A NOTE ON ESSENTIAL SMOOTHNESS IN THE HESTON MODEL

MARTIN FORDE, ANTOINE JACQUIER, AND ALEKSANDAR MIJATOVIĆ

Abstract. This note studies an issue relating to essential smoothness that can arise when the theory of large deviations is applied to a certain option pricing formula in the Heston model. The note identifies a gap, based on this issue, in the proof of Corollary 2.4 in [2] and describes how to circumvent it. This completes the proof of Corollary 2.4 in [2] and hence of the main result in [2], which describes the limiting behaviour of the implied volatility smile in the Heston model far from maturity.

1. Introduction

In [2] the authors study the limiting behaviour of the implied volatility in the Heston model as maturity tends to infinity. The main aim of this note is to give a rigorous account of the relationship between the concept of essential smoothness and the large deviation principle for the family of random variables $(X_t/t \pm E_\lambda/t)_{t\geq 1}$, where the process X denotes the log-spot in Heston model (5) and E_λ is an exponential random variable with parameter $\lambda > 0$ independent of X. This note fills a gap in the proof of Corollary 2.4 in [2] and hence completes the proof of the main result in [2], which describes the limiting behaviour of the implied volatility smile in the Heston model far from maturity.

The note is organized as follows. Section 2 describes the relevant concepts of the large deviation theory and discusses how the effective domain changes when a family of random variables is perturbed by an independent exponential random variable. Section 3 discusses the failure of essential smoothness when the Heston model is perturbed by an independent exponential, which is what causes the gap in the proof of Corollary 2.4 in [2]. Section 3 also proves Theorem 3, which fills the gap.

2. The large deviation principle for random variables in $\mathbb{R}$

We briefly recall the basic facts of the large deviation theory in $\mathbb{R}$ (see monograph [1, Ch. 2] for more details). Let $(Z_t)_{t\geq 1}$ be a family of random variables with $Z_t \in \mathbb{R}$. J is a *rate function* if it is lower semicontinuous and $J(\mathbb{R}) \subset [0,\infty]$ holds. The family $(Z_t)_{t\geq 1}$ satisfies the *large deviation principle (LDP)* with the *rate function* J if for every Borel set $B \subset \mathbb{R}$ we have

$$(1) \qquad -\inf_{x\in B^\circ} J(x) \leq \liminf_{t\to\infty} \frac{1}{t}\log \mathsf{P}\left[Z_t \in B\right] \leq \limsup_{t\to\infty} \frac{1}{t}\log \mathsf{P}\left[Z_t \in B\right] \leq -\inf_{x\in \overline{B}} J(x),$$

2000 *Mathematics Subject Classification.* 60G44.

Key words and phrases. Essential smoothness, large deviation principle, Heston model.

with the convention $\inf\emptyset = \infty$ the relative notions of interior (interior B°, closure $\overline{B}$ and boundary $\overline{B}\setminus B^\circ$ are in the topology of $\mathbb{R}$).

The Gärtner-Ellis theorem (Theorem 1 below) gives sufficient conditions for a family $(Z_t)_{t\geq 1}$ to satisfy the LDP (see monograph [1, Section 2.3] for details). Let $\Lambda_t(u) := \log \mathsf{E}\left[\mathrm{e}^{uZ_t}\right] \in (-\infty,\infty]$ be the cumulant generating function of Z_t. Assume that for every $u\in\mathbb{R}$

$$\Lambda(u) := \lim_{t\to\infty}\Lambda_t(tu)/t \quad \text{exists in } [-\infty,\infty] \qquad \text{and} \qquad 0\in\mathcal{D}_\Lambda^\circ, \tag{2}$$

where $\mathcal{D}_\Lambda := \{u\in\mathbb{R} : \Lambda(u)<\infty\}$ is the *effective domain* of Λ and $\mathcal{D}_\Lambda^\circ$ is its interior. The *Fenchel-Legendre transform* Λ^* of the convex function Λ is defined by the formula

$$\Lambda^*(x) \quad := \quad \sup\{ux-\Lambda(u)\ :\ u\in\mathbb{R}\} \quad \text{for} \quad x\in\mathbb{R}. \tag{3}$$

Under the assumption in (2), Λ^* is lower semicontinuous with compact level sets $\{x : \Lambda^*(x)\leq\alpha\}$ (see [1, Lemma 2.3.9(a)]) and $\Lambda^*(\mathbb{R})\subset[0,\infty]$ and hence satisfies the definition of a *good rate function*. We now state the Gärtner-Ellis theorem (see [1, Section 2.3] for its proof).

Theorem 1. *Let the random variables $(Z_t)_{t\geq1}$ satisfy the assumption in (2). If Λ is essentially smooth and lower semicontinuous, then LDP holds for $(Z_t)_{t\geq1}$ with the good rate function Λ^*.*

The function $\Lambda:\mathbb{R}\to(-\infty,\infty]$ defined in (2) is *essentially smooth* if it is (a) differentiable in $\mathcal{D}_\Lambda^\circ$ and (b) *steep*, i.e. $\lim_{n\to\infty}|\Lambda'(u_n)|=\infty$ for every sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_\Lambda^\circ$ that converges to a boundary point of $\mathcal{D}_\Lambda^\circ$. If $\mathcal{D}_\Lambda^\circ$ is a strict subset of $\mathbb{R}$, which is the case in the setting of [2] (see also Section 3 below), essential smoothness, which plays a key role in the proof of Theorem 1, is not automatic.

The following question is of central importance in [2]: does the LDP persist if a family of random variables $(Z_t)_{t\geq1}$ is perturbed by an independent exponential random variable E_1? It is implicitly assumed in the proof of Corollary 2.4 in [2] (see the last line on page 17 and lines 4 and 14 on page 18) that if $(Z_t)_{t\geq1}$ satisfies the assumptions of Theorem 1, then so do the families $(Y_t^{1+})_{t\geq1}$ and $(Y_t^{1-})_{t\geq1}$, where $Y_t^{1\pm}=Z_t\pm E_1/t$, and the LDP is applied. In particular the authors in [2] assume that the limiting cumulant generating functions of $(Y_t^{1\pm})_{t\geq1}$ are essentially smooth. However the following simple lemma holds.

Lemma 2. *Let $(Z_t)_{t\geq1}$ satisfy the assumption in (2) with a limiting cumulant generating function Λ. Let $\lambda>0$ and E_λ an exponential random variable independent of $(Z_t)_{t\geq1}$ with $\mathsf{E}[E_\lambda]=1/\lambda$ and let $Y_t^{\lambda\pm}:=Z_t\pm E_\lambda/t$. Then the families of random variables $(Y_t^{\lambda\pm})_{t\geq1}$ satisfy the assumption in (2) and the corresponding limiting cumulant generating functions are given by*

$$\Lambda^{\lambda+}(u)=\begin{cases}\Lambda(u), & \text{if } u\in\mathcal{D}_\Lambda\cap(-\infty,\lambda),\\ \infty, & \text{otherwise,}\end{cases} \qquad \text{and} \quad \Lambda^{\lambda-}(u) \quad = \begin{cases}\Lambda(u), & \text{if } u\in\mathcal{D}_\Lambda\cap(-\lambda,\infty),\\ \infty, & \text{otherwise.}\end{cases}$$

Remarks. **(a)** Let $(Z_t)_{t\geq 1}$ satisfy the assumption in (2) and assume further that Λ is differentiable in $\mathcal{D}_\Lambda^\circ$. If $1 \in \mathcal{D}_\Lambda^\circ$, then the right-hand boundary point of the interior of the effective domain $\mathcal{D}_{\Lambda^{1+}}^\circ$ is equal to 1 and Lemma 2 implies that the limiting cumulant generating function Λ^{1+} of $(Y_t^{1+})_{t\geq 1}$ is

- neither essentially smooth, since Λ^{1+} is not steep at 1,
- nor lower semicontinuous at 1, since it is differentiable in $\mathcal{D}_{\Lambda^{1+}}^\circ$ with $\Lambda^{1+}(1) = \infty$.

Loss of steepness and lower semicontinuity occurs also for $(Y_t^{1-})_{t\geq 1}$ in the case where $-1 \in \mathcal{D}_\Lambda^\circ$.

(b) Lemma 2 implies that if $(Z_t)_{t\geq 1}$ satisfies the assumptions of Theorem 1 *and* $\mathcal{D}_\Lambda$ is contained in $(-\infty, \lambda)$, for some $\lambda > 0$, then $(Y_t^{\lambda+})_{t\geq 1}$ also satisfies the assumptions of Theorem 1 and hence the LDP with a good rate function Λ^*. An analogous statement holds for $(Y_t^{\lambda-})_{t\geq 1}$.

Proof. Note that $\log \mathsf{E}\left[\mathrm{e}^{uE_\lambda}\right]$ is finite and equal to $\log\left(\lambda/(\lambda-u)\right)$ if and only if $u \in (-\infty, \lambda)$. For all large t and $u \in \mathcal{D}_\Lambda \cap (-\infty, \lambda)$, the assumption in (2) implies that $\Lambda_t^{\lambda+}(tu) = \log \mathsf{E}\left[\exp\left(tuY_t^{\lambda+}\right)\right]$ is finite and that the formula holds

$$\Lambda_t^{\lambda+}(tu) = \Lambda_t(tu) + \log\frac{\lambda}{\lambda-u}, \qquad \text{where} \qquad \Lambda_t(tu) = \log \mathsf{E}\left[\exp\left(tuZ_t\right)\right]. \tag{4}$$

The inequality $u \geq \lambda$ implies that, since $\Lambda_t(tu) > -\infty$, we have $\Lambda_t^{\lambda+}(tu) = \infty$ for all t and hence $\Lambda^{\lambda+}(u) = \infty$. If $u \in (\mathbb{R} \setminus \mathcal{D}_\Lambda) \cap (-\infty, \lambda)$, then (4) yields $\Lambda^{\lambda+}(u) = \lim_{t\nearrow\infty} \Lambda_t^{\lambda+}(tu)/t = \infty$. This proves the lemma for $(Y_t^{\lambda+})_{t\geq 1}$. The case of $(Y_t^{\lambda-})_{t\geq 1}$ is analogous. □

3. Essential smoothness can fail

The Heston model $S = \mathrm{e}^X$ is a stochastic volatility model with the log-stock process X given by

$$\mathrm{d}X_t = -\frac{Y_t}{2}\mathrm{d}t + \sqrt{Y_t}\mathrm{d}W_t^1 \quad \text{and} \quad \mathrm{d}Y_t = \kappa(\theta - Y_t)\mathrm{d}t + \sigma\sqrt{Y_t}\mathrm{d}W_t^2, \tag{5}$$

where $\kappa, \theta, \sigma > 0$, $Y_0 = y_0 > 0$, $X_0 = x_0 \in \mathbb{R}$ and W^1, W^2 are standard Brownian motions with correlation $\rho \in (-1, 1)$. The standing assumption

$$\rho\sigma - \kappa < 0, \tag{6}$$

is made in [2] (see equation (2.2) in Theorem 2.1 on page 5 of [2]). In particular the inequality in (6) implies that S is a strictly positive true martingale and allows the definition of the share measure $\widetilde{\mathsf{P}}$ via the Radon-Nikodym derivative $\mathrm{d}\widetilde{\mathsf{P}}/\mathrm{d}\mathsf{P} = \mathrm{e}^{X_t - x_0}$.

The authors' aim in [2] is to obtain the limiting implied volatility smile as maturity tends to infinity at the strike $K = S_0\mathrm{e}^{xt}$ for any $x \in \mathbb{R}$ in the Heston model. Their main formula is given in Corollary 3.1 of [2]. A key step in the proof of [2, Corollary 3.1] is given by [2, Corollary 2.4]. In the proof of [2, Corollary 2.4] (see last line on page 17 and lines 4 and 14 on page 18) it is implicitly assumed that the LDP for $(X_t/t_{t\geq 1}$ implies the LDP for the family $(X_t/t \pm E_1/t)_{t\geq 1}$. However, as we have seen in Section 2 (see remarks following Lemma 2), Theorem 1 cannot be applied directly to the family $(X_t/t \pm E_1/t)_{t\geq 1}$, even if $(X_t/t)_{t\geq 1}$ satisfies its assumptions. We start with a precise description of the problem and present the solution in Theorem 3.

Remarks. **(i)** Under (6), a simple calculation implies that Λ and $\mathcal{D}_\Lambda$ of the family $(X_t/t)_{t\geq 1}$ are:

$$\Lambda(u) = -\frac{\theta\kappa}{\sigma^2}\left(u\rho\sigma - \kappa + \sqrt{\Delta(u)}\right) \quad \text{for} \quad u \in \mathcal{D}_\Lambda \quad \text{and} \quad \mathcal{D}_\Lambda = [u_-, u_+] \quad \text{where} \tag{7}$$

$$u_\pm = \left(1/2 - \rho\kappa/\sigma \pm \sqrt{(\kappa/\sigma - \rho)\,\kappa/\sigma + 1/4}\right) / \left(1 - \rho^2\right) \quad \text{with} \quad u_- < 0 < 1 < u_+. \tag{8}$$

In (7) the function Δ is a quadratic $\Delta(u) = (u\rho\sigma - \kappa)^2 - \sigma^2(u^2 - u)$ and the boundary points u_+ and u_- of the effective domain $\mathcal{D}_\Lambda$ are its zeros. Elementary calculations show that Λ is essentially smooth and that the unique minimum of Λ^* is attained at $\Lambda'(0) = -\theta/2$. Therefore $(X_t/t)_{t\geq 1}$ satisfies the LDP with the good rate function Λ^*, defined in (3), by Theorem 1.

(ii) Under the share measure $\widetilde{\mathsf{P}}$, given by $\mathrm{d}\widetilde{\mathsf{P}}/\mathrm{d}\mathsf{P} = \mathrm{e}^{X_t - x_0}$, we have $\widetilde{\mathsf{E}}\left[\mathrm{e}^{uX_t}\right] = \mathrm{e}^{-x_0}\mathsf{E}\left[\mathrm{e}^{(u+1)X_t}\right]$ for all $u \in \mathbb{R}$ and $t > 0$ and hence the family $(X_t/t)_{t\geq 1}$ under $\widetilde{\mathsf{P}}$ satisfies the assumption in (2) with the limiting cumulant generating function $\widetilde{\Lambda}(u) = \Lambda(u+1)$, $\mathcal{D}_{\widetilde{\Lambda}} = [u_- - 1, u_+ - 1]$. As before, $(X_t/t)_{t\geq 1}$ satisfies the LDP under $\widetilde{\mathsf{P}}$ with the strictly convex good rate function $\widetilde{\Lambda}$, which satisfies $\widetilde{\Lambda}^*(x) = \Lambda^*(x) - x$ for all $x \in \mathbb{R}$ and attains its unique minimum at $\widetilde{\Lambda}'(0) = \Lambda'(1) = \theta\kappa/(\kappa - \rho\sigma)$.

Theorem 3. *Let the process X be given by (5) and assume that (6) holds. Let E_1 be the exponential random variable with $\mathsf{E}[E_1] = 1$, which is independent of X. Then the following limits hold:*

$$\lim_{t\nearrow\infty} \frac{1}{t}\log \mathsf{P}\left[X_t - x_0 + E_1 < xt\right] = -\Lambda^*(x) \quad \textit{for} \quad x \leq \Lambda'(0) = -\theta/2; \tag{9}$$

$$\lim_{t\nearrow\infty} \frac{1}{t}\log \widetilde{\mathsf{P}}\left[X_t - x_0 - E_1 > xt\right] = x - \Lambda^*(x) \quad \textit{for} \quad x \geq \Lambda'(1) = \theta\kappa/(\kappa - \rho\sigma); \tag{10}$$

$$\lim_{t\nearrow\infty} \frac{1}{t}\log \widetilde{\mathsf{P}}\left[X_t - x_0 - E_1 \leq xt\right] = x - \Lambda^*(x) \quad \textit{for} \quad x \in \left[\Lambda'(0), \Lambda'(1)\right]; \tag{11}$$

where Λ is given in (7), its Fenchel-Legendre transform Λ^* is defined in (3) and $\mathrm{d}\widetilde{\mathsf{P}}/\mathrm{d}\mathsf{P} = \mathrm{e}^{X_t - x_0}$.

Remark. The limits in Theorem 3 are precisely the limits that arise in the proof of [2, Corollary 2.4] (see the last line on page 17 and lines 4 and 14 on page 18) and are claimed to hold since the family $(X_t/t)_{t\geq 1}$ satisfies the LDP under P and $\widetilde{\mathsf{P}}$ by Remarks (i) and (ii) above and Theorem 1. However Lemma 2 implies that the limiting cumulant generating function Λ^{1+} of the family of random variables $(Z_t + E_1/t)_{t\geq 1}$, where $Z_t = (X_t - x_0)/t$, is neither lower semicontinuous nor essentially smooth. Hence Theorem 1 cannot be applied to $(Z_t + E_1/t)_{t\geq 1}$. An anologous issue arises under the measure $\widetilde{\mathsf{P}}$.

Proof. The basic idea of the proof is simple: for (9) we sandwich the probability $\mathsf{P}\left[X_t - x_0 + E_1 < xt\right]$ between two tail probabilities of two families of random variables, which satisfy the LDP with the same rate function Λ^* by Lemma 2 and Theorem 1. The limits in (10) and (11) follow similarly.

For given parameter values in the Heston model pick $\lambda > u_+$, where u_+ is defined in (8). Let E_λ be an exponential random variable with $\mathsf{E}[E_\lambda] = 1/\lambda$, defined on the same probability space as X and E_1 and independent of both. Since $u_+ > 1$, we have the elementary inequality

$$\mathsf{P}\left[E_\lambda < \alpha\right] = I_{\{\alpha>0\}}\left(1 - \mathrm{e}^{-\lambda\alpha}\right) \le I_{\{\alpha>0\}}\left(1 - \mathrm{e}^{-\alpha}\right) = \mathsf{P}\left[E_1 < \alpha\right] \qquad \text{for any} \quad \alpha \in \mathbb{R}. \tag{12}$$

The inequality

$$\mathsf{P}\left[X_t - x_0 + E_\lambda < xt\right] \quad\le\quad \mathsf{P}\left[X_t - x_0 + E_1 < xt\right] \tag{13}$$

follows by conditioning on X_t and applying (12). On the other hand, since $E_1 > 0$ a.s., we have

$$\mathsf{P}\left[X_t - x_0 + E_1 < xt\right] \quad\le\quad \mathsf{P}\left[X_t - x_0 < xt\right]. \tag{14}$$

Lemma 2 implies that the families of random variables $(Z_t + E_\lambda/t)_{t\ge1}$ and $(Z_t)_{t\ge1}$, where $Z_t = (X_t - x_0)/t$, both have the limiting cumulant generating function equal to Λ given in (7) with the effective domain $\mathcal{D}_\Lambda = [u_-, u_+]$. Since Λ is essentially smooth and lower semicontinuous on $\mathcal{D}_\Lambda$ and the assumption in (2) is satisfied, Theorem 1 implies that $(Z_t + E_\lambda/t)_{t\ge1}$ and $(Z_t)_{t\ge1}$, satisfy the LDP with the good rate function Λ^*. Since x in (9) is assumed to be less or equal to the unique minimum $\Lambda'(0) = -\theta/2$ of Λ^* (see Remark (i) above) and Λ^* is non-negative and strictly convex, the LDP (see the inequalities in (1)) and the inequalities in (13) and (14) imply the limit in (9).

To prove (10) pick $\lambda > 1 - u_-$ and note that the inequality in (12) and conditioning on X_t yield

$$\widetilde{\mathsf{P}}\left[X_t - x_0 > xt\right] \ge \widetilde{\mathsf{P}}\left[X_t - x_0 - E_1 > xt\right] \ge \widetilde{\mathsf{P}}\left[X_t - x_0 - E_\lambda > xt\right]. \tag{15}$$

As before, Lemma 2 and Theorem 1 imply that $(Z_t - E_\lambda/t)_{t\ge1}$ and $(Z_t)_{t\ge1}$ satisfy the LDP with the convex rate function $\widetilde{\Lambda}^*$, which by Remark (ii) above attains its unique minimum at $\Lambda'(1) = \theta\kappa/(\kappa - \rho\sigma)$. Since $x \ge \Lambda'(1)$ in (10), the limit follows. A similar argument implies the limit in (11) for all $x \in [\Lambda'(0), \Lambda'(1)]$, which concludes the proof. □

References

[1] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*. Springer-Verlag, New York, 2nd ed., 1998.

[2] M. Forde and A. Jacquier. The large-maturity smile for the Heston model. To appear in *Finance & Stochastics*.

Department of Mathematical Sciences, Dublin City University, Ireland
E-mail address: martin.forde@dcu.ie

Department of Mathematics, TU Berlin, Germany
E-mail address: jacquier@math.tu-berlin.de

Department of Statistics, University of Warwick, UK
E-mail address: a.mijatovic@warwick.ac.uk