Modular curves, forms, elliptic curves, and symbols – Exercises 2

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1. Show that for $g \in \Gamma_0(N)$ the modular symbol $\{\alpha, g(\alpha)\} \in H(N)$ is independent of $\alpha \in \mathbb{P}^1(\mathbf{Q})$.

2. If $\alpha = a/b \in \mathbb{P}^1(\mathbf{Q})$, show that, as cusps of $X_0(p)$,

$$[\alpha] = \begin{cases} [\infty] & \text{iff } p \mid b, \\ [0] & \text{iff } p \nmid b. \end{cases}$$

3. (a) Show that $(c : d) \in \mathbb{P}^1(\mathbf{Z}/N\mathbf{Z})$ is fixed by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ iff $c^2 + d^2 \equiv 0 \bmod N$, and is fixed by $TS = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ iff $c^2 + cd + d^2 \equiv 0 \bmod N$.

   (b) Let $N = p \geq 5$. Count the number of 2-term relations and the number of 3-term relations in $H(p)$ (which depend on $p \bmod 4$ and $p \bmod 3$ respectively). Hence show that the system of relations has rank

$$\begin{cases} \frac{1}{6}(5p+13) & \text{if } p \equiv 1 \bmod 12 \\ \frac{1}{6}(5p+5) & \text{if } p \equiv 5 \bmod 12 \\ \frac{1}{6}(5p+7) & \text{if } p \equiv 7 \bmod 12 \\ \frac{1}{6}(5p-1) & \text{if } p \equiv 11 \bmod 12 \end{cases}$$

   and hence that the dimension of $H(p)$ is

$$\frac{1}{6}(p-13), \frac{1}{6}(p-5), \frac{1}{6}(p-7), \frac{1}{6}(p+1)$$

   respectively. Compare with the formula for $g(X_0(p))$.

4. (a) Compute $T_3\left\{0, \frac{1}{2}\right\}$ using continued fractions and hence find the Hecke eigenvalues a_3 for $N = 11$.

   (b) Write down the extended CH-matrices for $p = 3$ and $p = 5$.

   (c) Recompute a_3 and compute a_5 using (b) for $N = 11$.

5. (a) Compute $H(17)$, $H^+(17)$ and the eigenvalues of T_2, T_3. Hence find the BSD ratios $L(E,1)/\Omega$ for the elliptic curves $X_0(17)$.

   (b) If you enjoyed that, repeat with $H(19)$.