Modular curves, forms, elliptic curves, and symbols – Exercises 1

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1. Convince yourself that
$$[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)] = N \prod_{p \mid N} \left(1 + \frac{1}{p}\right);$$
or at least that $\Gamma_0(p)$ has index $p+1$.

2. Prove the genus formula for $\Gamma_0(p)$ by showing that:
   - above ∞ there are two cusps: 0 with width p, and ∞ with width 1.
   - above i there are
   $$\begin{cases} \frac{1}{2}(p+1) \text{ points of multiplicity 2, if } p \equiv 3 \bmod 4 \\ \frac{1}{2}(p-1) \text{ points of multiplicity 2, and 2 of multiplicity 1, if } p \equiv 1 \bmod 4 \end{cases}$$
   - above ρ there are
   $$\begin{cases} \frac{1}{3}(p+1) \text{ points of multiplicity 3, if } p \equiv 2 \bmod 3 \\ \frac{1}{3}(p-1) \text{ points of multiplicity 3, and 2 of multiplicty 1, if } p \equiv 1 \bmod 3 \end{cases}$$

3. Check that $(t+a)^3 = 1728t + (t+b)^2(t+c)$ (as an identity in t) forces $(a,b,c) = \pm(16,-8,64)$.

4. For $p = 3, 5$ verify the formula for the map $X_0(p) \to X(1)$ given in lectures. If possible, derive it yourself.

5. Find all imaginary quadratic algebraic integers α of norm 2 (up to multiplication by units). [Hint: The answer is $1+i, \sqrt{-2}, (1 \pm \sqrt{-7})/2$.] What are the associated j-invariants?

6. Find the genus of $X_0(p)$ for primes $p < 20$. How could you determine the genus of $X_0^+(p)$? [Hint: It is 0 for just 10 primes p s.t. $g(X_0(p)) \geq 1$].