2^{ND} EU/US SUMMER SCHOOL ON AUTOMORPHIC FORMS
SINGULAR MODULI AND MODULAR FORMS – EXERCISE SHEET 1

LECTURERS: WILLIAM DUKE, PAUL JENKINS
ASSISTANT: YINGKUN LI

Exercise 1. Define the Eisenstein series of weight 2 by

$$G_2(z) := \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}'_m} \frac{1}{(mz+n)^2} = 2\zeta(2) + \sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2},$$

where $\mathbb{Z}'_m = \mathbb{Z}\backslash\{0\}$ if $m = 0$ and $\mathbb{Z}$ otherwise. Note that the order of the summation matters here since the sum does not converge absolutely. The Fourier expansion of $E_2(z) := \frac{1}{2\zeta(2)} G_2(z)$ is analogous to those of $E_4(z)$ and $E_6(z)$ and given by

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n.$$

(1) Show that

$$\frac{1}{z^2} G_2(-1/z) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}'_n} \frac{1}{(mz+n)^2} = 2\zeta(2) + \sum_{n\in\mathbb{Z}}\sum_{m\neq 0} \frac{1}{(mz+n)^2}.$$

(2) Show that

$$\sum_{m\neq 0}\sum_{d\in\mathbb{Z}} \frac{1}{(mz+d)(mz+d+1)} = 0$$

and subtract it from $G_2(z)$ to show that $G_2(z)$ has the absolutely convergent expression

$$G_2(z) = 2\zeta(2) + \sum_{\substack{m\neq 0\\ n\in\mathbb{Z}}} \frac{1}{(mz+d)^2(mz+d+1)}.$$

(3) Show that

$$G_2(z) - \frac{1}{z^2}G_2(-1/z) = -\sum_{n\in\mathbb{Z}}\sum_{m\neq 0}\frac{1}{(mz+d)} - \frac{1}{(mz+d+1)}$$
$$= -\lim_{N\to\infty}\sum_{n=-N}^{N-1}\sum_{m\neq 0}\frac{1}{(mz+d)} - \frac{1}{(mz+d+1)}$$
$$= -\frac{2}{z}\lim_{N\to\infty}\sum_{m=1}^{\infty}\frac{1}{(-N/z+m)} + \frac{1}{(-N/z-m)}.$$

(4) Prove the identity

$$\frac{1}{z} + \sum_{d=1}^{\infty}\left(\frac{1}{z-d} + \frac{1}{z+d}\right) = \pi\cot(\pi z) = \pi i - 2\pi i\sum_{m=0}^{\infty} q^m. \tag{1}$$

and use it to deduce that

$$E_2(z) - \frac{1}{z^2}E_2(-1/z) = -\frac{12}{2\pi i z}. \tag{2}$$

(5) Show that $\hat{E}(z) := E(z) - \frac{3}{\pi \mathrm{Im}(z)}$ satisfies the transformation

$$\left(\hat{E}_2 \mid_2 \gamma\right)(z) = \hat{E}_2(z)$$

for all $\gamma \in \Gamma$.

Exercise 2. Define the function $\tilde{\Delta} : \mathcal{H} \longrightarrow \mathbb{C}$ by

$$\tilde{\Delta}(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24}. \tag{3}$$

(1) Prove that

$$\frac{1}{2\pi i}\frac{d}{dz}\log\tilde{\Delta}(z) = 1 - 24\sum_{m=1}^{\infty}\sigma(m)e^{2\pi i m z} = E_2(z).$$

(2) Using equation (2), show that

$$\frac{1}{2\pi i}\frac{d}{dz}\log\left(\frac{\tilde{\Delta}(-1/z)}{z^{12}\tilde{\Delta}(z)}\right) = 0.$$

(3) Use the previous part and the valence formula to show that $\tilde{\Delta}(z) = \Delta(z) = \frac{E_4^3(z)-E_6^2(z)}{1728} \in S_{12}$.

(4) Suppose that $f(z) \in M_k^!$ has integral Fourier coefficients and leading coefficient 1. Prove that if $f(z_0) = 0$ for some $z_0 \in \mathcal{H}$, then $j(z_0)$ is an algebraic integer.

Exercise 3. The polynomial $\Psi_m(x,y)$ is reducible when m is not squarefree. Instead, one could consider

$$\begin{aligned} \Phi_m(x, j(z)) &:= \prod_{\gamma\in\Gamma\backslash\Delta_m} (x - j(\gamma z)), \\ \Delta_m &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_m : \gcd(a,b,c,d) = 1 \right\}. \end{aligned} \tag{4}$$

(1) Show the following subset of Δ_m consists of coset representatives of $\Gamma\backslash\Delta_m$

$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Delta_m : \gcd(a,b,d) = 1, 0 \le b \le d-1 \right\}$$

and that it has size

$$\psi(m) := m \prod_{p|m \text{ prime}} \left(1 + \frac{1}{p}\right).$$

(2) Prove that

$$\Psi_m(x,y) = \prod_{d^2|m} \Phi_{m/d^2}(x,y).$$

In particular, $\Psi_m(x,y) = \Phi_m(x,y)$ when m is squarefree. Furthermore, one has

$$\sigma(m) = \sum_{d^2|m} \psi\left(\frac{m}{d^2}\right),$$

$$\frac{\zeta(s)\zeta(s-1)}{\zeta(2s)} = \sum_{m\ge1} \frac{\psi(m)}{m^s}$$

when $\mathrm{Re}(s)$ is large enough.

(3) Show that as a polynomial in x, the coefficients $\Phi_m(x, j(z))$ are modular functions in $M_0^!$ with integral Fourier coefficients. From this, deduce that $\Phi_m(x,y) \in \mathbb{Z}[x,y]$.

(4) Show that $\Phi_m(x,y)$ is irreducible as a polynomial in x over the field $\mathbb{C}(y)$.

(5) Prove that $\Phi_m(x,y) = \Phi_m(y,x)$.

Exercise 4. Let $D < 0$ be a discriminant. In this exercise, we will prove the finiteness of the size of $\Gamma\backslash\mathcal{Q}_D$. For convenience, we will use (A,B,C) to denote a binary quadratic form $Ax^2 + Bxy + Cy^2 \in \mathcal{Q}_D$ and $\sim$ to represent equivalence under the action of Γ. We say a form (A,B,C) is *reduced* if

$$|B| \le A \le C.$$

(1) For any reduced form (A, B, C) of a fixed discriminant D, prove that $|B| < \sqrt{|D|/3}$. From this, deduce that the number of reduced forms of a fixed discriminant $D < 0$ is finite.
(2) Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and calculate $\gamma \cdot (A, B, C)$. What is the result when $a = 0$?
(3) If $(A, B, C), (A', B', C') \in \mathcal{Q}_D$ are equivalent, prove that

$$\{Ax^2 + Bxy + Cy^2 : x, y \in \mathbb{Z}\} = \{A'x^2 + B'xy + C'y^2 : x, y \in \mathbb{Z}\}.$$

(4) Prove that two forms (A_1, B_1, C_1) and (A_2, B_2, C_2) of the same discriminant are equivalent if and only if there exist $a, c \in \mathbb{Z}$ such that

$$A_1a^2 + B_1ac + C_1c^2 = A_2,$$
$$2A_1a + (B_1 + B_2)c \equiv 0 \pmod{2A_2},$$
$$(B_1 - B_2)a + 2C_1c \equiv 0 \pmod{2A_2}.$$

(5) Find all pairs of distinct reduced forms that are equivalent to each other.
(6) Show that every binary quadratic form $(A, B, C) \in \mathcal{Q}_D$ is equivalent to a reduced form and prove that $h(D)$ is finite.
(7) Show that every quadratic form $(A, B, C) \in \mathcal{Q}_{-163}$ is equivalent to $(1, 1, 41)$ and that the size of $\Gamma\backslash\mathcal{Q}_{-23}$ is 3.

Exercise 5. For a fixed *fundamental* discriminant $D < 0$, let $C(D)$ denote the set of coset representatives of $\Gamma\backslash\mathcal{Q}_D$. We will define the composition law after Dirichlet in this exercise. Two forms $Q_j = (A_j, B_j, C_j) \in \mathcal{Q}_D, j = 1, 2$ are called *united* if $\gcd(A_1, A_2, (B_1+B_2)/2) = 1$.

(1) If $Q_1, Q_2 \in \mathcal{Q}_D$ are united forms, show that there exists $b, c \in \mathbb{Z}$ such that $Q_1 \sim (A_1, b, A_2c)$ and $Q_2 \sim (A_2, b, A_1c)$. In this way, we define the composition of Q_1 and Q_2 to be

$$Q_1 \circ Q_2 := (A_1A_2, b, c) \in \mathcal{Q}_D.$$

(2) Prove that the composition respects the equivalence relation $\sim$, i.e. if $Q_1, Q_2 \in \mathcal{Q}_D$ and $Q_1', Q_2' \in \mathcal{Q}_D$ are two pairs of united forms such that $Q_j \sim Q_j'$ for $j = 1, 2$, then

$$Q_1 \circ Q_2 \sim Q_1' \circ Q_2'.$$

This then defines a composition law on $C(D)$.
(3) Deduce that

$$(1, B, C) \circ (A', B', C') \sim (A', B', C'),$$
$$(A, B, C) \circ (A, -B, C) \sim (A, B, C) \circ (C, B, A) \sim (AC, B, 1),$$

and conclude that $C(D)$ is an abelian group under composition.
(4) Show that $h(D)$ is even if $D < 0$ is the product of a positive and a negative discriminant.

Exercise 6. For a discriminant $D < 0$, define the Hurwitz class number $H(D)$ by

$$H(D) := \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \frac{1}{w_Q}.$$

It is the "degree" of the "modified class polynomial" $\mathcal{H}_D(x) := \prod_{Q \in \Gamma \backslash \mathcal{Q}_D} \left(x - j(\tau_Q)\right)^{1/w_Q}$.

(1) When m is not a perfect square, prove that

$$\Psi_m(x, x) = \pm \prod_{|t| < 2\sqrt{m}} \mathcal{H}_{t^2 - 4m}(x).$$

(2) When m is a perfect square, prove that

$$\left. \frac{\Psi_m(x, y)}{\Psi_1(x, y)} \right|_{x=y} = \pm \sqrt{m} \frac{\mathcal{H}_{t^2-4m}(x)}{\mathcal{H}_{t^2-4}(x)}.$$

(3) Set $H(0) := -\frac{1}{12}$ and prove the following result due to Hurwitz

$$\sum_{d \mid m} \max\left(d, \tfrac{m}{d}\right) = \sum_{\substack{t \in \mathbb{Z} \\ |t| \leq 2\sqrt{m}}} H(t^2 - 4m). \tag{5}$$

(4) Make a table of the Hurwitz class numbers $H(D)$ up to $|D| \leq 24$.