LECTURE 3: FUNCTORIALITY

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1. Introduction

Langlands always viewed the "principle of functoriality" as central to his view of automorphic representations. It is both a consequence of the local and global Langlands correspondences and encompasses them as a special case. It gives a *conceptual framework* for the transferring of automorphic or admissible representations.

Functoriality is mediated by admissible homomorphisms between L-groups. Let K denote a local or global field. Let H and G be connected reductive groups over K and ${}^LH$ and ${}^LG$ their L-groups.

Definition. *A homomorphism $u : {}^LH \to {}^LG$ is called an* L-homomorphism *if*

(i) *it is a homomorphism over $Gal(\bar{K}/K)$, that is, the following commutes:*

$$\begin{array}{ccc} {}^LH & \xrightarrow{\quad u \quad} & {}^LG \\ & \searrow \quad \swarrow & \\ & Gal(\bar{K}/K) & \end{array}$$

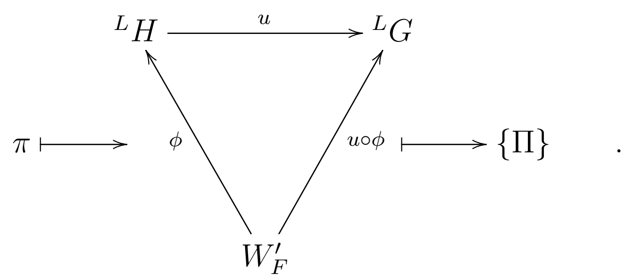
(ii) *u is continuous*
(iii) *The restriction of u to $\widehat{H}$ is a complex analytic $u : \widehat{H} \to \widehat{G}$.*

A consequence of this definition is that if G *is quasisplit* and $\phi \in \Phi(H)$ then the composition $u \circ \phi \in \Phi(G)$ is also admissible. So L-homomorphisms are those that induce maps $\Phi(u) : \Phi(H) \to \Phi(G)$ on parameter spaces, as long as G is quasisplit. (It is possible that they can induce maps on parameter spaces even if G is not quasisplit. This is a matter of preserving "relevancy".)

2. Local Functoriality

Now take $K = F$ a local field. If we assume the LLC for G and H (which we know for many representations and completely in some cases) then it is easy to see how an L-homomorphism gives a transfer of admissible representations.

Local Functoriality Diagram. *Let F be local, H and G defined over F, and G quasisplit. Let $u : {}^L H \to {}^L G$ be an L-homomorphism. Then for $\pi \in \mathcal{A}_\phi(H)$ we have*



where $\{\Pi\} = \mathcal{A}_{u\circ\phi}(G) \subset \mathcal{A}(G(F))$.

So each $\pi \in \mathcal{A}(H(F))$ determines a local L-packet $\mathcal{A}_{u\circ\phi}(G(F))$ of representations of $G(F)$. What we really transfer is the Langlands parameter and hence their associated L-packets. (However, remember that if π is unramified, the L-packet is a singleton.)

One of the important aspects of Functoriality is that it preserves Langlands L-functions and ε-factors, essentially by definition. If $u : {}^L H \to {}^L G$ and $r : {}^L G \to GL_n(\mathbb{C})$ then the composition $r \circ u : {}^L H \to GL_n(\mathbb{C})$ and

$$L(s, \pi, r \circ u) = L(s, (r \circ u) \circ \phi) = L(s, r \circ (u \circ \phi)) = L(s, \Pi, r)$$

and similarly

$$\varepsilon(s, \pi, r \circ u, \psi) = \varepsilon(s, \Pi, r, \psi).$$

So we can view Functoriality as giving a map or transfer $\mathcal{A}(H(F)) \to \mathcal{A}(G(F))$ mediated by the preservation of Langlands L- and ε-factors (through local Artin L- and ε-factors).

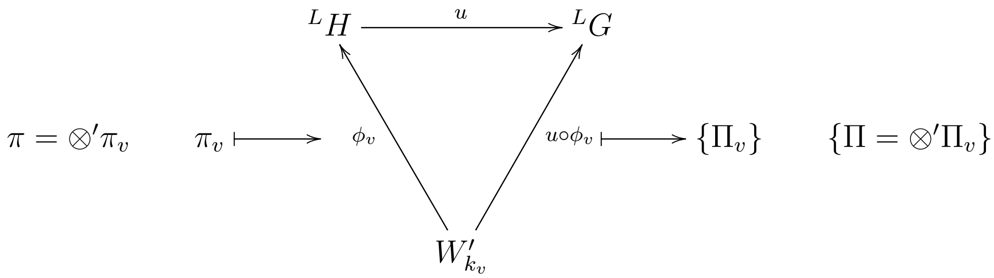
If we take $G = GL_n$ so that ${}^L G = GL_n(\mathbb{C}) \times W'_F$ then the L-packets for GL_n are singletons, and if we take $r = id$ we get functoriality to GL_n in terms of the analytic L-functions on GL_n that appeared in the LLC.

3. Global Functoriality

If we had a global version of the Weil-Deligne group (or maybe when we do) we could write down a similar global functoriality diagram. In its absence we can think of a functoriality in terms of a local/global compatibility as follows.

Let k be a global field, $u : {}^L H \to {}^L G$ an L-homomorphism. The local/global L-group constructions are compatible, so for all places v of k u induces a L-homomorphism, which we still denote by $u : {}^L H_v \to {}^L G_v$. Then we can piece together the various local functoriality diagrams to obtain

Global Functoriality Conjecture. *Let $u : {}^L H \to {}^L G$ be an L-homomorphism. Let $\pi \simeq \otimes' \pi_v$ be an irreducible automorphic representation of $H(\mathbb{A})$. Then applying our local* *functoriality diagram at each place we obtain a "global diagram"*



Then at least one (if not all) of the Π *formed this way should be an automorphic representation of* $G(\mathbb{A})$.

(Remember that if $u \circ \phi_v$ is unramified, which will be the case for almost all v, then $\mathcal{A}_{u\circ\phi_v}$ is a singleton.)

Again, this global functoriality should preserve global Langlands L-functions, that is, for all $r : {}^LG \to GL_n(\mathbb{C})$ we have

$$L(s, \pi, r \circ u) = \prod_v L(s, \pi_v, r \circ u) = \prod_v L(s, r \circ u \circ \phi_v) = \prod_v L(s, \Pi_v, r) = L(s, \Pi, r)$$

and similarly

$$\varepsilon(s, \pi, r \circ u) = \varepsilon(s, \pi, r).$$

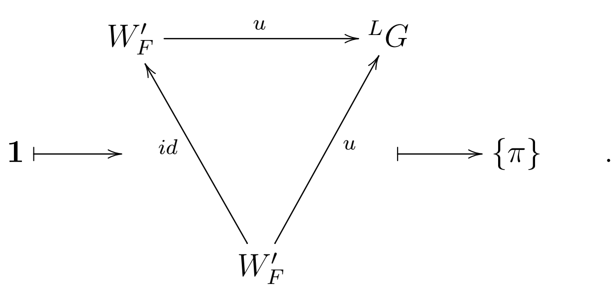
So, once again, you can view global functoriality as a transfer now of automorphic representations (or L-packets) $\mathcal{A}(H) \to \mathcal{A}(G)$ mediated by the equality of Langlands L-functions and ε-factors.

4. The local Langlands correspondence revisited

Let us now consider Functoriality in the case of $H = \{e\}$, the trivial group. Then there is no restriction on the G we can use as the target since H has no internal structure.

If we let F be a local field and take the Weil-Deligne from of the L-group then ${}^LH = \{e\} \times W'_F = W'_F$. The only representation of $H(F)$ is the trivial one, $\mathbf{1}$, and the only admissible homomorphism of W'_F to ${}^LH$ is the identity map.

Now an L-homomorphism $u : {}^LH \to {}^LG$ is simply an admissible homomorphism from W'_F to ${}^LG$ and our local functoriality diagram takes the form



where $\{\pi\} = \mathcal{A}_u(G)$ is the L-packet associated to u and we have recovered the local Langlands correspondence for G.

5. Galois theoretic examples (for GL_n)

If L/K is a finite extension of local or global fields then

$$Gal(\bar{L}/L) \subset Gal(\bar{K}/K) \quad \text{or} \quad W_L \subset W_K \quad \text{or in the local case } W'_L \subset W'_K.$$

If we look at the complex representations of these groups, say in Weil group setting, we have, following Artin, induction and restriction operations

$$Rep(W_L) \xrightarrow{Ind} Rep(W_K) \xrightarrow{Rest} Rep(W_L)$$

In terms of the LLC and GLC for GL_n there should be analogous operations on admissible or automorphic representations of GL_n. These should all be expressible in terms of functoriality. Now, the formalism of functoriality has a fixed base field, so there are some technical aspects to changing fields that are handled by Weil's restriction of scalars.

(a) Base change or automorphic restriction. This would correspond to the restriction of representations from W_K to W_L above. We would need a transfer $\mathcal{A}(GL_n(K)) \to \mathcal{A}(GL_n(L))$. Weil's restriction of scalars gives a group $G = Res_{L/K}(GL_n)$ such that $G(K) = GL_n(L)$. We have

$$G = \left(\prod_{Gal(L/K)} (GL_n)^\sigma \right) \quad \text{and} \quad {}^LG = \left(\prod_{Gal(L/K)} GL_n(\mathbb{C}) \right) \rtimes Gal(\bar{K}/K)$$

where the Galois action factors through $Gal(L/K)$ where it acts by permuting the factors in the product. Then the required L homomorphism $u : {}^LGL_n \to {}^LG$ is simple the diagonal embedding on the $GL_n(\mathbb{C})$. The map $\Phi(u) : \Phi(GL_n) \to \Phi(G)$ indeed corresponds to the restriction $W_K \to W_L$.

In the case of L/K solvable, Arthur and Clozel established this functoriality via the twisted trace formula in both the local and global contexts.

(b) Automorphic induction. This would correspond to the induction of representations from W_L to W_K. Suppose that L/K is separable of degree d. Then if $\rho : W_L \to GL_n(\mathbb{C})$ we have $Ind(\rho) : W_K \to GL_{nd}(\mathbb{C})$. So we need a functoriality of the form $\mathcal{A}(GL_n(L)) \to \mathcal{A}(GL_{nd}(K))$.

To achieve this, we need to take $H = Res_{L/K}(GL_n)$ and $G = GL_{dn}$. The map on L groups

$$u : {}^LH = \left(\prod_{Gal(L/K)} GL_n(\mathbb{C}) \right) \rtimes Gal(\bar{K}/K) \to {}^LG = GL_{dn}(\mathbb{C}) \times Gal(\bar{K}/K)$$

is given by the block diagonal embedding on $\widehat{H}$

$$\widehat{H} = GL_n(\mathbb{C}) \times \cdots \times GL_n(\mathbb{C}) \to \begin{pmatrix} GL_n(\mathbb{C}) & & \\ & \ddots & \\ & & GL_n(\mathbb{C}) \end{pmatrix} \subset \widehat{GL_{dn}}$$

and $Gal(L/K)$ maps into the Weyl group $W(GL_{dn})$ as permutations of the blocks.

This was established for L/K solvable, both locally and globally, by Arthur and Clozel via the trace formula. Local automorphic induction in general was established by Henniart and Herb by character identities.

Note that while both base change and automorphic induction are best motivated for GL_n, the formalism works for any reductive algebraic group G.

6. Group theoretic examples

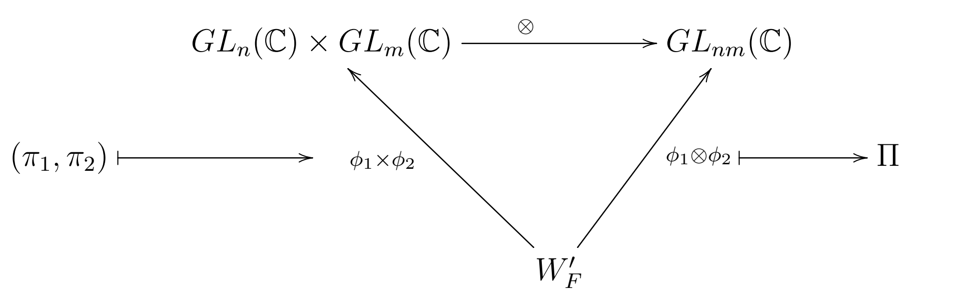
In these examples we take H split and $G = GL_n$. The Galois group plays a minimal role.

(a) Tensor products. Let $H = GL_n \times GL_m$ and $G = GL_{nm}$. Then the tensor product of matrices gives us a homomorphism of dual groups

$$\otimes : \widehat{H} = GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \to \widehat{G} = GL_{nm}(\mathbb{C})$$

which we extend trivially on $G_K = Gal(\bar{K}/K)$ to a L-homomorphism.

Over a local field $K = F$ we can define this by the LLC



Over a global field k, we would use a local/global principle. If we have $\pi_1 \simeq \otimes' \pi_{1,v}$ a cuspidal automorphic representation of $GL_n(\mathbb{A})$ and $\pi_2 \simeq \otimes' \pi_{2,v}$ a cuspidal automorphic representation of $GL_m(\mathbb{A})$, then for each pair of local components $(\pi_{1,v}, \pi_{2,v})$ we can apply the local diagram to obtain a representation Π_v of $GL_{mn}(k_v)$. According to the global functoriality conjecture, $\Pi = \otimes' \Pi_v$ should be an automorphic representation of $GL_{nm}(\mathbb{A})$.

We only know this in the trivial case of $m = 1$ and then in the cases

- $n = m = 2$, so $\mathcal{A}_0(GL_2) \times \mathcal{A}_0(GL_2) \to \mathcal{A}(GL_4)$, due to Ramakrishnan,
- $n = 3, m = 2$, so $\mathcal{A}_0(GL_3) \times \mathcal{A}_0(GL_2) \to \mathcal{A}(GL_6)$, due to Kim and Shahidi.

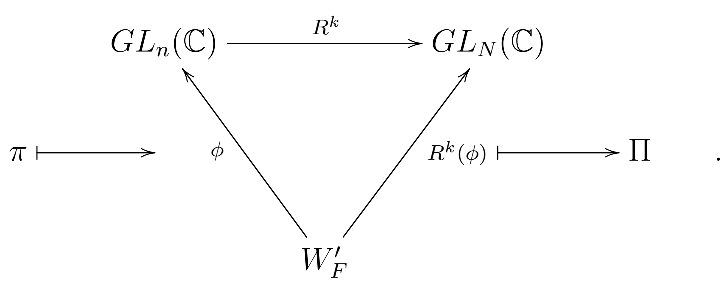
Both of these were obtained by the method of L-functions, i.e, the converse theorem for GL_n.

(b) Symmetric and exterior powers. In this case both H and G are linear, say $H = GL_n$ and $G = GL_N$. There are two natural maps between linear groups, derived from successive tensors, namely the symmetric and exterior powers:

$$Sym^k : GL_n(\mathbb{C}) \to GL_N(\mathbb{C}) \quad \text{and} \quad \Lambda^k : GL_n(\mathbb{C}) \to GL_N(\mathbb{C})$$

where the choice of N depends on both n and k. We can extend these to L-homomorphisms by making them trivial on $G_K = Gal(\bar{K}/K)$.

For F a local field we can define these by the LLC, which remember is known for GL,



where $R^k = Sym^k$ or Λ^k and N is taken appropriately.

Over a global field k, we would use a local/global principle. If we have $\pi \simeq \otimes' \pi_v$ a cuspidal automorphic representation of $GL_n(\mathbb{A})$ then for each local component π_v we can apply the local diagram to obtain a representation Π_v of $GL_N(k_v)$. According to the global functoriality conjecture, $\Pi = \otimes' \Pi_v$ should be an automorphic representation of $GL_N(\mathbb{A})$.

For the symmetric power functoriality, all we only have results for $n = 2$ and $k = 2, 3, 4$. These are:

- $Sym^2 : \mathcal{A}_0(GL_2) \to \mathcal{A}(GL_3)$ due to Gelbart and Jacquet,
- $Sym^3 : \mathcal{A}_0(GL_2) \to \mathcal{A}(GL_4)$ due to Kim and Shahidi,
- $Sym^4 : \mathcal{A}_0(GL_2) \to \mathcal{A}(GL_5)$ due to Kim.

For exterior powers, in the case of Λ^n this functoriality should map $\mathcal{A}(GL_n) \to \mathcal{A}(GL_1)$ and this is just the map $\pi \mapsto \omega_\pi$ the central character. Otherwise, the only non-trivial case we know is for $n = 4$ and $k = 2$, so

- $\Lambda^2 : \mathcal{A}_0(GL_4) \to \mathcal{A}(GL_6)$ due to Kim.

These were all obtained by the method of L-functions and converse theorems.

Note that if we knew the symmetric power liftings $Sym^k : \mathcal{A}_0(GL_2) \to \mathcal{A}(GL_{k+1})$ for all k, then both the Ramanujan conjecture and Selberg's eigenvalue conjecture for GL_2 would follow. Similarly for GL_n.

(c) Classical groups (endoscopy). In this family of examples we take H to be a split or quasisplit classical group (so Sp, SO, or U) and $G = GL_N$ for an appropriate N. Then we will have natural embeddings ${}^L H \hookrightarrow {}^L G$:

H	${}^L H$	${}^L GL_N$	GL_N
Sp_{2n}	$SO_{2n+1}(\mathbb{C}) \times G_K$	$GL_{2n+1}(\mathbb{C}) \times G_K$	GL_{2n+1}
SO_{2n+1}	$Sp_{2n}(\mathbb{C}) \times G_K$	$GL_{2n}(\mathbb{C}) \times G_K$	GL_{2n}
SO_{2n}	$SO_{2n}(\mathbb{C}) \times G_K$	$GL_{2n}(\mathbb{C}) \times G_K$	GL_{2n}
SO^*_{2n}	$SO_{2n}(\mathbb{C}) \rtimes G_K$	$GL_{2n}(\mathbb{C}) \times G_K$	GL_{2n}
U_n	$GL_n(\mathbb{C}) \rtimes G_K$	$(GL_n(\mathbb{C}) \times GL_n(\mathbb{C})) \rtimes G_K$	$Res_{E/K}(GL_n)$

In each case ${}^L H$ occurs as the fixed point set of an involution in ${}^L G$. This corresponds to Langlands notion of endoscopic transfer.

These have been (somewhat) established by both methods.

• Arthur, in his book "The Endoscopic Classification of Representations" established this in full generality, locally and globally, for the split classical groups. This was (is being) done via various forms of the trace formula, as various Fundamental Lemmas get established. The methods have been carried over to the quasisplit situation by Mok. I am not sure how this is related to knowledge of the LLC and GLC for these groups.

• The L-function method carried this out for generic cuspidal representations of H.

The image on GL_N can be characterized in term of L-functions.

7. Concluding unscientific postscript

• There was a recent conference on "The Future of the Trace Formula" in Banff. In its current shape, it seems the trace formula could analyze the endoscopic transfer from "quasi-classical" groups (i.e., such that $\widehat{H}$ is a classical group). This always relies on various forms of the trace formula and fundamental lemmas.

• The L-function method of establishing functoriality to GL_n relies on the converse theorem and the analytic control of twisted L-functions. The cases where the analytic properties of the L-functions can be handled by the Langlands–Shahidi method (Fourier coefficients of Eisenstein series) have mostly been done. (Asgari and Shahidi are finishing up $GSpin$.) New cases of functoriality by this method would rely on new integral representations for twisted L-functions. As RPL would probably say, "that way madness lies" ...

Exercises

1. Let $\pi \simeq \otimes' \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$ for some number field k. At almost all places π_v is unramified and there is an unramified Langlands parameter ϕ_v associated to it. The Satake parameter(s) for π_v is either the matrix, which we can take to be diagonal,

$$\phi_v(\Phi_v) = A_{\pi_v} = \begin{pmatrix} \alpha_{v,1} & \\ & \alpha_{v,2} \end{pmatrix} \in \widehat{T} \subset GL_2(\mathbb{C})$$

or its entries, and for any representation $r : GL_2(\mathbb{C}) \to GL_n(\mathbb{C})$ we have

$$L(s, \pi_v, r) = \det(I_n - r(A_{\pi_v}) q_v^{-s})^{-1} \quad \text{and} \quad L^S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r).$$

(a) What are the Satake parameters, local L-functions and the partial L-functions for $\Lambda^2(\pi)$? Can you see why this is $L^S(s, \omega_\pi)$?

(b) What are the Satake parameters, local L-functions and the partial L-functions for $Sym^k(\pi)$?

2. Now let $\pi \simeq \otimes' \pi_v$ be an unitary automorphic representation of $GL_n(\mathbb{A})$ for some number field k. At almost all places π_v is unramified and there is an unramified Langlands parameter ϕ_v associated to it. The Satake parameter(s) for π_v is either the matrix, which we can take to be diagonal,

$$\phi_v(\Phi_v) = A_{\pi_v} = \begin{pmatrix} \alpha_{v,1} & & \\ & \ddots & \\ & & \alpha_{v,n} \end{pmatrix} \in \widehat{T} \subset GL_n(\mathbb{C}).$$

A theorem of Jacquet and Shalika, coming from the theory of integral representations, says that there is a uniform bound on the Satake parameters of unitary automorphic representations of $GL_n(\mathbb{C})$ of the form

$$q_v^{-1/2} \leq |\alpha_{v,i}| \leq q_v^{1/2}.$$

Note that n does not appear ... this is a uniform bound.

(a) Combining the Sym^4 lift from GL_2 to GL_5 with the Jacquet–Shalika bound on Satake parameters for GL_5, what bounds on the Satake parameters for unitary cusp forms on GL_2 do you get?

(b) If we knew the existence of all symmetric power lifts $Sym^k : \mathcal{A}_0(GL_2) \to \mathcal{A}(GL_{k+1})$ what bounds on the Satake parameters for unitary cusp forms of GL_2 would we have? This is the Ramanujan conjecture for GL_2. (Note, this is over any number field.)

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See the papers of Langlands given in the previous lectures as well as the two surveys:

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