Lecture 1: Exercises

1. Derive Artin's factorization of $\zeta_L(s)$ from the formal properties of the Artin L-function. (Use the decomposition of the regular representation.)

2. For F a local field, show $W_F^{ab} =\simeq F^\times$. For $F = \mathbb{R},\ \mathbb{C}$ you can do this by hand. For F non-archimedean, you can only do this if you know LCFT.

Lecture 2: Exercises

1. Take the following two low rank algebraic groups and work out the roots, positive roots, and simple roots and simple coroots.

(a) Sp_4. Take the symplectic form to be represented by $J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$ and define

$$Sp_4(K) = \{g \in GL_4(K) \mid {}^t g J g = J\}$$

With this form, one can obtain a Borel subgroup by intersecting with the Borel of GL_4.

(b) SO_5. Now take the symmetric form to be represented by $S = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$ and then

$$SO_5(K) = \{g \in GL_5(K) \mid {}^t g S g = S\}$$

Again, with this form, one can obtain a Borel subgroup by intersecting with the Borel of GL_5.

2. Can you see any hints of the Langlands duality between Sp_4 and SO_5.

Lecture 3: Exercises

1. Let $\pi \simeq \otimes' \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$ for some number field k. At almost all places π_v is unramified and there is an unramified Langlands parameter ϕ_v associated to it. The Satake parameter(s) for π_v is either the matrix, which we can take to be diagonal,

$$\phi_v(\Phi_v) = t_{\pi_v} = \begin{pmatrix} \alpha_{v,1} & \\ & \alpha_{v,2} \end{pmatrix} \in \widehat{T} \subset GL_2(\mathbb{C})$$

or its entries, and for any representation $r : GL_2(\mathbb{C}) \to GL_n(\mathbb{C})$ we have

$$L(s, \pi_v, r) = \det(I_n - r(t_{\pi_v}) q_v^{-s})^{-1} \quad \text{and} \quad L^S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r).$$

(a) What are the Satake parameters, local L-functions and the partial L-functions for $\Lambda^2(\pi)$? Can you see why this is $L^S(s, \omega_\pi)$?

(b) What are the Satake parameters, local L-functions and the partial L-functions for $Sym^k(\pi)$?

2. Now let $\pi \simeq \otimes' \pi_v$ be an unitary automorphic representation of $GL_n(\mathbb{A})$ for some number field k. At almost all places π_v is unramified and there is an unramified Langlands parameter ϕ_v associated to it. The Satake parameter(s) for π_v is either the matrix, which we can take to be diagonal,

$$\phi_v(\Phi_v) = t_{\pi_v} = \begin{pmatrix} \alpha_{v,1} & & \\ & \ddots & \\ & & \alpha_{v,n} \end{pmatrix} \in \widehat{T} \subset GL_n(\mathbb{C}).$$

A theorem of Jacquet and Shalika, coming from the theory of integral representations, says that there is a uniform bound on the Satake parameters of unitary automorphic representations of $GL_n(\mathbb{C})$ of the form

$$q_v^{-1/2} \leq |\alpha_{v,i}| \leq q_v^{1/2}.$$

Note that n does not appear ... this is a uniform bound.

(a) Combining the Sym^4 lift from GL_2 to GL_5 with the Jacquet–Shalika bound on Satake parameters for GL_5, what bounds on the Satake parameters for unitary cusp forms on GL_2 do you get?

(b) If we knew the existence of all symmetric power lifts $Sym^k : \mathcal{A}_0(GL_2) \to \mathcal{A}(GL_{k+1})$ what bounds on the Satake parameters for unitary cusp forms of GL_2 would we have? This is the Ramanujan conjecture for GL_2. (Note, this is over any number field.)