THE LANGLANDS PROGRAM: NOTES, DAY I

SOLOMON FRIEDBERG

ABSTRACT. These are notes for the first of a two-day series of lectures introducing graduate students to (parts of) the Langlands Program, delivered in the Building Bridges: 2nd EU/US Summer School on Automorphic Forms and Related Topics, July 2014.

INTRODUCTION: THE BIG PICTURE

There are two different kinds of number theoretic L-functions that have been studied extensively. The first are *Artin L-functions*. Let F be a number field, and G_F be the Galois group of an algebraic closure $\bar{F}$ of F over F. A *Galois representation* is a continuous homomorphism $\rho : G_F \to \mathrm{Aut}(V)$ where V is a finite dimensional complex vector space. Here continuous means there exists a finite Galois extension K/F such that ρ factors through the finite Galois group $\mathrm{Gal}(K/F)$. For each unramified prime ideal p of F, there is a conjugacy class Fr_p, the Frobenius class, in $\mathrm{Gal}(K/F)$ that determines how p factors in K. (If p is ramified then one gets a class modulo the inertia subgroup I_p.) Then Artin defined the L-function, given as an infinite product absolutely convergent for $\Re(s) > 1$:

$$L(s,\rho) = \prod_p \det\left(I_V - \rho(Fr_p)|V^{I_p}\,\mathrm{N}(p)^{-s}\right)^{-1}.$$

Here V^{I_p} is the subspace of V fixed by the inertia subgroup; for an unramified place I_p is trivial so $V^{I_p} = V$. Also $\mathrm{N}(p)$ denotes the absolute norm of p. *Artin's Conjecture* states that if ρ does not contain a copy of the trivial representation then $L(s,\rho)$ is entire.

The second class of L-functions is that of *automorphic L-functions*. These are functions attached to harmonic analysis on groups modulo discrete subgroups in a way that we shall make precise (in certain cases) today. The first examples of these functions are Dirichlet L-functions (and more generally Hecke L-functions with Grossencharacter) and the L-functions attached to modular forms. Notice that for

Date: Lectures on July 4, 2014. Revised version of these notes, July 9, 2014.

Hecke L-functions almost all Euler factors have degree one in $\mathrm{N}(p)^{-s}$ while for modular forms almost all Euler factors have degree two in p^{-s}. They are known to be entire in "most" cases (non-trivial Dirichlet character, cusp forms).

Artin reciprocity is a precise formulation of statement that in the abelian case, these two classes of L-functions are related. Namely, *every* $L(s, \rho)$, ρ one-dimensional, is in fact a Hecke L-function. This is a key part of class field theory. But what about the non-abelian case? Then there are irreducible representations of $\mathrm{Gal}(K/F)$ of dimension greater than 1.

Langlands envisions a strong generalization. Namely, he conjectures that *every Artin L-function is an automorphic L-function.* If true, then Artin's Conjecture would follow. Moreover, he conjectures that operations in linear algebra that are natural for Artin L-functions should have analogues for *all* automorphic L-functions (and for automorphic representations, as will be explained in this mini-course). For example, given a second complex vector space W and a complex analytic representation $\sigma : \mathrm{Aut}(V) \to \mathrm{Aut}(W)$, one may compose to make a new Galois representation $\sigma \circ \rho : \mathrm{Gal}(K/F) \to \mathrm{Aut}(W)$. Finding an analogue of this on the automorphic side turns out to be a deep problem.

We mention that other connections also fall under the rubric of the Langlands program. The above relations are global, but there are also *Local Langlands Conjectures* describing the representations of local analogues of the absolute Galois group, namely representations of the Weil-Deligne group, in terms of representations of p-adic groups. And one may replace Galois representations by representations of the fundamental group of an algebraic curve and relate them to automorphic sheaves, a program known as the geometric Langlands program. (For a gentle but far reaching introduction to this program written for a general audience which also explains the way that the Langlands program connects to physics, see Edward Frenkel's charming book "Love and Math.")

1. Lecture 1: Automorphic Forms

1.1. **The General Set Up.** Suppose that G is a *topological group*. That is, G is a group and also a topological space, and the product and inverse maps

$$p : G \times G \to G \quad p(g_1, g_2) = g_1 g_2, \qquad i : G \to G \quad i(g) = g^{-1}$$

are continuous. (Here $G \times G$ is endowed with the product topology.) Let Γ be a *discrete* subgroup of G which is not too small. Then experience suggests that: *the study of left Γ-invariant functions on G is of interest.* As a slight generalization, one may study functions that satisfy

$$f(\gamma g) = \chi(\gamma) f(g) \qquad \text{for all } \gamma \in \Gamma,\ g \in G$$

where χ is a character of Γ.

The two examples of automorphic L-functions mentioned above both arise from functions of this type. Recall that a number field F embeds discretely in its ring of adeles $\mathbb{A}_F$ by the diagonal embedding $f \mapsto (f, f, \dots)$. Similarly $\Gamma = F^\times$ embeds discretely in $G = \mathbb{A}_F^\times$. (The topology on $\mathbb{A}_F^\times$ is the relative topology after embedding $\mathbb{A}_F^\times$ into $\mathbb{A}_F \times \mathbb{A}_F$ by $a \mapsto (a, a^{-1})$.) Classical Hecke characters turn out to be equivalent to continuous functions ξ on G that are Γ-invariant. We will make the correspondence explicit when $F = \mathbb{Q}$ later in this lecture.

Similarly, if $G = SL(2, \mathbb{R})$, then $\Gamma = SL(2, \mathbb{Z})$ is discrete. Classical modular forms are functions on the upper half plane but they are related to functions on the group G, as we now explain.

1.2. **Modular Forms.** Classical modular forms (including Maass forms) are functions on the complex upper half plane $\mathfrak{h} = \{x + iy \in \mathbb{C} \mid y > 0\}$. Our first step is to view them as functions on the group $G = SL(2, \mathbb{R})$. To do so, suppose that $f : \mathfrak{h} \to \mathbb{C}$ transforms by the equation

$$f(\gamma \circ z) = \chi(\gamma) j(\gamma, z) f(z) \quad \text{for all} \quad \gamma \in \Gamma, \tag{1.1}$$

where Γ is a congruence subgroup of $SL(2, \mathbb{Z})$, χ is a character of Γ, and where $j : G \times \mathfrak{h} \to \mathbb{C}$ satisfies the cocycle equation

$$j(\gamma_1 \gamma_2, z) = j(\gamma_1, \gamma_2 \circ z)\, j(\gamma_2, z).$$

For example, if f is a classical modular form of weight k with character χ one may take $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) = (cz + d)^k$. Define $F : G \to \mathbb{C}$ by $F(g) = j(g, i)^{-1} f(g \circ i)$. Recall that the group $K = SO(2, \mathbb{R})$ is the stabilizer of i in G.

Exercise 1. *The function F satisfies the properties*

1. $F(g\kappa) = j(\kappa, i)^{-1} F(g)$ *for all* $\kappa \in K$.
2. $F(\gamma g) = \chi(\gamma)\, F(g)$ *for all* $\gamma \in \Gamma$.

Moreover, the map $f \mapsto F$ is a one-to-one correspondence between functions on $\mathfrak{h}$ satisfying (1.1) *and functions F on G satisfying Properties 1, 2 above.*

Note that if $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) = (cz + d)^k$, then we have

$$j(\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right), i)^{-1} = e^{ik\theta},$$

that is F transforms on the right by a character of K.

Observe that in particular the function G transforms under the center by a character, and indeed if $-I_2 \in \Gamma$, then comparing (1),(2) above we arrive at the familiar requisite parity condition $\chi(-1) = (-1)^k$. Also, since every matrix in $GL(2, \mathbb{R})$ of positive determinant can be adjusted by a positive scalar to give one of determinant 1, we may canonically extend F to a function on $GL^+(2, \mathbb{R})$, the subgroup of

$g \in GL(2,\mathbb{R})$ such that $\det g > 0$, that is invariant under the subgroup of scalar matrices rI_2, $r > 0$.

To work on the group, we must also translate the analytic conditions (that is, the property that f is either holomorphic or more generally an eigenfunction of the Laplacian) and the growth conditions. The differential operator on the group that is needed is one described Lie-theoretically, namely it generates the center of the universal enveloping algebra of the associated Lie algebra. (Recall that vectors X in the Lie algebra act on differentiable functions by $g \mapsto \frac{d}{dt} f(g \exp(tX)) \mid_{t=0}$, and this extends to the universal enveloping algebra.) The growth condition may be taken to be the condition that $F(g)$ grows by at most a power of $||g||$ where $||g||$ is the length of the vector given by the entries of g together with $\det(g)$.

Our goal is to pass from a function on $GL^+(2,\mathbb{R})$ to a function on the adelic group $GL(2,\mathbb{A})$ (where $\mathbb{A} = \mathbb{A}_\mathbb{Q}$). First, though, let us handle the analogous problem on GL_1. Given a Dirichlet character χ, we attach a character ω of the idele class group $\mathbb{Q}^\times \backslash \mathbb{A}^\times$. Suppose first that χ is a Dirichlet character modulo p^f with p a prime and $f > 0$. If v is a finite place prime to p (we regard v as both a place and a prime integer), let $\omega_v : F_v^\times \to \mathbb{C}^\times$ be the character which is trivial on units and whose value on a local uniformizer ϖ_v is given by $\chi(v)$. If $v = p$, define ω_v by $\omega_v(p^k(j + p^f\mathbb{Z}_p)) = \chi(j)^{-1}$ when $j, k \in \mathbb{Z}$ and $(j,p) = 1$. And define $\omega_\infty(r) = -1$ if χ is odd and $r < 0$ and 1 otherwise. Then define $\omega = \prod_v \omega_v$. Finally, if χ has conductor N which is expressed in terms of distinct prime powers as $N = \prod p_i^{f_i}$, to lift χ to an idele class character first factor χ as a product of characters of conductors $p_i^{f_i}$ and then lift each factor to an idele class character as above.

Exercise 2. (1) *Confirm that the ω so-obtained is an idele class character.*
(2) *Let $S_f(N) = \{v \in S_f \mid ord_v(N) \neq 0\}$. Show that*
$$\chi(d) = \prod_{v \notin S_f(N)} \omega_v(d) = \prod_{v \in S_f(N)} \omega_v(d)^{-1}.$$

We are now ready to pass from modular forms f to functions on the adele group $GL(2,\mathbb{A})$. So let $f(z)$ be a holomorphic modular form of weight k, level N, and character χ. That is,
$$f(\gamma \circ z) = \chi(d)\,(cz+d)^k f(z) \qquad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$
Let $F(g)$ be the associated function on $GL_2^+(\mathbb{R})$. The passage from the real points to the adelic points is accomplished by the strong approximation theorem.

Let us state this result more generally and then apply it in our case. Fix F a number field with archimedean places S_∞ and finite places S_f, and let $\mathbb{A}$ now denote the adeles of F. The *finite adeles* of F is the restricted direct product of the completions of F at places in S_f: $\mathbb{A}_f = \prod_{v \in S_f} F_v$. Similarly we set $F_\infty = \prod_{v \in S_\infty} F_v$. Both $\mathbb{A}_f$ and

F_∞ canonically embed in the full ring of adeles. If $v \in S_f$, let O_v denote the ring of integers of F_v and $O_v^\times$ the subgroup of units. Then we have

Theorem 1. *Let $n \geq 1$. Let K_0 be a compact open subgroup of $GL_n(\mathbb{A}_f)$ such that* $\det : K_0 \to \mathbb{A}_f^\times$ *has image $\prod_{v \in S_f} O_v^\times$. Then the quotient*

$$GL_n(F)GL_n(F_\infty)\backslash GL_n(\mathbb{A})/K_0$$

has cardinality equal to the class number of F.

Exercise 3. *Confirm the theorem above when $n = 1$.*

We return to the case of a modular form, so once again $\mathbb{A}$ denotes the adeles of $\mathbb{Q}$. Let us introduce the compact open subgroup $K_0(N)$ of $GL_2(\mathbb{A}_f)$ consisting of matrices $g_v = \left(\begin{smallmatrix} a_v & b_v \\ c_v & d_v \end{smallmatrix}\right)$ such that at each v, $g_v \in GL_2(O_v)$ and $\mathrm{ord}_v(c_v) \geq \mathrm{ord}_v(N)$. (This last condition is that $c_v \equiv 0 \bmod NO_v$ for all finite places.) Then as a Corollary of the Theorem above, we see that $GL_2(\mathbb{A}) = GL_2(\mathbb{Q})GL_2(\mathbb{R})K_0(N)$.

Let $\lambda : K_0(N) \to \mathbb{C}^\times$ be given by

$$\lambda((g_v)) = \prod_{v \notin S_f(N)} \omega_v(d_v).$$

Let $K_\infty = SO_2(\mathbb{R})$. Then we note that a full maximal compact subgroup of $GL_2(\mathbb{A})$ is given by $K := K_0(1)K_\infty$.

Suppose that f is a modular form as above, and F is the corresponding function on $GL^+(2, \mathbb{R})$. Define $\phi : GL_2(\mathbb{A}) \to \mathbb{C}$ by writing $g \in GL_2(\mathbb{A})$ in the form $g = \gamma g_\infty k_0$ with $\gamma \in GL_2(\mathbb{Q})$, $g \in GL(2, \mathbb{R})^+$, and $k_0 \in K_0(N)$. Such a factorization of g exists by the Strong Approximation Theorem. Then we define $\phi(g) = F(g_\infty)\lambda(k_0)$.

Proposition 1.1.
(1) *The function $\phi : GL_2(\mathbb{A}) \to \mathbb{C}$ is well-defined.*
(2) *$\phi(\gamma g) = \phi(g)$ for all $\gamma \in GL_2(F)$.*
(3) *$\phi\left(\left(\begin{smallmatrix} z & \\ & z \end{smallmatrix}\right) g\right) = \omega(z)\phi(g)$ for all $z \in \mathbb{A}^\times$.*
(4) *For each $g_f \in GL_2(\mathbb{A}_f)$, the function $g_\infty \mapsto \phi(g_f g_\infty)$ is smooth.*
(5) *The span of the right translates of K_∞ on ϕ is a finite dimensional space of functions. (That is, ϕ is right K_∞-finite.)*
(6) *There is a compact open subgroup K' of $GL_2(\mathbb{A}_f)$ such that ϕ is invariant under right translation by K'.*
(7) *Let $\mathcal{Z}$ denote the center of the universal enveloping algebra of $\mathfrak{gl}(2, \mathbb{R})$, realized as differential operators that act on ϕ through their action g_∞. Then ϕ is $\mathcal{Z}$-finite.*
(8) *ϕ is of moderate growth.*

Exercise 4. *Confirm this Proposition.*

In view of Property 3, ω is called the *central character* of ϕ.

Note that Property 2 above shows that ϕ is indeed a function on a group invariant under a discrete subgroup, as described in subsection 1.1. Also, since the subgroup K' above necessarily satisfies $[K_0(1) : K_0(1) \cap K'] < \infty$, Property 6 is equivalent to ϕ being right $K_0(1)$-finite. Thus Properties 5 and 6 are equivalent to the property that ϕ is right K-finite for the maximal compact group K.

To be sure, if ϕ arose from a holomorphic modular form, then a stronger property (which may be phrased in terms of the complexification of the universal enveloping algebra of $\mathfrak{gl}(2, \mathbb{R})$) holds. However, we shall focus on the properties above.

Definition . A function $\phi : GL(2, \mathbb{A}) \to \mathbb{C}$ satisfying the properties above is called a (K-finite) automorphic form on $GL_2(\mathbb{A})$.

1.3. **Automorphic Forms on More General Groups.** Proposition 1.1 suggests directly the definition of automorphic form for other groups. If G is a reductive algebraic group defined over the number (or function) field F and $\mathbb{A}$ denotes the adeles of F, an *automorphic form* is a function $\phi : G(\mathbb{A}) \to \mathbb{C}$ satisfying the properties of Proposition 1.1 above. Here K is taken to be a maximal compact subgroup of $G(\mathbb{A})$ and $\mathcal{Z}$ is in general a finitely generated algebra.

We note that this general definition includes many other classical kinds of modular forms. If F is a totally real number field, then it includes the adelizations of Hilbert modular forms (and Hilbert modular forms of Maass type). If G is a symplectic group, it includes adelizations of Siegel modular forms. But we should emphasize what is not included. For GL_2, modular forms transforming under non-congruence subgroups are not included (the Strong Approximation Theorem may not be applied). Modular forms of half-integral weight and their generalizations are not included; these can be included by passing to covers of $G(\mathbb{A})$, the so-called metaplectic groups. And mock modular forms violate the requisite growth properties.

Before going farther, let us ask: what's the point? To be sure, working with Hilbert modular forms when the class number is not one is very technical. But why go to the adeles? There are several advantages. One is that cusps are easier to work with; essentially we need only worry about one cusp. But this is minor. The key advantage is that it allows us to break global problems into local problems, and apply methods and facts from the representation theory of p-adic groups. We shall see this below.

In working with classical modular forms, one often focuses on cusp forms, identified by their growth properties as one approaches the cusps. It then develops that cusp forms have Fourier expansions with constant term zero. For more general groups, the notion of Fourier expansion is more difficult, and it turns out to be better to *define* a cusp form by the vanishing of its constant terms. For simplicity we give the definition for GL_n but it applies to other groups with obvious minor modifications.

For $1 \leq r \leq n-1$ let $U_r \subset GL_n$ be the subgroup of block upper-triangular unipotent matrices

$$U_r = \left\{ \begin{pmatrix} I_r & X \\ 0 & I_{n-r} \end{pmatrix} \right\}.$$

Definition . A function ϕ on $GL_n(F)\backslash GL_n(\mathbb{A})$ is *cuspidal* if for each r, $1 \leq r \leq n-1$,

$$\int_{U_r(F)\backslash U_r(\mathbb{A})} \phi\,(ug)\;du = 0$$

for almost all g.

A classical cusp form is rapidly decreasing as one approaches all cusps. More generally, Gelfand and Piatetski-Shapiro proved that a cuspidal automorphic form on $GL_n(\mathbb{A})$, defined as above, is rapidly decreasing. We won't formulate this precisely but it is this property that makes integrals involving cusp forms that arise in the theory converge.

1.4. **Eisenstein series.** Let us return momentarily to the classical language of the upper half plane. Suppose that ϕ is a Maass cusp form on $\mathfrak{h}$ for $\Gamma = SL(2, \mathbb{Z})$. Then one may check that ϕ is square-integrable with respect to the natural invariant measure $y^{-2}dx\,dy$ on $\mathfrak{h}$:

$$\int_{\Gamma\backslash\mathfrak{h}} |\phi(z)|^2\,\frac{dx\,dy}{y^2} < \infty.$$

It is then natural to consider the entire space $L^2(\Gamma\backslash\mathfrak{h})$. One can try to decompose an arbitrary function in this space by writing it as a sum of cusp forms (and the trivial function, say), but this does not turn out to fill up the entire space. The missing piece is given by integrals of Eisenstein series. To illustrate, we next develop the simplest non-holomorphic Eisenstein series and compute its Fourier expansion. This will turn out to give a hint of a far larger picture.

For each fixed $s \in \mathbb{C}$ with $\Re(s)$ sufficiently large, define the *Eisenstein series*

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s = \sum_{\substack{(c,d)=1, \\ (c,d) \mod \pm 1}} \frac{y^s}{|cz+d|^{2s}}.$$

(The conditon $\Re(s)$ sufficiently large guarantees convergence.) Then

$$E(\gamma z, s) = E(z, s)$$

for any $\gamma \in \Gamma$. In particular, since

$$E(z+1, s) = E\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) z, s\right) = E(z, s),$$

we see $E(z, s)$ is periodic so it has a Fourier expansion.

How can we find the Fourier coefficients of $E(z, s)$? One approach is to pass to the "normalized Eisenstein series"

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s),$$

where

$$\begin{aligned}\zeta(2s)E(z,s) &= \sum_{\substack{(m,n)\in\mathbb{Z}^2,\\ (m,n)\neq(0,0) \mod \pm 1}} \frac{y^s}{|mz+n|^{2s}} \\ &= \underbrace{y^s\zeta(2s)}_{m=0} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} \frac{y^s}{((mx+n)^2+(my)^2)^s}.\end{aligned}$$

The $m = 0$ term above does not depend on x and so does not contribute to the non-zero Fourier coefficients. To compute the r-th Fourier coefficient of $E^*(z,s)$, $r \neq 0$:

$$\begin{aligned}&\pi^{-s}\Gamma(s)y^s \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} \int_0^1 \frac{e^{-2\pi i r x}}{((mx+n)^2+(my)^2)^s}\,dx \\ &\qquad = \pi^{-s}\Gamma(s)y^s \sum_{m=1}^{\infty}\sum_{n \bmod m} \int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{((mx+n)^2+(my)^2)^s}\,dx.\end{aligned}$$

In the above we have taken $n \bmod m$ and so adjusted the region of integration. (This is a first example of the "unfolding" technique that is common in the theory of automorphic forms.) Next, change $x \mapsto x - \frac{n}{m}$, we have

$$\pi^{-s}\Gamma(s)y^s \sum_{m=1}^{\infty} \frac{\sum_{n \bmod m} e^{2\pi i r \frac{n}{m}}}{m^{2s}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x^2+y^2)^s}\,dx.$$

Now we may compute the summation over $n \bmod m$:

$$\sum_{n \bmod m} e^{2\pi i r \frac{n}{m}} = \begin{cases} m, & \text{if } m \mid r \\ 0, & \text{if } m \nmid r. \end{cases}$$

We recall that the K-Bessel function is given for $y > 0$ and $s \in \mathbb{C}$ by

$$K_s(y) = \tfrac{1}{2}\int_0^{\infty} e^{-y(t+t^{-1})/2} t^s \frac{dt}{t}. \tag{1.2}$$

(This integral is easily seen to be convergent for all s.) From this definition, it is an exercise (or see [3], pg. 67) to check that if $r \neq 0$ then

$$\pi^{-s}\Gamma(s)y^s \int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x^2+y^2)^s}\,dx = 2|r|^{s-\frac{1}{2}}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|r|y). \tag{1.3}$$

Conclusion: For $n \geq 1$, let

$$\sigma_w(n) := \sum_{0<d|n} d^w.$$

Then the r-th Fourier coefficient of $E^*(z,s)$ is

$$2|r|^{s-\frac{1}{2}}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|r|y)$$

if $r \neq 0$, and

$$\pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{1-s}\Gamma(1-s)\zeta(2-2s)y^{1-s}$$

if $r = 0$.

Exercise 5. *Redo the Fourier expansion argument above more group theoretically using the unnormalized Eisenstein series and the Bruhat decomposition. (To do this, break the big Bruhat cell up into pieces such that Γ_∞ acts properly on the right on each piece.)*

It is not difficult to observe that each non-zero Fourier coefficient of $E^*(z,s)$ is defined for all complex s and is symmetric under $s \mapsto 1-s$ (this uses the symmetry of the K-Bessel function $K_a(y) = K_{-a}(y)$). The same properties of the $r=0$ coefficient are true by the analytic continuation of the Riemann zeta function (with poles at $s = 0, 1$). Moreover the Fourier series is convergent for all s. Thus we have

Corollary 1.2. *$E^*(z,s)$ has analytic continuation to all $s \in \mathbb{C}$, except for simple poles at $s = 0$ and 1, and has functional equation under $s \mapsto 1-s$.*

Remark 1.4. In fact, one can establish this continuation by spectral methods (and this is true much more generally). Thus one can give another proof of the meromorphic continuation and functional equation of the Riemann zeta function, a proof that hinges on the occurrence of the Riemann zeta function in the constant term of the Eisenstein series. Applied to higher rank groups, this leads to the Langlands-Shahidi method for studying certain Langlands L-functions.

To conclude, we write the spectral expansion, due to Selberg, for $L^2(\Gamma\backslash\mathfrak{h})$, which requires the integrals of the Eisenstein series $E(z,s)$. Let η_j, $j \geq 1$, be a basis for the space of Maass cusp forms on $\Gamma = SL(2,\mathbb{Z})$ that is orthonormal with respect to the Petersson inner product

$$< f, g >= \int_{\Gamma\backslash\mathfrak{h}} f(z)\,\overline{g(z)}\,\frac{dx\,dy}{y^2}.$$

Let $\eta_0(z)$ be the constant function $\sqrt{3/\pi}$. Then one has:

Theorem 2. *Suppose that $f(z) \in L^2(SL(2,\mathbb{Z})\backslash\mathfrak{h})$. Then*

$$f(z) = \sum_{j=0}^{\infty} < f, \eta_j > \eta_j(z) + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} < f, E(*,s) > E(z,s)\,ds.$$

2. Lecture 2: Automorphic Representations

Let us return to the more general situation that F is a number field. Let G be a reductive group that is defined over F. This is a technical notion , but for this course it is sufficient to think of G as being either GL_n with $n > 1$ or else Sp_{2n}, the group of $2n \times 2n$ symplectic matrices, with $n \geq 1$. Then there is an invariant measure on $G(\mathbb{A})$ that is defined up to a constant, so we may consider square integrable functions on $G(F)\backslash G(\mathbb{A})$. Let ω be a fixed unitary character, and let $L^2(G(F)\backslash G(\mathbb{A}), \omega)$ denote the space of square integrable functions that transform with central character ω. Then $G(\mathbb{A})$ acts on this space by right translation. By analogy with the spectral decomposition above, it is natural to ask for decomposition of this right regular representation. We shall return to this question later in this lecture.

2.1. **Definition of Automorphic Representation.**

We could try to consider a variation on this question. Fix ω and let $\mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$ denote the space of all automorphic forms that have central character ω. Then we could hope that $G(\mathbb{A})$ acts by right translation on this space. The property of being left-$G(F)$ invariant is indeed preserved by this right action. However, *the property of right K-finiteness is not preserved*. It is close to being preserved: If $\phi \in \mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$ and $g_0 \in G(\mathbb{A}_f)$, then indeed the function $\rho(g_0)\phi$, that is, the function $g \mapsto \phi(gg_0)$, is again in $\mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$. This is true since if $K' \subset G(\mathbb{A}_f)$ fixes ϕ then $g_0^{-1}K'g_0$ fixes $\rho(g_0)\phi$ and the subgroup $g_0^{-1}K'g_0 \cap K'$ is of finite index in K'. However, it is not true that right translation by $G(F_v)$ fixes $\mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$ if v is archimedean. Indeed, in general $\rho(g_v)\phi$ is not K_∞-finite, where K_∞ is a maximal compact subgroup of $G(F_\infty) = \prod_{v|\infty} G(F_v)$.

There are two solutions to this problem. The first is to *weaken the requirement that ϕ be K_∞-finite*. Let $\mathfrak{g}$ be the Lie algebra of $G(F_\infty)$.

Definition . A smooth automorphic form on G is a function $\phi : G(\mathbb{A}) \to \mathbb{C}$ such that

(1) $\phi(\gamma g) = \phi(g)$ for all $\gamma \in G(F)$.
(2) For each $g_f \in G(\mathbb{A}_f)$, the function $g_\infty \mapsto \phi(g_f g_\infty)$ is smooth.
(3) There is a compact open subgroup K' of $G(\mathbb{A}_f)$ such that ϕ is invariant under right translation by K'.
(4) Let $\mathcal{Z}$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, realized as differential operators that act on ϕ through their action g_∞. Then ϕ is $\mathcal{Z}$-finite.
(5) ϕ is of uniform moderate growth: there exists a positive integer r and such that for all $X \in \mathcal{U}(\mathfrak{g})$ (the universal enveloping algebra of $\mathfrak{g}$), $|X\phi(g)| \leq C_X ||g||^r$.

(For the general notion of norm $||g||$, see Borel and Jacquet [2], pg. 189.)

We write the space of smooth automorphic forms with central character ω as $\mathcal{A}^\infty(G(F)\backslash G(\mathbb{A}), \omega)$. The space $\mathcal{A}^\infty$ has a limit Fréchet topology coming from the uniform growth seminorms. (For the exact description, see [5].) We have

Proposition 2.1. (1) *$\mathcal{A} \subseteq \mathcal{A}^\infty$; in fact $\mathcal{A} = \{\phi \in \mathcal{A}^\infty | \phi$ is K-finite$\}$.*
(2) *$\mathcal{A}$ is dense in $\mathcal{A}^\infty$.*

Finally, we give one definition of an automorphic representation. Recall that in a category, a subquotient is a quotient of a subobject.

Definition . A smooth automorphic representation (π, V) of $G(\mathbb{A})$ is a closed irreducible sub-quotient of $\mathcal{A}^\infty(G(F)\backslash G(\mathbb{A}), \omega)$.

Remark 2.1. Instead of enlarging $\mathcal{A}$, we could instead have worked directly with $\mathcal{A}$ and simply acknowledged that we did not have a right regular action of all of $G(\mathbb{A})$ but only of $G(\mathbb{A}_f)$. Though we do not have a right action of $G(F_\infty)$, we have some structure. Namely, there is an action of the Lie algebra $\mathfrak{g}$ and also of the maximal compact group K_∞ on $\mathcal{A}$, and the two actions are compatible. Thus at the archimedean places we have a structure known as a $(\mathfrak{g}, K_\infty)$ module. One can define automorphic representations of $\mathcal{A}$ as subquotients in the category of objects with these actions. This can also be rephrased in terms of the global Hecke algebra.

We may also consider the subspace $\mathcal{A}_0$ (resp. $\mathcal{A}_0^\infty$) of $\mathcal{A}$ (resp. of $\mathcal{A}^\infty$) consisting of cusp forms. Since the condition for cuspidality is the vanishing of integrals of the form $\int \phi(ug)\,du$ while the actions we are considering are right actions, these spaces are indeed submodules. An irreducible sub-quotient of $\mathcal{A}_0^\infty$ is called a *cuspidal automorphic representation*.

In fact, there are a number of subspaces of interest in the theory, and we wish to set them out now. First, let Z denote the center of G, and fix a unitary character ω of $Z(F)\backslash Z(\mathbb{A})$. We write $\mathcal{A}(\omega)$ and $\mathcal{A}^\infty(\omega)$ to indicate the subspaces with central character ω.

Second, recall that every cusp form is rapidly decreasing, hence L^2, but the full space of L^2 functions is larger. Let $L^2(\omega) = L^2(G(F)\backslash G(\mathbb{A}), \omega)$ denote the Hilbert space of all measurable functions on $G(F)\backslash G(\mathbb{A})$ such that $\phi(zg) = \omega(z)\,\phi(g)$ for all $z \in Z(\mathbb{A})$ and such that

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} |\phi(g)|^2\,dg < \infty.$$

Also, let $L_0^2(\omega)$ be the subspace of $L^2(\omega)$ consisting of L^2 functions which are cuspidal. (Note that these functions are not required to be right K_f-finite.) Then $G(\mathbb{A})$ acts via the right regular representation on $L^2(\omega)$ and on $L_0^2(\omega)$. Then we have

$$\mathcal{A}_0(\omega) \subseteq \mathcal{A}_0^\infty(\omega) \subseteq L_0^2(\omega) \subseteq L^2(\omega).$$

It is a consequence of the generalization of Theorem 2 that the subspace of smooth vectors in $L^2_0(\omega)$ is the space $\mathcal{A}^\infty_0(\omega)$, and thus the subspace of K-finite vectors in $L^2_0(\omega)$ is $\mathcal{A}_0(\omega)$. (Here we omit some details related to Frechet topologies.)

We have the fundamental decomposition theorem for L^2 cusp forms due to Gelfand and Piatetski-Shapiro with respect to the right action of $G(\mathbb{A})$.

Theorem 3. *The space $L^2_0(\omega)$ decomposes into a discrete Hilbert space direct sum of irreducible unitary sub-representations*

$$L^2_0(\omega) = \oplus m(\pi) V_\pi \tag{2.2}$$

with the multiplicities $m(\pi)$ finite.

Taking the subspaces of smooth (resp. K-finite) vectors in the decomposition (2.2) we arrive at decompositions

$$\mathcal{A}^\infty_0(\omega) = \oplus m(\pi) V^\infty_\pi, \qquad \mathcal{A}_0(\omega) = \oplus m(\pi)(V_\pi)_K.$$

The irreducible constituents are called the unitary smooth (resp. K-finite) cuspidal representations of $G(\mathbb{A})$.

Suppose one starts with a classical cusp form $f(z)$ (either holomorphic or a Maass form) for a congruence subgroup of $SL_2(\mathbb{Z})$. Then the closure of the span of the right action of $GL_2(\mathbb{A})$ on the associated adelic function ϕ gives a subspace V_f of the space of cuspidal automorphic form on $GL_2(\mathbb{A})$. In general, it is not irreducible. However, *if f is a Hecke eigenform* then in fact V_f is irreducible.

2.2. **Unramified Principal Series.** Suppose that f is a classical cusp form which is also a Hecke eigenform. Then all Fourier coefficients of f may be determined by the coefficients of f at primes p. Our next task is to make an analogous statement for automorphic representations, decomposing them in terms of representations of $G(F_v)$ as v runs over the places of F. We shall focus on what happens at the nonarchimedean places, suppressing some technicalities at the archimedean places.

For convenience, in this subsection we take F to be a nonarchimedean local field with ring of integers O and local uniformizer ϖ (dropping the subscript v). We show how to construct a family of representations that will turn out to be the local constituents of an automorphic representation at primes not dividing the level. For simplicity we restrict to the groups $GL_n(F)$, $n > 1$, but the ideas work more generally.

Let T be the subgroup of $GL_n(F)$ consisting of diagonal matrices. Recall that a character $\chi : F^\times \to \mathbb{C}^\times$ is called *unramified* if it is trivial on the subgroup of units $O^\times$. Such a character is determined by its value on ϖ, so that in general $\chi(a) = |a|^c$ for some complex constant c. If $\chi_1, \cdots, \chi_n$ are unramified quasicharacters, let χ be the character of T given by

$$\chi(\mathrm{diag}(t_1, \dots, t_n)) = \chi_1(t_1) \dots \chi_n(t_n).$$

Let B denote the (standard Borel) subgroup of $GL_n(F)$ consisting of upper triangular matrices, and let U denote the subgroup of B consisting of unipotent matrices (that is, upper triangular matrices whose diagonal entries are all 1). Note that $B = TU$. Let δ_B be the modular function of the Borel subgroup of GL_n, that is the quotient of right Haar measuer by left Haar measure on B, given explicitly by the formula

$$\delta_B(t_1, \dots, t_n) = |t_1|^{n-1}|t_2|^{n-3} \dots |t_n|^{1-n}.$$

Let $I(\chi)$ denote the space of locally constant functions $\varphi : GL_n(F) \to \mathbb{C}$ such that

$$\varphi(tug) = \delta_B(t)^{1/2}\chi(t)\varphi(g) \text{ for all } t \in T, u \in U.$$

(Incorporating the $\delta_B^{1/2}$ into the definition turns out to be convenient, and is standard.) Then $GL_n(F)$ acts on $I(\chi)$ by the right regular representation. The module $I(\chi)$ is called the *unramified principal series*.

Fix $K = G(O)$. Then it can be shown that K is a maximal compact subgroup of $GL_n(F)$ (called the hyperspecial maximal compact).

Definition . Suppose F is non-archimedean. An (irreducible) representation (π, V) of $GL_n(F)$ is *class one* if $V^K \neq 0$. It is *admissible* if the subgroup of $GL_n(F)$ fixing any vector $v \in V$ is open and the subspace of vectors fixed by any compact open subgroup of $GL_n(F)$ is finite dimensional.

Exercise 6. (1) *Show that $I(\chi)$ is class one, and moreover that the space of K-fixed vectors of $I(\chi)$ is exactly one-dimensional. (We call the unique vector ϕ in $I(\chi)$ such that $\phi(e) = 1$ the normalized spherical vector.)*

(2) *Show that $I(\chi)$ is admissible.*

(3) (Whittaker function on $GL_2(F)$) *For $i = 1, 2$, let $\chi_i(a) = |a|^{z_i}$, where z_i are complex numbers, and let ϕ be the normalized spherical vector in $I(\chi)$. Let ψ be an additive character on F that is trivial on O but non-trivial on $\varpi^{-1}O$. Show that the p-adic integral*

$$W(g) = \int_F \phi\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(x)\, dx$$

converges absolutely for $\Re(z_1 - z_2) > 0$. Compute $W(diag(y, 1))$. Hint: Break the integral up into pieces. For example, in the case $|y| = 1$, break the integral over F into the pieces $x \in O$ and $x \in \varpi^{-m}O^\times$ with $m \geq 1$.

If $w \in GL_n(F)$ normalizes T then one can define a new character χ^w by $\chi^w(t) = \chi(wtw^{-1})$. Of course, if $w \in T$, then $wtw^{-1} = t$ for all $t \in T$, and so $\chi^w = \chi$. A character χ is said to be in *general position* if $\chi \neq \chi^w$ for all w which normalize T such that $w \notin T$. Then one has the following general theorem [1].

Theorem 4. *Suppose that χ is in general position. Then $I(\chi)$ is irreducible.*

2.3. **Tensor Product Theorem.** We return to the global situation. We are familiar with the tensor product of a finite number of representations. To adopt this notion to an infinite number of representations, given a collection of class one representations (π_v, V_v) for v in an (infinite) set S of places of F let $\xi_v \in V_v^{K_v}$ be a nonzero vector. Then one may form the restricted tensor product $V := \otimes'_{v\in S} V_v$ relative to the vectors $\{\xi_v\}$, consisting of the vector space spanned by tensors $\otimes_{v\in S} x_v$ where $x_v \in V_v$ for all $v \in S$ and $x_v = \xi_v$ for almost all v. The restricted product $\prod'_{v\in S} G(F_v)$ acts on V since by definiton if $(g_v)_{v\in S}$ is in this product, then $g_v \in K_v$ for almost all v and so g_v fixes ξ_v for almost all v. We write $\pi = \otimes_{v\in S}\pi_v$ for the resulting representation.

In the next theorem we explain that an irreducible smooth automorphic representation of $GL_n(\mathbb{A})$ has such a structure.

Theorem 5. *Let (π, V_π) be an irreducible smooth automorphic representation of $GL_n(\mathbb{A})$. Then there exist irreducible admissible smooth representations (π_v, V_v) of $GL_n(F_v)$ for all places v such that π_v is (a component of) an unramified principal series representation for $GL_n(F_v)$ for almost all v and such that π is the restricted tensor product $\pi = \otimes'\pi_v$. (Here we take ξ_v to be the normalized spherical vector.)*

When we decompose an automorphic representation corresponding to a newform f into local constituents, the places v for which the representation π_v is unramified principal series are exactly the places v corresponding to primes p_v that do not divide the level of f. The character χ_v at such a place v is determined by the Hecke eigenvalue at p_v.

3. Lecture 3: Automorphic Forms and L-Functions on GL_n

Dirichlet invented the L-functions that bear his name in the course of proving his theorem on primes in arithmetic progressions. Since an idele class character is an automorphic form on GL_1, the theory of Dirichlet and, more generally, Hecke L-functions is tantamount to a theory of L-functions attached to an automorphic form on GL_1. We begin by briefly summarizing aspects of this theory.

Tate showed how to use adelic and p-adic integrals to establish (and better understand) the properties of the Hecke L-functions $L(s,\omega)$ attached to an idele class character $\omega = \prod_v \omega_v$ of F. A key aspect of Tate's approach is the passage between global and local: Tate's global (adelic) integral representing $L(s,\omega)$ may be expressed in terms of integrals over the p-adic groups $F_v^\times$ where v runs over the places of F. An important consequence is that the *global epsilon factor* that appears in the functional equation

$$L(s,\omega) = \epsilon(s,\omega)\, L(1-s,\omega^{-1})$$

is seen to factor as a product of *local epsilon factors* attached to the local characters ω_v. This factorization is not quite canonical but depends on a choice of global additive character ψ of $F\backslash\mathbb{A}$. Write $\psi = \prod_v \psi_v$. Then Tate shows that

$$\epsilon(s,\omega) = \prod_v \epsilon_v(s,\omega_v,\psi_v),$$

where the product is finite since if ω_v is unramified and the conductor of ψ_v is O_v then $\epsilon(s,\omega_v,\psi_v) = 1$.

The theory of L-functions attached to classical modular forms goes back to Ramanujan and Hecke. Recall that the L-function may be given as a Mellin transform and its analytic properties obtained from this expression and modularity. This can be extended to other classes of automorphic forms on GL_2 over an arbitrary number field F. However, the relation to Tate's work on GL_1 is not immediately apparent.

Jacquet and Langlands were the first to systematically address the theory of automorphic forms over $GL_2(\mathbb{A})$ in the style of Tate, using the tensor product theorem, Theorem 5. If $\pi = \otimes' \pi_v$ then they attach L-factors to each π_v such that the global L-function $L(s,\pi) = \prod_v L(s,\pi_v)$ has analytic continuation and functional equation. The local L-factors are easiest when π_v is unramified principal series. Once again, the epsilon factor also is seen to be a product of local epsilon factors. This theory was generalized to GL_n by Jacquet, Piatetski-Shapiro and Shalika. In this lecture, we recap the generalization of the theory of L-functions attached to automorphic forms on GL_n.

3.1. **The Partial L-function.**

Let π be an irreducible cuspidal smooth automorphic representation of $GL_n(\mathbb{A})$. Then there is a finite set S of places of F containing all archimedean places such that for all $v \notin S$, π_v is the unramified principal series $I(\chi_v)$. For convenience we shall always assume that $I(\chi_v)$ is irreducible.

Choose a local uniformizer ϖ_v for F_v, and let q_v be the cardinality of $O_v/\varpi_v O_v$. Suppose that on the torus T,

$$\chi_v(\mathrm{diag}(t_1,\dots,t_n)) = \prod_{i=1}^{n} \chi_{i,v}(t_i),$$

where the χ_i are unramified quasicharacters, that is homomorphisms $F_v^\times \to \mathbb{C}^\times$ which are trivial on $O_v^\times$. Then we define the *local L-factor*

$$L(s,\pi_v) = \prod_{i=1}^{n} \left(1 - \chi_i(\varpi_v) q_v^{-s}\right)^{-1}.$$

Note that since χ_i is unramified, this is well-defined. (In fact, the character χ is itself determined only up to a permutation of the χ_i but the product is independent of this choice as well.) Then the partial *global L-function* for π is given by

$$L^S(s,\pi) = \prod_{v\notin S} L(s,\pi_v).$$

This product may be seen to converge for $\Re(s)$ sufficiently large.

Exercise 7. (1) *Suppose that (π, V) is as above and ω is an idele class character. Define the twist of π by ω to be the representation whose functions are given by $g \mapsto \phi(g)\omega(\det(g))$ for $\phi \in V$. Show that the twist is again a smooth automorphic representation. How is the partial L-function of the twist of π by ω related to the partial L-function for π?*

(2) *Suppose that (π, V) is as above. For ϕ in V, let $\tilde{\phi}$ denote the function $g \mapsto \phi({}^T g^{-1})$ where ^{T} denotes the transpose. Let $\widetilde{V}$ denote the space of functions $\tilde{\phi}$ for $\phi \in V$. Show that $\widetilde{V}$ is a space of automorphic forms. The associated automorphic representation is written $\widetilde{\pi}$ and is called the* contragredient *automorphic representation. How is the partial L-function for $\widetilde{\pi}$ related to the partial L-function for π?*

3.2. **Rankin-Selberg L-functions.** Classically, given two modular cusp forms f_1 and f_2, the Rankin-Selberg method attaches an L-function that is of degree 4 in p^{-s} at almost all places. In fact, this L-function may be obtained by integrating $f_1(z)\overline{f_2}(z)$ against the Eisenstein series $E(z, s)$ described in Lecture 1. The analytic continuation and functional equation of the resulting Dirichlet series then follows from Corollary 1.2. The decomposition into local L-factors is accomplished by Hecke theory.

Similarly, given two cuspidal automorphic representations π and π' on $GL_n(\mathbb{A})$ and $GL_m(\mathbb{A})$, resp., let S be a set of places as above such that if $v \notin S$, then both π_v and π'_v are unramified principal series: $\pi_v = I(\chi_v)$, $\pi'_v = I(\chi'_v)$. Here χ_v is as above and χ'_v is obtained from quasicharacters $\chi'_{j,v}$, $1 \leq j \leq m$. Then one defines the local and partial Rankin-Selberg L-functions by

$$L(s, \pi_v, \pi'_v) = \prod_{i=1}^{n} \prod_{j=1}^{m} \left(1 - \chi_{i,v}(\varpi_v)\chi'_{j,v}(\varpi_v)q_v^{-s}\right)^{-1}$$

and

$$L^S(s, \pi, \pi') = \prod_{v \notin S} L(s, \pi_v, \pi'_v).$$

Once again the partial L-function converges for $\Re(s)$ sufficiently large.

A helpful way to view these expressions is as follows. If $\pi_v = I(\chi_v)$ as above, let A_{π_v} be the semisimple conjugacy class in $GL_n(\mathbb{C})$ containing the diagonal matrix $\operatorname{diag}(\chi_{1,v}(\varpi_v), \ldots, \chi_{n,v}(\varpi_v))$. Similarly define the conjugacy class $A_{\pi'_v}$ in $GL_m(\mathbb{C})$. Then

$$L(s, \pi_v) = \left(\det(I_n - A_{\pi_v} q_v^{-s})\right)^{-1}, \qquad L(s, \pi_v, \pi'_v) = \left(\det(I_{nm} - A_{\pi_v} \otimes A_{\pi'_v} q_v^{-s})\right)^{-1}.$$

Here the local factors are realized as (the reciprocals of) characteristic polynomials attached to the semisimple conjugacy classes A_{π_v} (resp. $A_{\pi_v} \otimes A_{\pi'_v}$), evaluated at

q_v^{-s}. For unramified local representations π_v of more general types of groups G, the semisimple conjugacy class attached to π_v sits in the *Langlands dual group* of G, rather than the complex points of G. This theme will be taken up further in the second day of this course.

Exercise 8. *Let F be a non-archimedean local field and let $W(g)$ be the Whittaker function on $GL_2(F)$ defined in Exercise 6. Compute the Mellin transform*

$$\int_{F^\times} W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) |y|^{s-1}\, dy$$

in terms of the local L-function for $I(\chi)$.

3.3. **Ramified L-factors.** To get a satisfactory theory, one would like to define $L(s,\pi_v)$ or $L(s,\pi_v,\pi'_v)$ at the remaining places. We will not have time to do this in any detail. Instead we describe the main steps without details. We focus on the theory for finite but ramified places. For archimedean places, see the discussion in [5], Ch. 8. At those places the local factors may be expressed in terms of Gamma functions.

First, generalizing the integrals of Hecke and Rankin-Selberg, one defines a *family* of global integrals $I(s,\phi,\phi')$ as ϕ runs over the space of V and ϕ' runs over the space of V'. (One proceeds analogously for $L(s,\pi)$.) From the integral expression the integrals giving $I(s,\phi,\phi')$ may be shown to be absolutely convergent for all complex s and satisfy the functional equation

$$I(s,\phi,\phi') = \widetilde{I}(1-s,\widetilde{\phi},\widetilde{\phi}'),$$

where $\widetilde{I}$ is another family of global integrals. Next, if ϕ and ϕ' are pure tensors (in the classical language, this means that they are both Hecke eigenforms) then the integrals $I(s,\phi,\phi')$ are seen to factor into local integrals. In this way one arrives at a *family of local integrals* attached to π_v and π'_v. (One could handle the family $\widetilde{I}(s,\widetilde{\phi},\widetilde{\phi}')$ in a similar way, but we focus on the first family of integrals for the moment.) The local integrals depend on a local additive character ψ_v but we suppress this from the notation.

One proceeds to analyze the space of local integrals. Suppose that v is a finite place of F. One shows that each local integral is a rational function in q_v^s, and moreover that the space of local integrals is a $\mathbb{C}[q_v^s,q_v^{-s}]$-*fractional ideal* in $\mathbb{C}(q_v^{-s})$ containing the constant 1. Now the ring $\mathbb{C}[q_v^s,q_v^{-s}]$ is a principal ideal domain, so this fractional ideal has a generator. Since it contains 1, we may choose the generator to be of the form $1/P_v(q^{-s})$ where $P_v(x) \in \mathbb{C}[x]$ has constant term 1. We then *define* the local L-factor $L(s,\pi_v,\pi'_v)$ to be $P_v(q_v^{-s})$. This works for π'_v on $GL_1(F_v)$ in particular. If we choose π'_v to be the trivial character we obtain, by definition, $L(s,\pi_v)$. It can be shown (this takes some work) that if π_v, π'_v are unramified principal series and the local additive character has conductor O_v then this local factor matches the definition given in the previous subsection.

We may now define the *complete L-functions* $L(s,\pi)$ and $L(s,\pi,\pi')$ to be the product of the local L-functions over *all* places of F.

As preparation for the global functional equation, one proves a local functional equation. This involves a local epsilon factor $\epsilon(s,\pi_v,\pi'_v,\psi_v)$ that is shown to be of the form $a_v q_v^{-f_v s}$ for some constants a_v and non-negative integers f_v. The local epsilon factor is the constant 1 when all data is unramified.

3.4. **Global Properties.** The global L-functions have analytic continuation and functional equation. Suppose $\pi = \otimes' \pi_v$, $\pi' = \otimes' \pi'_v$ are cuspidal automorphic representations of $GL_n(\mathbb{A})$, $GL_m(\mathbb{A})$ resp., and let $\psi = \prod_v \psi_v$ be an additive character on $F\backslash\mathbb{A}$. Define the global epsilon factor

$$\epsilon(s,\pi,\pi') = \prod_v \epsilon(s,\pi_v,\pi'_v,\psi_v).$$

Since almost all factors in this product are 1, the product makes sense. It may be seen that the product is independent of the choice of ψ. In fact from the local analysis, it follows that the global epsilon factor is of the form $WN^{1/2-s}$ where W is a complex number of absolute value 1 and N is a positive integer. Then one has

Theorem 6. *The L-function $L(s,\pi,\pi')$, defined by a convergent Euler product for $\Re(s)$ sufficiently large, has meromorphic continuation to all complex s and satisfies the functional equation*

$$L(s,\pi,\pi') = \epsilon(s,\pi,\pi')\, L(1-s,\widetilde{\pi},\widetilde{\pi'}).$$

Moreover, $L(s,\pi,\pi')$ is entire if $m \neq n$, while if $m = n$ then $L(s,\pi,\pi')$ is entire except for two possible simple poles, which occur if and only if there exists $\sigma \in \mathbb{R}$ such that $\widetilde{\pi} \cong \pi' \otimes |\det|^{\sigma}$. In that case the simple poles are at $s = i\sigma$ and $s = 1 - i\sigma$. Moreover, the function $L(s,\pi,\pi')$ is bounded in vertical strips of finite width away from its poles.

In the special case $m = 1$, we get

Corollary 3.1. *The L-function $L(s,\pi)$, which is defined by an Euler product (with almost all Euler factors of degree n in q_v^{-s}) that is absolutely convergent for $\Re(s)$ sufficiently large, has analytic continuation to an entire function of s, and satisfies the functional equation*

$$L(s,\pi) = \epsilon(s,\pi)L(1-s,\widetilde{\pi}).$$

Remark 3.1. Corollary 3.1 is due to Godement and Jacquet, who showed that the L-function $L(s,\pi)$ could be represented by another integral, based on matrix coefficients, that is a close generalization of Tate's thesis. A key step is a Poisson summation formula. An account of the Godement-Jacquet theory may be found in [7], Vol. II. Theorem 6 and Corollary 3.1 hold for all number fields F.

Remark 3.2. *(Another approach to L-functions.)* It is also possible to establish the properties of the L-functions treated here by a different method, the *Langlands-Shahidi method*. In this method, one use π and π' to construct a family of Eisenstein series $E(s,\phi,\phi')$ on the group $GL_{n+m}(\mathbb{A})$. This series is realized as a sum over $P_{n,m}(F)\backslash GL_{n+m}(F)$, where $P_{n,m}(F)$ is the subgroup

$$P_{n,m} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mid A \in GL_n(F), B \in \mathrm{Mat}_{n\times m}(F), D \in GL_m(F) \right\}.$$

Here one takes advantage of the automorphicity of $\phi \in V_\pi$ and $\phi' \in V_{\pi'}$ to build them into the Eisenstein series. On the one hand, one may continue this series by a subtle argument involving spectral theory. On the other, the Whittaker coefficients of this series may be seen to involve the L-functions $L(s,\pi,\pi')$. The desired properties follow. The local factors in this case are obtained by the study of intertwining operators and make use of the local uniqueness of the Whittaker model.

To conclude this lecture, let us return to Artin's Conjecture. If ρ is an n-dimensional irreducible Galois representation (and in particular one that does not contain the trivial representation) then Langlands conjectures that the Artin L-function $L(s,\rho)$ is an automorphic L-function $L(s,\pi)$ attached to a cuspidal automorphic representation π of $GL_n(\mathbb{A})$, where $\mathbb{A}$ is the adeles of the base field F. If this is known, then Artin's Conjecture that $L(s,\rho)$ is entire follows from Corollary 3.1.

References

[1] Bernstein, Joseph and Zelevinsky, Andrei, Representations of $GL(n,F)$ where F is a non-archimedean local field. Russian Math. Surveys 31 (1976), 1–68.

[2] Borel, Armand and Jacquet, Hervé. Automorphic forms and automorphic representations. Proc. Symp. Pure Math. 33, part 1, 189–207.

[3] Bump, Daniel. Automorphic Forms and Representations. Cambridge Studies in Advanced Mathematics 55. Cambridge University Press, 1997.

[4] Bump, Daniel; Cogdell, James W.; de Shalit, Ehud; Gaitsgory, Dennis; Kowalski, Emmanuel and Kudla, Stephen S. An introduction to the Langlands program. Lectures presented at the Hebrew University of Jerusalem, Jerusalem, March 12–16, 2001. Edited by Joseph Bernstein and Stephen Gelbart. Birkhäuser Boston, Inc., Boston, MA, 2003.

[5] Cogdell, James W. Lectures on L-functions, converse theorems, and functoriality for GL_n, in: Fields Inst. Monogr., 20, Amer. Math. Soc., Providence, RI, 2004, 1–96.

[6] Gelbart, Stephen. An elementary introduction to the Langlands Program, Bull. A.M.S. 10 (no. 2), 1984, 177–218.

[7] Goldfeld, Dorian and Hundley, Joseph. Automorphic representations and L-functions for the general linear group, Vols. I, II. Cambridge Studies in Advanced Mathematics 129. Cambridge University Press, 2011.