SHORT MATURITY ASYMPTOTICS FOR A FAST MEAN-REVERTING HESTON STOCHASTIC VOLATILITY MODEL *

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Abstract. In this paper, we study the Heston stochastic volatility model in the regime where the maturity is small but large compared to the mean-reversion time of the stochastic volatility factor. We derive a large deviation principal and compute the rate function by a precise study of the moment generating function and its asymptotic. We then obtain asymptotic prices for Out-of-The-Money call and put options, and their corresponding implied volatilities.

Key words. Stochastic volatility, Heston model, multiscale asymptotics, large deviation principle, implied volatility smile/skew.

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1. Introduction. Large deviations theory provides a natural framework for approximating the exponentially small probabilities associated with the behaviour of a diffusion process over a small time interval. In the context of financial Mathematics, large deviations theory arises in the computation of small-maturity, out-of- the-money call or put option prices, or the probability of reaching a default level in a small time period. The theory of large deviations has been recently applied to local and stochastic volatility models [3, 4, 5, 6, 14, 23], and has given very interesting results on the behavior of implied volatilities near maturity (an implied volatility is the volatility parameter needed in the Black-Scholes formula in order to match a call option price. It is common practice to quote prices in volatility through this transformation). In the context of stochastic volatility models, the rate function involved in the large deviation estimates is given in terms of a distance function, which in general cannot be calculated in closed-form. For particular models, such as the SABR model [13, 15], approximations obtained by expansion techniques have been proposed (see also [9, 12, 18]).

Multi-factors stochastic volatility models have been studied during the last ten years by many authors (se for instance [8, 10, 12, 19, 20]). They are quite efficient in capturing the main features of implied volatilities known as smiles and skews. They are usually not simple to calibrate, in particular with respect to the stability of parameter estimation. In the presence of separated time scales, an asymptotic theory has been proposed in [10, 11]. It has the advantage of capturing the main effects of stochastic volatility through a small number of group parameters arising in the asymptotic. The fast time scale expansion is related to the ergodic property of the corresponding fast mean reverting stochastic volatility factor.

In this paper, we study the Heston stochastic volatility model in the regime in which the maturity is small but large compared to the mean-reversion time of the stochastic volatility factor. This is a realistic situation where for instance the maturity

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is one month and the volatility mean-reversion time is of the order of a few days. We derive a large deviation principal and compute the rate function by a precise study of the moment generating function and its asymptotic.

1.1. The Heston model. We consider the risk-neutral Heston stochastic volatility model for the price S_t and its square-volatility Y_t:

$$dS_t = rS_t dt + S_t\sqrt{Y_t}dW_t^1\,, \tag{1.1}$$

$$dY_t = \kappa(\theta - Y_t)dt + \nu\sqrt{Y_t}dW_t^2\,, \tag{1.2}$$

where W^1, W^2 are two standard Brownian motions with co-variation $d\langle W^1, W^2\rangle_t = \rho dt$, where ρ is constant such that $|\rho| < 1$. The short rate r is constant, and throughout, we assume that $2\kappa\theta > \nu^2$, $\nu, \kappa, \theta, Y_0 > 0$, so that the square-root (or CIR) process (Y_t) stays positive at all times (see for instance [17]). In this paper, we are mainly interested in small-time asymptotics for S_t, when the stochastic volatility factor Y_t runs on a fast scale.

1.2. Fast mean-reverting stochastic volatility scaling. By fast mean-reverting stochastic volatility, we mean that the rate of mean reversion κ is large. In order to ensure that volatility is not "dying" or "exploding" we also impose that the *volatility of volatility* parameter ν is large of the order of square root of κ. In order to achieve this scaling, we introduce a small parameter $0 < \epsilon \ll 1$, and we replace (κ, ν) by $(\kappa/\epsilon^2, \nu/\epsilon)$ in (1.1–1.2) so that the model becomes:

$$dS_t = rS_t dt + S_t\sqrt{Y_t}dW_t^1\,, \tag{1.3}$$

$$dY_t = \frac{\kappa}{\epsilon^2}(\theta - Y_t)dt + \frac{\nu}{\epsilon}\sqrt{Y_t}dW_t^2\,. \tag{1.4}$$

The small quantity ϵ^2 represents the intrinsic time scale of the volatility process (Y_t), or, in other words, its de-correlation time (we refer to [10] for more details). Observe that the condition $2\left(\frac{\kappa}{\epsilon^2}\right)\theta > \left(\frac{\nu}{\epsilon}\right)^2$ is equivalent to $2\kappa\theta > \nu^2$ and therefore independent of ϵ. Derivatives and implied volatilities have been studied extensively in [10] for a range of maturities and for general stochastic volatility processes by means of singular perturbation techniques around the Black-Scholes model. In this regime, as $\epsilon \to 0$ and fixed maturity, a call option can be approximated by its Black-Scholes price at a constant *effective volatility* plus a small correction of order ϵ and proportional to ρ, involving the *Delta* and *Gamma* of the leading order Black-Scholes price. In terms of the implied volatility surface, it turns out that the *skew* is asymptotically affine in Log-Moneyness-to-Maturity-Ratio $\log(K/S)/T$ which leads to a particularly simple calibration procedure.

1.3. Short-maturity scaling. Since here, we are interested in short maturities, but long compared with the volatility time scale ϵ^2, we rescale time by the change of variable $t \mapsto \epsilon t$, so that typical maturities will be of the order of ϵ. This regime will be of practical use when, for instance, the maturity is a couple of weeks and the time-scale for the volatility to mean revert is of the order of a couple of days.

Performing the change of variable $t \mapsto \epsilon t$ in (1.3–1.4) gives rise to the rescaled process denoted by $(S_{\epsilon,t}, Y_{\epsilon,t})$, and defined by

$$dS_{\epsilon,t} = \epsilon r S_{\epsilon,t}dt + S_{\epsilon,t}\sqrt{\epsilon Y_{\epsilon,t}}dW_t^1\,, \tag{1.5}$$

$$dY_{\epsilon,t} = \frac{\kappa}{\epsilon}(\theta - Y_{\epsilon,t})dt + \frac{\nu}{\sqrt{\epsilon}}\sqrt{Y_{\epsilon,t}}dW_t^2, \tag{1.6}$$

where we have used that $(W^1_{\epsilon t}, W^2_{\epsilon t}) = (\sqrt{\epsilon}\, W^1_t, \sqrt{\epsilon}\, W^2_t)$ in distribution, therefore preserving the constant correlation ρ.

We will use the discounted price $\tilde{S}_{\epsilon,t} = e^{-r\epsilon t} S_{\epsilon,t}$ which satisfies

$$d\tilde{S}_{\epsilon,t} = \tilde{S}_{\epsilon,t}\sqrt{\epsilon Y_{\epsilon,t}}dW^1_t. \tag{1.7}$$

It will also be useful to consider the log-price $X_{\epsilon,t} = \log S_{\epsilon,t}$ which satisfies

$$dX_{\epsilon,t} = r\epsilon dt - \frac{1}{2}\epsilon Y_{\epsilon,t}dt + \sqrt{\epsilon Y_{\epsilon,t}}dW^1_t\,. \tag{1.8}$$

In both cases $Y_{\epsilon,t}$ satisfies (1.6), and we note that

$$\tilde{S}_{\epsilon,t} = x\exp\left(-\frac{\epsilon}{2}\int_0^t Y_{\epsilon,s}ds + \sqrt{\epsilon}\int_0^t \sqrt{Y_{\epsilon,s}}dW^1_s\right). \tag{1.9}$$

1.4. Main results. In Section 2 we derive the following result which describes the asymptotic behavior of $X_{\epsilon,t}$ as $\epsilon \to 0$ for fixed $t > 0$.

THEOREM 1.1. *Assume $X_{\epsilon,0} = x_0$. For each $t > 0$, $\{X_{\epsilon,t} : \epsilon > 0\}$ satisfies the large deviation principle with good rate function*

$$I(q; x_0, t) = \Lambda^*(q - x_0; 0, t),$$

where Λ^* is the Legendre transform of Λ

$$\Lambda^*(q; x, t) \equiv \sup_{p\in R}\{qp - \Lambda(p; x, t)\},$$

and $\Lambda(p; x, t) : R \times R \times R_+ \mapsto R \cup \{+\infty\}$ is given explicitly by:

$$\begin{aligned}\Lambda(p; x, t) &= xp + \frac{\kappa\theta t}{\nu^2}\left((\kappa - \nu\rho p) - \sqrt{(\kappa - \rho\nu p)^2 - \nu^2p^2}\right), \\ &\qquad \textit{for } -\frac{\kappa}{\nu(1-\rho)} \le p \le \frac{\kappa}{\nu(1+\rho)}\,, \\ &= +\infty \qquad \textit{otherwise.}\end{aligned} \tag{1.10}$$

The function $\Lambda(p; x, t)$, and the rate function $\Lambda^*(q)$ given below, are plotted in Figure 2.1 in the three cases, $\rho > 0$, $\rho = 0$, and $\rho < 0$.

LEMMA 1.2. *The rate function Λ^* is given explicitly by*

$$\Lambda^*(q; 0, t) = qp(q; t) - \Lambda(p(q; t); 0, t),$$

where $p(q; t)$ is defined by

$$\begin{aligned}p(q; t) &= \frac{\kappa}{\nu(1-\rho^2)}\left(-\rho + \frac{q\nu + \kappa\theta t\rho}{\sqrt{(q\nu + \kappa\theta t\rho)^2 + (1-\rho^2)\kappa^2\theta^2t^2}}\right) \\ &\in \textit{int}(\textit{Dom}(\Lambda)) = \left(-\frac{\kappa}{\nu(1-\rho)}, \frac{\kappa}{\nu(1+\rho)}\right).\end{aligned} \tag{1.11}$$

$\Lambda^*(q; 0, t)$ is finite for all $q \in R$; it is strictly increasing for $q > 0$ and strictly decreasing for $q < 0$; and $\Lambda^*(0; 0, t) = 0$.

$\Lambda^*(q; 0, t)$ is continuous in $(q, t) \in R \times R_+$.

Remarks.

1. $\Lambda^*(q;x,t) = \Lambda^*(q-x;0,t)$ since the only x dependence in Λ is the linear term xp.
2. Note also the scaling property $\Lambda(p;x,t) = t\,\Lambda(p;\frac{x}{t},1)$. In the following we choose to keep the t-dependence.
3. In this asymptotic regime, the limiting quantities Λ and Λ^* do not depend on the starting level of volatility y, and they depend on the κ (mean-reversion rate) and ν (volatility-of-volatility) parameters only through their ratio ν/κ.
4. The previous remark will also apply to asymptotic option prices and implied volatilities described below. In this regime, therefore, the relevant features of the Heston model are captured by just three parameters: the ergodic mean θ, the correlation ρ, and the ratio ν/κ. They control, respectively, the implied volatility skew's *level, slope*, and *convexity*.

The proof of Theorem 1.1 is the object of Section 2 and is concluded after Lemma 2.4. The proof of Lemma 1.2 is given at the end of Section 2.

A practical application of this result is the following rare event estimate for pricing out-of-the-money options of small maturity, derived in Section 2.2.

COROLLARY 1.3. *Suppose that log-moneyness is positive,* $\log(\frac{K}{S_0}) > 0$*, and* $t > 0$ *fixed. Then*

$$\lim_{\epsilon\to 0^+} \epsilon \log E[e^{-r\epsilon t}(S_{\epsilon,t} - K)^+ | S_{\epsilon,0} = S_0, Y_{\epsilon,0} = y_0] = -\Lambda^*\left(\log(\frac{K}{S_0}); 0, t\right),$$

independently of the initial square-volatility level y_0*. Note that the maturity of the option is* $T = \epsilon t$ *which goes to zero in the limit. The discounting factor* $e^{-r\epsilon t}$ *plays no role in this asymptotic result.*

Moreover, the asymptotic implied volatility can be computed. Let $\sigma_\epsilon(t,x)$ denote the Black-Scholes implied volatility for the European call option with strike price K, out-of-the-money so that $x = \log(K/S_0) > 0$, with short maturity $T = \epsilon t$ for $t > 0$ fixed, and computed under the dynamics given by (1.3,1.4). In Section 2.3 we prove the following result:

COROLLARY 1.4.

$$\lim_{\epsilon\to 0^+} \sigma_\epsilon^2(t,x) = \frac{x^2}{2\Lambda^*(x;0,t)t}, \quad x = \log(\frac{K}{S_0}) > 0.$$

Similarly, by considering Out-of-The-Money put options, one obtains the same formula for $x < 0$. The At-The-Money volatility is obtained by taking the limit $x \to 0$ (a precise statement is given in Lemma 2.6).

In fact, the results in Corollaries 1.3 and 1.4 hold for any fast mean-reverting stochastic volatility model (other than Heston's) which satisfies a large deviation principle, as in Theorem 1.1, provided the asymptotic rate function satisfies: $\Lambda^*(q;0,t)$ is finite for all $q \in R$; it is strictly increasing for $q > 0$ and strictly decreasing for $q < 0$; and $\Lambda^*(0;0,t) = 0$. This last remark is easily justified by going through the proofs of these results given in Sections 2.2 and 2.3.

2. Moment generating function and its asymptotic. Much of our analysis relies on an explicit calculation of a moment generating function, and evaluating its limit. First we define the quantity

$$\Lambda_\epsilon(p) = \Lambda_\epsilon(p;x,y,t) = \epsilon \log E[e^{\frac{p}{\epsilon}X_{\epsilon,t}} | X_{\epsilon,0} = x, Y_{\epsilon,0} = y]$$

$$
\begin{aligned}
&= \epsilon \log E[S_{\epsilon,t}^{\frac{p}{\epsilon}} | S_{\epsilon,0} = e^x, Y_{\epsilon,0} = y] \\
&= \epsilon r p t + \epsilon \log E[\tilde{S}_{\epsilon,t}^{\frac{p}{\epsilon}} | \tilde{S}_{\epsilon,0} = e^x, Y_{\epsilon,0} = y], \qquad (2.1)
\end{aligned}
$$

where $S_{\epsilon,t}, Y_{\epsilon,t}, X_{\epsilon,t}$ and $\tilde{S}_{\epsilon,t}$ are defined in Section 1.3. Using (1.9) and introducing a Brownian motion W^3 independent of W^2, the moments of $\tilde{S}_{\epsilon,t}$ can be formally rewritten as follows:

$$
\begin{aligned}
E[\tilde{S}_{\epsilon,t}^{\frac{p}{\epsilon}} &| \tilde{S}_{\epsilon,0} = e^x, Y_{\epsilon,0} = y] \\
&= e^{\frac{xp}{\epsilon}} E[e^{-\frac{p}{2}\int_0^t Y_{\epsilon,s} ds + \frac{p}{\sqrt{\epsilon}}\int_0^t \sqrt{Y_{\epsilon,s}} dW_s^1} | Y_{\epsilon,0} = y] \\
&= e^{\frac{xp}{\epsilon}} E[e^{-\frac{p}{2}\int_0^t Y_{\epsilon,s} ds + \frac{p\rho}{\sqrt{\epsilon}}\int_0^t \sqrt{Y_{\epsilon,s}} dW_s^2 + \frac{p\sqrt{1-\rho^2}}{\sqrt{\epsilon}}\int_0^t \sqrt{Y_{\epsilon,s}} dW_s^3} | Y_{\epsilon,0} = y] \\
&= e^{\frac{xp}{\epsilon}} E[e^{-\frac{p}{2}\int_0^t Y_{\epsilon,s} ds + \frac{p\rho}{\sqrt{\epsilon}}\int_0^t \sqrt{Y_{\epsilon,s}} dW_s^2 + \frac{p^2(1-\rho^2)}{2\epsilon}\int_0^t Y_{\epsilon,s} ds} | Y_{\epsilon,0} = y] \\
&= e^{\frac{xp}{\epsilon}} E[e^{\frac{p\rho}{\sqrt{\epsilon}}\int_0^t \sqrt{Y_{\epsilon,s}} dW_s^2 - \frac{p^2\rho^2}{2\epsilon}\int_0^t Y_{\epsilon,s} ds} e^{\frac{p(p-\epsilon)}{2\epsilon}\int_0^t Y_{\epsilon,s} ds} | Y_{\epsilon,0} = y],
\end{aligned}
$$

where we integrated with respect to the independent Brownian motion W^3 and redistribute the bounded variation terms. Using Girsanov tranform, one obtains that

$$
E[\tilde{S}_{\epsilon,t}^{\frac{p}{\epsilon}} | \tilde{S}_{\epsilon,0} = e^x, Y_{\epsilon,0} = y] = e^{\frac{xp}{\epsilon}} E^Q[e^{\frac{p(p-\epsilon)}{2\epsilon}\int_0^t Z_{\epsilon,s} ds} | Z_{\epsilon,0} = y], \qquad (2.2)
$$

where, under the measure Q, the process $Z_{\epsilon,t}$ satisfies the equation

$$
dZ_{\epsilon,t} = \frac{1}{\epsilon}\left(\kappa\theta - (\kappa - \nu\rho p) Z_{\epsilon,t}\right) dt + \frac{\nu}{\sqrt{\epsilon}} \sqrt{Z_{\epsilon,t}} dW_t^Q, \qquad (2.3)
$$

driven by a Brownian motion W^Q. The result (2.2–2.3) is given in Lemma 2.3 in Andersen and Piterbarg [2] (with a proof in B.1 of their supplementary material). Note that the proof of (2.2) in Andersen and Piterbarg [2] allows the possibility of "$+\infty = +\infty$". Although the statement of their Lemma 2.3 is limited to the case of $p(p-\epsilon) > 0$, the proof is not limited to that case, allowing $p \in R$.

2.1. Explicit evaluation of Λ_ϵ. The following two inequalities play important roles:

$$
(\rho\nu p - \kappa)^2 \geq p(p-\epsilon)\nu^2, \qquad (2.4)
$$

$$
\rho\nu p < \kappa. \qquad (2.5)
$$

When (2.4) and (2.5) are both satisfied, then by results concerning exponential functionals of CIR processes (e.g. Corollary 3 of Albanese and Lawi [1], or Theorem 3.1 of Hurd and Kuznetsov [16]), we have

$$
E^Q[e^{\frac{p(p-\epsilon)}{2\epsilon}\int_0^t Z_{\epsilon,s} ds} | Z_{\epsilon,0} = y] = e^{m(t) - n(t) y},
$$

where

$$
\begin{aligned}
m(t) = m_\epsilon(t) &= \frac{\kappa\theta t}{\nu^2}(b - \bar{b}) + \frac{2\kappa\theta}{\nu^2} \log\left(\frac{\bar{b} e^{\bar{b}t/2}}{\bar{b}\cosh(\frac{\bar{b}t}{2}) + b\sinh(\frac{\bar{b}t}{2})}\right), \\
n(t) = n_\epsilon(t) &= \frac{-p(p-\epsilon)}{\epsilon}\left(\frac{\sinh(\frac{\bar{b}t}{2})}{\bar{b}\cosh(\frac{\bar{b}t}{2}) + b\sinh(\frac{\bar{b}t}{2})}\right), \\
\bar{b} &= \frac{1}{\epsilon}\sqrt{(\kappa - \nu\rho p)^2 - \nu^2 p(p-\epsilon)}\,, \\
b &= \frac{\kappa - \nu\rho p}{\epsilon}\,,
\end{aligned}
$$

and consequently, Λ_ϵ defined in (2.1), is given explicitly by

$$\Lambda_\epsilon(p; x, y, t) = \epsilon rpt + xp + \epsilon\left(m_\epsilon(t) - n_\epsilon(t)y\right). \tag{2.6}$$

Note that when the limit exists as $\epsilon \to 0^+$, the only contribution from $\epsilon\left(m_\epsilon(t) - n_\epsilon(t)y\right)$ comes from the first term of $m(t)$ which leads to formula (1.10) for $\Lambda(p; x, t)$.

Next, we show that if (2.4) or (2.5) is violated, then $\Lambda_\epsilon = +\infty$. First, we sort out (2.4–2.5) more explicitly. The inequality (2.4) is equivalent to

$$c_{1,\epsilon} \le p \le c_{2,\epsilon}\,,$$

where

$$\begin{aligned} c_{1,\epsilon} &= \frac{(\epsilon\nu - 2\kappa\rho) - \sqrt{(\epsilon\nu - 2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)}}{2\nu(1-\rho^2)} \le 0,\\ c_{2,\epsilon} &= \frac{(\epsilon\nu - 2\kappa\rho) + \sqrt{(\epsilon\nu - 2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)}}{2\nu(1-\rho^2)} \ge 0. \end{aligned}$$

We denote the case $\epsilon = 0$ as follows

$$c_1 = -\frac{\kappa}{\nu(1-\rho)}, \quad c_2 = \frac{\kappa}{\nu(1+\rho)}.$$

Then in the limit $\epsilon \to 0^+$, (2.4) becomes

$$(\rho\nu p - \kappa)^2 \ge p^2\nu^2 \Leftrightarrow c_1 \le p \le c_2. \tag{2.7}$$

LEMMA 2.1. *For ϵ small enough,*

$$c_1 < c_{1,\epsilon} < 0 < c_2 < c_{2,\epsilon}. \tag{2.8}$$

Moreover, (2.5) always holds if (2.4) is satisfied for ϵ small enough. In fact,

1. *if $0 < \rho < 1$, then $c_2 < c_{2,\epsilon} < \frac{\kappa}{\rho\nu}$.*
2. *if $-1 < \rho < 0$, then $\frac{\kappa}{\rho\nu} < c_1 < c_{1,\epsilon}$.*
3. *if $\rho = 0$, then (2.5) always holds.*

Proof. (2.8) follows from the definition by direct verification.

Assume that $\rho > 0$. Then $(1+\rho)^{-1} < \rho^{-1}$, and therefore $c_2 = \frac{\kappa}{\nu(1+\rho)} < \frac{\kappa}{\rho\nu}$, since $c_2 < c_{2,\epsilon}$ and $\lim_\epsilon c_{2,\epsilon} = c_2$, $c_2 < c_{2,\epsilon} < \frac{\kappa}{\rho\nu}$ when ϵ is small enough.

The other case follows by a similar computation (note that, if $-1 < \rho < 0$, $\rho^{-1} < -(1-\rho)^{-1}$ implying $\frac{\kappa}{\rho\nu} < c_1$). ∎

We have

LEMMA 2.2. *$\Lambda_\epsilon(p)$ is lower semicontinuous and convex in p. For $\epsilon > 0$ small enough, (2.6) holds when $c_{1,\epsilon} \le p \le c_{2,\epsilon}$.*

Proof. The lower semi-continuity and convexity of $\Lambda_\epsilon(p)$ follow from its definition as a logarithmic transform of moment generating function for $X_{\epsilon,t}$.

The other conclusion follows from Lemma 2.1. ∎

Using the convexity of Λ_ϵ, we conclude next that $\Lambda_\epsilon(p) = +\infty$ whenever $p \notin [c_{1,\epsilon}, c_{2,\epsilon}]$ (again, when $\epsilon > 0$ is small enough). This is implied by the behavior of $\partial_p\Lambda_\epsilon$ for $p \in (c_{1,\epsilon}, c_{2,\epsilon})$. From (2.6), we get:

$$\partial_p\Lambda_\epsilon = \epsilon rt + x + \epsilon\partial_p m - \epsilon y \partial_p n$$

$$= \epsilon rt + x + \epsilon\frac{2\kappa\theta}{\nu^2}\left(\frac{\partial\bar{b}}{\bar{b}} + \frac{t}{2}\partial b - \frac{(\partial\bar{b})(1+\frac{t}{2}b)\cosh(\frac{\bar{b}t}{2}) + (\partial b)(1+\frac{t}{2}\bar{b})\sinh(\frac{\bar{b}t}{2})}{\bar{b}\cosh(\frac{\bar{b}t}{2}) + b\sinh(\frac{\bar{b}t}{2})}\right)$$
$$+y\left(\frac{(2p-\epsilon)\sinh(\frac{\bar{b}}{2}t) + (p^2-p\epsilon)(\partial\bar{b})\frac{t}{2}\cosh(\frac{\bar{b}}{2}t)}{\bar{b}\cosh(\frac{\bar{b}t}{2}) + b\sinh(\frac{\bar{b}t}{2})}\right.$$
$$\left.-(p^2-p\epsilon)\frac{(1+\frac{t}{2}\bar{b})(\partial b)\sinh^2(\frac{\bar{b}}{2}t) + (\partial\bar{b})(1+\frac{t}{2}b)\sinh(\frac{\bar{b}}{2}t)\cosh(\frac{\bar{b}}{2}t)}{(\bar{b}\cosh(\frac{\bar{b}t}{2}) + b\sinh(\frac{\bar{b}t}{2}))^2}\right).$$

When $p \to c_{1,\epsilon}^+$, or $c_{2,\epsilon}^-$, then $\bar{b} \to 0^+$ and $\partial_p\bar{b} \to \pm\infty$. By Lemma 2.1, in both cases we have $\lim_p b > 0$. Therefore, by the above expression for $\partial_p\Lambda_\epsilon$, we conclude that, for $\epsilon > 0$ small enough,

$$\lim_{p\to c_{1,\epsilon}^+}\partial_p\Lambda_\epsilon(p) = -\infty, \qquad \lim_{p\to c_{2,\epsilon}^-}\partial_p\Lambda_\epsilon(p) = \infty. \tag{2.9}$$

Since Λ_ϵ is convex and lower semicontinuous, the limits (2.9) imply

LEMMA 2.3. *For $\epsilon > 0$ small enough, Λ_ϵ is given by (2.6) when $c_{1,\epsilon} \le p \le c_{2,\epsilon}$, and $+\infty$ otherwise.*

Proof. Follows from Lemma 3.1 given in the Appendix. ◻

The following result follows easily.

LEMMA 2.4. *The function Λ given in (1.10) is lower semicontinuous and essentially smooth in p. Moreover, $\Lambda_\epsilon(\cdot;x,y,t)$ Γ-converges to $\Lambda(\cdot;x,t)$ (see Definition 3.2). In particular, for each $x \in R$, $y > 0$, $t > 0$, we have:*

1. *For every $p \in R$, there exists $\{p_\epsilon\}$ with $p_\epsilon \to p$ such that*
$$\lim_{\epsilon\to 0^+}\Lambda_\epsilon(p_\epsilon;x,y,t) = \Lambda(p;x,t).$$
2. *For every $p \in R$ and every $p_\epsilon \to p$,*
$$\liminf_{\epsilon\to 0^+}\Lambda_\epsilon(p_\epsilon;x,y,t) \ge \Lambda(p;x,t).$$

By Lemma 3.3, Theorem 1.1 follows.

We now give the proof of Lemma 1.2, where we derive an explicit formula for Λ^*, the Legendre transform of Λ defined by (1.10).

Proof of Lemma 1.2. By the essential smoothness property of $\Lambda(p) = \Lambda(p;0,t)$ in p, the equation

$$\frac{\partial}{\partial p}(qp - \Lambda(p)) = 0$$

has a solution $p \in \mathrm{int}(\mathrm{Dom}(\Lambda)) = (-\frac{\kappa}{\nu(1-\rho)}, \frac{\kappa}{\nu(1+\rho)})$, which equivalently solves

$$q = \partial_p\Lambda = \frac{\kappa\theta t}{\nu}\left(-\rho + \frac{(\kappa-\rho\nu p)\rho + \nu p}{\sqrt{(\kappa-\rho\nu p)^2 - \nu^2p^2}}\right). \tag{2.10}$$

If $\nu q = -\rho\kappa\theta t$, it follows from (2.10) that $p = -\frac{\kappa\rho}{\nu(1-\rho^2)}$, and therefore that (1.11) is satisfied.

If $\nu q \ne -\rho\kappa\theta t$, then (2.10) is a quadratic equation in p with the sign condition

$$\frac{\kappa\rho + \nu(1-\rho^2)p}{\nu q + \rho\kappa\theta t} > 0$$

One can easily verify that p given by (1.11) is the only root satisfying the sign condition. Consequently, the expression of $\Lambda^*(q;0,t)$ follows.

It follows by direct verification that $\Lambda^*(0;0,t)=0$, and that $\Lambda^*(q;0,t)$ is continuous, finitely defined for all $q\in R$.

Also by direct calculation, $\partial_p\Lambda(0;0,t)=0$ and

$$\partial_{pp}\Lambda(p;0,t)=\frac{\kappa^3\theta t}{((\kappa-\rho\nu p)^2-\nu^2p^2)^{3/2}}>0.$$

Therefore, for $p\in \mathrm{int}(\mathrm{Dom}(\Lambda))$, $\partial_p\Lambda(p;0,t)$ is negative when $p<0$, and is positive when $p>0$. By a convex analysis result, $q=\partial_p\Lambda(p;0,t)$, $p\in\mathrm{int}(\mathrm{Dom}(\Lambda))$ if and only if $p=\partial_q\Lambda^*(q;0,t)$. Consequently, $\Lambda^*(q;0,t)$ is strictly increasing when $q>0$ and strictly decreasing for $q<0$; and achieves its minimum (zero) when $q=0$.

2.2. Pricing. We now prove Corollary 1.3.

Recall that $S_{\epsilon,t}=e^{X_{\epsilon,t}}$ and $S_{\epsilon,0}=S_0$. For $\delta>0$ we have

$$\begin{aligned}E[(S_{\epsilon,t}-K)^+]&\geq E[\mathbf{1}_{\{S_{\epsilon,t}-K>\delta\}}(S_{\epsilon,t}-K)^+]\\&\geq \delta P(S_{\epsilon,t}>K+\delta).\end{aligned}\tag{2.11}$$

By Theorem 1.1, it follows that

$$\begin{aligned}\liminf_{\epsilon\to0^+}\epsilon\log E[(S_{\epsilon,t}-K)^+]&\geq\liminf_{\epsilon\to0^+}\epsilon\log P(X_{\epsilon,t}>\log(K+\delta))\\&\geq-\inf_{q>\log(K+\delta)}\Lambda^*(q-\log S_0;0,t)=-\Lambda^*\left(\log\left(\frac{K+\delta}{S_0}\right);0,t\right).\end{aligned}$$

The last equality follows from the fact that $\log(\frac{K}{S_0})>0$ and that $\Lambda^*(q;0,t)$ is non-decreasing for q in the region $q\geq0$ (see Lemma 1.2). Taking $\delta\to0^+$, by continuity of Λ^*, we obtain the desired lower bound.

To show the upper bound, we note that for $p,q>1$ such that $p^{-1}+q^{-1}=1$,

$$E[(S_{\epsilon,t}-K)^+]\leq E^{1/p}[|(S_{\epsilon,t}-K)^+|^p]E^{1/q}[\mathbf{1}_{\{S_{\epsilon,t}-K\geq0\}}]\,.$$

Therefore

$$\begin{aligned}\epsilon\log E[(S_{\epsilon,t}-K)^+]&\leq\frac{\epsilon}{p}\log E[(S_{\epsilon,t})^p]+\epsilon(1-\frac{1}{p})\log P(S_{\epsilon,t}\geq K)\\&\leq\frac{1}{p}\Lambda_\epsilon(\epsilon p)+(1-\frac{1}{p})\epsilon\log P(S_{\epsilon,t}\geq K)\,.\end{aligned}$$

Taking $\lim_{p\to+\infty}\limsup_{\epsilon\to0^+}$ on both sides, and noting that $\lim_{\epsilon\to0^+}\Lambda_\epsilon(\epsilon p)=0$, we deduce (by Theorem 1.1) the desired upper bound

$$\limsup_{\epsilon\to0^+}\epsilon\log E[(S_{\epsilon,t}-K)^+]\leq-\Lambda^*\left(\log(\frac{K}{S_0});0,t\right).$$

2.3. Implied volatility. We prove Corollary 1.4 which gives the asymptotic behavior of the implied volatility $\sigma_\epsilon(t,x)$. Throughout, we denote the log-moneyness by $x=\log(K/S_0)>0$, and for simplicity $\sigma_\epsilon(t,x)=\sigma_\epsilon$, t and x being fixed in the following analysis.

First, we show that

$$\lim_{\epsilon\to0^+}\sigma_\epsilon\sqrt{\epsilon t}=0.\tag{2.12}$$

By Lemma 1.2, $\Lambda^*(x;0,t) > 0$. Let $0 < \delta < \Lambda^*(x;0,t)$. By definition of σ_ϵ, and Corollary 1.3, for $\epsilon > 0$ small enough

$$\begin{aligned} e^{-(\Lambda^*(x;0,t)-\delta)/\epsilon} &\geq E[(S_{\epsilon,t}-K)^+] \\ &= e^{r\epsilon t} S_0 \Phi\left(\frac{-x + r\epsilon t + \frac{1}{2}\sigma_\epsilon^2 \epsilon t}{\sigma_\epsilon\sqrt{\epsilon t}}\right) - K\Phi\left(\frac{-x + r\epsilon t - \frac{1}{2}\sigma_\epsilon^2 \epsilon t}{\sigma_\epsilon\sqrt{\epsilon t}}\right), \end{aligned}$$

where we have used the Black-Scholes formula and denoted by Φ the $\mathcal{N}(0,1)$ cdf. Since $E[(S_{\epsilon,t}-K)^+] \geq 0$, one deduces that the right-hand side must converge to zero as $\epsilon \to 0^+$. Let $l \geq 0$ be the limit of $\sigma_\epsilon\sqrt{\epsilon t}$ along a converging subsequence, then l must satisfy

$$S_0\Phi\left(-\frac{x}{l} + \frac{l}{2}\right) - K\Phi\left(-\frac{x}{l} - \frac{l}{2}\right) = 0\,,$$

with $x = \log(K/S_0) > 0$. One can easily check that $l = 0$ is the only solution, and therefore (2.12) holds.

The following estimate on Φ using its derivative denoted by ϕ is classical, and will be useful (we refer to [21] Section 14.8 for instance).

LEMMA 2.5. *For $x > 0$,*

$$(x + \frac{1}{x})^{-1}\phi(x) \leq 1 - \Phi(x) \leq \frac{1}{x}\phi(x). \tag{2.13}$$

Next, we establish the lower bound for the limit in Corollary 1.4. We will use the classical notation

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + r\epsilon t + \frac{\sigma_\epsilon^2}{2}\epsilon t}{\sigma_\epsilon\sqrt{\epsilon t}}\,.$$

Let $\delta > 0$, by the definition of $\sigma_\epsilon(t)$ and Corollary 1.3, for $\epsilon > 0$ small enough, we have

$$\begin{aligned} e^{-(\Lambda^*(x;0,t)+\delta)/\epsilon} &\leq E[(S_{\epsilon,t}-K)^+] \\ &\leq e^{r\epsilon t} S_0 \Phi(d_1) = e^{r\epsilon t} S_0 \left(1 - \Phi(-d_1)\right) \\ &\leq e^{r\epsilon t} S_0 \left(\frac{1}{-d_1}\right)\phi(-d_1)\,, \end{aligned}$$

where the last line follows from (2.13). By (2.12) and $S_0 < K$, we know that $\lim_{\epsilon\to 0^+} d_1 = -\infty$. Taking $(\epsilon\log)$ on both sides, one sees that the leading order term on the right-hand side is given by

$$-\epsilon\frac{\left(\log(\frac{S_0}{K})\right)^2}{2(\sigma_\epsilon\sqrt{\epsilon t})^2} = -\frac{x^2}{2\sigma_\epsilon^2 t}\,.$$

Therefore any limit point of σ_ϵ along a converging subsequence (ϵ_n) will satisfy

$$-(\Lambda^*(x,;0,t)+\delta) \leq -\frac{x^2}{2\lim_{\epsilon_n\to 0^+}\sigma_{\epsilon_n}^2 t}\,, \tag{2.14}$$

for all $\delta > 0$, and consequently the desired lower bound.

Next, we justify the upper bound. To avoid confusion, we denote by P the measure under which S_ϵ is defined in Section 1.3, and by $P_{BS} = P_{BS}(\sigma_\epsilon)$ the measure under which S_ϵ follows the Black-Scholes model with constant volatility $\sigma_\epsilon = \sigma_\epsilon(t,x)$:

$$dS_{\epsilon,s} = S_{\epsilon,s}\left(rds + \sigma_\epsilon dW_s\right) \,,$$

where W is a Brownian motion under P_{BS} (note that here t is fixed and the maturity of the call option is ϵt). Then, using the classical notation

$$d_2 = \frac{\log\left(\frac{S_0}{K+\delta}\right) + r\epsilon t - \frac{\sigma_\epsilon^2}{2}\epsilon t}{\sigma_\epsilon\sqrt{\epsilon t}} \,,$$

one obtains

$$\begin{aligned} e^{-(\Lambda^*(x;0,t)-\delta)/\epsilon} &\geq E^P[(S_{\epsilon,t}-K)^+] = E^{P_{BS}}[(S_{\epsilon,t}-K)^+] \\ &\geq \delta P_{BS}(S_{\epsilon,t} > K+\delta) \\ &= \delta\left(1 - \Phi\left(-d_2\right)\right) \\ &\geq \delta\left(\frac{-d_2}{1+d_2^2}\right)\phi(-d_2), \end{aligned}$$

where the second inequality follows by (2.11). Arguing as above, in the case of the lower bound, we know that $\lim_{\epsilon\to 0^+} d_2 = -\infty$. Taking $(\epsilon \log)$ on both sides, the leading order term on the right-hand side is given by

$$-\frac{\left(\log(\frac{S_0}{K+\delta})\right)^2}{2\sigma_\epsilon^2 t} \,,$$

and therefore, along any converging subsequence,

$$-(\Lambda^*(x;0,t) - \delta) \geq -\frac{\left(\log(\frac{K+\delta}{S_0})\right)^2}{2\lim_{\epsilon_n\to 0^+}\sigma_{\epsilon_n}^2 t} \,.$$

Sending $\delta \to 0^+$ gives the desired upper bound, which concludes the proof of Corollary 1.4.

To summarize, we proved that in this regime (fast mean reverting volatility and short maturity) the asymptotic implied volatility of an OTM call option ($x > 0$) is given by

$$\sigma(t,x)^2 = \frac{x^2}{2\Lambda^*(x;0,t)t} \,,$$

where Λ^* is given in Lemma 1.2. The same formula for $x < 0$ is derived similarly by considering OTM put options. Using the explicit formula for $\Lambda^*(x;0,t)$, one can derive the At-The-Money limit:

$$\lim_{x\to 0}\sigma(t,x)^2 = \theta \,,$$

by checking that near zero $p(q;t) = \frac{q}{\theta t} + O(q^2)$, $\Lambda(p;0,t) = \frac{\theta t}{2}p^2 + O(p^3)$, and consequently

$$\Lambda^*(q;0,t) = q\left(\frac{q}{\theta t}\right) - \frac{\theta t}{2}\left(\frac{q}{\theta t}\right)^2 + O(q^3) = \frac{q^2}{2\theta t} + O(q^3).$$

In fact, we can also derive the limit as $\epsilon \to 0^+$ of the At-The-Money volatility $\sigma_\epsilon(t,0)$:

LEMMA 2.6. *The asymptotic At-The-Money volatility is given by*

$$\lim_{\epsilon\to 0^+} \sigma_\epsilon(t,0)^2 = \lim_{x\to 0} \sigma(t,x)^2 = \theta\,.$$

Proof. This is not a large deviation result but rather an averaging result of the type studied in [10]. Since it involves convergence in distribution, it is more convenient to work with put options whose payoffs are continuous and bounded. The ATM volatility is defined by the unique positive number $\sigma_\epsilon(0,t)$ satisfying

$$E[(S_0 - S_{\epsilon,t})^+] = S_0\Phi(-d_2) - e^{r\epsilon t} S_0 \Phi(-d_1)\,,$$

where here

$$d_{1,2} = \frac{(r \pm \frac{1}{2}\sigma_\epsilon(0)^2)\sqrt{\epsilon t}}{\sigma_\epsilon(0)}\,, \tag{2.15}$$

and we have denoted $\sigma_\epsilon(0,t) = \sigma_\epsilon(0)$ since t is fixed. Using equation (1.5) and dividing on both sides by $\sqrt{\epsilon}\, S_0$, on gets

$$\begin{aligned} E\left[\left(-\sqrt{\epsilon}\int_0^t r\,\frac{S_{\epsilon,s}}{S_0}ds - \int_0^t \frac{S_{\epsilon,s}}{S_0}\sqrt{Y_{\epsilon,s}}dW_s^1\right)^+\right] \\ = \frac{1}{\sqrt{\epsilon}}\left(\Phi(-d_2) - e^{r\epsilon t}\Phi(-d_1)\right). \end{aligned} \tag{2.16}$$

One easily obtains the convergence in probability to zero of the following integrals:

$$\sqrt{\epsilon}\int_0^t r\,\frac{S_{\epsilon,s}}{S_0}ds \quad \text{and} \quad \int_0^t \left(\frac{S_{\epsilon,s}}{S_0} - 1\right)\sqrt{Y_{\epsilon,s}}dW_s^1\,.$$

The convergence of the quadratic variation of the martingale term, $\int_0^t Y_{\epsilon,s}\,ds \to \bar\sigma^2 t$, implies the convergence in distribution

$$\left(-\sqrt{\epsilon}\int_0^t r\,\frac{S_{\epsilon,s}}{S_0}ds - \int_0^t \frac{S_{\epsilon,s}}{S_0}\sqrt{Y_{\epsilon,s}}dW_s^1\right) \to \int_0^t \bar\sigma dW_s^1 = \bar\sigma W_t^1\,,$$

where $\bar\sigma^2$ is the ergodic average of the square volatility $Y_{\epsilon,\cdot}$, that is:

$$\bar\sigma^2 = \int_0^{+\infty} y\Gamma(dy)\,,$$

where Γ is the invariant distribution of the ergodic process Y defined by (1.2). A complete proof of this result involves introducing a solution ψ of the Poisson equation

$$\mathcal{L}\psi(y) = y - \bar\sigma^2\,,$$

where $\mathcal{L}$ is the infinitesimal generator of the process Y, and using Ito's formula to show that

$$\int_0^t \left(Y_{\epsilon,s} - \bar\sigma^2\right) ds = \int_0^t \mathcal{L}\psi(Y_{\epsilon,s})ds = \epsilon\left(\psi(Y_{\epsilon,t}) - \psi(Y_0)\right) - \sqrt{\epsilon}\int_0^t \sigma\psi'(Y_{\epsilon,s})Y_{\epsilon,s}dW_s^2$$

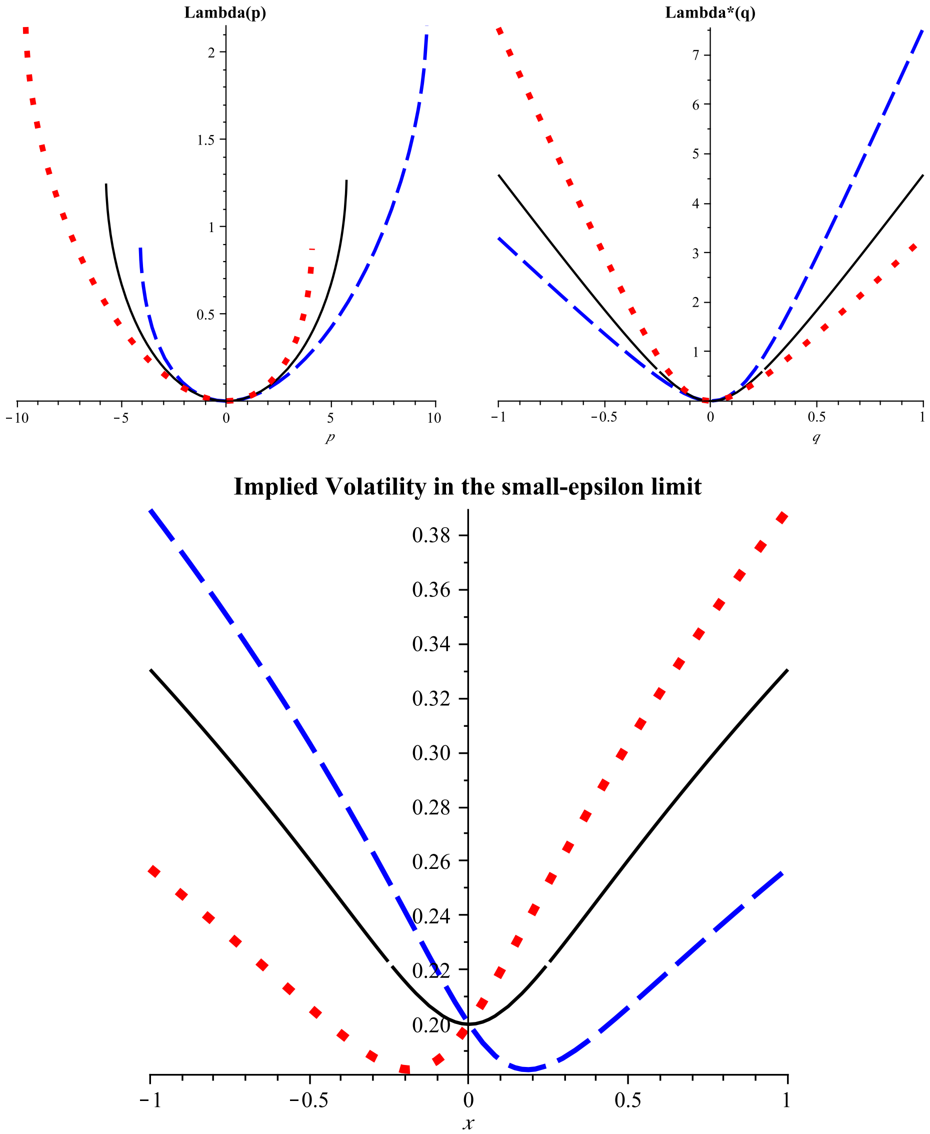

FIG. 2.1. Here we have plotted Λ, Λ^*, and the implied volatility in the small-ϵ limit as a function of the log-moneyness $x = \log(K/S_0)$. The parameters are $t = 1$, *ergodic mean* $\theta = .04$, *convexity* $\nu/\kappa = 1.74$ ($\kappa = 1.15$, $\nu = .2$), and *skew* $\rho = -.4$ (dashed blue), $\rho = 0$ (solid black), $\rho = +.4$ (dotted red).

converges to zero (we refer to [10] for details).

In this case, the invariant distribution is a *Gamma* with mean θ and consequently $\bar{\sigma}^2 = \theta$. Therefore, the left-hand side of (2.16) converges to $E[(\bar{\sigma}W_t^1)^+] = \bar{\sigma}\sqrt{t}/\sqrt{2\pi} = \sqrt{\theta t}/\sqrt{2\pi}$. By direct inspection of the right-hand side of (2.16) and the relation (2.15) between $d_{1,2}$ and $\sigma_\epsilon(0)$, one deduces that $\sigma_\epsilon(0)$ must converge to θ as $\epsilon \to 0^+$. $\square$

In Figure 2.1 we show plots of the functions Λ and Λ^*, and of the implied volatility smile/skew obtained in the limit $\epsilon \to 0^+$.

3. Appendix.

3.1. A property of convex functions in R.

LEMMA 3.1. *Suppose $\Lambda : R \mapsto \bar{R}$ is convex and for some $c \in R$*

$$\liminf_{x\to c^-} \Lambda(x) > -\infty \quad \textit{and} \quad \lim_{x\to c^-} \partial\Lambda(x) = +\infty\,,$$

then $\Lambda(y) = +\infty$ for all $y > c$. Similarly, if for some $c \in R$

$$\limsup_{x\to c^+} \Lambda(x) > -\infty \quad \textit{and} \quad \lim_{x\to c^+} \partial\Lambda(x) = -\infty$$

then $\Lambda(y) = +\infty$ for all $y < c$.

Proof. Let $y > c > x$ and denote $\delta = y - c > 0$. Then

$$\Lambda(y) \geq \Lambda(x) + \partial\Lambda(x)(y - x).$$

Taking $x \to c^-$ gives

$$\Lambda(y) \geq \liminf_{x\to c^-} \Lambda(x) + \limsup_{x\to c^-} \partial\Lambda(x)\delta = +\infty.$$

□

3.2. Gärtner-Ellis theorem via Γ-convergence.

We generalize the Gärtner-Ellis theorem (e.g. Theorem 2.3.6 in Dembo and Zeitouni [7]) for Euclidean space valued random variables.

DEFINITION 3.2. *Let sequence $\Lambda_n, \Lambda : R^d \mapsto \bar{R}$. We say that Λ_n Γ-converges to Λ (denoted $\Lambda_n \xrightarrow{\Gamma} \Lambda$), if for all $p \in R^d$,*

1. *(limsup inequality) there exists a sequence of $\{p_n\}$ converging to p such that*

$$\Lambda(p) \geq \limsup_{n\to\infty} \Lambda_n(p_n).$$

2. *(liminf inequality) for every sequence $\{p_n\}$ converging to p, we have*

$$\Lambda(p) \leq \liminf_{n\to\infty} \Lambda_n(p_n).$$

Let $\{X_n : n = 1, 2, \ldots\}$ be a sequence of R^d-valued random variables, and denote

$$\Lambda_n(p) = \frac{1}{n} \log E[e^{npX_n}], \quad p \in R^d.$$

LEMMA 3.3. *Suppose that the limsup property in Γ-convergence holds for Λ_n to a $\Lambda : R^d \mapsto \bar{R}$. Then the large deviation upper bound holds for all compact $F \subset R^d$*

$$\limsup_{n\to\infty} \frac{1}{n} \log P(X_n \in F) \leq -\inf_{x\in F} \Lambda^*(x). \tag{3.1}$$

Furthermore, if $0 \in interior(D(\Lambda))$, then $\{X_n\}$ is exponentially tight and (3.1) holds for all closed $F \subset R^d$.

In addition to the above, suppose that the liminf property in Γ-convergence holds for Λ_n to a $\Lambda : R^d \mapsto \bar{R}$, and assume $\Lambda(0) = 0$. Then the following upper bound holds

$$\liminf_{n\to\infty} \frac{1}{n} \log P(X_n \in G) \geq -\inf_{x\in G\cap\mathcal{F}} \Lambda^*(x), \quad G \textit{ open in} R^d. \tag{3.2}$$

where $\mathcal{F}$ is the set of exposed points of Λ^* with exposing hyperplane in interior$(D(\Lambda))$ where $D(\Lambda) = \{x : \Lambda(x) < \infty\}$.

If Λ is lower semicontinuous and essentially smooth (e.g. Definition 2.3.5), then $\inf_{x\in G\cap\mathcal{F}} \Lambda^*(x) = \inf_{x\in G} \Lambda^*(x)$ *for all G open, and Λ^* is a good rate function.*

Proof. The upper bound (3.1) for compact set F has been shown in Theorem 1.2 of Zabell [22] for more general case. Under the condition $0 \in \text{interior}(D(\Lambda))$, the exponential tightness follows (e.g. the proof on page 48-49 of [7]).

We prove the lower bound (3.2) by highlighting the new ingredients needed to modify [7]. For each $y \in \mathcal{F}$ and $\eta \in \text{interior}(D(\Lambda))$ the exposing hyperplane for y, by the limsup inequality of Λ_n to Λ, there exists $\eta_n \to \eta$ such that $\Lambda_n(\eta_n) < \infty$. We define a new probability measure

$$\frac{d\tilde{\mu}_n}{dPX_n^{-1}}(z) = e^{n(\eta_n\cdot z - \Lambda_n(\eta_n))}.$$

Using the liminf inequality of Λ_n to Λ, then as in [7],

$$\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log P(X_n \in B(y,\delta)) \geq -\Lambda^*(y) + \lim_{\delta\to 0+}\liminf_{n\to\infty}\frac{1}{n}\log\tilde{\mu}_n(B(y,\delta))$$

Now let $\tilde{\Lambda}(\cdot) = \Lambda(\cdot + \eta) - \Lambda(\eta)$. Then $\tilde{\Lambda}(0) = 0$ and $0 \in \text{interior}(D(\tilde{\Lambda}))$. Let

$$\tilde{\Lambda}_n(p) = \frac{1}{n}\int_{R^d} e^{npx}\tilde{\mu}_n(dx).$$

Then $\tilde{\Lambda}_n \xrightarrow{\Gamma} \tilde{\Lambda}$. The rest of the proof follows verbatim of that in [7], concluding

$$\liminf_{n\to\infty}\frac{1}{n}\log\tilde{\mu}_n(B(y,\delta)) = 0.$$

Hence (3.2) follows. □

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