Large deviations for the boundary local time of doubly reflected Brownian motion

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Abstract
We compute a closed-form expression for the moment generating function \( \hat{f}(x; \lambda, \alpha) = \frac{1}{\lambda} \mathbb{E}_x(e^{\alpha L_t}) \), where \( L_t \) is the local time at zero for standard Brownian motion with reflecting barriers at 0 and \( b \), and \( \tau \sim \text{Exp}(\lambda) \) is independent of \( W \). By analyzing how and where \( \hat{f}(x; \cdot, \alpha) \) blows up in \( \lambda \), a large-time large deviation principle (LDP) for \( L_t/t \) is established using a Tauberian result and the Gärtner-Ellis Theorem.

Keywords: Brownian motion, Large deviation, Local time.

1. Introduction

Diffusion processes with reflecting barriers have found many applications in finance, economics, biology, queueing theory, and electrical engineering. In a financial context, we recall the currency exchange rate target-zone models in [KRU91] (see also [SVE91, BER92, DJ94], and [BAL98]), where the exchange rate is allowed the float within two barriers; asset pricing models with price caps (see [HAN99]); interest rate models with targeting by the monetary authority (e.g.[FAR03]); short rate models with reflection at zero (e.g. [GOL97, GOR04]); and stochastic volatility models (most notably the Heston and Schöbel-Zhu models). In queueing theory, diffusions with reflecting barriers arise as heavy-traffic approximations of queueing systems and reflected Brownian motions is ubiquitous in queueing models [HAR85, ABA87a, ABA87b]. More recently, reflected Ornstein-Uhlenbeck(OU) and reflected affine processes have been studied as approximations of queueing systems with reneging or balking [WAR03a, WAR03b]. Applications of reflected OU processes in mathematical biology are discussed in [RIC87]. Doubly reflected Brownian motion also arises naturally in the solution for the optimal trading strategy in the large-time limit for an investor who is permitted to trade a safe and a risky asset under the Black-Scholes model, subject to proportional transaction costs with exponential or power utility (see [GM13] and [GGMS12] respectively).

The asymptotics in this article are obtained using a Tauberian theorem. Tauberian results typically allow us to deduce the large-time or tail behavior of a quantity of interest based on the behavior of its Laplace transform (see Feller[FE71] or the excellent monograph of Bingham et al.[BGT87] for details or [BF08] for applications to tail asymptotics for time-changed exponential Lévy models). In this article, we compute a closed-form expression for the moment generating function (mgf) \( \hat{f}(x; \lambda, \alpha) = \frac{1}{\lambda} \mathbb{E}_x(e^{\alpha L_t}) \), where \( L_t \) is the local time at zero for standard Brownian motion with reflecting barriers at 0 and \( b \), and \( \tau \) is an independent exponential random variable with parameter \( \lambda \). We do this by first deriving the relevant ODE and boundary conditions for \( \hat{f}(x; \lambda, \alpha) \) using an augmented filtration and computing the optional projection, and we then solve this ODE in closed form. \( \hat{f}(x; \lambda, \alpha) \) does not appear amenable to Laplace inversion; however from an analysis of the location of the pole of \( \hat{f}(x; \cdot, \alpha) \), we can compute the re-scaled log mgf limit \( V(\alpha) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_x(e^{\alpha L_t}) \) for \( \alpha \in \mathbb{R} \) using the Tauberian result in Proposition 4.3 in [KOR02] via the so-called Fejér kernel. From this we then establish a large deviation principle for \( L_t/t \) as \( t \to \infty \) using the Gärtner-Ellis Theorem from large deviations theory.

Throughout the paper, we let \( P_x(\cdot) = P(\cdot|X_0 = x) \) denote the law of \( X \) given its initial value at time 0 for any \( x \in [0, b] \), and by \( \mathbb{E}_x(\cdot) \) the expectation under \( P_x \). Further, we let \( \mathbb{E} \equiv \mathbb{E}_0 \).

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2. The modelling set up

We begin by defining the Brownian motion \( X \) with two reflecting boundaries. Let \( W_t \) be standard Brownian motion starting at 0. Then for any \( x \in [0, b] \), there is a unique pair of non-decreasing, continuous adapted processes \((L, U)\), starting at 0, such that

\[
X_t = x + W_t + L_t - U_t \quad \in \quad [0, b], \quad \forall t \geq 0.
\]

such that \( L \) can only increase when \( X = 0 \) and \( U \) can only increase when \( X = b \). Existence and uniqueness follow easily from the more general work of Lions&Sznitman [LS84] the earlier work of Skorokhod [SKO62], or a bare-hands proof can be given by successive applications of the standard one-sided reflection mapping using a sequence of stopping times (see [WIL92].)

It can be shown that

\[
\lim_{t \to \infty} L_t/t = \mathbb{E}(L_{\tau^b+\tau^U})/\mathbb{E}(\tau^b + \tau^U), \quad \text{and} \quad \lim_{t \to \infty} U_t/t = \mathbb{E}(U_{\tau^b+\tau^U})/\mathbb{E}(\tau^b + \tau^U),
\]

\[
\lim_{t \to \infty} \frac{1}{t} \text{Var}(L_t) = \sigma_L^2, \quad \lim_{t \to \infty} \frac{1}{t} \text{Var}(U_t) = \sigma_U^2,
\]

where \( \tau^b = \inf\{ t : X_t = b \} \), \( \tau^U = \inf\{ t \geq \tau^b : X_t = 0 \} \) (see [WIL92]) for some non-negative constants \( \sigma_L, \sigma_U \).

**Proposition 2.1.** Let \( \tau \) denote an independent exponential random variable with parameter \( \lambda \). Then for \( \alpha < 0 \),

\[
\hat{f}(x) \equiv \hat{f}(x; \lambda, \alpha) := \frac{1}{\lambda} \mathbb{E}_x(e^{\alpha L_{\tau}}) = \int_0^\infty e^{-\lambda t} \mathbb{E}_x(e^{\alpha L_t}) dt
\]

is smooth on \((0, b)\) and satisfies the following ODE

\[
\frac{1}{2} \hat{f}_{xx} = \lambda \hat{f} - 1, \quad \hat{f}_x(0) + \alpha \hat{f}(0) = \hat{f}_x(b) = 0.
\]

**Proof.** We first show that \( \hat{f} \in C^\infty(0, b) \). To this end, note that for \( x \in [0, b] \),

\[
\mathbb{E}_x(e^{\alpha L_{\tau}}) = \mathbb{P}_x(\tau > H_0) \mathbb{E}_0(e^{\alpha L_{\tau}}) + \mathbb{P}_x(\tau \leq H_0)
\]

where \( H_x = \inf\{ t : X_t = x \} \) is the first hitting time to \( x \). The law of \((b - X_t; t \in [0, H_0])\) given \( X_t = x \) is the same as that of \((|W_t|; t \in [0, H_0])\) given \(|W_0| = b - x \). Thus by Eq. 2.0.1 on page 355 of [BS02] we have

\[
\mathbb{P}_x(\tau > H_0) = \mathbb{E}_x(e^{-\lambda H_0}) = \frac{\cosh((b - x)\sqrt{2\lambda})}{\cosh(\sqrt{2\lambda})}.
\]

It follows that

\[
\mathbb{E}_x(e^{\alpha L_{\tau}}) = \frac{\cosh((b - x)\sqrt{2\lambda})}{\cosh(\sqrt{2\lambda})} \left[ \mathbb{E}_0(e^{\alpha L_{\tau}}) - 1 \right] + 1.
\]

That is,

\[
\hat{f}(x) = \frac{\cosh((b - x)\sqrt{2\lambda})}{\cosh(\sqrt{2\lambda})} (\hat{f}(0) - \frac{1}{\lambda}) + \frac{1}{\lambda}, \quad \forall x \in [0, b].
\]  

It can then be easily seen from (2) that \( \hat{f} \in C^\infty(0, b) \).

To show that \( \hat{f} \) satisfies (1) and the boundary conditions, we construct a martingale that is adapted to the filtration generated by \( X \). More specifically, we introduce the natural filtration \( \mathcal{F}_t = \sigma(X_s; s \leq t) \) and the augmented filtration \( \mathcal{F}_t = \mathcal{F}_t \vee \sigma(1_{\{t < \tau\}}) \), where \( \sigma(1_{\{t < \tau\}}) \) is the sigma algebra generated by \( 1_{\{t < \tau\}} \). Then we have a uniformly bounded, and hence uniformly integrable \( \mathcal{F}_t \)-martingale:

\[
\overline{M}_t := \mathbb{E}(e^{\alpha L_{\tau}}|\mathcal{F}_t) = 1_{\{t < \tau\}}e^{\alpha L_t} + 1_{\{t \geq \tau\}}e^{\alpha L_t}X_t(e^{\alpha L_{\tau}}) = 1_{\{t < \tau\}}e^{\alpha L_t} + 1_{\{t \geq \tau\}}e^{\alpha L_t} \lambda \hat{f}(X_t).
\]

We now define the optional projection of \( \overline{M}_t \): using the fact that \( X \) and \( \tau \) are independent, we have

\[
M_t = \mathbb{E}(\overline{M}_t|\mathcal{F}_t) = \lambda \int_0^t e^{\alpha L_s - \lambda s} ds + e^{\alpha L_t - \lambda t} \lambda \hat{f}(X_t).
\]
Further, \( M_t \) is a \( \mathcal{F}_t \)-martingale, in that for all \( t > s \) we have
\[
\mathbb{E}(M_t|\mathcal{F}_s) = \mathbb{E}(\mathbb{E}(M_t|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}(\mathbb{E}(M_t|\mathcal{F}_s)|\mathcal{F}_s) = \mathbb{E}(M_s|\mathcal{F}_s) = M_s.
\]

Applying Itô’s lemma to \( M_t \), we have that
\[
dM_t = e^{\alpha L_t - \lambda t} \left[ \lambda dt + \lambda \dot{f}(X_t)(\alpha dL_t - \lambda dt) + \frac{1}{2} \lambda f_{xx}(X_t) dt + \lambda f_x(X_t)(dW_t + dL_t - dU_t) \right].
\]
But for \( M_t \) to be a martingale, we must have
\[
\frac{1}{2} f_{xx}(x) - \lambda \dot{f}(x) + 1 = 0, \quad \dot{f}(0) + \alpha \dot{f}(0) = 0, \quad \dot{f}(b) = 0.
\]
This completes the proof. \( \square \)

Solving the ODE in Proposition 2.1 we obtain the following result:

**Proposition 2.2.**
\[
\dot{f}(x; \lambda, \alpha) = \frac{1}{\lambda} + e^{x\sqrt{2\lambda}} A_\lambda(\alpha) + e^{-x\sqrt{2\lambda}} B_\lambda(\alpha)
\]
for \( \lambda > 0, \alpha < 0 \), where
\[
A_\lambda(\alpha) = \frac{\alpha e^{-b\sqrt{2\lambda}} / \cosh(b\sqrt{2\lambda})}{2\lambda |\alpha^*(\lambda) - \alpha|}, \quad B_\lambda(\alpha) = e^{2\sqrt{2\lambda} b} A_\lambda(\alpha), \quad \alpha^*(\lambda) = \sqrt{2\lambda} \tanh(b\sqrt{2\lambda}).
\]

**Remark 2.3.** Observe that the expression for \( \dot{f}(x) \) involves \( \sqrt{\lambda} \), which has a branch point at \( \lambda = 0 \). However, \( \dot{f} \) remains a continuous function across the branch cut at \( \lambda = 0 \); thus \( \dot{f} \) is an analytic function of \( \lambda \) in some punctured disc about \( \lambda = 0 \). As \( \lim_{\lambda \to 0} \lambda \cdot \dot{f}(\lambda) = 0 \), we conclude that \( \lambda = 0 \) is a removable singularity.

**Remark 2.4.** It can be verified that \( \alpha^*(\cdot) \) in (4) is a strictly increasing mapping from \([0, \infty)\) onto \([0, \infty)\). Further, we may analytically extend \( \alpha^*(\cdot) \) to get a strictly increasing, strictly concave, smooth real-valued function that maps \((\frac{\pi^2}{8b^2}, \infty)\) onto \( \mathbb{R} \).

### 3. Large-time asymptotics

In this section, we characterize the large-time behaviour of \( L_t \). To this end, let us consider the inverse of \( \alpha^* \), \( V(\alpha) := (\alpha^*)^{-1}(\alpha) \) for \( \alpha \in \mathbb{R} \). From Remark 2.4, we know that \( V \) is a strictly increasing, strictly convex smooth function, with range \((\frac{\pi^2}{8b^2}, \infty)\).

**Lemma 3.1.** The equality (3) also holds for all \( \alpha \in \mathbb{R}, \lambda \in \mathbb{C} \) such that \( \Re(\lambda) > V(\alpha) \).

*Proof.* See Appendix A. \( \square \)

**Proposition 3.2.** We have the following large-time behaviour for the moment generating function of \( L_t \):
\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_x(e^{\alpha L_t}) = V(\alpha) < \infty \quad \forall \alpha \in \mathbb{R}.
\]

*Proof.* See Appendix B. \( \square \)

**Remark 3.3.** Note that \( V(\cdot) \) does not depend on the starting value \( x \), due to the ergodicity of \( X \).

The following lemma will be needed in the statement of the large deviation principle in the theorem that follows.

**Lemma 3.4.** (a) Define \( V^*(x) := \sup_{\alpha \in \mathbb{R}} [\alpha x - V(\alpha)] \) for all \( x \geq 0 \). Then we have
\[
V^*(x) = \begin{cases} 
\alpha^*(\lambda^*) - \lambda^* & \text{for } x > 0 \\
\pi^2/(8b^2) & \text{for } x = 0
\end{cases}
\]
where \( \lambda^* = \lambda^*(x) \) is the unique solution of \( \alpha^*(\cdot)|'(\lambda) = 1/x \) for fixed \( x > 0 \).
(b) \( V^* \in C([0, \infty)) \cap C^1((0, \infty)) \) and \( V^* \) is a strictly convex function on \((0, \infty)\).
(c) \( V^* \) attains its minimum value of zero uniquely at \( x^* = \frac{1}{2\pi} \).
Proof. See Appendix C. \hfill \Box

**Theorem 3.5.** $L_t/t$ satisfies a large deviation principle on $[0, \infty)$ as $t \to \infty$ with a strictly convex rate function $V^*(x)$.

**Proof.** From Lemma 3.4, we know that $V^*$ is a strictly convex function on $(0, \infty)$. Hence the set of exposed points of $V^*$ is $(0, \infty)$ (see Definition 2.3.3 in [DZ98]), and since $D_V^0 = (-\infty, \infty)$, the exposing hyperplane will always lie in $D_V^0$. Therefore, by the Gärtner-Ellis Theorem (see Theorem 2.3.6 in [DZ98]), $L_t/t$ satisfies the LDP with convex rate function $V^*(x)$. \hfill \Box

**Appendix A. Proof of Lemma 3.1**

Recall from Propositions 2.1 and 2.2 that, for $\alpha < 0$ and $\lambda > 0$,

$$
\int_0^\infty e^{-\lambda t} \mathbb{E}_x(e^{\alpha L_t})dt = \hat{f}(x; \lambda, \alpha) = \frac{1}{\lambda} + e^{x \sqrt{\lambda}} A_\lambda(\alpha) + e^{-x \sqrt{\lambda}} B_\lambda(\alpha)
$$

(B-1)

where $A_\lambda(\alpha) = \frac{\alpha e^{-b_1 \lambda^{1/2}}/\cosh(b_1 \sqrt{\lambda})}{2\lambda(\alpha^*(\lambda) - \alpha)}$ and $B_\lambda(\alpha) = e^{2 \sqrt{\lambda} b_2} A_\lambda(\alpha)$. We wish to show that (B-1) still holds for a wider range of $\alpha$ and $\lambda$ values using analytic continuation. We first note that $\hat{f}$ has a singularity when $\alpha = \alpha^*(\lambda)$, and by Theorems 5a and 5b on page 57 in [WID46], we know that the abscissa of convergence for a Laplace transform is a point of singularity and the Laplace transform is analytic in its region of convergence.

We are interested in the values of $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ such that the Laplace transform $\hat{f}(x; \lambda, \alpha) = \int_0^\infty e^{-\lambda t} \mathbb{E}_x(e^{\alpha L_t})dt$ is finite. We recall the following fact: for any fixed $x \in [0, b]$, 

(ï) $f(x; \lambda, \alpha) < \infty$ for $\alpha < 0$ and $\lambda > 0$.

We now proceed in three stages:

- **Fix $\lambda > 0$ (so $\lambda \in \mathbb{R}$).** We apply the Widder results with $\alpha$ as the Laplace variable, i.e. we consider

$$
\mathbb{E}(e^{\alpha L_t}) = \int_0^\infty e^{\alpha y} dF(y),
$$

where $F(y)$ is the distribution function of $L_t$. By (ï), the region of convergence is non-empty. We can then extend the region of convergence up to $\alpha^*(\lambda) > 0$, as $\alpha^*(\lambda)$ is the point of singularity.

- **Fix $\alpha < 0$.** We apply the Widder results again, but we now take $\lambda$ as the Laplace variable. By (ï), the region of convergence is non-empty. According to Widder, the abscissa of convergence (say $\lambda_c$) is a point of singularity and $\hat{f}(x; \lambda, \alpha)$ is analytic in $\lambda$ when $\Re(\lambda) > \lambda_c$. So we are looking at a point of singularity on the real line, and this is the value of $\lambda_c$ that satisfies $\alpha^*(\lambda_c) = \alpha$. Or, in other words, $\lambda_c = (\alpha^*)^{-1}(\alpha) = V(\alpha)$. Thus, by Widder, $\hat{f}(x; \lambda, \alpha)$ is finite when $\Re(\lambda) > V(\alpha)$.

- **Fix $\alpha > 0$.** We apply Widder’s theorem using $\lambda$ as the Laplace variable. By the first bullet point, we know that there exists some $\lambda \in \mathbb{R}$ (such that $\alpha < \alpha^*(\lambda)$), for which $\hat{f}(x; \lambda, \alpha)$ is finite. Hence, the region of convergence of $\hat{f}(x; \lambda, \alpha)$ is non-empty for this $\alpha$. Then, by Widder, the abscissa of convergence $\lambda_c$ is a point of singularity and $\hat{f}(x; \lambda, \alpha)$ is analytic for $\Re(\lambda) > \lambda_c$. The singularity is at $\alpha = \alpha^*(\lambda_c)$. Solving for points of singularity on the real line i.e. solving for $\lambda_c$ in $\alpha = \alpha^*(\lambda_c)$, gives us $\lambda_c = V(\alpha)$ and so $\hat{f}(x; \lambda, \alpha)$ converges when $\Re(\lambda) > \lambda_c = V(\alpha)$.

This gives the region of $\lambda$ and $\alpha$ for which $\hat{f}(x; \lambda, \alpha)$ converges: for every $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ such that $\Re(\lambda) > V(\alpha)$, and $\hat{f}(x; \lambda, \alpha)$ is analytic in this region.

**Appendix B. Proof of Proposition 3.2**

From the known large-time behaviour of the local time of standard Brownian motion, we expect that $\mathbb{E}_x(e^{\alpha L_t}) \sim \text{const.} \times e^{U(\alpha)t}$ as $t \to \infty$, for some non-decreasing function $U(\alpha)$ to be determined. Then as $t \to \infty$,

$$
\hat{f}(x; \lambda, \alpha) = \int_0^\infty e^{-\lambda t} \mathbb{E}_x(e^{\alpha L_t})dt \sim \int_0^\infty e^{-\lambda t} \text{const.} \times e^{U(\alpha)t} dt,
$$

(C-1)
and \( \hat{f}(x; \lambda, \alpha) \) blows up when \( \lambda = U(\alpha) \) (for \( \alpha \) fixed). But we know that \( \hat{f}(x; \lambda, \alpha) \) blows up at \( \alpha = \alpha^*(\lambda) \); thus we expect that \( \lambda = U(\alpha^*(\lambda)) \), i.e. \( U(\alpha) = (\alpha^*)^{-1}(\alpha) = V(\alpha) \). We now make this statement rigorous using a variant of Ikehara’s Tauberian Theorem (see e.g. Theorem 17 on page 233 in Widder[1WID46]).

We first define a positive function \( v \) on \( \mathbb{R} \):

\[
v(t) \equiv v(t; x, \alpha) := 1_{t \geq 0} e^{-V(\alpha)t} e^{x\alpha L t}.
\]

Then the Laplace transform of \( v \) is given by

\[
\hat{v}(\lambda) = \int_0^\infty e^{-\lambda t} v(t) dt = \int_0^\infty e^{-(\lambda+V(\alpha)t)} e^{x\alpha L t} dt = \hat{f}(x; \lambda + V(\alpha), \alpha),
\]

which, by Lemma 3.1 is analytic for all \( \lambda \in \mathbb{C} \) such that \( \Re(\lambda) > 0 \). We now need to characterize how \( \hat{v}(\lambda) \) blows up as \( \Re(\lambda) \downarrow 0 \). To this end, looking at the expression for \( A_2(\alpha) \), we notice that \( A_2(\alpha) \) has a pole at \( \lambda = V(\alpha) \in (-\frac{\pi^2}{a^2}, \infty) \), and is analytic elsewhere for \( \Re(\lambda) > -\frac{\pi^2}{a^2} \) (see Remarks 2.3 and 2.4). It is also easily seen that, \( \alpha^*(\lambda) > 0 \) for all \( \lambda \in (-\frac{\pi^2}{a^2}, \infty) \). Hence, by the Laurent expansion of \( \hat{v}(\lambda) \) at 0, there exists a function \( g(\lambda) \), which is analytic for all \( \lambda \in \mathbb{C} \) with \( \Re(\lambda) > -\varepsilon \) and \( |\Im(\lambda)| \leq c \) for some constants \( \varepsilon, c > 0 \), such that

\[
\hat{v}(\lambda) = \frac{C}{\lambda} + g(\lambda)
\]

for some constant \( C \) which we find to be positive (\( C \) is the residue of \( \hat{v} \) at \( \lambda = 0 \)). \( g(x + iy) \) is continuous on \( \mathcal{D} := \{(x, y) : |x| \leq \varepsilon, |y| \leq c\} \), thus \( g(x + iy) \) is uniformly continuous on \( \mathcal{D} \), so \( g(x + iy) \to g(iy) \) uniformly as \( x \downarrow 0 \) for any fixed \( y \in [-c, c] \). Moreover, for any \( x > 0 \)

\[
\int_{-c}^c |\hat{v}(x + iy) - \frac{C}{x + iy} - g(iy)| dy = \int_{-c}^c |g(x + iy) - g(iy)| dy
\]

Since \( g \) is analytic everywhere and uniformly continuous, if we take the limit as \( x \to 0 \), the above integral converges to 0, so the function \( g(x + iy) \) also converges to \( g(iy) \) in \( L^1([-c, c]) \), as \( x \downarrow 0 \).

We can now apply Proposition 4.3 in [KOR02] to obtain that for the “Fejér kernel" \( K(t) = \frac{\sin t}{\pi t} \),

\[
\lim_{t \to \infty} \int_{-\infty}^{ct} v(t - \frac{s}{c}) \cdot K(s) ds = C.
\]

(C-2)

We now proceed as in the proof of Theorem 4.2 in [KOR02] to show that \( v(t) = O(1) \) as \( t \to \infty \).

1. \( \alpha > 0 \). In this case we know that \( E_x(e^{x\alpha L t}) \) is non-decreasing, so \( v(t) \geq v(s) e^{V(\alpha)(s-t)} \) for all \( t \geq s \geq 0 \). For any fixed \( a > 0 \), using (C-2) we have that

\[
C = \lim_{t \to \infty} \int_{-\infty}^{ct} v(t - \frac{s}{c}) \cdot K(s) ds \geq \limsup_{t \to \infty} \int_{-a}^{at} v(t - \frac{s}{c}) \cdot K(s) ds \geq \limsup_{t \to \infty} v(t - \frac{a}{c}) e^{-2V(\alpha)\frac{s}{c}} \int_{-a}^{a} K(s) ds,
\]

which implies that

\[
\limsup_{t \to \infty} v(t) \leq \frac{e^{2V(\alpha)\frac{s}{c}}}{\int_{-a}^{a} K(s) ds} C < \infty.
\]

Hence, there exists a constant \( M > 0 \) such that \( v(t) \leq M \) for all \( t \). Similarly, for any fixed \( a > 0 \), we have

\[
\liminf_{t \to \infty} v(t + \frac{a}{c}) e^{2V(\alpha)\frac{s}{c}} \int_{-a}^{a} K(s) ds = \liminf_{t \to \infty} \int_{-a}^{a} v(t - \frac{s}{c}) K(s) ds
\]

\[
= \liminf_{t \to \infty} \left( \int_{-\infty}^{ct} + \int_{ct}^{\infty} - \int_{-ct}^{-\infty} - \int_{-\infty}^{-ct} \right) v(t - \frac{s}{c}) K(s) ds \geq \liminf_{t \to \infty} \left( \int_{-\infty}^{ct} + \int_{ct}^{\infty} \right) v(t - \frac{s}{c}) K(s)
\]

\[
- \limsup_{t \to \infty} \int_{-\infty}^{-ct} v(t - \frac{s}{c}) K(s) - \limsup_{t \to \infty} \int_{ct}^{\infty} v(t - \frac{s}{c}) K(s) \geq C - \frac{4M}{\pi} \int_{\frac{1}{2}}^{\infty} \frac{1}{s^2} ds = C - \frac{4M}{\pi a},
\]

where we have used (C-2) and the fact that \( 0 \leq K(t) \leq \frac{2}{\pi t} \) in the last inequality. Hence, for \( a > 0 \) sufficiently large, we have

\[
\liminf_{t \to \infty} v(t) \geq \frac{e^{-2V(\alpha)\frac{s}{c}}}{\int_{-a}^{a} K(s) ds} (C - \frac{4M}{\pi a}) > 0.
\]
2. \( \alpha < 0 \). In this case we know that \( \mathbb{E}_x(e^{\alpha L_t}) \) is non-increasing, so \( v(t) \leq v(s)e^{V(\alpha)(s-t)} \) for all \( t \geq s \geq 0 \). Using the same argument as above, we have, for any fixed \( a > 0 \),

\[
Ce^{2V(\alpha)x} \geq \limsup_{t \to \infty} v(t + \frac{a}{c}) \int_{-a}^{a} K(s)ds,
\]

\[
(C - \frac{4M}{\pi a}) e^{-2V(\alpha)x} \leq \liminf_{t \to \infty} v(t - \frac{a}{c}) \int_{-a}^{a} K(s)ds.
\]

Hence for \( a > 0 \) sufficiently large, we have

\[
0 \leq \frac{e^{-2V(\alpha)x}}{\int_{-a}^{a} K(s)ds} (C - 4M/\pi a) \leq \liminf_{t \to \infty} v(t) \leq \limsup_{t \to \infty} v(t) \leq \frac{e^{2V(\alpha)x}}{\int_{-a}^{a} K(s)ds} C < \infty.
\]

Hence, by Proposition 4.3 in [KOR02], the result follows.

**Appendix C. Proof of Lemma 3.4**

We break the proof into three parts:

(a) Computing the Legendre transform of \( V \) boils down to solving \( V'(\alpha) = x \). But this is the same as solving \( (V^{-1})'(\lambda) = \frac{1}{x} \) for \( \lambda \), when \( x > 0 \). Recall that \( V^{-1}(\cdot) = \alpha^* (\cdot) \) is known in closed form. Since \( (\alpha^*)''(\lambda) < 0 \) for all \( \lambda \) in the domain of \( \alpha^* \), i.e. \( \lambda > -\frac{2}{\sqrt{\pi}} \) (from Remark 2.4), by the Inverse function theorem, \( \lambda^*(x) := (\alpha^*)'^{-1}(1/x) \) is well-defined and \( \lambda^* \in C^1((0, \infty)) \). Using the fact that \( \alpha^*(\lambda^*) = V^{-1}(\lambda^*) \), we have

\[
V^*(x) = x\alpha^* - V(\alpha^*) = x\alpha^*(\lambda^*(x)) - \lambda^*(x).
\]

When \( x = 0 \), the definition of \( V^* \) in Lemma 3.4 gives us \( V^*(0) = \sup_{\alpha \in \mathbb{R}} \{-V(\alpha)\} = -\inf_{\alpha \in \mathbb{R}} \{V(\alpha)\} = -\lim_{\alpha \to -\infty} V(\alpha) = \pi^2/(8b^2) \), where the last two equalities hold because \( V \) is a monotonically increasing function with range \((-\pi^2/(8b^2), \infty)\).

(b) By the Inverse function theorem, we know that \( \lambda^* \in C^1((0, \infty)) \) and so is \( \alpha^* \), thus \( V^* \in C^1((0, \infty)) \). It is easy to check that \( \lim_{x \to 0} \{x\alpha^*(\lambda^*(x)) - \lambda^*(x)\} = \pi^2/(8b^2) = V^*(0) \), which gives continuity of \( V^* \) up to the boundary \( x = 0 \). Using (5), we obtain

\[
(V^*)'(x) = \alpha^*(\lambda^*(x)) + x \cdot (\alpha^*)'(\lambda^*(x)) \cdot (\lambda^*)'(x) - (\lambda^*)'(x) = x\alpha^*(\lambda^*(x)) + \frac{1}{x} 
\]

\[ (V^*)'(0) = \alpha^*(\lambda^*(0)). \] Thus we have (using again \( (\alpha^*)'' < 0 \))

\[
(V^*)''(x) = (\alpha^*)'(\lambda^*(x)) \cdot (\lambda^*)'(x) = \frac{1}{x} \cdot ((\alpha^*)'^{-1})'(\frac{1}{x}) \cdot (-\frac{1}{x^2}) = \frac{1}{x^3} \cdot \frac{1}{(\alpha^*)''(\lambda^*(x))} > 0.
\]

(c) Since \( V^* \) is strictly convex, it has a unique minimum. The unique minimum of \( V^* \) occurs at \( x^* = (V^*)'^{-1}(0) = V'(0) = 1/\alpha''(0) = \frac{1}{2b^2} \).

**References**


