text{for } p \in \left[p_-^S, p_+^S\right] \\ \infty, & \text{for } p \notin \left[p_-^S, p_+^S\right], \end{cases} \end{aligned}$$

where $p_\pm^S = \frac{2\bar{\kappa}\rho+\sigma\pm\eta_S}{2(1-\rho^2)\sigma}$ *and* $\eta_S = \sqrt{\sigma^2+4\bar{\kappa}^2+4\rho\sigma\bar{\kappa}}$. *Furthermore,* $p_-^S < 0$, $p_+^S > 1$, $V_S^*(-x)$ *attains its minimum value at* $x_S^* = \frac{1}{2}\bar{\theta}$, *and for all* $a < b$, *we have*

$$-\lim \frac{1}{t}\log \mathbb{P}^*\left(\frac{X_t - x_0}{t} \in (a,b)\right) = \inf_{x\in(a,b)} V_S^*(-x) = \begin{cases} 0, & \text{for } a \le x^* \le b, \\ V_S^*(-b), & \text{for } a \le b \le x^*, \\ V_S^*(-a), & \text{for } x^* \le a \le b. \end{cases}$$

Proof. The corollary just follows by applying Theorem 2.1 to (3.13) under the measure $\mathbb{P}^*$ after noticing that the condition $\bar{\kappa} = \kappa - \rho\sigma > -\rho\sigma$ is trivially satisfied because $\kappa > 0$ by assumption. □

We also require the following lemma.

Lemma 3.2. *For all* $x \in \mathbb{R}$, *we have* $V_S^*(-x) = V^*(x) - x$.

Proof. Let $B_\delta(x)$ denote a ball of radius δ centered at $x \in \mathbb{R}$. The functions V^* and V_S^* are continuous, so for any $\delta > 0$, there exists an $\epsilon = \epsilon(\delta)$ such that $V_S^*(-x) - \epsilon < V_S^*(y) < V_S^*(-x) + \epsilon$ for $y \in B_\delta(x)$. Applying the large-time large deviation principle under $\mathbb{P}^*$, we have

$$\begin{aligned} e^{-(V_S^*(-x)+2\epsilon)t} &\le \mathbb{P}^*\left(-\frac{X_t - x_0}{t} \in B_\delta(-x)\right) \\ &= \mathbb{P}^*\left(\frac{X_t - x_0}{t} \in B_\delta(x)\right) \\ &= \mathbb{E}^{\mathbb{P}}\left(e^{X_t - x_0} 1_{\left\{\frac{X_t - x_0}{t} \in B_\delta(x)\right\}}\right) \\ &\le e^{(x+\delta)t}\mathbb{P}\left(\frac{X_t - x_0}{t} \in B_\delta(x)\right) \\ &\le e^{(x+\delta)t}e^{-(V^*(x)+2\epsilon)t}. \end{aligned}$$

We proceed similarly for the lower bound. □

Corollary 3.3. *For all* $K = S_0 e^x > 0$, *we have the following large-time asymptotic behaviour for* S_t *under* $\mathbb{P}^*$,

$$\begin{aligned} -\lim_{t\to\infty}\frac{1}{t}\log \mathbb{P}^*(S_t > K) &= -\lim_{t\to\infty}\frac{1}{t}\log \mathbb{P}^*(X_t - x_0 > x) = V^*(0) \\ &= \frac{\kappa\theta}{2\sigma^2(1-\rho^2)}(-2\kappa + \rho\sigma + \eta) > 0, \end{aligned}$$

We now prove Corollary 2.4.

Proof. We first consider $x > \frac{1}{2}\bar{\theta}$. From page 4 in Carr & Madan [8], we have

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0 e^{xt})^+ = \mathbb{P}^*\left(\log S_t - E > \log\left(S_0 e^{xt}\right)\right) = \mathbb{P}^*(X_t - x_0 - E > xt), \tag{3.14}$$

where E is an independent Exponential random variable under $\mathbb{P}^*$ with parameter $\lambda = 1$. By Corollary 3.1 and the independence assumption, we see that

$$\begin{aligned} \frac{1}{t}\log \mathbb{E}^{\mathbb{P}^*}\left(e^{p(-(X_t - x_0 - E))}\right) &= \frac{1}{t}\log\left(\mathbb{E}^{\mathbb{P}^*}\left(e^{-p(X_t - x_0)}\right)\mathbb{E}^{\mathbb{P}^*}\left(e^{pE}\right)\right) \\ &= V_S(p) + O(1/t), \quad \text{as } t \to \infty, \end{aligned}$$

i.e. E does not affect the leading order asymptotics of $X_t - x_0$. Thus, using Corollary 3.1 and the continuity of V_S^*, we have

$$\lim_{t\to\infty} -\frac{1}{t}\log \mathbb{P}^*(X_t - x_0 - E > xt) = \lim_{t\to\infty} -\frac{1}{t}\log \mathbb{P}^*(-(X_t - x_0 - E) < -xt) = V_S^*(-x).$$

Now, for $-\frac{1}{2}\theta < x < \frac{1}{2}\bar{\theta}$, using (3.14) we have

$$1 - \frac{1}{S_0}\mathbb{E}(S_t - S_0 e^{xt})^+ = \mathbb{P}^*(X_t - x_0 - E \leq xt).$$

Taking logs and dividing by t, we obtain

$$\lim_{t\to\infty} \frac{1}{t}\log\left(1 - \frac{1}{S_0}\mathbb{E}(S_t - S_0 e^{xt})^+\right) = \lim_{t\to\infty} \frac{1}{t}\log \mathbb{P}^*(X_t - x_0 - E \leq xt) = V_S^*(-x).$$

Finally, for $x < -\frac{1}{2}\theta$, proceeding along similar lines to Carr & Madan [8], we have

$$\begin{aligned}\frac{1}{K}\mathbb{E}(K - S_t)^+ = \mathbb{E}(1 - \frac{S_t}{K})^+ &= \int_0^\infty (1 - e^{-y}) f(y) dy \\ &= \int_0^\infty (1 - F(y)) e^{-y} dy,\end{aligned}$$

where $S/K = e^{-y}$, f is the density of y, and F is the corresponding distribution function. But e^{-y} is the density of a positive Exponential random variable E with parameter 1, so we can rewrite this expression as

$$\mathbb{P}\left(\log\frac{K}{S_t} > E\right) = \mathbb{P}(X_t - x_0 + E < x).$$

Setting $K = S_0 e^{xt}$ and using the Gärtner-Ellis theorem under $\mathbb{P}$ and the continuity of V, we find that

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\log\left(\frac{1}{S_0 e^{xt}}\mathbb{E}(S_0 e^{xt} - S_t)^+\right) &= -x + \frac{1}{t}\lim_{t\to\infty} \log \mathbb{E}(S_0 e^{xt} - S_t)^+ \\ &= \lim_{t\to\infty} \frac{1}{t}\log \mathbb{P}(X_t - x_0 + E < xt) = -V^*(x),\end{aligned}$$

and the result follows from Lemma 3.2. □

3.3. **Proof of Corollaries 2.7 and 2.9.** We first prove Corollary 2.7.

Proof. Consider $x > 0$. By Theorem 2.1, we know that for all $\epsilon, \delta > 0$, there exists a $t^* = t^*(\delta, \epsilon, x)$ such that for all $t > t^*$ we have

$$\begin{aligned}e^{-(V^*(\delta)+\epsilon)t} &\leq \mathbb{P}\left(\frac{X_t - x_0}{t} > \delta\right) \leq \mathbb{P}(X_t - x_0 > x) \\ &= \mathbb{P}\left(\frac{X_t - x_0}{t} > \frac{x}{t}\right) \leq \mathbb{P}\left(\frac{X_t - x_0}{t} > 0\right) \leq e^{-(V^*(0)-\epsilon)t}.\end{aligned}$$

The result then follows from the continuity of V^*. We proceed similarly for $x < 0$. □

We now proove Corollary 2.9.

Proof. Consider $x > 0$. By Corollary 2.4 in the case $-\frac{1}{2}\theta \leq x \leq \frac{1}{2}\bar{\theta}$, we know that for all $\delta \in (0, \frac{1}{2}\bar{\theta})$ and all $\epsilon > 0$, there exists a $t^* = t^*(\delta, \epsilon, x)$ such that for all $t > t^*$ we have

$$\begin{aligned}e^{-(V_S^*(-\delta)-\epsilon)t} &\geq S_0 - \mathbb{E}(S_t - S_0 e^{\delta t})^+ \geq S_0 - \mathbb{E}(S_t - S_0 e^{x})^+ \\ &\geq S_0 - \mathbb{E}(S_t - S_0)^+ \geq e^{-(V_S^*(0)+\epsilon)t}.\end{aligned}$$

The result then follows from the continuity of V_S^*, and the fact that $V_S^*(0) = V^*(0)$ by Lemma 3.2. We proceed similarly for $x < 0$. $\square$

3.4. **Proof of Theorem 2.11.** We proceed similarly to Corollary 2.4. For the Black-Scholes model with volatility $\sigma > 0$, the moment generating function of $X_t - x_0$ has the closed-form expression

$$\mathbb{E}\left(e^{p(X_t - x_0)}\right) = \exp\left(V_{\text{BS}}(p,\sigma)t\right),$$

where $V_{\text{BS}}(p,\sigma) = \frac{1}{2}\sigma^2(p^2 - p)$. The function $p \mapsto V_{\text{BS}}(p,\sigma)$ is convex, lower semicontinuous and essentially smooth (see also Figure 1) so by the Gärtner-Ellis theorem, the process $\frac{1}{t}(X_t - x_0)$ satisfies a large deviation principle with rate function $V_{\text{BS}}^*(\cdot,\sigma)$ equal to the Fenchel-Legendre transform of $V_{\text{BS}}(\cdot,\sigma)$. By the essential smoothness property of $V_{\text{BS}}(\cdot,\sigma)$, we see that the equation

$$\frac{\partial}{\partial p}(px - V_{\text{BS}}(p,\sigma))|_{p^*} = 0$$

has a unique solution $p^* = p^*(x)$ given by $p^*(x) = \frac{1}{2} + \frac{x}{\sigma^2}$.

3.5. **Proof of Theorem 2.12.** We can then compute $V_{\text{BS}}^*(x,\sigma)$ as

$$V_{\text{BS}}^*(x,\sigma) = p^*(x)x - V_{\text{BS}}(p^*(x),\sigma) = \frac{(x + \frac{1}{2}\sigma^2)^2}{2\sigma^2}, \quad \text{for all } x \in \mathbb{R},\ \sigma > 0.$$

We can easily show that Lemma 3.2 also holds for the Black-Scholes model, so we can then compute the rate function $\tilde{V}_{\text{BS}}^*(x,\sigma)$ for $-\frac{1}{t}(X_t - x_0)$ under $\mathbb{P}^*$ as

$$\tilde{V}_{\text{BS}}^*(x,\sigma) = V_{\text{BS}}^*(x,\sigma) - x = V_{\text{BS}}^*(-x,\sigma) = \frac{(-x + \frac{1}{2}\sigma^2)^2}{2\sigma^2}.$$

3.6. **Proof of Corollary 2.14.** Let $\sigma_\infty(x)$ be given by (2.6), and we first assume that $x > \frac{1}{2}\bar{\theta}$ so the negative square root applies. Both roots are the solutions to the equation

$$V^*(x) - x = V_{\text{BS}}^*(x,\sigma_\infty(x)) - x = \frac{(-x + \frac{1}{2}\sigma_\infty^2(x))^2}{2\sigma_\infty^2(x)},$$

and clearly the expression with the negative square root is the smaller of the two solutions. By Theorem 2.1, we know that for all $\epsilon > 0$, there exists a $t^*(\epsilon)$ such that for all $t > t^*(\epsilon)$ we have

$$\mathbb{E}(S_t - S_0 e^{xt})^+ \leq e^{-(V^*(x) - x - \epsilon)t} = e^{-(V_{\text{BS}}^*(x,\sigma_\infty(x)) - x - \epsilon)t}. \tag{3.15}$$

$\sigma_\infty(x)$ is continuous as a function of x, and

$$\frac{1}{2}\sigma_\infty^2(x) - x = \left(2(V^*(x) - x) - 2\sqrt{(V^*(x) - x)^2 + (V^*(x) - x)x}\right) < 0,$$

because $V^*(x) - x > 0$. For x fixed, $V_{\text{BS}}^*(x,\sigma) - x = \frac{(-x + \frac{1}{2}\sigma^2)^2}{2\sigma^2}$ is a continuous and strictly decreasing function of $\sigma > 0$ for $\frac{1}{2}\sigma^2 < x$ (see also Figure 1). Thus, for any $\delta > 0$ such that $\frac{(\sigma_\infty(x) + \delta)^2}{2} < x$, we can choose an $\epsilon = \epsilon(\delta) > 0$ such that

$$V_{\text{BS}}^*(x,\sigma_\infty(x)) - \epsilon = V_{\text{BS}}^*(x,\sigma_\infty(x) + \delta) + \epsilon, \tag{3.16}$$

where $\epsilon(\delta) \to 0$ as $\delta \to 0$. Combining (3.15), (3.16) and Proposition 2.12, we have

$$\mathbb{E}(S_t - S_0 e^{xt})^+ \leq e^{-(V_{\text{BS}}^*(x,\sigma_\infty(x)+\delta) - x + \epsilon)t} \leq \mathbb{E}^{\mathbb{P}_{\sigma_\infty(x)+\delta}^{\text{BS}}}(S_t - S_0 e^{xt})^+,$$

for t sufficiently large, where $\mathbb{P}_\sigma^{\text{BS}}$ denotes the probability measure under which the log stock price follows the Black-Scholes model with volatility σ. Thus, by the monotonicity of the Black-Scholes call option formula as a function of the volatility, we have the the following upper bound for the implied volatility $\sigma_t(x)$ at maturity t,

$$\sigma_t(x) \leq \sigma_\infty(x) + \delta.$$

We proceed similarly for the lower bound, so we have $|\sigma_t(x) - \sigma_\infty(x)| < \delta$ for t sufficiently large. We proceed similarly for the lower bound, and for the cases $-\frac{1}{2}\theta < x < \frac{1}{2}\bar{\theta}$ and $x < \frac{1}{2}\theta$.

3.7. **Proof of Corollary 2.17.** Let $K > 0$. Recall the well known identity

$$\mathbb{E}(S_t - K)^+ = S_0\mathbb{P}^*(S_t > K) - K\mathbb{P}(S_t > K).$$

From Corollaries 2.7 and 3.3, we know that $\mathbb{P}(S_t > K) \to 0$ and $\mathbb{P}^*(S_t > K) \to 1$ as $t \to \infty$; thus we see that $\mathbb{E}(S_t - K)^+$ converges from below to S_0 as $t \to \infty$. By inspection of the Black-Scholes call option formula, we see that this can only happen if the dimensionless implied variance $\hat{\sigma}_t(x)^2 t$ tends to infinity as t tends to infinity. Using the classical notation for the Black-Scholes formula, we set $d_1 = \left(-x + \frac{1}{2}\hat{\sigma}_t^2(x)t\right) / \left(\hat{\sigma}_t(x)\sqrt{t}\right)$ and $d_2 = d_1 - \hat{\sigma}_t(x)\sqrt{t}$, and we see that $d_1 \to \infty$ and $d_2 \to -\infty$ as $t \to \infty$. By Corollary 2.9, we know that for any $\delta, \epsilon > 0$, there exists a $t^* = t^*(\delta, \epsilon)$ such that for all $t > t^*$, we have

$$\begin{aligned}
S_0 - e^{-(V_S^*(0)+\epsilon)t} &\geq \mathbb{E}(S_t - K)^+ \\
&= S_0\Phi(d_1) - K\Phi(d_2) \\
&= S_0(1 - \Phi^c(d_1)) - K\Phi^c(-d_2) \\
&\geq S_0\left(1 - \frac{1}{d_1}n(d_1)\right) - K\frac{1}{|d_2|}n(d_2), \text{ using Appendix C} \\
&= S_0\left(1 - \frac{1}{d_1}n(d_1) - \frac{1}{|d_2|}n(d_1)\right), \text{ because } Kn(d_2) = S_0 n(d_1) \\
&\geq S_0\left(1 - \frac{1+\delta}{\hat{\sigma}_t(x)\sqrt{t}}n(d_1)\right) \\
&\geq S_0(1 - (1+\delta)n(d_1)).
\end{aligned}$$

Subtracting S_0 from both sides, taking logs and dividing by t, we obtain

$$V_S^*(0) + \epsilon \geq \frac{\left(-x + \frac{1}{2}\hat{\sigma}_t(x)^2\right)^2}{2\hat{\sigma}_t(x)^2 t} - \epsilon \geq \frac{1}{8}\hat{\sigma}_t(x)^2 - \epsilon.$$

We proceed similarly for the other bound and the corollary follows.

Appendix A. The large deviation principle and the Gärtner-Ellis theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ on Ω.

Definition A.1. A rate function I is a lower semicontinuous mapping $I : \Omega \to [0, \infty)$, such that for all $\alpha \geq 0$, the level set $\Psi_\alpha = \{x : I(x) \leq \alpha\}$ is closed. A rate function is said to be good if all the level sets are compact subsets of Ω.

Definition A.2. (see page 5 in Dembo & Zeitouni [10]).
The sequence $(X_n)_{n\in\mathbb{N}}$ satisfies a large deviation principle with rate function I if, for all $\Gamma \in \mathcal{F}$,

$$-\inf_{x\in\Gamma^0} I(x) \leq \liminf_{n\to\infty} \frac{1}{n}\log\mathbb{P}(X_n \in \Gamma) \leq \limsup_{n\to\infty} \frac{1}{n}\log\mathbb{P}(X_n \in \Gamma) \leq -\inf_{x\in\bar{\Gamma}} I(x). \tag{A.1}$$

Lemma A.3. *If the rate function I is continuous on a set $B \in \mathcal{F}$, then*

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in B) = \inf_{x\in B} I(x).$$

Proof. The function I is continuous, so $\inf_{x\in\Gamma^0} I(x) = \inf_{x\in\bar{\Gamma}} I(x)$, hence the liminf equals the limsup in (A.1); see also the discussion on I-continuity sets on page 5 in [10]. □

We now assume that $\Omega = \mathbb{R}^d$.

Definition A.4. (Definition 2.3.5 in Dembo & Zeitouni [10]).
Consider a convex function $\Lambda : \mathbb{R}^d \to (-\infty, \infty]$, and let $\mathcal{D}_\Lambda = \{\lambda \in \mathbb{R}^d : \Lambda(\lambda) < \infty\}$. Λ is said to be essentially smooth if

- The interior $\mathcal{D}_\Lambda^0$ of $\mathcal{D}_\Lambda$ is non-empty.
- Λ is differentiable throughout $\mathcal{D}_\Lambda^0$.
- Λ is steep, namely $\lim_{n\to\infty} |\nabla\Lambda(\lambda_n)| = \infty$ whenever $\{\lambda_n\}$ is a sequence in $\mathcal{D}_\Lambda^0$ converging to a boundary point of $\mathcal{D}_\lambda^o$.

Assumption A.5. (Assumption 2.3.2 in Dembo & Zeitouni [10]).
For each $\lambda \in \mathbb{R}^d$, we assume that the logarithmic moment generating function, defined as the limit

$$\Lambda(\lambda) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left(e^{n\langle\lambda, X_n\rangle}\right)$$

exists as an extended real number. Further, the origin belongs to the interior of $\mathcal{D} = \{\lambda \in \mathbb{R}^d : \Lambda(\lambda) < \infty\}$.

We recall part (c) of the Gärtner-Ellis theorem (Theorem 2.3.6 in Dembo & Zeitouni [10]):

Theorem A.6. . *Let Assumption A.5 hold. If Λ is lower semicontinuous and essentially smooth, then the sequence of random variables $(X_n)_{n\in\mathbb{N}}$ satisfies a large deviation principle with rate function Λ^*, which is the Fenchel-Legendre transform of Λ, defined by the variational formula*

$$\Lambda^*(x) = \sup_{\lambda\in\mathbb{R}^d} \{\langle\lambda, x\rangle - \Lambda(\lambda)\}, \quad \textit{for all } x \in \mathbb{R}^d. \tag{A.2}$$

Lemma A.7. *(Lemma 2.3.9 (a) in Dembo & Zeitouni* [10]*).*
Let Assumption A.5 hold. Then Λ is a convex function, $\Lambda(\lambda) > -\infty$ everywhere, and Λ^* is a good rate function and is convex.

Remark A.8. Since for any $n \in \mathbb{N}$, $\mathbb{P}(X_n \in \Omega) = 1$, it is necessary that $\inf_{x\in\Omega} I(x) = 0$. If Λ^* is a good rate function, then the level set $\{x : \Lambda^*(x) \leq \alpha\}$ is compact, so we know that the infimum is attained on this compact set because Λ^* is lower semicontinuous, i.e. there exists at least one point x^* for which $\Lambda^*(x^*) = 0$ (see pages 5 and 6 in Dembo & Zeitouni [10]).

Appendix B. Proof that $p_+ > 1$

Recall that

$$p_\pm = \frac{\sigma - 2\kappa\rho \pm \eta}{2(1-\rho^2)\sigma},$$

where $\eta^2 = (\sigma - 2\kappa\rho)^2 + 4\kappa^2(1-\rho^2) = \sigma^2 + 4\bar{\kappa}\kappa$, with $\bar{\kappa} = \kappa - \rho\sigma > 0$ by assumption. We have

$$p_+ > 1 \Leftrightarrow \frac{\sigma - 2\kappa\rho + \eta}{2\sigma(1-\rho^2)} > 1 \Leftrightarrow \sigma - 2\kappa\rho + \eta > 2\sigma(1-\rho^2)$$

$$\Leftrightarrow \eta > \sigma + 2\kappa\rho - 2\sigma\rho^2 = \sigma + 2\rho(\kappa - \rho\sigma) = \sigma + 2\bar{\kappa}\rho. \tag{B.1}$$

But

$$\eta^2 = \sigma^2 + 4\bar{\kappa}\kappa = (\sigma + 2\bar{\kappa}\rho)^2 + 4\bar{\kappa}(\kappa - \rho\sigma - \bar{\kappa}\rho^2)$$
$$= (\sigma + 2\bar{\kappa}\rho)^2 + 4\bar{\kappa}^2(1-\rho^2) > (\sigma + 2\bar{\kappa}\rho)^2,$$

and hence Inequality (B.1) is always satisfied.

Appendix C. Estimates for the Standard Normal distribution function

Let $n(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ for all $z \in \mathbb{R}$ and $\forall x \in \mathbb{R}$, $\Phi^c(x) = \int_x^\infty n(z)dz$. Then when $x > 0$, we have the following bounds (see Williams [49]):

$$(x + \frac{1}{x})^{-1} n(x) \leq \Phi^c(x) \leq \frac{1}{x} n(x)$$

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