Problem 1

Let K be the quadratic field $\mathbb{Q}(i)$.

1. Describe how primes of $\mathbb{Q}$ decompose in $K/\mathbb{Q}$.

2. Use this to write down the Dedekind ζ-function $\zeta_K(s)$.

3. Find a Dirichlet character χ such that

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(\chi, s),$$

Use this to describe the functional equation for $L(\chi, s)$.

4. Let ρ_K be the Galois representation associated to $K/\mathbb{Q}$, coming from the action of $G_{\mathbb{Q}}$ on embeddings $K \hookrightarrow \mathbb{C}$. Prove that it is the same as the regular representation of $\mathrm{Gal}(K/\mathbb{Q})$ (viewed as a representation of $G_{\mathbb{Q}}$), and that it decomposes as $\mathbf{1} \oplus \rho_\chi$.

Problem 2

Let E be the elliptic curve $y^2 = x^3 - 5^2$ over $\mathbb{Q}_5$, of discriminant $\Delta = -2^4 \cdot 3^3 \cdot 5^4$. The aim is to give a complete description of V_2E as a $G_{\mathbb{Q}_5}$-module.

1. Prove that $K = \mathbb{Q}_5(E[2])$ is a Galois extension of $\mathbb{Q}_5$ with $\mathrm{Gal}(K/\mathbb{Q}_5) \cong S_3$.

2. Consider the action of $\mathrm{Gal}(K/\mathbb{Q}_5) \cong S_3$ on $E[2]$ as a representation

$$\mathrm{Gal}(K/\mathbb{Q}_5) \longrightarrow \mathrm{Aut}\, E[2] \cong \mathrm{GL}_2(\mathbb{F}_2).$$

Show that in some basis of $E[2]$ it is given by

$$\text{elt of order } 3 \longmapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right), \qquad \text{elt of order } 2 \longmapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right).$$

3. Show that E has additive reduction over $\mathbb{Q}_5$.

4. Show that E has good reduction over $\mathbb{Q}_5(\sqrt[3]{5})$. What is the reduced curve $\tilde{E}/\mathbb{F}_5$ over that field? What is $\#\tilde{E}(\mathbb{F}_5)$?

5. Prove that $\mathbb{Q}_5(E[2^n])$ has ramification degree 3 over $\mathbb{Q}_5$, for every $n \geq 1$.

6. Let $F = K^{nr} = \mathbb{Q}_5^{nr}(\sqrt[3]{5})$. Show that $G_{\mathbb{Q}_5}$ acts on $T_2(E)$ through its quotient $\mathrm{Gal}(F/\mathbb{Q}_5)$, and inertia $I \lhd G_{\mathbb{Q}_5}$ through $\mathrm{Gal}(F/\mathbb{Q}_5^{nr}) \cong C_3$.
[Draw the picture how $\mathbb{Q}_5$, $\mathbb{Q}_5^{nr}$, K and F fit together — it should help.]

7. Let σ be a generator of $\mathrm{Gal}(F/\mathbb{Q}_5^{nr}) \cong C_3$. Find its eigenvalues on V_2E. (Note that you know the determinant.) Deduce that V_2E and V_2E^* have trivial inertia invariants, and so $E/\mathbb{Q}_5$ has trivial Euler factor $F_5(T) = 1$.

7*. An ambitious version of (7) is to prove that *every* elliptic curve over $\mathbb{Q}_p$ with additive reduction has $F_p(T) = 1$. (It might help to consider potentially good and potentially multiplicative reduction separately — for the former the argument is as in (7).)

8. Let Φ be the Frobenius element of $F/\mathbb{Q}_5(\sqrt[3]{5})$. Explain why σ and Φ generate $\mathrm{Gal}(F/\mathbb{Q}_5)$, and $\Phi\sigma\Phi^{-1} = \sigma^{-1}$. What is the characteristic polynomial of Φ on V_2E? Use this to find the action of $\mathrm{Gal}(F/\mathbb{Q}_5)$, and therefore of $G_{\mathbb{Q}_5}$, on V_2E in some basis.