# X0(11)

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27/06/2014

Finding an equation for $X_0(11)$ as a hyperelliptic curve, so that the hyperelliptic involution matches the Fricke involution. Using $q$-expansions, and following Elkies.

```
M2_11 = ModularForms(11); M2_11
```

```
Modular Forms space of dimension 2 for Congruence Subgroup Gamma0(11) of weight 2 over
Rational Field
```

This space is 2-dimensional, spanned by a cusp form $\omega$ and an Eisenstein series (in that order). The cusp form $\omega = (\eta\eta_{11})^2$ (though we do not use this).

```
omega, eis = M2_11.basis()
print(omega)
eis = (5/12)*eis
print(eis)
```

```
q - 2*q^2 - q^3 + 2*q^4 + q^5 + O(q^6)

5/12 + q + 3*q^2 + 4*q^3 + 7*q^4 + 6*q^5 + O(q^6)
```

The combination $\varepsilon$ defined here is chosen to be "anti-invariant", i.e. in the $-1$-eigenspace for the Fricke involution, normalised so that $\varepsilon(\infty) = 1$.

```
eps = (3/5)*(4*eis + omega); eps
```

```
1 + 3*q + 6*q^2 + 9*q^3 + 18*q^4 + 15*q^5 + O(q^6)
```

```
prec = 100
eps = eps.q_expansion(prec)
omega = omega.q_expansion(prec)
```

Since $\omega$ and $\varepsilon$ both have weight 2 and are anti-invariant, their ratio $u = \varepsilon/\omega$ is invariant and of weight 0, so is a modular function for $X_0(11)^+$.

```
uq = eps/omega
uq
```

```
q^-1 + 5 + 17*q + 46*q^2 + 116*q^3 + 252*q^4 + 533*q^5 + 1034*q^6 + 1961*q^7 + 3540*q^8 +
6253*q^9 + 10654*q^10 + 17897*q^11 + 29284*q^12 + 47265*q^13 + 74868*q^14 + 117158*q^15 +
180608*q^16 + 275562*q^17 + 415300*q^18 + 620210*q^19 + 916860*q^20 + 1344251*q^21 +
1953974*q^22 + 2819664*q^23 + 4038300*q^24 + 5746031*q^25 + 8122072*q^26 + 11413112*q^27 +
15943576*q^28 + 22153909*q^29 + 30620666*q^30 + 42118002*q^31 + 57654984*q^32 +
78572714*q^33 + 106612356*q^34 + 144066806*q^35 + 193899752*q^36 + 259984292*q^37 +
```

```
347304576*q^38 + 462327209*q^39 + 613339252*q^40 + 811024026*q^41 + 1069013852*q^42 +
1404781194*q^43 + 1840532364*q^44 + 2404575004*q^45 + 3132744362*q^46 + 4070498790*q^47 +
5275162528*q^48 + 6819123727*q^49 + 8793330264*q^50 + 11312138480*q^51 + 14518717280*q^52
+ 18592370019*q^53 + 23756804606*q^54 + 30291149097*q^55 + 38542515224*q^56 +
48942470286*q^57 + 62026009696*q^58 + 78456076244*q^59 + 99051996964*q^60 +
124825832891*q^61 + 157024747466*q^62 + 197184622687*q^63 + 247192843664*q^64 +
309367035050*q^65 + 386547572642*q^66 + 482212730000*q^67 + 600614281848*q^68 +
746945176660*q^69 + 927537356500*q^70 + 1150104904690*q^71 + 1424031255192*q^72 +
1760720414752*q^73 + 2174012098122*q^74 + 2680687048012*q^75 + 3301064549872*q^76 +
4059726733634*q^77 + 4986375019400*q^78 + 6116864490456*q^79 + 7494426711616*q^80 +
9171141557605*q^81 + 11209676564144*q^82 + 13685373963210*q^83 + 16688715678952*q^84 +
20328272749626*q^85 + 24734186469916*q^86 + 30062322710283*q^87 + 36499171513860*q^88 +
44267679822862*q^89 + 53634124701508*q^90 + 64916277214824*q^91 + 78493013669156*q^92 +
94815707495615*q^93 + 114421627976672*q^94 + 137949789612063*q^95 + 166159574862040*q^96 +
199952721936124*q^97 + O(q^98)
```

In fact as $u$ has a simple pole at $\infty$ it has degree 1, i.e. it generates teh function field for $X_0(11)^+$.

```
(5*uq-3)*omega/12 == eis.q_expansion(50)
True
```

Differentiating $u$ gives something invariant of weight 2, so dividing by $\omega$ we obtain an anti-invariant function (weight 0). Here we introduce a minus sign compared with Elkies' $v$, so as to make the $q^{-2}$ coefficient $-1$ (so that $b_{11}$ below has leading coefficient $1/2$ rather than $-1/2$).

```
q = uq.parent().gen()
vq = q*uq.derivative()/omega
vq
-q^-2 - 2*q^-1 + 12 + 116*q + 597*q^2 + 2298*q^3 + 7616*q^4 + 22396*q^5 + 60732*q^6 +
153682*q^7 + 368584*q^8 + 843150*q^9 + 1855509*q^10 + 3943764*q^11 + 8136120*q^12 +
16338586*q^13 + 32038879*q^14 + 61475598*q^15 + 115674856*q^16 + 213775728*q^17 +
388636425*q^18 + 695849836*q^19 + 1228523208*q^20 + 2140715850*q^21 + 3684990683*q^22 +
6271127674*q^23 + 10558430496*q^24 + 17598152686*q^25 + 29053725785*q^26 +
47536435974*q^27 + 77116991920*q^28 + 124096327434*q^29 + 198165353562*q^30 +
314133732470*q^31 + 494501176392*q^32 + 773252291130*q^33 + 1201442062530*q^34 +
1855362844984*q^35 + 2848450725624*q^36 + 4348544164222*q^37 + 6602824330966*q^38 +
9973674691206*q^39 + 14990081019120*q^40 + 22420975180468*q^41 + 33379570735443*q^42 +
49471182496840*q^43 + 73002026047808*q^44 + 107273058548292*q^45 + 156992295641448*q^46 +
228851396952244*q^47 + 332329704085632*q^48 + 480811904807656*q^49 + 693137464903379*q^50
+ 995742278645436*q^51 + 1425611221617296*q^52 + 2034337368045386*q^53 +
2893688797535424*q^54 + 4103223002226436*q^55 + 5800677235001344*q^56 +
8176111878989148*q^57 + 11491118474322133*q^58 + 16104845115889630*q^59 +
22509181667497464*q^60 + 31376223089306734*q^61 + 43622160980574054*q^62 +
60493107724993050*q^63 + 83680149954059976*q^64 + 115473274757328988*q^65 +
158966904025682874*q^66 + 218333810309774638*q^67 + 299189486102879136*q^68 +
409075940178647658*q^69 + 558102916267518746*q^70 + 759796250018432810*q^71 +
1032218346255320808*q^72 + 1399445540005274776*q^73 + 1893512783411777924*q^74 +
2556969288445756848*q^75 + 3446231700056486368*q^76 + 4635976735709940720*q^77 +
6224886648243585135*q^78 + 8343152708636449492*q^79 + 11162260046095626256*q^80 +
14907728724995875260*q^81 + 19875680343568575154*q^82 + 26454348206529981128*q^83 +
35151967413916207152*q^84 + 46632887542356440384*q^85 + 61764269171724037774*q^86 +
81676386009000650538*q^87 + 107840395258628150360*q^88 + 142168507513839466738*q^89 +
```

```
187142844741238564308*q^90 + 245980996025906396640*q^91 + 322848461807122469688*q^92 +
423130937080034495616*q^93 + 553782873658809598146*q^94 + 723773167890518667890*q^95 +
944654380000838072928*q^96 + O(q^97)
```

Now $v^2$ is a rational function of $u$, and in fact a polynomial since $u$ and $v$ only have poles at the cusps. Some work with $q$-expansions reveals this polynomial (details omitted, but we check the result: note that the following expression must be identically 0 since it has no poles and is 0 at $\infty$). We call the polynomial $f_{11}$, so $v^2 = f_{11}(u)$.

```
QU.<U> = QQ[]
f11 = U^4-16*U^3+2*U^2+12*U-7
```

```
vq^2 == f11(uq)
True
```

Now we wish to express $j$ as a polynomial in $u, v$. It is a polynomial as it has no poles except at the cusps 0 and $\infty$ which is where $u, v$ have their poles.

```
jq = j_invariant_qexp(100)
j11q = jq(q^11)
kq = jq-1728
k11q = j11q - 1728
```

The following function takes a $q$-expansion known to be a polynomial in $u$ and returns that polynomial, as an element of the polynomial ring $\mathbb{Q}[U]$.

```
def u_poly_conv(g,u=uq):
    h = 0
    while g.list() != []:
        e = -g.valuation()
        c = g[-e]
        g -= c*u^e
        h += c*U^e
    return h
```

Write $j = a_{11}(u) + vb_{11}(u)$. Then $j' = j(q^{11}) = a_{11}(u) - vb_{11}(u)$. We first find $a_{11}$ by recognising $j + j'$ as a polynomial in $u$. We want to find a quadratic polynomial in $\mathbb{Q}[U][X]$ whose roots are $j$ and $j'$, so the coefficients are $P_{11}(U)$ and $Q_{11}(U)$ where $P_{11}(u) = j + j'$ and $Q_{11}(u) = jj'$. Comparing coefficients gives $2a_{11} = -P_{11}$ and $a_{11}^2 - f_{11}b_{11}^2 = Q_{11}$.

```
P11 = -u_poly_conv(jq+j11q)
print P11.factor()
Q11 = u_poly_conv(jq*j11q)
print Q11.factor()
(-1) * (U^11 - 55*U^10 + 1188*U^9 - 12716*U^8 + 69630*U^7 - 177408*U^6 + 133056*U^5 +
132066*U^4 - 187407*U^3 + 40095*U^2 + 24300*U - 6750)

(U^4 + 228*U^3 + 486*U^2 - 540*U + 225)^3
```

```
u_poly_conv(kq*k11q).factor()
(U^6 - 522*U^5 - 10017*U^4 + 2484*U^3 - 5265*U^2 + 12150*U - 5103)^2
```

We create a polynomial in two variables $U$, $J$ which vanishes identically at $U = u$, $J = j$.

```
QUJ.<U,J> = QQ[]
pol = J^2 + P11*J + Q11
```

```
pol(uq,jq) == 0
True
```

Now we find $a_{11}$ and $b_{11}$. The first is easy.

```
a11 = u_poly_conv(jq+j11q)/2
print a11.factor()
b11 = u_poly_conv((jq-j11q)/vq)/2
print b11.factor()
(1/2) * (U^11 - 55*U^10 + 1188*U^9 - 12716*U^8 + 69630*U^7 - 177408*U^6 + 133056*U^5 +
132066*U^4 - 187407*U^3 + 40095*U^2 + 24300*U - 6750)

(1/2) * U * (U - 15) * (U - 6) * (U - 3) * (U - 1) * (U^2 - 12*U - 9) * (U^2 - 10*U + 5)
```

```
assert -2*a11 == P11
assert a11^2-f11*b11^2 == Q11
```

```
QUV.<U,V> = QQ[]
```

```
a11 = QUV(a11)
b11 = QUV(b11)
juv = a11+V*b11
juv.factor()
(1/2) * (U^11 - 55*U^10 + U^9*V + 1188*U^9 - 47*U^8*V - 12716*U^8 + 843*U^7*V + 69630*U^7
- 7187*U^6*V - 177408*U^6 + 29313*U^5*V + 133056*U^5 - 48573*U^4*V + 132066*U^4 +
10665*U^3*V - 187407*U^3 + 27135*U^2*V + 40095*U^2 - 12150*U*V + 24300*U - 6750)
```

We define the map from the $(u, v)$ curve to the $j$-line, and apply it to the noncuspidal points:

```
def j_map(u,v):
    return juv(u,v)
```

```
[j_map(u,v) for u,v in [(-1,0),(-2,-11),(-2,11)]]
[-32768, -121, -24729001]
```

The first of these is the $j$-invariant of $(1 + \sqrt{-11})/2$:

```
[(d,f) for d,f,j in cm_j_invariants_and_orders(QQ) if j==-32768]
[(-11, 1)]
```

```
E = EllipticCurve(j=-121); E.label()
'121c1'
```

The other two are linked by 11-isogenies defined over $\mathbb{Q}$:

```
EllipticCurve(j=-121).isogenies_prime_degree(11)[0].codomain().\
    j_invariant()
```

```
-24729001
```

```
EllipticCurve(j=-32768).isogenies_prime_degree(11)[0].codomain().\
    j_invariant()
```

```
-32768
```

```
EllipticCurve(j=-24729001).isogenies_prime_degree(11)[0].codomain().\
    j_invariant()
```

```
-121
```