The conditional law of the Bacry-Muzy and Riemann-Liouville log correlated Gaussian fields and their GMC, via Gaussian Hilbert and fractional Sobolev spaces

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Abstract

We compute $\mathbb{E}(X_t | (X_s)_{0 \leq s \leq L})$ for the standard Bacry-Muzy log-correlated Gaussian field $X$ with covariance $\log^+ \frac{1}{|y-x|}$, which corrects the finite-horizon prediction formula in Vargas et al. [DRV12]. The problem can be viewed as a linear filtering problem, and we solve the problem by showing that the $L^2(\mathbb{P})$ closure of $\{ \int_{[0,L]} \phi(s) X_s ds : \phi \in \mathcal{S}, \text{supp}(\phi) \subseteq [0,L] \}$ is equal to $\{ X(\phi) : \phi \in H^{-\frac{1}{2}}, \text{supp}(\phi) \subseteq [0,L] \}$, where $X(\phi)$ is defined as a continuous linear extension of $X$ acting on $\mathcal{S} \subset H^0$, $H^s$ denotes the fractional Sobolev space of order $s$ and $\mathbb{P}$ is the law of the field $X$ on the space of tempered distributions. The explicit formula for the filter is obtained as the solution to a Fredholm integral equation of the first kind with logarithmic kernel. From this we characterize the conditional law of the Gaussian multiplicative chaos (GMC) $M_\gamma$ generated by $X$, using that $M_\gamma$ is measurable with respect to $X$. We also outline how one can adapt this result for the Riemann-Liouville GMC introduced in [FFGS19], which has a natural application to the Rough Bergomi volatility model in the $H \to 0$ limit.1

1 Introduction

Originally pioneered by Kahane [Kah85], Gaussian multiplicative chaos (GMC) is a random measure on a domain of $\mathbb{R}^d$ that can be formally written as

$$M_\gamma(dx) = e^{\gamma X_x - \frac{1}{2} \gamma^2 \mathbb{E}(X_x^2)} dx$$

(1)

where $X$ is a Gaussian field with zero mean and covariance $K(x,y) := \mathbb{E}(X_x X_y) = \log^+ \frac{1}{|y-x|} + g(x,y)$ for some bounded continuous function $g$. $X$ is not defined pointwise because there is a singularity in its covariance, rather $X$ is a random tempered distribution, i.e. an element of the dual of the Schwartz space $\mathcal{S}$ under the locally convex topology induced by the Schwartz space semi-norms. For this reason, making rigorous sense of (1) requires a regularizing sequence $X^\varepsilon$ of Gaussian processes (with the singularity removed, see e.g. [BBM13] and [BM03] for a description of such a regularization in 1d based on integrating a Gaussian white noise over truncated triangular region, which is summarized in Section 2.3 in [FFGS19], or page 17 in [RV10] and section 3.4 in [Sha16] for a general method in $\mathbb{R}^d$ using a convolution to smooth $X$). In most of the literature on GMC, the choice of $X^\varepsilon$ is a martingale in $\varepsilon$, from which we can then easily verify that $M_\gamma^\varepsilon(A) = \int_A e^{\gamma X^\varepsilon_x - \frac{1}{2} \gamma^2 \mathbb{E}(X^\varepsilon_x^2)} dx$ is a martingale, and then obtain a.s. convergence of $M_\gamma^\varepsilon(A)$ using the martingale convergence to a random variable $M_\gamma(A)$ with $\mathbb{E}(M_\gamma(A)) = \text{Leb}(A)$, and with a bit more work we can verify that $M_\gamma(\cdot)$ defines a random measure (see the end of Section 4 on page 18 in [RV10]).

If $\gamma^2 < 2d$, $M_\gamma^\varepsilon(dx) = e^{\gamma X_x - \frac{1}{2} \gamma^2 \mathbb{E}(X_x^2)} dx$ tends weakly to a multifractal random measure $M_\gamma$ with full support a.s. which satisfies the multifractal property

$$\mathbb{E}(M_\gamma([0,t])^q) = c_q t^{\zeta(q)}$$

(2)

for $q \in (1,q^*)$ for some constant $c_q = \mathbb{E}(M_\gamma([0,1]^q))$, where $q^* = \frac{d}{2\gamma^2}$ and

$$\zeta(q) = q - \frac{1}{2} \gamma^2 (q^2 - q)$$

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2see Lemma 3 in [BM03] to see why the critical $q$ value is $q^*$
and \(\mathbb{E}(M_\gamma([0, t])) = \infty\) if \(q > \frac{\gamma}{2}\), see Theorem 2.13 in [RV14] and Lemma 3 in [BM03]). Moreover, we can show that the support of \(M_\gamma\) is a so-called \(\gamma\)-thick points of \(X\), i.e. points such that \(\lim_{\varepsilon \to 0} \frac{X_s^\varepsilon}{\log \varepsilon} = \gamma\) (see e.g. section 2 in [Aru17], [Ber17] and page 7 in [RV16] for more on this), and for \(g \equiv 0\), explicit expressions are known for the Mellin transform of the law of \(M_\gamma([0, 1])\) (see e.g. [Ost09], [Ost13], [Ost18]), which show that \(\log M_\gamma([0, 1])\) has an infinitely divisible law, and an explicit formula for sampling the law of the total mass of the GMC on the interval is given in [RZ17].

\(M_\gamma\) is the zero measure for \(\gamma^2 = 2d\) and \(\gamma^2 > 2d\); in these cases a different re-normalization is required to obtain a non-trivial limit. Specifically, for \(\gamma^2 = 2d\), we obtain a non-trivial limit by considering \(\sqrt{\log \frac{1}{\varepsilon}} \cdot M^\varepsilon = \frac{2}{\gamma^2} e^{\gamma X^\varepsilon - \frac{\gamma^2}{2} \text{Var}(X^\varepsilon)}|_{\gamma = \sqrt{2d}}\) [DRSV14] show that both these objects tend weakly to the same measure \(\mu^\varepsilon\) as \(\varepsilon \to 0\), and in 2d Aru et al. [APS19] have shown that \(M^\varepsilon \to 2\mu^\varepsilon\) in probability as \(\gamma\) tends to the critical value of 2, and the critical \(\gamma\)-value is particularly important in Liouville quantum gravity (again see [DRSV14] for further discussion). One can also construct a GMC for the super-critical phase \(\gamma > \sqrt{2}\), using an independent stable subordinator time-changed by a sub-critical GMC (see section 3 in [BJRV14]) to construct an atomic GMC with the correct (locally) multifractal exponent for \(\gamma\)-values greater than \(\sqrt{2}\), which is closely related to the non-standard branch of gravity in conformal field theory.

In the sub-critical case, using a limiting argument it can be shown that \(M_\gamma\) satisfies the “master equations”: \(M(X + f, dz) = e^{\gamma f(z)} M(X, dz)\) and \(\mathbb{E}(\int_D f(X, z) M_\gamma(dz)) = \mathbb{E}(\int_D f(X + \gamma^2 K(z, .), z)dz)\) for any measurable function \(F\) and any interval \(D\), which comes from the Cameron-Martin theorem for Gaussian measures and the notion of rooted measures and the disintegration theorem (see section 2.1 in [Aru17] for a nice discussion on this). Moreover, either of these two can equations can be taken as the definition of GMC, and they uniquely determine \(M_\gamma\) as a measurable function of \(X\), and hence also uniquely fix its law (see also Theorem 6 in Shamov [Sha16]).

GMC also has a natural and important application in Liouville Quantum Field Theory; LQFT is a 2d model of random surfaces, which (formally) we can view as a random metric in the context of quantum gravity, where we weight the classical free field action with an interaction term given by the exponential of a GMC and can be viewed as a toy model to understand in quantum gravity how the interaction with matter influences the geometry of space-time.

## 2 Construction of the standard Bacry-Muzy GMC on the line

Define the Gaussian process \(\omega_\varepsilon(t)\) as in Eq 7 in [BBM13] with \(\lambda = 1\) (except here we use \(\varepsilon\) instead of \(l\)), and set \(X^\varepsilon_t := \omega_\varepsilon(t) - \mathbb{E}(\omega_\varepsilon(t))\), so

\[
X^\varepsilon_t = \int_{(u,s)\in A^\varepsilon(t)} dW(u,s) \tag{3}
\]

where \(dW(u,s)\) is 2-dimensional Gaussian white noise with variance \(s^{-2} du ds\), and \(A^\varepsilon(t)\) is triangular region defined in Eq 8 (and Figure 1) in [BBM13]. Then

\[
R^\varepsilon(s,t) := \mathbb{E}(X^\varepsilon_s X^\varepsilon_t) = \begin{cases} 
\log \frac{T}{\tau} & \varepsilon \leq \tau \leq T \\
\log \frac{T}{\varepsilon} + 1 - \frac{T}{\tau} & \tau \leq \varepsilon \\
0 & \tau > T 
\end{cases} \tag{4}
\]

where \(\tau = |t-s|\) (see Eq 10 in [BBM13]), and one can easily verify that

\[
R^\varepsilon(s,t) \leq \log \frac{T}{\tau} = R(s,t) \tag{5}
\]

for \(s, t \in [0, T]\) (see Eq 25 in [BM03]). Using (3), we also see that

\[
\mathbb{E}(X^\varepsilon_s X^\varepsilon_{s'}) = \mathbb{E}\left(\int_{(u,v)\in A^\varepsilon(s)} dW(u,v) \int_{(u,v)\in A^\varepsilon(s')} dW(u,v)\right) = \int_{A^\varepsilon(s)\cap A^\varepsilon(s')} \frac{1}{2} dv du = \mathbb{E}(X^\varepsilon_s X^\varepsilon_{s'})
\]

for \(0 < \varepsilon' \leq \varepsilon\) (i.e. the answer does not depend on \(\varepsilon'\)). We now define the measure

\[
M^\varepsilon_\gamma(dt) = e^{\gamma X^\varepsilon_t - \frac{\gamma^2}{2} \text{Var}(X^\varepsilon_t)} dt.
\]

One can easily verify that \(M^\varepsilon_\gamma(A)\) is a backwards martingale with respect to the filtration \(\mathcal{F}_\varepsilon := \sigma(W(A, B) : A \subset \mathbb{R}^+, B \subseteq [\varepsilon, \infty])\) (see e.g. subsection 5.1 in [BM03] and page 17 in [RV10]) and

\[
\sup_{\varepsilon>0} \mathbb{E}(M^\varepsilon_\gamma(A)^q) < \infty
\]

for all \(q > 0\).
(Lemma 3 i) in [BM03], so from the martingale convergence theorem, \( M^*_q(A) \) converges to some random variable \( M_q(A) \) in \( L^q \) for \( q \in (1, q^*) \), and from the reverse triangle inequality this implies that

\[
\lim_{\varepsilon \to 0} \mathbb{E}((M^*_q(A))^q) = \mathbb{E}(M_q(A))^q)
\]

Moreover, one can show that \( M_q(\cdot) \) defines measure (see e.g. end of Section 4 on page 18 in [RV10]), and since \( M^*_q(A) \to M_q(A) \) a.s. for any Borel set \( A \) this implies weak convergence of \( M^*_q \) to \( M_q \) a.s. (from e.g. Theorem 3.1 parts a) and f) in Ethier&Kurtz[EK86]).

Moreover \( M_q \) is multifractal, i.e. \( \mathbb{E}([M_q([0,t])]^q) = c_{q,T} T^{\zeta(q)} \) (see e.g. Lemma 4 in [BM03]) for some finite constant \( c_{q,T} > 0 \), depending only on \( q \) and \( T \). For integer \( q \geq 1 \), we also note that

\[
\mathbb{E}(M_q(A)^q) = \int_A \cdots \int_A e^{\sum_{1 \leq i < j \leq q} \log \frac{\|T_{i,j}\|}{c}} du_i \cdots du_q
\]

so we see that

\[
c_{q,T} = c_q T^{\gamma(q-1)}
\]

where \( c_q := c_{q,1} \), and this also holds for non-integer \( q \) (see e.g. Theorem 3.16 in [Koz06]).

### 3 The conditional law of the standard log correlated Gaussian field

Consider a standard log-correlated Gaussian field \( Z \) on \( \mathbb{R} \) with covariance \( R(s, t) = \log^+ \frac{T}{|t-s|} \). From the Minlos-Bochner theorem, we know that the law of \( Z \) is a Gaussian measure on the space \( S' \) of tempered distributions (see e.g. [DRSV17] and Appendix A in [FFGS19] for more details on tempered distributions) which is the dual of the Schwartz space \( S \) (see e.g. section 2.2 in [DRSV14] and Theorem 2.1 in [BDW17]). Moreover, \( S \) is a Montel space and thus is reflexive, i.e. \( (S')' \) is isomorphic to \( S \) using the canonical embedding of \( S \) into its bi-dual \( (S')' \). From here on, we are only concerned with the restriction of \( Z \) to \( [0,T] \) (on which the covariance of \( Z \) is just \( \log^+ \frac{T}{|t-s|} \)), so we set \( Z \) equal to zero outside this interval for simplicity.

**Proposition 3.1** \( X^\varepsilon \) tends to \( X \) in distribution with respect to the strong and weak topology (see page 2 in [BDW17] for definitions), where \( X \) has the same law as the field \( Z \) defined above.

**Proof.** \( 0 \leq R_\varepsilon(s, t) \leq R(s, t) \) for \( s, t \in [0, T] \) (see (5)), so from the dominated convergence theorem, we have

\[
\lim_{\varepsilon \to 0} \int_{[0,T]^2} \phi_1(s) \phi_2(t) R_\varepsilon(s, t) ds dt = \int_{[0,T]^2} \phi_1(s) \phi_2(t) R(s, t) ds dt
\]

for any \( \phi_1, \phi_2 \in \mathcal{S} \), where \( R_\varepsilon(s, t) \) is defined as in (4). Similarly, for any sequence \( \phi_k \in \mathcal{S} \) with \( \|\phi_k\|_{m, j} \to 0 \) for all \( m, j \in \mathbb{N}_0 \) for any \( n \in \mathbb{N} \) (i.e. under the Schwartz space semi-norm defined in Eq 1 in [BDW17])

\[
\lim_{k \to \infty} \int_{[0,T]^2} \phi_k(s) \phi_k(t) R(s, t) ds dt = 0
\]

since \( \nu(A) := \int_A R(s, t) ds dt \) is a bounded non-negative measure (since \( \int_0^T \int_0^T R(s, t) ds dt < \infty \)), and the convergence here implies in particular that \( \phi_k \) tends to \( \phi \) pointwise, so we can use the bounded convergence theorem. Thus if we define

\[
\mathcal{L}_{X^\varepsilon}(f) := \mathbb{E}(e^{f(X^\varepsilon)}) = e^{-\frac{1}{2} \int_{[0,T]^2} f(s) f(t) R_\varepsilon(s, t) ds dt}
\]

\[
\mathcal{L}(f) := e^{-\frac{1}{2} \int_{[0,T]^2} f(s) f(t) R(s, t) ds dt}
\]

for \( f \in \mathcal{S} \), then from (8) and (9) and Lévy’s continuity theorem for generalized random fields in the space of tempered distributions (see Theorem 2.3 and Corollary 2.4 in [BDW17]), we see that \( \mathcal{L}_{X^\varepsilon}(f) \) tends to \( \mathcal{L}(f) \) pointwise and \( \mathcal{L}(\cdot) \) is continuous at zero, then there exists a generalized random field \( X \) (i.e. a random tempered distribution, such that \( L_X = L \) and \( X^\varepsilon \) tends to \( X \) in distribution with respect to the strong and weak topology (see page 2 in [BDW17] for definition). □
In general, the conditional expectation of a random variable is equal to its projection onto the Gaussian Hilbert space (sub-Hilbert space of \(L^2(\Omega, \mathcal{F}, \mathbb{P})\)) generated by the variables on which we are conditioning. To this end, we let \(\tilde{F}\) denote the Hilbert space given by the \(L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})\) closure of
\[
\tilde{F} = \{X(\phi) : \phi \in \mathcal{S}, \text{supp}(\phi) \subseteq [0, L]\}
\]
where \(\mathcal{F}_L = \sigma((X_u)_{0 \leq u \leq L})\). The closure here is necessary because the notion of orthogonal projection requires the Hilbert space structure, and there is no guarantee that the conditional expectation \(\mathbb{E}(X(\psi)|\mathcal{F}_L)\) will be a random variable of the form \(\int_{[0, L]} X_\phi(s)ds\) with \(\phi \in \mathcal{S}\).

In order to characterize \(\tilde{F}\), we first note that
\[
\mathbb{E}\left(\left(\int X_\phi(s)ds\right)^2\right) = \int \int R(s, t)\phi(s)\phi(t)dsdt.
\]
From Eqs 2.1 in [DRV12], we also know that
\[
c\|\phi\|_{H^{-\frac{1}{2}}} \leq \int \int R(s, t)\phi(s)\phi(t)dsdt \leq C\|\phi\|_{H^{-\frac{1}{2}}}
\]
where \(0 < c < C < \infty\). Let \(H^s\) denotes the fractional Sobolev space of order \(s\) (see e.g. Section 2.2 in [JSW18] for definitions). Then we can put two inner products on the linear space \(\mathcal{S}\) of Schwarz functions:
\begin{enumerate}
  \item \(\langle \phi, \psi \rangle_{H^{-\frac{1}{2}}} := \int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}} \hat{\phi}(k)\hat{\psi}(k)dk\) (i.e. the standard inner product on \(H^{-\frac{1}{2}}\))
  \item \(\langle \phi, \psi \rangle := \mathbb{E}[X(\phi)X(\psi)] = \int \int \phi(s)\psi(t)R(s, t)dsdt\)
\end{enumerate}
Eq 2.2 in [DRV12] shows that these two inner products are equivalent and thus generate the same topologies on \(\mathcal{S}\).

We now make the following observations:
\begin{itemize}
  \item Let \(\phi \in H^{-\frac{1}{2}}\), with \(\text{supp}(\phi) \subseteq [0, L]\). \(\mathcal{S}\) is dense in \(H^{-\frac{1}{2}}\), so there exists a sequence \(\phi_n \in \mathcal{S}\) with \(\text{supp}(\phi_n) \subseteq [0, L]\) such that \(\|\phi_n - \phi\|_{H^{-\frac{1}{2}}} \to 0\), and \(\phi\) is a Cauchy sequence in \(H^{-\frac{1}{2}}\) so (by the equivalence of norms) \(X(\phi_n)\) is a Cauchy sequence in \(\tilde{F}\), and thus converges to some \(Y\) in \(\tilde{F}\). This defines \(X(\phi) := Y\) as a continuous linear extension of \(X\) from \(\mathcal{S}\) to the larger space \(H^{-\frac{1}{2}}\), which we will also often write as \(\int \phi(t)X_dt\). To check that \(X(\phi)\) is uniquely specified, consider two such sequences \(\phi_n\) and \(\phi'_n\). Then from the triangle inequality
\[
\|\phi_n - \phi'_n\|_{H^{-\frac{1}{2}}} \leq \|\phi_n - \phi\|_{H^{-\frac{1}{2}}} + \|\phi' - \phi_n\|_{H^{-\frac{1}{2}}} \to 0
\]
and thus (by the equivalence of norms) we have \(\|X(\phi_n) - X(\phi'_n)\|_{L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})} = \|X(\phi_n) - X(\phi'_n)\|_{\tilde{F}} \to 0\).

  \item Conversely, for any \(Z \in \tilde{F}\), there exists a sequence \(\phi_n \in \mathcal{S}\) such that \(X(\phi_n)\) converges to \(Z \in L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})\), so \(\phi_n\) is a Cauchy sequence with respect to the second norm defined above, and hence also a Cauchy sequence with respect to the \(H^{-\frac{1}{2}}\) norm (by the equivalence of the two norms). \(H^{-\frac{1}{2}}\) is a Hilbert space so Cauchy sequences in \(H^{-\frac{1}{2}}\) converge i.e. there exists a \(\phi\) in \(H^{-\frac{1}{2}}\) such that \(\phi_n \to \phi \in H^{-\frac{1}{2}}\).
\end{itemize}

Thus we have shown that
\[
\tilde{F} = \{X(\phi) : \phi \in H^{-\frac{1}{2}}, \text{supp}(\phi) \subseteq [0, L]\}
\]
where we are using the extension of \(X\) to \(H^{-\frac{1}{2}}\) on the right hand side here as defined in the first bullet point above.

Moreover (since \(\mathbb{E}(X(\psi)|\mathcal{F}_L) \in \tilde{F}\)) we see that for any \(\psi \in \mathcal{S}\)
\[
\mathbb{E}(X(\psi)|\mathcal{F}_L) = \int_{[0, L]} X_\psi(s)ds := X(k_\psi)
\]
for some \(k_\psi(s) \in H^{-\frac{1}{2}}([0, L])\), where \(X(.)\) in the final expression is the linear extension we have just defined. This analysis shows that \(\tilde{F}\) is isometrically isomorphic to the set of functions in \(H^{-\frac{1}{2}}\) with support in \([0, L]\).

Moreover, we can now extend the inner product to \(H^{-\frac{1}{2}}\) as
\[
\langle \phi, \psi \rangle = \lim_{n \to \infty} \mathbb{E}[X(\phi_n)X(\psi_n)] = \lim_{n \to \infty} \int \int \phi_n(s)\psi_n(t)R(s, t)dsdt
\]
where \(\phi_n, \phi_n \in \mathcal{S}\) and \(\phi_n \to \phi\) in \(H^{-\frac{1}{2}}\) and \(\psi_n \to \psi\) in \(H^{-\frac{1}{2}}\).
Proposition 3.2 \( X \in H^{-\frac{1}{2} - \delta} \) a.s. for any \( \delta > 0 \).

**Proof.** The proof is almost identical to Proposition 2.1 in [FFGS19], but since some of its arguments are needed for the next Proposition as well, we have put a proof in Appendix A. 

**Remark 3.1** One can actually show the stronger result that \( X \in H^{-\delta} \subset H^{-\frac{1}{2}} \) a.s. for any \( \delta > 0 \), but we will not need this here (see also [BDW17]).

Proposition 3.3 \( X^\varepsilon \to X \) in \( H^{-\frac{1}{2} - \delta} \) in probability for any \( \delta > 0 \), where \( X^\varepsilon \) is defined as in (3).

**Proof.** See Appendix B. 

We know that for any \( \psi \in \mathcal{S} \) with \( \text{supp}(\psi) \subset [L, T] \), the conditional expectation \( \mathbb{E}(X(\psi)|\mathcal{F}_L) = X(k_\psi) \) minimizes

\[
\mathbb{E}((X(\psi) - Y)^2)
\]

over all \( Y \in L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P}) \), and \( \mathbb{E}((\int_{[L,T]} X_t \psi(t)dt - \mathbb{E}(\int_{[L,T]} X_t \psi(t)dt | \mathcal{F}_L))Z) = 0 \) for all \( Z \in \mathcal{F}_L \), so in particular setting \( Z = \int_{[0,L]} \psi_2(s)X_sds \) for \( \psi_2 \in \mathcal{S} \) with \( \text{supp}(\psi_2) \subset [0, L] \), we see that

\[
0 = \mathbb{E}((X(\psi) - X(k_\psi))X(\psi_2)) = \mathbb{E}((\int_{[L,T]} \psi(t)X_tdt - \int_{[0,L]} k_\psi(u)X_udu)\int_{[0,L]} \psi_2(s)X_sds)
\]

\[
= \int_{[L,T]} \int_{[0,L]} \psi(t)\psi_2(s)R(t-s)dsdt - \int_{[L,T]} \int_{[0,L]} R(s-u)k_\psi(u)\psi_2(s)duds . \tag{11}
\]

In (13) below we construct an explicit solution \( k_t(\cdot) \) to

\[
0 = \mathbb{E}((X_t - \int_{[0,L]} k(u)X_udu)X_s) = R(s,t) - \int_{[0,L]} R(u,s)k(u)du \tag{12}
\]

for \( s \in [0, L] \), with \( k_t \in \text{supp}(\psi) \subset [t, T] \), which implies that (11) holds if we set \( k_\psi(u) = \int_{[L,T]} \psi(t)k_t(u)dt \).

Proposition 3.4 The covariance operator \( R\phi = \int_0^T R(s,t)\phi(s)ds \) acting on \( H^{-\frac{1}{2}} \) is positive definite, and \( \int_0^T R(s,t)\phi(s)ds = 0 \) if and only if \( \phi \equiv 0 \) Lebesgue a.e.

**Proof.** From the discussion on page 4, we know that bilinear form \( R \) is (up to an equivalence) the inner product on \( H^{-\frac{1}{2}} \) so it has to be positive definite (from the definition of a norm), and thus \( \int_0^T R(s,t)\phi(s)ds \neq 0 \) if \( \phi \neq 0 \), since otherwise \( R(\phi, \phi) = \int_0^T \int_0^T R(s,t)\phi(s)d\phi(t)dt = 0 \).

The integral equation in (12) (with \( t \) fixed) is the well known Wiener-Hopf equation. We refer the reader to [Poor94] for more details on the Wiener-Hopf equation in the context of ordinary Gaussian processes.

Corollary 3.5 Proposition 3.4 shows that the Wiener-Hopf equation in (12) has a unique solution.

If \( t \leq T \) (so we can replace \( \log^+ \) with \( \log \)), we can re-write (11) as

\[
\int_{[0,L]} k_t(u)\log\frac{T}{s-u}du = f(s) := \log\frac{T}{t-s}
\]

and we see that this is now a Fredholm integral equation of the 1st kind with logarithmic kernel, which can be solved explicitly by a minor extension of page 299 in [EK00] (who consider \( T = 1 \)) to give

\[
k_t(u) = \frac{1}{\pi^2} \int_0^L \frac{\sqrt{v(L-v)}}{u(L-u)} f'(v) \frac{du}{u-v} + \frac{c_t}{\pi \sqrt{u(L-u)}} = \frac{(c_t-1)u + t - c_t t - \sqrt{t(t-L)}}{\pi(u-t)\sqrt{u(L-u)}} \tag{13}
\]

where the integral in the second expression is understood in the principal value sense, and

\[
c_t = \int_0^L k_t(u)du = \frac{1}{\pi(\log(\frac{1}{4}L) - \log T)} \int_0^L \frac{\log\frac{t-v}{T}}{\sqrt{v(L-v)}}dv < \infty.
\]
We now verify that \( k_\psi(u) \in H^{-\frac{1}{2}} \). To this end, we first note that
\[
\pi \log \frac{L}{4} - \pi \log T = \int_0^L \frac{\log \frac{L-v}{T-v}}{\sqrt{v(L-v)}} \, dv \leq \int_0^L \frac{\log \frac{L-v}{T-v}}{\sqrt{v(L-v)}} \, dv \leq \int_0^L \frac{\log \frac{L}{T}}{\sqrt{v(L-v)}} \, dv \leq \pi \log t - \pi \log T.
\]
\[
\frac{(c_1 - 1)u + t - c_1 t - \sqrt{t(T-L)}}{\pi(u-t)\sqrt{u(L-u)}} 1_{u \in [L,T]} 1_{t \in [L,T]} \leq h(u,t) = \frac{c_1}{(t-u)\sqrt{u(L-u)}} 1_{u \in [L,T]} 1_{t \in [L,T]}
\]
for some constant \( c_1 \). We know that
\[
\int_{[L,T]} \left( \int_{[0,L]} |\psi(t)h(u,t)|^p \, du \right)^{\frac{2}{p}} \, dt \leq \|\psi\|_L^\infty \int_{[L,T]} \left( \int_{[0,L]} |h(u,t)|^p \, du \right)^{\frac{2}{p}} \, dt
\]
and setting \( p = \frac{3}{2} \) we find that
\[
\int_{[0,L]} |h(u,t)|^p \, du = G(t) := \text{const.} \times \frac{2t-L}{t(t-L)^{\frac{3}{2}}}
\]
which implies that
\[
\int_{[L,T]} G(t)^{\frac{2}{3}} \, dt < \infty
\]
so for \( p = \frac{3}{2} \) the double integral in (14) is finite, so (from the Minkowski integral inequality) \( \int_{[L,T]} h(\cdot,t) \, dt \) and thus \( \int_{[L,T]} |\psi(t)k_t(\cdot)| \, dt \in L^p \), and hence its Fourier transform is in \( L^q = L^3 \) where \( 1/p + 1/q = 1 \), and thus is \( O(|\xi|^{-\frac{3}{2}}) \) for \( \xi \gg 1 \) and \( O(|\xi|^{-\frac{3}{2} + q}) \) for \( \xi \ll 1 \).

Hence
\[
\|k_\psi\|_{H^{-\frac{1}{2}}} = \int_{-\infty}^{\infty} (1 + |\xi|^2)^{-\frac{1}{2}} \\int_{[L,T]} e^{i\xi u} \psi(t)k_t(u)1_{u \in [0,L]} \, dt \, du \, d\xi
\]
\[
= \int_{-\infty}^{\infty} (1 + |\xi|^2)^{-\frac{1}{2}} \\int_{[0,L]} e^{i\xi u} \psi(t)k_t(u) \, du \, d\xi < \infty
\]
which verifies the validity of our explicit solution for \( k_u(t) \).

**Remark 3.2** Corollary 3.3 in [DRV12] gives the following nice prediction formula for a log-correlated Gaussian field \( X \) with covariance \( \log \frac{t}{\pi} \): \(^3\)
\[
\mathbb{E}(X_t | (X_s)_{s \leq 0}) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\sqrt{t}}{(t-s)\sqrt{-s}} X_s \, ds
\]
which we can verify satisfies the associated Wiener-Hopf equation (and is also very similar to the prediction formula for the Riemann-Liouville process in Proposition 2.9 in [FSV19] in the \( H \to 0 \) limit). However the prediction formula for the finite history case stated in Theorem 3.5 in [DRV12] appears to be wrong since numerical tests confirm that it does not satisfy the Wiener-Hopf equation. Our linear filter \( \int_{[0,L]} k_t(u)X_u \, du \) corrects this formula for the case when \( L + t \leq T \).

**Remark 3.3** Clearly if \( t - L > T \), the history of \( X \) over \([0,L]\) is of no use for prediction since in this case \( \mathbb{E}(X_sX_t) = 0 \) for \( s \in [0,L] \), and the conditioned process then has the same law as the unconditioned process.

### 3.1 The conditional covariance

We use \( \mathbb{E}_L(\cdot) \) as shorthand for \( \mathbb{E}(\cdot | (X_u)_{0 \leq u \leq L}) \). Then from the tower property we see that
\[
\mathbb{E}(X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s)) = \mathbb{E}_L((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s))) = \mathbb{E}_L((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s)))
\]

\(^3\)i.e. \( \log \) not \( \log^+ \)
and the final equality follows since the conditional covariance of a Gaussian process or field is deterministic, and does not depend on its history. Given \( k_t(u) \), we can now compute the conditional covariance in the final line explicitly (for \( s, t \in [L,T] \)) as

\[
R_L(s,t) := \mathbb{E}_L((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s))) = \mathbb{E}((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s)))
= \mathbb{E}((X_t - \int_{[0,L]} k_t(u)X_u du)(X_s - \int_{[0,L]} k_s(v)X_v dv))
= R(s,t) - \int_{[0,L]} k_t(u)R(u,s) du - \int_{[0,L]} k_s(v)R(v,t) dv + \int_{[0,L]} \int_{[0,L]} k_t(u)k_s(v)R(u,v)du dv .
\]

4 Application to Gaussian multiplicative chaos

4.1 Rooted measures

**Proposition 4.1** (see also Lemma 2.1 in [Aru17] and Theorems 4 and 17 in [Sha16]). We have the following “master equation” for any bounded continuous function \( F \) on \( H^{-\frac{1}{2} - \delta} \times [0,T] \) (under the product topology induced by the Hilbert space norm on \( H^{-\frac{1}{2} - \delta} \) and the usual Euclidean metric on \( [0,T] \)):

\[
\frac{1}{T} \mathbb{E} \left( \int_0^T F(X,t)M_\gamma(dt) \right) = \frac{1}{T} \mathbb{E} \left( \int_0^T F(X + \gamma R(t,\cdot),t)dt \right) .
\]

**Proof.** See Appendix C. \( \square \)

**Corollary 4.2** \( M_\gamma \) is measurable with respect to \( X \).

**Proof.** \( \mathcal{H} = H^{-\frac{1}{2} - \delta} \times [0,T] \) is a metric space, so if \( \mu \) and \( \nu \) are two finite Borel measures on \( \mathcal{H} \) then \( \int fd\mu = \int fd\nu \) for all \( f \in C_b(\mathcal{H}) \) means that \( \mu = \nu \), so the left hand side of (15) uniquely defines a measure \( \mathbb{P}^* \) on \( \mathcal{H} \times [0,D] \) which satisfies

\[
\frac{1}{T} \mathbb{E} \left( \int_0^T F(X,t)M_\gamma(dt) \right) = \int \int F(\omega,t)\mathbb{P}^*(d\omega,dt)
\]

where

\[
\mathbb{P}^*(d\omega,dt) := \frac{1}{T} \mathbb{E} (1_{X \in d\omega} M_\gamma(\omega,dt)) = \frac{1}{T} \mathbb{E} (1_{X + \gamma R(\cdot,\cdot) \in d\omega}) dt
\]

where \( Q^X \) denotes the law of \( X \) on \( H^{-\frac{1}{2} - \delta} \).

Moreover, if \( F \equiv 1 \), \( \frac{1}{T} \mathbb{E} (\int_0^T F(X,t)M_\gamma(dt)) = 1 \), so \( \mathbb{P}^*(d\omega,dt) \) is a probability measure, known as a rooted or Peglière measure (see [Aru17] and [Sha16] for more on this). Moreover, using a similar argument to the third bullet point in Appendix C, we know that the conditional law of \( \mathbb{P}^* \) given \( X \) is \( M_\gamma(dt)/M_\gamma([0,T]) \) and from the disintegration theorem, we know that this (probability) measure is a measurable with respect to \( R^\gamma \). Then using a similar argument to the second bullet point in Appendix C, if we take the sample space \( \Omega \) to be \( H^{-\frac{1}{2} - \delta} \) with \( \sigma \)-algebra \( \sigma(H^{-\frac{1}{2} - \delta}) \), then the “tilted” probability measure \( Q^X_\gamma(d\omega) := \frac{1}{T} M_\gamma([0,T])Q^X(\omega,dt) \) on \( (\Omega,\mathcal{F}) \) is the marginal law of \( \mathbb{P}^* \) on \( H^{-\frac{1}{2} - \delta} \) (where \( Q^X \) is the law of \( X \) on \( H^{-\frac{1}{2} - \delta} \) and \( Q^X_\gamma \ll Q^X \), so \( \frac{1}{\gamma} M_\gamma([0,T](\omega)) \) is the (a.s.) unique Radon-Nikodym derivative of \( Q^X_\gamma \) with respect to \( Q^X \), which is a measurable function of \( \omega \). Thus we have shown that \( M_\gamma(dt)/M_\gamma([0,T]) \) and \( M_\gamma([0,T]) \) are measurable w.r.t \( X \) and thus so is \( M_\gamma \). \( \square \)

4.2 The conditional law of \( M_\gamma \)

From the Corollary above, \( M_\gamma(dt) \) is a measurable w.r.t \( X \), so \( M_\gamma \) given \( (X)_{0 \leq s \leq L} \) is just obtained as

\[
M_\gamma((X(0 \leq s \leq L) \oplus X', dt)
\]

where \( \oplus \) denotes concatenation, and \( X' \) is a Gaussian field (which is also a random element of \( S' \)) on \([L,T]\) with mean \( \mathbb{E}_L(X_L) \) and covariance \( R_L(s,t) \). This then uniquely specifies the law of \( M_\gamma \) conditioned on its history over \([0,L]\).}

\footnote{We thank Juhan Aru for his help with multiple parts of this proof.
Figure 1: Here we have plotted a Monte Carlo simulation of the multifractal random measure $M_t(dt)$ on $[0,1]$ with $\gamma = 0.20$, 0.45 and 1 using the regularized autocovariance $\log^+ \frac{r}{|r|+\varepsilon}$ for $\varepsilon = .000001$, and we see greater intermittency as $\gamma$ increases.

Figure 2: In the first three graphs we have plotted the optimal linear filter $k(u)$ in (13) associated with the multifractal random walk with $L = 1$, $T = 2$ for $t = 2, 1.5$ and 1.00001 respectively, and the numerics confirm that the Wiener-Hopf equation is satisfied (Mathematica code available on request), and $k(u)$ is U-shaped and strictly positive for all $u \in [0, L]$ for $t$ sufficiently small.
4.3 Conditional law of the Riemann-Liouville field

Formally letting $H \to 0$ in the prediction formula for the Riemann-Liouville process in Proposition 2.9 in [FSV19] in the $H \to 0$ limit, we obtain the following conditional law for the Riemann-Liouville field $Z$ defined in section 2 in [FFGS19]:

**Proposition 4.3** $Z$ has conditional mean and covariance given by

\[
E(Z_u|(Z_v)_{0 \leq v \leq t}) = \int_0^t \tilde{k}(s)Z_s ds
\]

\[
\text{Cov}(Z_s,Z_u|(Z_v)_{0 \leq v \leq t}) = \int_t^{s \wedge u} (u - v)^{-\frac{1}{2}}(s - v)^{-\frac{1}{2}} dv
\]

for $u \geq t$, where $\tilde{k}(s) = \frac{1}{\pi} \left( \frac{u - t}{s - t} \right)^{\frac{1}{2}} \frac{1}{u - s}$.

**Remark 4.1** This is essentially the same type of linear filter that we have obtained in section 3 for the Bacry-Muzy field. To make this rigorous, we can consider $Y_t = e^{z_t}$; then one can verify that $Y$ is a strictly stationary Gaussian field with covariance $R_Y(s,t) = R(\tau) = 2\tanh^{-1}(e^{-\frac{1}{2}\tau})$ where $\tau = t - s$, and from Parseval’s theorem (similar to Eq 2.1 in [DRV12]) we obtain

\[
\int \int \phi(t)\phi(s)R_Y(s,t)dsdt = \int \dot{R}(k)|\hat{\phi}(k)|^2 dk
\]

where $\dot{R}(k) = \frac{-iH_{-\frac{1}{2}-ik}+iH_{-\frac{1}{2}+ik}+2\pi \tanh(k\pi)}{k\sqrt{2\pi}}$ and $H_n$ denotes the $n$th harmonic number. Then $\dot{R}(|k|)$ is continuous, strictly positive and decreasing with $\dot{R}(0) < \infty$ and $\dot{R}(|k|) \sim \frac{\sqrt{\pi}}{|k|\sqrt{2}} \sim \text{const.} \times (1 + |k|^2)^{-\frac{1}{2}}$ as $|k| \to \infty$. Hence (10) still holds with $R$ replaced by $R_Y$ and we can then repeat our previous arguments to make (17) rigorous (after transforming back from $Y$ to $Z$). In [FFGS19] we define the GMC associated with $Z$ (which we call $\xi_1$) and one can show that $\xi_1$ is also measurable with respect to $Z$ so (16) still holds with $M_1$ replaced by $\xi_1$ and $X$ replaced by $Z$.

**References**


A Proof of Proposition 3.2

\[ \mathbb{E}(\|X\|_{H^{-\frac{1}{2}-\delta}}^2) = \mathbb{E}\left(\int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} |\hat{X}_k|^2 dk \right) \]
\[ = \mathbb{E}\left(\int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} \hat{X}_k \bar{\hat{X}}_k dk \right) \]
\[ = \mathbb{E}\left(\int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} \int_0^T e^{ikt} X_t dt \int_0^T e^{-iks} X_s ds dk \right) \]
\[ = \mathbb{E}\left(\int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} \int_0^T \int_0^T e^{ik(t-s)} X_s X_t ds dt dk \right) \]
\[ = \int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} \int_0^T \int_0^T e^{ik(t-s)} R(s, t) ds dtdk \]

Using that \( R \in L^1([0, T]^2) \), we see that \( \int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} \int_0^T \int_0^T \mathbb{E}(X_s X_t) ds dt dk = \int_0^T \int_0^T R(s, t) ds dt \).

\( \int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} dk < \infty \) if \( \delta > 0 \), so by Fubini we have

\[ \mathbb{E}(\|Z\|_{H^{-\frac{1}{2}-\delta}}^2) = \mathbb{E}\left(\int_0^T \int_0^T R(s, t) \int_{-\infty}^{\infty} e^{ik(t-s)} (1 + |k|^2)^{-\frac{1}{2}-\delta} dk ds dt \right) \]
\[ = 2c_\delta \int_0^T \int_0^T R(s, t)(t-s)^\delta \text{BesselK}(\delta, t-s) ds dt \]
\[ < c_\delta \int_{[0, T]^2} R(s, t) ds dt < \infty \quad \text{(A-1)} \]

where we have used that the Fourier transform of \( \hat{f} (k) := (1 + |k|^2)^{-\frac{1}{2}-\delta} \) is \( f(t) = c_\delta |t|^\delta \text{BesselK}(\delta, |t|) \) for some real constant \( c_\delta \), and that \( t^\delta \text{BesselK}(\delta, t) \) is bounded on \([0, T]\) if \( \delta > 0 \). For \( \delta \leq 0 \), the integrand in the triple integral in the first line is not absolutely integrable.

B Proof of Proposition 3.3

Using that

\[ \chi(s, t, \varepsilon, \varepsilon_2) := \mathbb{E}((X_\varepsilon^{s_2} - X_t^{s_2})(X_\varepsilon^{s_2} - X_{s_2}^{s_2})) = R_{\varepsilon_2}(s, t) - \mathbb{E}(X_\varepsilon^{s_2} X_t^{s_2}) - \mathbb{E}(X_\varepsilon^{s_2} X_{s_2}^{s_2}) + R_{\varepsilon_2}(s, t) \rightarrow 0 \]
\[ = R_{\varepsilon_2}(s, t) - \mathbb{E}(X_\varepsilon^{s_2} X_t^{s_2}) - \mathbb{E}(X_\varepsilon^{s_2} X_{s_2}^{s_2}) + R_{\varepsilon_2}(s, t) \]

as \( \varepsilon, \varepsilon_2 \to 0 \) and that \( |\chi(s, t, \varepsilon, \varepsilon_2)| \leq 4R(s, t) \), we can use a similar argument to (A-1) and the dominated convergence theorem to show that

\[ \mathbb{E}(\|X_\varepsilon^{s_2} - X_\varepsilon^{s_2}\|^2_{H^{-\frac{1}{2}-\delta}}) \leq c_\varepsilon \int_{[0, T]^2} \chi(s, t, \varepsilon, \varepsilon_2) ds dt \rightarrow 0 \quad \text{(B-1)} \]

as \( \varepsilon, \varepsilon_2 \to 0 \), so \( X_\varepsilon \) is a Cauchy sequence in the Hilbert space \( L^2(\Omega, \mathcal{F}, \mathbb{P}; H^{-\frac{1}{2}-\delta}) \) of \( H^{-\frac{1}{2}-\delta} \)-valued random variables \( X \) with \( \mathbb{E}(\|X\|_{H^{-\frac{1}{2}-\delta}}^2) < \infty \), and thus converges in this space. Using that

\[ \mathbb{P}(\|X_\varepsilon^{s_2} - X_\varepsilon^{s_2}\|_{H^{-\frac{1}{2}-\delta}} > k) \leq \frac{1}{k^2} \mathbb{E}(\|Z_\varepsilon^{s_2} - X_\varepsilon^{s_2}\|^2_{H^{-\frac{1}{2}-\delta}}) \]

the claim is proved.

C Proof of Proposition 4.1

Similar to the analysis before Lemma 2.1 in [Aru17] with rooted measures, we let \( D = [0, T] \) and we can define a sequence of approximate “rooted” probability measures \( \mathbb{P}_\varepsilon^r \) on \( D \times H^{-\frac{1}{2}-\delta} \) as

\[ \mathbb{P}_\varepsilon^r(dt, d\omega) = \frac{dt}{\text{Leb}(D)} e^{\gamma_\omega(t)-\frac{1}{2} \gamma_2^2 \mathbb{E}(X_\varepsilon^{s_2})} \mathbb{Q}^X(\omega) \]

where \( \mathbb{Q}^X \) denotes the law of \( X_\varepsilon \) on \( H^{-\frac{1}{2}-\delta} \), and \( X_\varepsilon \) is defined as in (3). Then
• The marginal law on $D$ is
  \[
  \frac{dt}{\text{Leb}(D)} \mathbb{E}^{Q^x}(e^{\gamma_{\omega}(t)} - \frac{1}{2} \gamma^2 \mathbb{E}(X^x)) = \frac{dt}{\text{Leb}(D)}
  \]
i.e. the uniform probability measure on $D$.

• The marginal law on $H^{-\frac{1}{2}-\delta}$ is
  \[
  \frac{\int_0^\infty e^{\gamma_{\omega}(t)} - \frac{1}{2} \gamma^2 \mathbb{E}(X^x) dt}{\text{Leb}(D)} \mathbb{Q}^x(d\omega) = M^x(D) \mathbb{Q}^x(d\omega),
  \]
i.e. the law of $X^x$ tilted by $M^x(D)/\text{Leb}(D)$.

• The conditional law on $D$ given $\omega$ is the probability measure:
  \[
  \frac{e^{\gamma_{\omega}(t)} - \frac{1}{2} \gamma^2 \mathbb{E}(X^x)}{M^x(D)} dt = \frac{M^x(dt)}{M^x(D)}.
  \]

• The conditional law on $H^{-\frac{1}{2}-\delta}$ given $t$ is $e^{\gamma_{\omega}(t)} - \frac{1}{2} \gamma^2 \mathbb{E}(X^x) \mathbb{Q}^x(d\omega)$. From Girsanov’s theorem (see e.g. section 6.1 in [Var17]), we can re-write this as
  \[
  \mathbb{Q}(X^x + \gamma R_{\varepsilon}(., t) \in d\omega)
  \]  
  (C-1)

Thus we can sample from $\mathbb{P}_x^*$ by either (i) sampling from $\frac{M^x(D)}{\text{Leb}(D)} \mathbb{Q}^x(d\omega)$ and then sampling a point according to $M^x(dt)/M^x(D)$, or ii) sampling $t$ from the uniform measure on $[0,T]$, and then sampling $X^x + \gamma R_{\varepsilon}(., t)$, with $X^x$ independent of $t$. Combining these two prescriptions, we see that

\[
\mathbb{E} \left( \frac{M^x(D)}{\text{Leb}(D)} \int_0^T F(X^x, t) M^x(dt) \right) = \frac{1}{\text{Leb}(D)} \mathbb{E} \left( \int_0^T F(X^x + \gamma R_{\varepsilon}(t, .), t) dt \right)
\]

which we can re-write as

\[
\mathbb{E} \left( \int_0^T F(X^x, t) M^x(dt) \right) = \mathbb{E} \left( \int_0^T F(X^x + \gamma R_{\varepsilon}(t, .), t) dt \right).
\]

(C-2)

We first consider the left hand side of this expression as $\varepsilon \to 0$. To begin with, we note that

\[
\begin{align*}
|\mathbb{E} \left( \int_0^T F(X^x, t) M^x(dt) - \int_0^T F(X, t) M^x(dt) \right)| \\
\leq \left| \mathbb{E} \left( \int_0^T (F(X^x, t) - F(X, t)) M^x(dt) \right) \right| + \left| \mathbb{E} \left( \int_0^T F(X, t) (M^x(dt) - M^x(dt)) \right) \right|
\end{align*}
\]

and we can bound the first term in the final expression using Hölder’s inequality as

\[
\mathbb{E} \left( \int_0^T (F(X^x, t) - F(X, t)) M^x(dt) \right) \leq \mathbb{E} \left( \sup_{t \in [0, T]} |F(X^x, t) - F(X, t)| \cdot M^x([0, T]) \right) \\
\leq \mathbb{E} \left( \left( \sup_{t \in [0, T]} |F(X^x, t) - F(X, t)|^p \right)^{\frac{1}{p}} \right) \cdot \mathbb{E} \left( \left( M^x([0, T])^q \right)^{\frac{1}{q}} \right)
\]

for $1/p + 1/q = 1$, and from (2) we know that

\[
\mathbb{E}(M^x([0, T])^q) = c_q T^{s(q)} < \infty
\]

for any $q \in (1, q^*) = \left[ \frac{2}{r}, \frac{2}{s} \right]$. We claim that $\sup_{t \in [0, T]} |F(X^x, t) - F(X, t)| \to 0$ a.s. Indeed, suppose to the contrary. Let $f_x(t) := F(X^x, t)$ and $f(t) := F(X, t)$. If the claim is false, $f_x$ does not tend to $f$ uniformly on $[0, T]$, so there exists a sequence $\varepsilon_n \to 0$, a $\delta > 0$ and a sequence $t_n \in [0, T]$ such that

\[
|f_{\varepsilon_n}(t_n) - f(t_n)| \geq \delta
\]

for all $n \in \mathbb{N}$. But by Bolzano-Weierstrass, we can choose a convergent subsequence $(t_{n_k})$ of $(t_n)$ with $t_{n_k} \to t_\infty \in [0, T]$. Then $f_{\varepsilon_{n_k}}(t_{n_k}) = F(X^{\varepsilon_{n_k}}, t_{n_k})$ and $f(t_{n_k}) = F(X, t_{n_k})$. From Proposition 3.3 we know that $X^x$ tends to $X$ in $H^{-\frac{1}{2}-\delta}$ in probability, and thus almost surely along a further subsequence $\varepsilon_{n_k}$, thus (by continuity of $F$ in both arguments) $F(X^{\varepsilon_{n_k}}, t_{n_k}) \to F(X, t_\infty)$ a.s. and hence

\[
|F(X^{\varepsilon_{n_k}}, t_{n_k}) - F(X, t_{n_k})| = |f_{\varepsilon_{n_k}}(t_{n_k}) - f(t_{n_k})| \to 0
\]

a.s., which violates (C-5). Hence the right hand side of (C-4) tends to zero (along any subsequence) for $q \in (1, q^*)$.
The term $\int_0^T F(X,t)(M^\xi(dt) - M_t(dt))$ inside the expectation on the right hand side of (C-3) converges to zero a.s. since $M^\xi$ tends weakly to $M_t$ a.s. (see top of page 3 for details) and the random $F(X,t)$ is continuous in $t$ for each $\omega$. Moreover

$$\int_0^T F(X,t)(M^\xi(dt) - M_t(dt)) \leq \|F\|_\infty(M^\xi([0,T]) + M_t([0,T])).$$

From (6) we also know that $M^\xi([0,T])$ is uniformly integrable, so by e.g. the Theorem in section 13.7 in [Wil91], the rightmost term of (C-3) tends to zero.

Finally, the right hand side of (C-2) converges by the a.s. convergence of $X^\xi$ to $X$ in Proposition 3.3 and the bounded convergence theorem.