2^{ND} EU/US SUMMER SCHOOL ON AUTOMORPHIC FORMS
NOTES ON SINGULAR MODULI AND MODULAR FORMS

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Lecture 1: Modular Forms and Singular Moduli

A weakly holomorphic modular form $f(z)$ of weight $k \in 2\mathbb{Z}$ for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ is a holomorphic function on the upper half-plane $\mathcal{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ such that

$$(f \mid_k \gamma)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and

$$f(z) = \sum_{n \geq n_0} a(n) q^n, \ q = e(z) = e^{2\pi i z}$$

with $n_0 = \mathrm{ord}_\infty f$. Let $M_k^!$ denote the $\mathbb{C}$ vector space of all such forms. Any $f \in M_k^!$ satisfies the valence formula

$$\frac{k}{12} = \mathrm{ord}_\infty f + \sum_{z \in \mathcal{F}} \frac{\mathrm{ord}_z f}{w_z}, \tag{1}$$

where $\mathcal{F}$ is the fundamental domain and w_z is the order of the subgroup of $\Gamma/\{\pm \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)\}$ stabilizing z and given by

$$w_z = \begin{cases} 2, & z = i, \\ 3, & z = \rho, \\ 1, & \textit{otherwise}. \end{cases} \tag{2}$$

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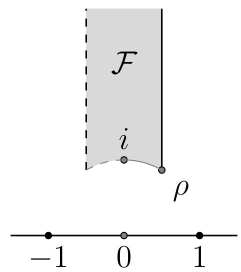


This is obtained by integrating $F(z) := \frac{f'(z)}{f(z)} - \frac{ik}{2y}$ around $\partial\overline{\mathcal{F}}$ and applying Green's Theorem.

Write $k = 12\ell + k'$ with $\ell \in \mathbb{Z}, k' \in \{0, 4, 6, 8, 10, 14\}$. Then equation (1) implies that $f(z) = 0$ whenever $\mathrm{ord}_\infty f > \ell$. Let $S_k \subset M_k \subset M_k^!$ be the subspaces of cusp forms and modular forms, i.e. $f \in S_k$, resp. $f \in M_k$, if $\mathrm{ord}_\infty f > 0$, resp. $\mathrm{ord}_\infty f \geq 0$.

To construct modular forms, one can start with Eisenstein series. For $k \geq 4$ even, let

$$E_k(z) := \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{1}{(cz+d)^k} = 1 + \frac{2}{\zeta(1-k)} \sum_{n\geq 1} \sigma_{k-1}(n) q^n \in M_k, \tag{3}$$

where $\zeta(s)$ is the Riemann zeta function and $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$ is the divisor sum function. The Fourier expansion of $E_k(z)$ can be derived from the following trigonometric identity

$$\frac{1}{z} + \sum_{d=1}^{\infty} \left(\frac{1}{z-d} + \frac{1}{z+d} \right) = \pi \cot(\pi z) = \pi i - 2\pi i \sum_{m=0}^{\infty} q^m. \tag{4}$$

When $k = 4$ and 6, we have

$$\begin{aligned} E_4(z) &= 1 + 240q + O(q^2), \\ E_6(z) &= 1 - 504q - O(q^2), \\ \mathrm{ord}_\rho E_4 &= \mathrm{ord}_i E_6 = 1. \end{aligned}$$

Define the modular discriminant function $\Delta(z) \in M_{12}$ and modular j-invariant $j(z)$ by

$$\begin{aligned} \Delta(z) &:= \frac{E_4^3(z) - E_6^2(z)}{1728} = q - 24q^2 + 252q^3 - 1472q^4 + \dots, \\ j(z) &:= \frac{E_4^3(z)}{\Delta(z)} = q^{-1} + 744 + \sum_{n\geq 1} c(n) q^n. \end{aligned}$$

Since $\mathrm{ord}_\infty \Delta(z) = 1$, equation (1) implies that $\Delta(z)$ has no zeros in $\mathcal{H}$ and $j(z) \in M_0^!$.

For each $m \geq 0$, there is a unique $j_m \in M_0^!$ such that

$$j_m(z) = q^{-m} + O(q) = F_m(j(z))$$

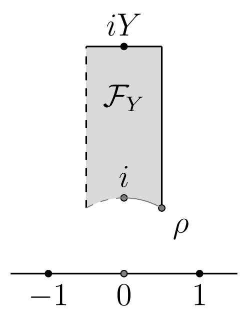
with $F_m(x) \in \mathbb{Z}[x]$ a monic polynomial. The existence is easily shown by induction, e.g.

$$F_0(x) = 1, F_1(x) = x - 744, F_2(x) = x^2 - 1488x + 159768.$$

The $F_m(x)$'s are called *Faber polynomials.* Their uniqueness follows from equation (1) and that $\{j_m\}_{m\geq 0}$ is a basis for $M_0^!$. So any $f(z) = \sum a(n)q^n \in M_0^!$ having $a(n) \in \mathbb{Z}$ can be expressed uniquely as $f(z) = P(j(z))$ with $P(x) \in \mathbb{Z}[x]$. This is called the "q-expansion principle". The generating function of $j_m(z)$ is given by

$$\frac{-j'(z)}{j(z) - j(\tau)} = \sum_{m\geq 0} j_m(\tau)q^m = \sum_{m\geq 0} F_m(j(\tau))q^m, \tag{5}$$

which comes from integrating $\frac{j_m(z)j'(z)}{j(z)-j(\tau)}$ around $\partial\overline{\mathcal{F}_Y}$, the boundary of the closure of the truncated fundamental domain $\mathcal{F}_Y$.



Letting $\tau = i$ and ρ yields the q-expansion of various modular forms

$$\sum_{m\geq 0} j_m(i)q^m = \frac{-j'(z)}{j(z) - 1728} = \frac{E_4^2(z)E_6(z)}{\Delta(z)} \cdot \frac{\Delta(z)}{E_6^2(z)} = \frac{E_4^2(z)}{E_6(z)},$$

$$\sum_{m\geq 0} j_m(\rho)q^m = \frac{-j'(z)}{j(z)} = \frac{E_4^2(z)E_6(z)}{\Delta(z)} \cdot \frac{\Delta(z)}{E_4^3(z)} = \frac{E_6(z)}{E_4(z)}.$$

Remark 1. The generating function generalizes to other nonzero weight k as well.

A CM point is $\tau \in \mathcal{H}$ such that $A\tau^2 + B\tau + C = 0$, $A, B, C \in \mathbb{Z}$, $A > 0$. Thus, $\tau = \frac{-B+\sqrt{D}}{2A}$ with $D = B^2 - 4AC \equiv 0, 1 \pmod 4$ negative and $|D|$ not a square.

Theorem 2. *If τ is a CM point, then $j(\tau)$ is an algebraic integer.*

Example 3. If D is even, then $\tau = \frac{\sqrt{D}}{2}$, otherwise $\tau = \frac{1+\sqrt{D}}{2}$.

D	-3	-4	-7	-8	-11	-12	-15
τ_0	ρ	i	$\frac{1+\sqrt{7}i}{2}$	$\sqrt{2}i$	$\frac{1+\sqrt{11}i}{2}$	$\sqrt{3}i$	$\frac{1+\sqrt{15}i}{2}$
$j(\tau_0)$	0	1728	-3375	8000	-32768	54000	$\frac{-191025-85995\sqrt{3}}{2}$

As preparation for the proof of Theorem 2, consider the following polynomial in x for $m \geq 1$

$$\prod_{\substack{ad=m \\ 1\leq b\leq d}} \left(x - j\left(\frac{az+b}{d}\right)\right) = \sum_{r=0}^{\sigma(m)} f_r(z)x^r.$$

Now the set $\{\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) : ad = m, 1 \leq b \leq d\}$ is a complete set of representatives for $\Gamma\backslash\Gamma_m$, where

$$\Gamma_m := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{Z}) : ad - bc = m \right\}.$$

Each $f_r(z)$ is a symmetric function in $\{j(\gamma z) : \gamma \in \Gamma\backslash\Gamma_m\}$. Since $\gamma'\Gamma_m(\gamma')^{-1} = \Gamma_m$ for any fixed $\gamma' \in \Gamma$ and that $f_r(z)$ has at most exponential growth at ∞, we know that $f_r(z) \in M_0^!$ for each $0 \leq r \leq \sigma(m)$.

For fixed a, d, consider

$$\begin{aligned} \prod_{1\leq b\leq d} \left(x - j\left(\frac{az+b}{d}\right)\right) &= (-1)^d \prod_{1\leq b\leq d} \left(e\left(\tfrac{-b}{d}\right) q^{-\frac{a}{d}} + (744 - x) + \sum_{n\geq 1} c(n) e\left(\tfrac{bn}{d}\right) q^{\frac{na}{d}}\right) \\ &= (-1)^d q^{-a} + \sum_{n\geq -d+1} A_n(x) q^{\frac{na}{d}}, \end{aligned} \tag{6}$$

where $A_n(x) \in R[x]$, $R = \mathbb{Z}[e\left(\frac{1}{d}\right)]$. From equation (6), we see that $A_n(x)$ is left invariant under all Galois automorphisms of $\mathbb{Q}(e^{2\pi i/d})$, which implies that $A_n(x) \in \mathbb{Z}[x]$. Also $A_n(x)$ vanishes unless $d \mid n$ since the left hand side of equation (6) is invariant under $z \mapsto z + 1$. Thus, we could write

$$\prod_{1\leq b\leq d} \left(x - j\left(\frac{az+b}{d}\right)\right) = (-1)^d q^a + \sum_{n\geq 0} A_{nd}(x) q^{na}.$$

It follows that $f_r(z) = \sum_n a_r(n) q^n$ with $a_r(n) \in \mathbb{Z}$ and that there exists $P_r(x) \in \mathbb{Z}[x]$ satisfying

$$f_r(z) = P_r(j(z))$$

by the q-expansion principle. In particular, $P_{\sigma(m)}(x) = 1$ and

$$\begin{aligned} -P_{\sigma(m)-1}(j(z)) &= \sum_{\substack{ad=m \\ 1\leq b\leq d}} j\left(\frac{az+b}{d}\right) = 744\sigma(m) + \sum_{\substack{ad=m \\ 1\leq b\leq d}} j_1\left(\frac{az+b}{d}\right) \\ &= 744\sigma(m) + j_m(z). \end{aligned} \tag{7}$$

For example when $m = 2$, $\sigma(m) = 3$ and

$$\left(x - j\left(\frac{z}{2}\right)\right)\left(x - j\left(\frac{z+1}{2}\right)\right)(x - j(2z)) = x^3 + P_2(j(z))x^2 + P_1(j(z))x + P_0(j(z)),$$

where

$$\begin{aligned}-P_2(j(z)) &= j\left(\frac{z}{2}\right) + j\left(\frac{z+1}{2}\right) + j(2z)\\ &= \left(q^{-\frac{1}{2}} + 744 + O(q)\right) + \left(-q^{-\frac{1}{2}} + 744 + O(q)\right) + (q^{-2} + 744 + O(q))\\ &= q^{-2} + 3\cdot 744 + O(q)\\ &= j(z)^2 - 1488j(z) + 162000.\end{aligned}$$

Similarly, we have

$$\begin{aligned}P_1(j(z)) &= 1488j(z)^2 + 40773375j(z) + 8748000000,\\ P_0(j(z)) &= j(z)^3 - 162000j(z)^2 + 8748000000j(z) - 157464000000000.\end{aligned}$$

We thus have

$$\prod_{\substack{ad=m\\1\le b\le d}} \left(x - j\left(\frac{az+b}{d}\right)\right) = \Psi_m(x, j(z)),$$

where $\Psi_m(x,y) \in \mathbb{Z}[x,y]$ has degree $\sigma(m)$ in x. For example, the calculations above tells us that

$$\begin{aligned}\Psi_2(x,y) = &-x^2y^2 + x^3 + y^3 + 1488(x^2y + xy^2) - 162000(x^2+y^2) + 40773375xy\\ &+ 8748000000(x+y) - 157464000000000.\end{aligned}$$

Remark 4. It turns out that $\Psi_m(x,y) = \Psi_m(y,x)$. Since CM points are fixed points of transformations in Γ_m, we will consider $\Psi_m(x,x)$.

Lemma 5. *For m not a square, the polynomial $\pm\Psi_m(x,x)$ is monic of degree*

$$G(m) := \sum_{d|m} \max\left(d, \tfrac{m}{d}\right). \tag{8}$$

Remark 6. When $m = 2$, we have

$$\Psi_2(x,x) = -(x+3375)^2(x-1728)(x-8000),$$

which has degree $G(2) = 4$.

Proof. By definition, we have

$$\begin{aligned}\Psi_m(j(z), j(z)) &= \prod_{ad=m}\prod_{1\le b\le d}\left(j(z) - j\left(\frac{az+b}{d}\right)\right)\\ &= \prod_{ad=m}\prod_{1\le b\le d}\left(q^{-1} - e\left(\tfrac{-b}{d}\right)q^{-\frac{a}{d}} + O(1)\right).\end{aligned}$$

Using the fact that $\prod_{1\le b\le d}\left(x-e\left(\frac{-b}{d}\right)y\right)=x^d-y^d$, we obtain

$$\Psi_m(j(z),j(z))=\prod_{ad=m}\left(q^{-d}-q^{-a}+\text{ lower order terms }\right),$$

which implies the lemma. Note that $q^{-d}-q^{-a}$ is nonzero since m is not a perfect square. $\square$

Turning now to the proof of Theorem 2, let τ be a CM point with

$$A\tau^2+B\tau+C=0,\ D=B^2-4AC<0.$$

To realize τ as a fixed point of $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_m$, observe that

$$M\cdot\tau=\tau\Leftrightarrow c\tau^2+(d-a)\tau-b=0.$$

This happens if

$$M=\begin{pmatrix}\frac{1}{2}(t-Bu)&-Cu\\Au&\frac{1}{2}(t+Bu)\end{pmatrix}$$

for $m=\frac{t^2-Du^2}{4}, t=\mathrm{Tr}M, u\in\mathbb{Z}$ nonzero and $t\equiv Du\pmod 2$. For a fixed D, choose some t,u such that $t\equiv Du\pmod 2$ and $t^2-Du^2>0$ is not a square. Since $j(M\tau)=j(\tau)$, we have $\Psi_m(j(\tau),j(\tau))=0$, where $m=\frac{t^2-Du^2}{4}$. Since $\pm\Psi_m(x,x)$ is monic integral, we see that $j(\tau)$ is an algebraic integer.

For any $D<0$, $D\equiv 0,1\pmod 4$, let

$$\mathcal{Q}_D:=\{Q(x,y)=Ax^2+Bxy+Cy^2: A>0, B^2-4AC=D, \gcd(A,B,C)=1\}.$$

The set $\mathcal{Q}_D$ is acted on by $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ via $(\gamma\cdot Q)(x,y)=A'x^2+B'xy+C'y^2$ such that

$$Q(x',y')=A'x^2+B'xy+C'y^2,$$
$$\begin{pmatrix}x'\\y'\end{pmatrix}:=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

This translates into linear fractional action on the associated CM point

$$\tau_Q=\frac{-B+\sqrt{D}}{2A}\in\mathcal{H}.$$

The number of equivalence classes is denoted by $h(D)$ and is finite. We may choose the representatives in $\mathcal{F}$. The class polynomial is defined by

$$H_D(x):=\prod_{Q\in\Gamma\backslash\mathcal{Q}_D}(x-j(\tau_Q)). \tag{9}$$

The following result is proven using the theory of CM elliptic curves.

Theorem 7. *The polynomial $H_D(x)\in\mathbb{Z}[x]$ is monic and irreducible.*

As a consequence, $j(\tau_Q)$ has degree $h(D)$ over $\mathbb{Q}$ with conjugates $j(\tau_{Q'})$, $Q' \not\sim Q$. A deeper fact is that the field $H := \mathbb{Q}(\sqrt{D}, j(\tau_Q))$ is abelian over $K = \mathbb{Q}(\sqrt{D})$. If D is fundamental, then H is the Hilbert class field of K, i.e. its maximal unramified abelian extension. Furthermore, the Galois group $\mathrm{Gal}(H/K)$ is isomorphic to the class group $\Gamma\backslash\mathcal{Q}_D$. For complete proofs see [6] or [16].

Lecture 2: Borcherds products

Recall that for any integer $D < 0$ such that $D \equiv 0, 1 \pmod 4$, we have the set

$$\mathcal{Q}_D = \{Q(x, y) = Ax^2 + Bxy + Cy^2 : A > 0, B^2 - 4AC = D, (A, B, C) = 1\},$$

and that the quadratic form $Q \in \mathcal{Q}_D$ has root $\tau_Q \in \mathcal{H}$. We have the class polynomial

$$H_D(x) = \prod_{Q\in\Gamma\backslash\mathcal{Q}_D} (x - j(\tau_Q)) = x^{h(D)} - \left(\sum_{Q\in\Gamma\backslash\mathcal{Q}_D} j(\tau_Q)\right) x^{h(D)-1} + \cdots .$$

We modify these definitions slightly now: let $\mathcal{Q}_D$ be the set of all positive definite binary quadratic forms of discriminant D, not just the primitive forms, and let $\omega_Q = 2$ if the form $Q \sim ax^2 + ay^2$, $\omega_Q = 3$ if $Q \sim ax^2 + axy + ay^2$, and $\omega_Q = 1$ otherwise. Modify the class polynomial to get

$$\mathcal{H}_D(x) = \prod_{Q\in\Gamma\backslash\mathcal{Q}_D} (x - j(\tau_Q))^{1/\omega_Q} .$$

This is still a polynomial unless $-D/3$ is a square (in which case it is a polynomial multiplied by $x^{1/3}$) or $-D$ is a square (in which case it is a polynomial multiplied by $(x - 1728)^{1/2}$). The class number $h(D)$ now becomes the Hurwitz class number $H(D) = \sum_{Q\in\Gamma\backslash\mathcal{Q}_D} \frac{1}{\omega_Q}$, where $H(0) = -\frac{1}{12}$.

Borcherds [3] proved the following theorem.

Theorem 8. *We have* $\mathcal{H}_D(j(z)) = q^{-H(D)} \prod_{n=1}^{\infty}(1 - q^n)^{a_{1/2}(-D,n^2)}$, *where* $a_{1/2}(-D, n^2)$ *is the Fourier coefficient of* q^{n^2} *of the unique weakly holomorphic modular form of weight* $1/2$ *and level* 4 *in the Kohnen plus space with Fourier expansion beginning* $q^D + O(q)$.

A more general version of the theorem is as follows. Let $f(z) = \sum c(n)q^n$ be a weakly holomorphic modular form with integer coefficients of weight $1/2$ and level 4 in the Kohnen plus space $M^!_{1/2}(4)$. Let h be the constant term of the product $f(z)\left(\sum_{n\geq 0} H(-n)q^n\right)$. Then

$$F(z) = q^{-h} \prod_{n>0}(1 - q^n)^{c(n^2)}$$

is a meromorphic modular form for some character of $\mathrm{SL}_2(\mathbb{Z})$ of weight $c(0)$; additionally, $F(z)$ has integer coefficients, and all of its zeros and poles are at cusps or imaginary quadratic irrationals. Specifically, if $\tau \in \mathcal{H}$ has discriminant $D < 0$, then $\mathrm{ord}_\tau(F) = \sum_{d>0} c(Dd^2)$.

This mapping is an isomorphism between the additive group of weakly holomorphic modular forms with integer coefficients in the Kohnen plus space $M^!_{1/2}(4)$ and the multiplicative group of meromorphic modular forms with integer coefficients, leading coefficient 1, and Heegner divisor.

Example. The Jacobi theta function $\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2} = 1+2q+2q^4+\dots$ is a modular form in $M_{1/2}(4)$. We look at $12\theta(z) = 12 + 24q + 24q^4 + 24q^9 + \dots$ and see that the isomorphism gives $h = -1$ and

$$F(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \Delta(z).$$

Example. The Eisenstein series $E_4(z)$ has a zero at $\rho = \frac{-1+i\sqrt{3}}{2}$ and has integer coefficients and leading coefficient 1. Expanding its Fourier series as an infinite product, we get

$$E_4(z) = 1 + 240q + 2160q^2 + 6720q^3 + \dots = (1-q)^{-240}(1-q^2)^{26760}(1-q^3)^{-4096240}\cdots.$$

Sure enough, there is a modular form of weight 1/2 with Fourier expansion

$$q^{-3} + 4 - 240q + 26760q^4 - 85995q^5 + 1707264q^8 - 4096240q^9 + \dots.$$

This can also be done for $E_6(z), E_8(z), E_{10}(z), E_{14}(z)$, and $j(z)$, among others, since we know exactly where their zeros and poles are in the fundamental domain. Note that the product $H_D(j(z)) = \prod(j(z) - j(\tau_Q))$ satisfies the conditions of the more general theorem.

Explicit bases for spaces of modular forms. To better understand this theorem, we now look at explicit bases for spaces of modular forms of integral and half integral weight.

For each even integer weight k, write $k = 12\ell + k'$, where $k', \ell \in \mathbb{Z}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. For each integer $m \geq -\ell$, there is a unique modular form $f_{k,m}(z) \in M^!_k$ with a Fourier expansion of the form

$$f_{k,m}(z) = q^{-m} + O(q^{\ell+1}).$$

We can construct such a form explicitly using Δ, j, and $E_{k'}$, where $E_0 = 1$; specifically, we have $f_{k,m} = \Delta^\ell E_{k'} F(j)$, where $F(x)$ is a monic polynomial in x of degree $\ell + m$ with integer coefficients. We write $f_{k,m}(z) = q^{-m} + \sum_{n>\ell} a_k(m,n)q^n$, so that all of the Fourier coefficients $a_k(m,n)$ of these basis elements are integers.

In SAGE, this is the Victor Miller basis for spaces of holomorphic modular forms; see also [9]. A generating function for these basis elements is given by

$$\sum_{m\geq -\ell} f_{k,m}(\tau)q^m = \frac{f_{k,-\ell}(\tau)f_{2-k,-1-\ell}(z)}{j(z)-j(\tau)}.$$

For half integral weight modular forms for $\Gamma_0(4)$, there is a similar canonical basis. For any integer s we let $M^!_{s+1/2}(4)$ be the vector space of holomorphic functions on $\mathcal{H}$ that transform like θ^{2s+1} under $\Gamma_0(4)$, may have poles at the cusps, and satisfy the Kohnen plus-space condition that their q-expansion is supported on integers n with $(-1)^s n \equiv 0, 1 \pmod 4$. (The automorphy factor for θ is $\left(\frac{c}{d}\right)\varepsilon_d^{-1}\sqrt{cz+d}$ for a matrix $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$.)

Recall that the Shimura correspondence gives an isomorphism between spaces of modular forms of integral and half integral weight; specifically, for positive integers s, the space of cusp forms of weight $s+1/2$ and level 4 in the Kohnen plus space is isomorphic to the space of cuspforms of weight $2s$ and level 1.

From this, we compute dimensions of spaces of cusp forms of half integral weight and level 4 in the Kohnen plus space. In level 4, the space of holomorphic modular forms (not necessarily in the plus space) can be constructed explicitly as products of powers of $\theta(z)$ and the weight 2, level 4 Eisenstein series given by

$$E(z) = \sum_{n=0}^{\infty} \sigma(2n+1) q^{2n+1}.$$

Once we have a set of holomorphic forms spanning the space, we take linear combinations to get into the plus space, and construct weakly holomorphic modular forms by multiplying by powers of $j(4z)$. To construct forms of negative weight, we can divide holomorphic forms of positive weight by powers of $\Delta(4z) \in M_{12}(4)$. It turns out that for every integer s, there is a canonical basis $\{f_{s+1/2,m}(z)\}$ for $M^!_{s+1/2}(4)$ where the basis elements have integral coefficients and a Fourier expansion of the form

$$q^{-m} + \sum a_{s+1/2}(m,n) q^n,$$

with the maximum possible gap in the Fourier expansion after the q^{-m} term. This is all constructed in [10].

For example, to get the form of weight 1/2 that corresponds to $E_4(z)$ under the Borcherds isomorphism, we begin with weight 25/2 and find the two forms $f_{25/2,-4} = E^4\theta^9 - 16E^5\theta^5 = q^4 + O(q^5)$ and $f_{25/2,-1} = E\theta^{21} - 42E^2\theta^{17} + 584E^3\theta^{13} - 2808E^4\theta^9 + 1792E^5\theta^5 + 2048E^6\theta = q + O(q^5)$. Divide both by $\Delta(4z)$ to get forms in the plus space of weight 1/2 beginning $q^{-3} + O(1)$ and $1 + O(q)$, and take the appropriate linear combination to get $q^{-3} + 4 + O(q)$, which must be the correct form by dimension considerations.

Just as in the integral weight case, there is a generating function for these basis elements. It is given by

$$\sum f_{k,m}(\tau) q^m = \frac{f_k(\tau) f^*_{2-k}(z) + f^*_k(\tau) f_{2-k}(z)}{j(4z) - j(4\tau)};$$

here, f_k and f^*_k are the first two basis elements of weight k. Zagier gave this generating function for $k = 1/2, 3/2$ in [20], and the generalization can be found in [10].

Examples. The first few coefficients of the first basis elements of weight 1/2 are

$$\begin{aligned}
f_{1/2,0}(z) &= 1 + 2q + 2q^4 + 0q^5 + 0q^8 + \cdots,\\
f_{1/2,3}(z) &= q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 + \cdots,\\
f_{1/2,4}(z) &= q^{-4} + 492q + 143376q^4 + 565760q^5 + 18473000q^8 + \cdots,\\
f_{1/2,7}(z) &= q^{-7} - 4119q + 8288256q^4 - 52756480q^5 + 5734772736q^8 + \cdots.
\end{aligned}$$

The first few basis elements of weight 3/2 are

$$\begin{aligned}
f_{3/2,1}(z) &= q^{-1} - 2 + 248q^3 - 492q^4 + 4119q^7 - \cdots,\\
f_{3/2,4}(z) &= q^{-4} - 2 - 26752q^3 - 143376q^4 - 8288256q^7 - \cdots,\\
f_{3/2,5}(z) &= q^{-5} + 0 + 85995q^3 - 565760q^4 + 52756480q^7 - \cdots,\\
f_{3/2,8}(z) &= q^{-8} + 0 - 1707264q^3 - 18473000q^4 - 5734772736q^7 - \cdots.
\end{aligned}$$

Looking at these Fourier expansions, we see that the coefficients of one basis element $f_{1/2,m}(z)$ appear as the negatives of the coefficients of a particular power of q in each of the $f_{3/2}$. This theorem is true more generally; in fact, for all integers m, n and any integral or half integral weight k we have the Zagier duality [20]

$$a_k(m,n) = -a_{2-k}(n,m).$$

This duality theorem can be proved from the generating function (replace k with $2-k$ and switch τ and z), or by noting that the product $f_{k,m}(z)f_{2-k,n}(z)$ has constant term equal to $a_k(m,n) + a_{2-k}(n,m)$, and that this product (hit with the U_4 operator if the weight is half integral) is of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$, and every such form is the derivative of a polynomial in j and thus has a constant term of 0.

Lecture 3: A modular form proof of Borcherds's theorem

Borcherds proved his theorem (of which even the more general theorem above is a special case) as a consequence of work on denominator formulas of infinite dimensional Lie algebras, but asked whether a proof of the isomorphism existed that used only modular forms for $\mathrm{SL}_2(\mathbb{Z})$ and not automorphic forms on larger groups. Work of Zagier [20] answered this question in the affirmative. The proof has two parts: first, showing that traces of singular moduli are coefficients of weight 3/2 modular forms for $\Gamma_0(4)$, and then using the duality theorem above.

We define traces of singular moduli of discriminant D for a $\mathrm{SL}_2(\mathbb{Z})$-invariant function $F(z)$ by

$$\mathrm{Tr}_D(F) = \sum_{Q\in\Gamma\backslash\mathcal{Q}_D} \frac{F(\tau_Q)}{\omega_Q}.$$

It is convenient to use the basis elements $f_{0,m}(z)$; for instance, if $F = f_{0,0}(z) = 1$, then $\mathrm{Tr}_D(F) = H(D)$, and if $F = f_{0,1}(z) = j(z) - 744$, then we get the algebraic trace of $j(\tau_Q) - 744$. For instance, we have the following traces of $f_{0,1}(z) = j(z) - 744$:

D	-3	-4	-7	-8	-11	-12	-15
$H(D)$	1/3	1/2	1	1	1	4/3	2
$\mathrm{Tr}_D(j-744)$	-248	492	-4119	7256	-33512	53008	-192513

Given a discriminant D, then, the class polynomial $\mathcal{H}_D(j(z))$ can be written as

$$\begin{aligned}\mathcal{H}_D(j(z)) &= \prod_{Q\in\Gamma\backslash\mathcal{Q}_D} (j(z) - j(\tau_Q))^{1/\omega_Q} \\ &= \prod_{Q\in\Gamma\backslash\mathcal{Q}_D} (q^{-1} - f_{0,1}(\tau_Q) + O(q))^{1/\omega_Q} = q^{-H(D)}(1 - \mathrm{Tr}_D(f_{0,1})q + O(q^2)).\end{aligned}$$

In fact, it turns out that the identity

$$\mathcal{H}_D(j(z)) = q^{-H(D)} \exp\left(-\sum_{m=1}^{\infty} \mathrm{Tr}_D(f_{0,m})\frac{q^m}{m}\right) \tag{10}$$

holds for all D.

Zagier showed that these traces appear as coefficients of half integral weight modular forms. Specifically, he showed that if the basis element $f_{3/2,m}(z)$ is hit with the weight 3/2 Hecke operator T_ℓ to give a modular form $T_\ell f_{3/2,m}(z) = \sum a_{3/2,\ell}(m,n)q^n$, then the trace $\mathrm{Tr}_D(f_{0,\ell})$ is equal to the coefficient $-a_{3/2,\ell}(1,-D)$. (Recall that the half integral weight Hecke operators T_ℓ, for a prime ℓ not dividing the level, act on a form $f(z) = \sum a(n)q^n$ of weight $s + 1/2$ to give

$$T_\ell f(z) = \sum \left(a(\ell^2 n) + \left(\frac{n}{\ell}\right)\ell^{s-1}a(n) + \ell^{2s-1}a(n/\ell^2)\right) q^n.$$

The image is a modular form of the same weight and level.)

As an example, the traces of $f_{0,1}(z) = j(z) - 744$ appear as Fourier coefficients of the form

$$f_{3/2,1}(z) = q^{-1} - 2 + 248q^3 - 492q^4 + 4119q^7 - 7256q^8 + 33512q^{11} - 53008q^{12} + 192513q^{15} - \cdots .$$

A straightforward calculation using explicit formulas for the action of the Hecke operators shows that if $T_\ell f_{1/2,m}(z) = \sum a_{1/2,\ell}(m,n)q^n$, then the stronger duality

$$a_{1/2,\ell}(m,n) = -a_{3/2,\ell}(n,m) \tag{11}$$

holds for positive m,n. (Similar results will hold for pairs of weights k and $2-k$ for $k = 5/2, 7/2, 9/2, 11/2$, and $15/2$, but not for higher weights where there are nonzero cuspforms of weight $2k-1$.) Taking $\ell = m$, $m = -D$, and $n = 1$, we see from the Hecke operator formulas that

$$a_{1/2,m}(-D,1) = \sum_{n|m} n a_{1/2}(-D,n^2).$$

Thus, we have the following computations, proving Borcherds's theorem:

$$\begin{aligned}
\mathcal{H}_D(j(z)) &= q^{-H(D)} \exp\left(-\sum_{m=1}^{\infty} \mathrm{Tr}_D(f_{0,m})\frac{q^m}{m}\right) \\
&= q^{-H(D)} \exp\left(\sum_{m=1}^{\infty} a_{3/2,m}(1,-D)\frac{q^m}{m}\right) \quad \text{(by modularity)} \\
&= q^{-H(D)} \exp\left(-\sum_{m=1}^{\infty} a_{1/2,m}(-D,1)\frac{q^m}{m}\right) \quad \text{(by duality)} \\
&= q^{-H(D)} \exp\left(-\sum_{m=1}^{\infty}(\sum_{n|m} n a_{1/2}(-D,n^2))\frac{q^m}{m}\right) \\
&= q^{-H(D)} \exp\left(\sum_{n\geq 1} a_{1/2}(-D,n^2)\left(-\sum_{m=1}^{\infty}\frac{q^{mn}}{m}\right)\right) \\
&= q^{-H(D)} \prod_{n\geq 1} \exp(a_{1/2}(-D,n^2)\log(1-q^n)) \\
&= q^{-H(D)} \prod_{n=1}^{\infty}(1-q^n)^{a_{1/2}(-D,n^2)}.
\end{aligned}$$

The more general isomorphism between spaces of modular forms can be built from this along with product formulas for $\Delta(z)$ and $E_{k'}(z)$; note that meromorphic modular forms satisfying the appropriate conditions can be built as a product of a power of Δ, an Eisenstein series, and an appropriate rational function in j (which will be a quotient of class polynomials).

The idea behind Zagier's proof of "modularity" is to show that the coefficients $a_{3/2}(1,-D)$ satisfy certain recursions, and then to show that the traces satisfy the same recursions; it is a generalization of Kronecker's class number relations (see exercises). However, this proof does not generalize so well.

A different proof is given by Duke [7] and Bruinier-Jenkins-Ono [5]. The ideas of the proofs generalize better to broader situations and is as follows:

- (BJO) Construct Poincaré series of weight 3/2 for $\Gamma_0(4)$ with a pole of order m at ∞; explicit formulas for the Fourier coefficients
- Project to the Kohnen plus space to get a Maass form $F_m^+(z)$ with slightly different coefficients, with a pole of order m at all cusps

- Compare the nonholomorphic part of $F_m^+(z)$ to the nonholomorphic part of Zagier's Eisenstein series $G(z) = \sum H(n)q^n + NH$ of weight 3/2
- Make the nonholomorphic parts cancel, and we've constructed $f_{3/2,m}(z)$
- Explicit formulas for coefficients $a_{3/2}(m,n)$ in terms of Bessel functions and Kloosterman sums

$$K(m,n;c) = \sum_{\substack{a \pmod c \\ (a,c)=1}} \left(\frac{c}{a}\right)\varepsilon_a^{-1} e\left(\frac{ma+n\bar{a}}{c}\right)$$

- (D) Traces can be written in terms of Bessel functions and Salie sums

$$S_d(m,c) = \sum_{x^2 \equiv -d \pmod c} e\left(\frac{2mx}{c}\right)$$

- Use an identity between Kloosterman and Salie sums to complete the proof

Applications of traces of singular moduli. The coefficients of the class polynomial $\mathcal{H}_D(x) = \prod(x - j(\tau_Q))$ are symmetric polynomials in the singular moduli $j(\tau_Q)$, and the coefficient of $x^{h(D)-1}$ is the trace $\mathrm{Tr}_D(j - 744) = \mathrm{Tr}_D(f_{0,1})$. It turns out that knowing the values of $\mathrm{Tr}_D(f_{0,m})$ for $0 \le m \le H(D)$ is enough to compute $\mathcal{H}_D(x)$. To do this, first note that $f_{0,m}(z)$ is a polynomial in $j(z)$; for instance,

$$f_{0,2}(z) = j(z)^2 - 1488j(z) + 159768.$$

Let $z = \tau_Q$ and sum over all $Q \in \Gamma\backslash\mathcal{Q}_D$ to see that

$$\mathrm{Tr}_D(f_{0,2}) = \sum j(\tau_Q)^2 - 1488\mathrm{Tr}_D(f_{0,1}) + 159768;$$

this lets us find $\sum j(\tau_Q)^2$. Similar computations allow us to recursively compute $\sum j(\tau_Q)^m$. The Newton-Girard formulas may then be used to change power sums into symmetric polynomials, allowing us to compute all coefficients of $\mathcal{H}_D(x)$ if we know enough coefficients of $f_{3/2,1}(z)$.

Another interesting application is the fact that

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999992....$$

We know that $\mathbb{Q}(\sqrt{-163})$ has class number 1, and the quadratic form $x^2 + xy + 41y^2$ has a root $\tau = \frac{-1+i\sqrt{163}}{2}$ in the fundamental domain. The trace of $j(\tau) - 744$ has only one term but must still be an integer; approximating $j(\tau)$ with just the q^{-1} term and noting that the tail of the series is small gives $-\exp(\pi\sqrt{163}) \sim$ (integer).

Generalizations. So far, we have interpreted the coefficients $a_{3/2,m}(1,-D)$ as traces of singular moduli; there are many more coefficients to these modular forms. Zagier showed that all of the coefficients $a_{3/2,\ell}(m,n)$ can be interpreted as twisted traces of modular functions by multiplying by a genus character $\psi(Q)$ (coming from ψ_d for some discriminant d which is a factor of D) in the sum over quadratic forms.

To take traces of a function F over $\Gamma\backslash\mathcal{Q}_D$, the function F must be invariant under $\mathrm{SL}_2(\mathbb{Z})$, so taking traces of functions of nonzero weight doesn't seem to make sense. However, it turns out that for modular forms of nonpositive weight, such a trace can be defined. If f is a modular form of integer weight k, the nonholomorphic differentiation operator

$$q\frac{d}{dq} - \frac{k}{4\pi y}$$

of weight k (here $z = x + iy$) raises the weight of f by 2 and preserves modularity but not holomorphicity. If $k = 2 - 2s$ is nonpositive, applying this operator a total of $s - 1$ times gives a weak Maass form of weight 0, and traces can be taken. This was done by Zagier for small negative weights and by [10] for all negative weights; in fact, it turns out that the map (the Zagier lift) given by differentiating to weight 0, taking traces, and adding an appropriate principal part takes modular forms of weight $2 - 2s$ with integer coefficients to modular forms of weight $s + 1/2$ or $3/2 - s$ with integer coefficients, and is compatible with Hecke operators of integral and half integral weight. It is a negative weight analogue of the classical Shintani lift, summing over traces instead of integrating over geodesics.

Zagier touches on this question of higher level at the end of his paper [20], and proves some results for small levels N where $\Gamma_0^*(N)$ has genus zero (so that there is still a nice explicit basis). Miller and Pixton [18] defined the Zagier lift for any weakly holomorphic modular form of nonpositive weight and general level N that has poles only at the cusp at ∞. Bruinier and Funke [4] obtained results giving such a lift for arbitrary genus. See also the more recent work of Alfes and Ehlen [2]. The integrality-preserving property is extended to small prime level by Green [13], and denominators for all levels are bounded in work of Alfes [1].

If the discriminant $D > 0$, the problem changes; rather than summing over values of functions at points of discriminant D, the appropriate thing to do is to integrate over a geodesic. Cycle integrals of the j-function appear in Fourier coefficients of certain mock modular forms. See the work of Duke-Imamoḡlu-Tóth [8].

Lecture 4: Harmonic Maass Forms of Weight One

In this lecture, we will consider the product

$$\prod_{Q_1\in\Gamma\backslash\mathcal{Q}_{D_1}} \prod_{Q_2\in\Gamma\backslash\mathcal{Q}_{D_2}} \left(j(\tau_{Q_1}) - j(\tau_{Q_2})\right)^{4/(w_{Q_1}w_{Q_2})}$$

for discriminants $D_1, D_2 < 0$. By the theory of complex multiplication, this product of algebraic integers is in fact a rational integer when $D_1, D_2 < -4$ are fundamental. Gross and Zagier studied these integers in [14] and gave an explicit factorization.

Let $k \in \frac{1}{2}\mathbb{Z}$ be a half integer. For a positive integer N, let $\Gamma_0(N) \subset \mathrm{SL}(2,\mathbb{Z})$ be the congruence subgroup defined by

$$\Gamma_0(N) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}(2,\mathbb{Z}) : N \mid c \} .$$

We say that $f(z)$ has weight k and level N if it transforms like $\theta^{2k}(z)$ with respect to $\Gamma_0(N)$, i.e.

$$\frac{f(\gamma z)}{f(z)} = \frac{\theta^{2k}(\gamma z)}{\theta^{2k}(z)}$$

for all $\gamma \in \Gamma_0(N)$.

Define the weight k Laplacian Δ_k by

$$\Delta_k := \xi_{2-k} \circ \xi_k$$

where $\xi_k := 2iy^k \overline{\frac{\partial}{\partial \bar{z}}}$. The operator ξ_k has the property that it commutes the slash operator by changing the weight from k to $2-k$. We are interested in studying real-analytic functions $F : \mathcal{H} \longrightarrow \mathbb{C}$ annihilated by the differential operator Δ_k. Such function has a Fourier expansion of the form

$$F(z) = \sum_{n \geq n_0} c^+(n) q^n - \sum_{n \geq 0} \overline{c(n)} \beta_k(n,y) q^{-n},$$

where

$$\beta_k(n,y) := \begin{cases} \int_y^\infty e^{-4\pi n t} t^{-k} dt & n > 0, \\ \frac{y^{1-k}}{k-1} & n = 0, k \neq 1, \\ -\log y & n = 0, k = 1. \end{cases}$$

In this form, one has $(\xi_k F)(z) = \sum_{n \geq 0} c(n) q^n$. The Fourier expansion $\sum_{n \geq n_0} c^+(n) q^n$ is called the holomorphic part of $F(z)$.

When $k = 2$, the non-holomorphic Eisenstein series $\hat{E}_2(z) := -\frac{3}{\pi \mathrm{Im}(z)} + 1 - 24 \sum_{n \geq 1} \sigma(n) q^n$ is a harmonic Maass form. When $k = 3/2$, there is an Eisenstein series $\hat{E}_{3/2}(z)$ studied by Hirzebruch and Zagier where $\xi_{3/2}(\hat{E}_{3/2})$ is the Jacobi theta function of weight $1/2$. When $k = 1/2$, examples are Ramanujan's mock theta functions.

When $k = 1$, Deligne and Serre attached irreducible, odd, 2-dimensional complex Galois representations to newforms $g(z)$, whose L-functions are then Artin L-functions. These were already studied by Hecke in the dihedral setting, i.e. the Galois representation is induced from a character of the absolute Galois group of a quadratic field. Let $Q \in \mathcal{Q}_D$ for a fundamental discriminant $D < 0$. Define the quantity

$$r_Q(n) = \#\{\pm(x,y) \in \mathbb{Z}^2 : Q(x,y) = n\}.$$

The theta function

$$\vartheta_Q(z) := \sum_{n\geq 0} r_Q(n)q^n$$

is a modular form of weight one on $\Gamma_0(|D|)$ with character $\left(\frac{D}{\cdot}\right)$. One could obtain an eigenform $g_\psi(z)$ by summing together $[Q] \in \Gamma\backslash\mathcal{Q}_D$ with a class group character ψ

$$g_\psi(z) := \sum_{[Q]\in\Gamma\backslash\mathcal{Q}_D} \psi(Q)\vartheta_Q(z).$$

When ψ is a genus character, $g_\psi(z)$ is an Eisenstein series. Otherwise, $g_\psi(z)$ is a dihedral cusp form.

We are interested in studying harmonic Maass forms $\hat{g}(z)$ whose image under ξ_1 is $g_\psi(z)$. When $D = -p < -3$ with $p \equiv 3 \pmod 4$ and ψ a trivial character, the modular form $g_\psi(z)$ is the Eisenstein series given by

$$E_p(z) := \sum_{[Q]\in\Gamma\backslash\mathcal{Q}_{-p}} \vartheta_Q(z) = \frac{h(-p)}{2} + \sum_{n\geq 1} R_p(n)q^n.$$

The harmonic Maass form $\hat{E}_p(z)$ was constructed by Kudla-Rapoport-Yang in [15] by taking a derivative in s of the real-analytic Eisenstein series $E_1(z,s)$. It satisfies $\xi_1(\hat{E}_p(z)) = E_p(z)$ and its holomorphic part is given by $\sum_{n\geq 0} R_p^+(n)q^n$ where

$$R_p^+(\ell) = -2\log \ell$$

for all ℓ satisfying $\left(\frac{\ell}{p}\right) = -1$. These coefficients $R_p^+(n)$ has the arithmetic interpretation as the degrees of certain special cycles on an arithmetic curve parametrizing CM elliptic curves. Zagier noticed that his result with Gross on the factorization of differences of singular moduli can be written as

$$\sum_{Q\in\Gamma\backslash\mathcal{Q}_{-p}} \log\left|\prod_{Q'\in\Gamma\backslash\mathcal{Q}_D} (j(\tau_Q) - j(\tau_{Q'}))^{2/w_{Q'}}\right| = -\frac{1}{2}\sum_{m\in\mathbb{Z}} \delta_p(m)R_p^+\left(\frac{-Dp-m^2}{4}\right), \tag{12}$$

where $D < 0$ is any discriminant relatively prime to p and $\delta_p(m)$ is 2 if $p \mid m$ and 1 otherwise.

For the newform $g_\psi(z)$, there also exists harmonic Maass forms $\hat{g}_\psi(z)$ with holomorphic part

$$\sum_{n\geq -\frac{p+1}{24}} r_\psi^+(n)q^n$$

such that $\xi_1(\hat{g}_\psi(z)) = g_\psi(z)$. Furthermore, one could use these coefficients give an analogue of equation (12) [11]

$$\sum_{Q\in\Gamma\backslash\mathcal{Q}_{-p}} \psi^2(Q)\log\left|\prod_{Q'\in\Gamma\backslash\mathcal{Q}_D}(j(\tau_Q)-j(\tau_{Q'}))^{2/w_{Q'}}\right| = -\frac{1}{2}\sum_{m\in\mathbb{Z}}\delta_p(m)r_\psi^+\left(\frac{-Dp-m^2}{4}\right). \tag{13}$$

Notice that the sum is finite and even though $\frac{-Dp-m^2}{4}$ could be negative. Similar result holds with $-p$ replaced by a fundamental discriminant [12].

One way to prove equation (13) is to construct modular functions by Green's function. Then its special value at CM points can be expressed as an infinite sum involving the following counting function

$$\rho_Q(k,-p) := \#\left\{Q'\in\mathcal{Q}_D : \cosh d(\tau_Q,\tau_{Q'}) = \frac{k}{\sqrt{|pD|}}\right\},$$

for $k\in\mathbb{N}$ and $Q\in\mathcal{Q}_{-p}$. Here for $z_1,z_2\in\mathcal{H}$, $\cosh d(z_1,z_2)$ is the hyperbolic cosine of the hyperbolic distance between two points defined by

$$\cosh d(z_1,z_2) := 1+\frac{|z_1-z_2|^2}{2y_1y_2},$$

where $y_j = \mathrm{Im}(z_j)$. An elementary argument using quadratic forms relates this to the Fourier coefficient of ϑ_{Q^2} by

$$\rho_Q(k,-p) = r_{Q^2}\left(\frac{k^2+pD}{4}\right).$$

The details are outlined in the exercise.

To give an example of the individual coefficients $r_\psi^+(n)$, let $p=23$ and ψ is a non-trivial character of the class group of $K:=\mathbb{Q}(\sqrt{-23})$, the newform $g_\psi(z)$ has the Fourier expansion

$$g_\psi(z) = q\prod_{n=1}^\infty(1-q^n)\prod_{n=1}^\infty(1-q^{23n}).$$

Let $H := K[X]/(X^3-X-1)$ be the Hilbert class field of K and $\alpha>1$ the real root of X^3-X-1. Then there exists a harmonic Maass form $\hat{g}_\psi(z)$ with $\xi_1(\hat{g}_\psi(z)) = g_\psi(z)$ and holomorphic part

$$\sum_{\substack{n\geq -1\\ (\frac{n}{23})\neq 1}} r_\psi^+(n)q^n$$

and $r_\psi^+(-1) = -3r_\psi^+(-1) = 3\log(\alpha)$. It turns out that this harmonic Maass form is unique and the Fourier coefficients $r_\psi^+(n)$ has the shape

$$r_\psi^+(n) = -2 \sum_{[Q]\in\Gamma\backslash\mathcal{Q}_{-23}} \psi^2(Q)\log|u(n,[Q])|$$

with $u(n,[Q]) \in H$ having compatible action under $\mathrm{Gal}(H/K) \cong C(-p)$. When $[Q] = [Q_0]$ is the principal class, we give some numerical values of the coefficients $u(n,[Q_0])$ here.

n	5	7	10	11
$u(n,Q_0)$	$\varpi_5 := 2\alpha^2 - \alpha - 1$	$\varpi_7 := \alpha^2 + \alpha - 2$	$5\alpha^{-6}\varpi_5^{-1}$	$\varpi_{11} := 2\alpha^2 - \alpha$
n	14	15	17	19
$u(n,Q_0)$	$7\alpha^{-4}\varpi_7^{-1}$	$5\alpha\varpi_5^{-1}$	$\varpi_{17} := 2\alpha^2 + 3\alpha + 3$	$\varpi_{19} := 3\alpha^2 + \alpha$
n	20	21	22	23
$u(n,Q_0)$	$5\alpha^{-7}$	$7\alpha^{-10}\varpi_7^{-1}$	$11\alpha^5\varpi_{11}^{-1}$	$\varpi_{23} := \frac{10\alpha^2+8\alpha+1}{\sqrt{-23}}$

Notice that for all prime ℓ satisfying $\left(\frac{\ell}{23}\right) = -1$, the table shows that $\mathrm{Nm}(\varpi_\ell) = \ell^2$. This and more general results about the individual coefficient $r_\psi^+(n)$ has been obtained in [12, 19] using the technique of theta-lifting. Furthermore, the valuations of the algebraic numbers $u(n,Q_0)$ have the arithmetic interpretation as the degrees of special cycles on arithmetic curves [12].

Beyond this, there are a lot of unknowns about the Fourier coefficients of weight one harmonic Maass forms $\hat{g}(z)$ when $\xi_1(\hat{g})$ is a holomorphic newform associated to non-dihedral complex Galois representations. In [11], there is a numerical example of the octahedral case, where the Fourier coefficients seem to be logarithms of algebraic numbers in the number field determined by the adjoint of the associated Galois representation. We expect these Fourier coefficients to be related to special values of automorphic forms, as well as to have arithmetic interpretations.

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