2^{ND} EU/US SUMMER SCHOOL ON AUTOMORPHIC FORMS SINGULAR MODULI AND MODULAR FORMS – EXERCISE SHEET 2

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Exercise 1. In this exercise, we will explore the connection between CM points and CM elliptic curves. For a fixed $\tau \in \mathcal{H}$, one could associate a lattice, or a $\mathbb{Z}$-module of rank 2,

$$L_\tau = \{u + v\tau : u, v \in \mathbb{Z}\}.$$

The complex torus $\mathbb{C}/L_\tau$ is the $\mathbb{C}$ points of an elliptic curve, denoted by E_τ. Let $\wp(z, L_\tau)$ be the Weiestrass $\wp$-function defined by

$$\wp(z, L_\tau) := \frac{1}{z^2} + \sum_{\substack{\ell \in L_\tau \\ \ell \neq 0}} \left(\frac{1}{(z+\ell)^2} - \frac{1}{\ell^2} \right). \tag{1}$$

(1) By considering the derivative $\wp'(z, L_\tau)$ in z, show that $\wp(z, L_\tau)$ is a doubly periodic function, i.e. $\wp(z+\tau, L_\tau) = \wp(z+1, L_\tau) = \wp(z, L_\tau)$.
(2) Prove that $\wp(z, L_\tau)$ and its derivative $\wp'(z, L_\tau)$ in z satisfy the equation

$$\wp'(z, L_\tau)^2 = f_\tau(\wp(z, L_\tau))$$

where $f_\tau(x)$ is a cubic polynomial in x given by

$$\begin{aligned} f_\tau(x) &= 4x^3 - g_2(\tau)x - g_3(\tau) = 4(x-e_1)(x-e_2)(x-e_3), \\ g_2(\tau) &= 60 \cdot 2\zeta(4) \cdot E_4(\tau),\ g_3(\tau) = 140 \cdot 2\zeta(6) \cdot E_6(\tau), \\ e_1 &= \wp(1/2, L_\tau),\ e_2 = \wp(\tau/2, L_\tau),\ e_3 = \wp((1+\tau)/2, L_\tau). \end{aligned}$$

Thus, the following map gives an embedding of the torus $\mathbb{C}/L_\tau$ into projective space

$$\begin{aligned} \mathbb{C}/L_\tau &\longrightarrow \mathbb{P}^2_{\mathbb{C}} \\ z &\mapsto (\wp(z, L_\tau), \wp'(z, L_\tau), 1) \\ 0 &\mapsto (0, 1, 0). \end{aligned}$$

(*Hint: Show that the difference between the two sides is a bounded, entire function in $z \in \mathbb{C}$, which must be constant.*).

(3) Recall the discriminant of a polynomial $f(x) = a_0x^n + \cdots + a_{n-1}x + a_n = a_0(x - e_1)(x - e_2)\dots(x - e_n)$ is given by

$$\Delta_f := a_0^{2n-2} \prod_{i<j} (e_i - e_j)^2,$$

which is nonzero precisely when $f(x)$ has distinct roots. Furthermore, Δ_f can be written in terms of a_k for $0 \le k \le n$. Calculate the discriminant of $f_\tau(x) = 4x^3 - g_2(\tau)x - g_3(\tau)$ in terms of the Eisenstein series E_4 and E_6.

(4) Show that $j(\tau)$ gives a bijection between $\Gamma\backslash\mathcal{H}$ and the $\mathbb{P}^1_{\mathbb{C}}\backslash\{\text{point}\}$.

(5) Use the previous part to show that for any $A, B \in \mathbb{C}$ satisfying $A^3 \neq 27B^2$, there exists $\lambda \in \mathbb{C}$ and $\tau \in \mathcal{H}$ such that

$$\lambda^{-4} g_2(L_\tau) = A, \ \lambda^{-6} g_3(L_\tau) = B.$$

(6) An endomorphism $\varphi : \mathbb{C}/L_\tau \longrightarrow \mathbb{C}/L_\tau$ lifts to a homomorphism $\tilde{\varphi} : \mathbb{C} \longrightarrow \mathbb{C}$, which is given by multiplication by $\alpha := \tilde{\varphi}(1)L_\tau$. Clearly, the endomorphisms of an elliptic curve contains $\mathbb{Z}$. Prove that there is extra endomorphism if and only if τ is a CM point.

Exercise 2. For any $f : \mathbb{C} \longrightarrow \mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{R})$, define the slash operator $f \mid_k \gamma$ of integral weight k by

$$(f \mid_k \gamma)(z) := \frac{\det(\gamma)^{k/2}}{(cz + d)^k} f\left(\frac{az + b}{cz + d}\right).$$

For a prime p and integer k, define the operators U_p, V_p, T_p on $f(z)$ by

$$\begin{aligned} U_p(f)(z) &:= p^{k/2-1} \sum_{\lambda=0}^{p-1} \left(f \mid_k \left(\begin{smallmatrix} 1 & \lambda \\ 0 & p \end{smallmatrix}\right)\right)(z), \\ V_p(f)(z) &:= p^{-k/2}(f \mid_k \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right))(z), \\ T_p(f)(z) &:= U_p(f)(z) + p^{k-1} V_p(f)(z). \end{aligned}$$

The T_p operator is called the Hecke operator.

(1) Check that $(f \mid_k (\gamma_1\gamma_2))(z) = ((f \mid_k \gamma_1) \mid_k \gamma_2)(z)$.

(2) If $f(z)$ has the Fourier expansion $f(z) = \sum_{n \in \mathbb{Z}} a(n, y)e(nx)$, write out the Fourier expansion of $U_p(f)(z)$ and $V_p(f)(z)$.

(3) Show that if $(f \mid_k \gamma)(z) = f(z)$ for all $\gamma \in \Gamma_0(N)$, then $T_p(f)$ has the same property for all $p \nmid N$.

(4) Prove that $E(z) := \sum_{n=0}^{\infty} \sigma(2n + 1)q^{2n+1}$ satisfies

$$(E \mid_2 \gamma)(z) = E(z)$$

for all $\gamma \in \Gamma_0(4)$.

(5) Let $k \in \{4,6,8,10,14\}$ and $f_{k,m}(z) = \sum_{n \geq -m} a_k(m,n)q^n = q^{-m} + O(q) \in M_k^!$. Write $T_p f_{k,m}(z)$ as a sum of basis elements $f_{k,m'}(z)$. Compare Fourier coefficients to show that if $p \nmid n$, then the Fourier coefficient $a_k(m, np)$ is divisible by p^{k-1}. Is a similar statement true for $k = 0$ or for other weights? What about in half integral weight?

Exercise 3. For each even integer weight k, write $k = 12\ell + k'$, where $k', \ell \in \mathbb{Z}$ and $k' \in \{0,4,6,8,10,14\}$. For each integer $m \geq -\ell$, there is a unique modular form $f_{k,m}(z) \in M_k^!$ with a Fourier expansion of the form

$$f_{k,m}(z) = q^{-m} + O(q^{\ell+1}).$$

We write $f_{k,m}(z) = q^{-m} + \sum_{n > \ell} a_k(m,n)q^n$, so that all of the Fourier coefficients $a_k(m,n)$ of these basis elements are integers.

(1) Express $f_{k,m}(z)$ in terms of $\Delta(z), j(z)$ and $E_{k'}(z)$ with $k' \in \{0,4,6,8,10,14\}$ for $(k,m) = (16,3), (24,2), (30,1)$.

(2) Prove the generating functions

$$\sum_{m \geq -\ell} f_{k,m}(\tau) q^m = \frac{f_{k,-\ell}(\tau) f_{2-k,-1-\ell}(z)}{j(z) - j(\tau)},$$

Specialize to $k = 0$ to obtain

$$\frac{-j'(z)}{j(z) - j(\tau)} = \sum_{m \geq 0} j_m(\tau) q^m = \sum_{m \geq 0} F_m(j(\tau)) q^m.$$

(3) Prove the identity

$$\mathcal{H}_D(j(z)) = q^{-H(D)} \exp\left(- \sum_{m=1}^{\infty} \mathrm{Tr}_D(f_{0,m}) \frac{q^m}{m} \right)$$

holds for all D.

Exercise 4.

(1) Find the image of E_6 in $M_{1/2}^!(4)$ under Borcherds's isomorphism.

(2) Use $E(z), \theta(z)$ and $\Delta(4z)$ to construct $f_{3/2,1}(z)$. Then apply the Hecke operators T_2, T_3 to this form and compute the class polynomials of discriminants $D = -15, -20, -23$.

(3) Using the class polynomial for $D = -23$ to show that

$$H^+ := \mathbb{Q}\left(j\left(\tfrac{1+\sqrt{-23}}{2} \right) \right) \cong \mathbb{Q}[X]/(X^3 - X - 1).$$

Show that H^+ is not Galois over $\mathbb{Q}$ and the Galois closure of H^+ is given by

$$H := H^+(\sqrt{-23}).$$

Exercise 5. For two points $z_1, z_2 \in \mathcal{H}$, the hyperbolic cosine of the hyperbolic distance between them is defined by

$$\cosh d(z_1, z_2) := 1 + \frac{|z_1 - z_2|^2}{2y_1 y_2},$$

where $y_j = \mathrm{Im}(z_j)$.

(1) Show that for any $z_1, z_2 \in \mathcal{H}$ and $\gamma \in \mathrm{SL}(2, \mathbb{R})$

$$\cosh d(\gamma z_1, \gamma z_2) = \cosh d(z_1, z_2).$$

(2) For discriminants $D_1, D_2 < 0$ and $Q_j = (A_j, B_j, C_j) \in \mathcal{Q}_{D_j}$, let $\tau_{Q_j} \in \mathcal{H}$ be the corresponding CM points for $j = 1, 2$. Show that $\cosh d(\tau_{Q_1}, \tau_{Q_2})$ is given by

$$\cosh d(\tau_{Q_1}, \tau_{Q_2}) := \frac{2A_1C_2 + 2C_1A_2 - B_1B_2}{\sqrt{D_1D_2}}.$$

(3) We say that a quadratic form $(A, B, C) \in \mathcal{Q}_D$ is primitive if $\gcd(A, B, C) = 1$. Prove that every primitive quadratic form $Q \in \mathcal{Q}_D$ is equivalent to a quadratic form $(A, B, AC) \in \mathcal{Q}_D$ with $\gcd(A, B) = 1$. Using the definition of composition to show that Q^2 is equivalent to (A^2, B, C).

(4) For a positive definite quadratic form $Q \in \mathcal{Q}_D$, $k \geq 0$ an integer and $D' < 0$ a discriminant, define the sets

$$S_Q(k, D') := \left\{ Q' = (A', B', C') \in \mathcal{Q}_{D'} : \cosh d(\tau_Q, \tau_{Q'}) = \frac{k}{\sqrt{DD'}} \right\},$$
$$T_Q(m) := \{(x, y) \in \mathbb{Z}^2 : Q(x, y) = m\}.$$

Suppose $Q = (A, B, AC) \in \mathcal{Q}_D$. Show that for any $Q' = (A', B', C') \in \mathcal{Q}_{D'}$

$$(C' - CA', BA' - AB') \in T_{Q^2}\left(\tfrac{k^2 - DD'}{4}\right).$$

(5) When $D = -p$ such that $p \equiv 3 \pmod 4$ is a prime. Show that the map

$$S_Q(m, D') \longrightarrow T_{Q^2}\left(\tfrac{k^2 - pD'}{4}\right)$$
$$(A', B', C') \mapsto (C' - CA', BA' - AB')$$

is a bijection when $p \nmid k$.

(6) Let $\rho_Q(k, D') := \#S_Q(k, D')$ and $r_Q(m) := \frac{1}{2}\#T_Q(m)$ for any $Q \in \mathcal{Q}_{-p}$. Prove that

$$\rho_Q(k, D') = r_{Q^2}\left(\frac{k^2 - pD'}{4}\right)\delta_p(k),$$

where $\delta_p(k) = 2$ if $p \mid k$ and 1 otherwise.

Exercise 6. Recall that the differential operator ξ_k is defined by

$$\xi_k := 2iy^k \overline{\frac{\partial}{\partial \overline{z}}}.$$

for any integer $k \in \mathbb{Z}$.

(1) Show that

$$\xi_{2-k} \circ \xi_k = \Delta_k := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$
$$\xi_k \left(f \mid_k \gamma \right) = \left(\xi_k f \right) \mid_{2-k} \gamma.$$

for any differentiable function $f : \mathbb{C} \longrightarrow \mathbb{C}$ and $\gamma \in \mathrm{SL}(2, \mathbb{R})$.

(2) Prove that for any $z \in \mathcal{H}$ and $n > 0$

$$\beta_k(n, y) q^{-n} = O(e^{-2\pi n y}).$$

(3) Let $f : \mathcal{H} \longrightarrow \mathbb{C}$ be a real-analytic function defined by the Fourier series

$$f(z) := \sum_{n \geq 0} a(n) \beta_k(n) q^{-n}.$$

Calculate $\xi_k(f)$ as a Fourier series.

(4) Calculate $\xi_2(\hat{E}_2(z))$, where $\hat{E}_2(z) = -\frac{3}{\pi \mathrm{Im}(z)} + 1 - 24 \sum_{n \geq 1} \sigma(n) q^n$.