LECTURE 2: LANGLANDS CORRESPONDENCE FOR G

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1. Introduction

If we view the flow of information in the Langlands Correspondence as

$$\text{Galois Representations} \longleftarrow \begin{array}{c}\text{automorphic/admissible}\\ \text{representations of } GL_n\end{array}$$

then this is the approach to non-abelian CFT: understand $Gal(\bar{k}/k)$. However we can view the flow in the other direction

$$\text{Galois Representations} \longrightarrow \begin{array}{c}\text{automorphic/admissible}\\ \text{representations of } GL_n\end{array}$$

this gives an arithmetic parametrization of automorphic or admissible representations of GL_n.

Question. What if we want to replace GL_n by some other group G? After all, this is the strength of the Langlands program. The RHS is "understood", but on the Galois side

$$\rho : Gal(\bar{k}/k) \to GL_n(\mathbb{C})$$

what plays the role of $GL_n(\mathbb{C})$ for general G?

This is the (Langlands) *L-group* ${}^LG$. The L-group has its origin in the theory of Eisenstein series and L-functions. Recall from SF that the classical Eisenstein series (un-normalized) had a Fourier expansion

$$E(z,s) = \left[y^s + \frac{Z(2-2s)}{Z(2s)}y^{1-s}\right] + \cdots = \left[y^s + \frac{Z(2s-1)}{Z(2s)}y^{1-s}\right] + \cdots$$

where $Z(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. Classically one could use the continuation and FE of $\zeta(s)$ to derive that of $E(z,s)$. Langlands had developed a theory of Eisenstein series for general (reductive algebraic) G. Representation theoretically, Eisenstein series correspond to induced representations and the formalization of Eisenstein series make these induced representations automorphic. Langlands continued and proved the FE for his Eisenstein series without recourse to zeta functions and then turned the classical process on its head and tried to understand the Euler products that occurred in the constant terms of his Eisenstein series. At his disposal he had

- Artin L-functions and their Euler products,
- Satake's parametrization of spherical (unramified) representations (SF),
- his raw calculations, expressed in terms of structure theory of G.

In his letter to Weil in 1967 and his *Euler Products* notes from the same year langlands explained his notion of the L-group and how it expressed the Euler products that appeared in his constant terms in terms of a new type of L-functions. This led to the LLC, GLC, Functoriality, ...

2. The L-group ${}^LG$

To explain the L-group we must have some *structure theory.* I will explain for GL_n but there is something similar for general G (reductive algebraic). For classical groups like Sp_{2n} or SO_{2n+1} you can think of them as linear groups in GL_n to calculate with.

For $G = GL_n$ we have its Borel subgroup $B = TU$ with T the diagonal torus and U the upper triangular unipotent group:

$$T = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \right\} \quad U = \left\{ \begin{pmatrix} 1 & u_{1,2} & \cdots & u_{1,n} \\ & 1 & * & u_{2,n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \right\}.$$

The Lie algebra of GL_n is $\mathfrak{g} = Lie(GL_n) = M_n$ the $n \times n$ matrices.

Let K be a field of definition of G. (K could be local or global, and for GL_n we can take $\mathbb{Q}$ if we want.) Then we have

$$\begin{aligned} X(T) &= Hom_K(T, \mathbb{G}_m) = \text{the } K\text{-rational characters of } T \\ \cup& \\ \Phi(T) &= \text{the roots of } T \text{ in } G = \text{the characters occurring in the adjoint action of } T \text{ on } \mathfrak{g} \\ \cup& \\ \Phi^+(T) &= \text{the positive roots} = \text{the characters occurring in the adjoint action of } T \text{ on } \mathfrak{u} = Lie(U) \\ \cup& \\ \Delta \quad &= \text{the simple roots} = \text{“basis” of } \Phi^+(T) \end{aligned}$$

For GL_n the simple roots are $\Delta = \{\alpha_1, \dots, \alpha_{n-1}\}$ where $\alpha_i(t) = t_i t_{i+1}^{-1}$, which occur on the first diagonal rank of $\mathfrak{u}$.

And these have dual structures:

$$\begin{aligned} X^\vee(T) &= Hom_K(\mathbb{G}_m, T) = \text{the } K\text{-rational one parameter subgroups of } T \\ \cup& \\ \Phi^\vee(T) &= \text{the coroots of } T \text{ in } G \\ \cup& \\ \Delta^\vee \quad &= \text{the simple coroots} \end{aligned}$$

For GL_n the simple coroots are the $\alpha_i^\vee$ where $\alpha_i^\vee(t) = \mathrm{diag}(1, \dots, t, t^{-1}, \dots, 1)$ with t in the i^{th} position.

With this notation, the *root datum* for G is

$$\Psi(G) = (X(T), \Phi(T), X^\vee(T), \Phi^\vee(T))$$

and the *based root datum* is

$$\Psi_0(G) = (X(T), \Delta, X^\vee(T), \Delta^\vee).$$

The reason this (essentially linear algebra) data is important is the following:

Theorem. [Chevalley, Steinberg] *The root datum $\Psi(G)$ determines G up to $\bar{K}$-isomorphism.*

Most groups are more complicated and involve a galois structure. (Think of unitary groups which involve a quadratic extension E/K and its Galois conjugation in its definition.) How to classify groups over non-algebraically closed fields? I believe this is due to Tits. You have to use the based root datum and build in the Galois action. There is a split exact sequence

$$1 \longrightarrow Int(G) \longrightarrow Aut(G) \longrightarrow Aut(\Psi_0(G)) \longrightarrow 1$$

The K structure is given by a morphism of $\mu : Gal(\bar{K}/K) \to Aut(G)$ which descends to $\mu : Gal(\bar{K}/K) \to Aut(\Psi_0(G))$.

Theorem. [Tits] *G is determined up to K-isomorphism by $\Psi(G)$ and the morphism $\mu : Gal(\bar{K}/K) \to Aut(\Psi_0(G))$.*

Given this structure theory for G and his Euler product computations, Langlands first defined a complex analytic dual group $\widehat{G}$ as a first approximation to $GL_n(\mathbb{C})$. If we begin with the root datum $\Psi(G)$ associated to G and "dualize" in the sense of simply interchanging $X(T) \leftrightarrow X^\vee(T)$ and $\Phi(T) \leftrightarrow \Phi^\vee(T)$ we get another root datum

$$\Psi(G)^\vee = (X^\vee(T), \Phi^\vee(T), X(T), \Phi(T))$$

which then determines a group $\widehat{G}$ over $\mathbb{C}$ such that $\Psi(G)^\vee = \Psi(\widehat{G})$. This is the *Langlands dual group* of G. [Langlands took $\widehat{G}$ as a complex group, but $\Psi(G)^\vee$ determines a group over any algebraically closed field.]

It is not hard to compute the dual group using structure theory, but note that this duality "flips" groups of type B and C:

G	$\widehat{G}$
GL_n	$GL_n(\mathbb{C})$
Sp_{2n}	$SO_{2n+1}(\mathbb{C})$
SO_{2n+1}	$Sp_{2n}(\mathbb{C})$
SO_{2n}	$SO_{2n}(\mathbb{C})$

Note that we have recovered $GL_n(\mathbb{C})$ when $G = GL_n$.

To obtain the L-group as a group that actually sees G and not just G over $\bar{K}$ he had to work in the Galois action of $Gal(\bar{K}/K)$ on the complex group $\widehat{G}$. But this is "easy". Since $Aut(\Psi_0(G)) \simeq Aut(\Psi_0(G)^\vee) \simeq Aut(\Psi_0(\widehat{G}))$ and $Aut(\Psi_0(\widehat{G}) \to Aut(\widehat{G})$ by our split exact sequence above then we have $\mu : Gal(\bar{K}/K) \to Aut(\psi_0(\widehat{G})) \to Aut(\widehat{G})$. So Langlands set

$$^LG = \widehat{G} \rtimes_\mu Gal(\bar{K}/K)$$

and this is *the L-group of G*. Note, for certain purposes we could take $^LG = \widehat{G} \rtimes_\mu W_K$ or $^LG = \widehat{G} \rtimes_\mu W'_K$ since the Weil and Weil-Deligne groups naturally map to the Galois group.

The L-group let Langlands explain the Euler products that occurred in the constant terms of his Eisenstein series. It also gave him a way to formulate the langlands correspondence for general G, namely *replace $GL_n(\mathbb{C})$ by ^LG.*

3. Langlands Correspondence for G

F a local field. We begin with the local Langlands correspondence. We now have the L-group of G at our disposal. What plays the role of the actual Galois representations ρ? These are the *admissible homomorphisms.*

Definition. *A homomorphism $\phi : W'_F \to {}^LG$ is called* admissible *if*

(i) *it is a homomorphism over $Gal(\bar{F}/F)$, that is the following commutes:*

$$\begin{array}{ccc} W'_F & \xrightarrow{\ \phi\ } & {}^LG \\ & \searrow \quad \swarrow & \\ & Gal(\bar{F}/F) & \end{array}$$

(ii) *ϕ is continuous, $\phi(\mathbb{G}_a)$ is unipotent in $\widehat{G}$, and $\phi(\Phi_F)$ is semisimple,*
(iii) *"relevance" (roughly, if $\phi(W_F)$ lies in the Levi subgroup of a parabolic P, it must be defined over F).*

For $G(F)$ in general we do not have an independent analytic theory of L-functions as for GL_n. So the original LLC was phrased in terms of representation theory.

Local Langlands Correspondence/Conjecture for G. *Let G be a connected reductive algebraic over F. There exists a surjective map*

$$\mathcal{A}(G) = \left\{ \begin{array}{c} \textit{irreducible admissible} \\ \textit{representations } \pi \textit{ of } G(F) \end{array} \right\} \to \Phi(G) = \left\{ \begin{array}{c} \phi : W'_F \to {}^LG \\ \textit{admissible} \end{array} \right\}$$

with finite fibres $\mathcal{A}_\phi$ such that $\mathcal{A}(G) = \coprod \mathcal{A}_\phi$ satisfying a list of 5 representation theoretic desiderata.

Among the desiderata are

- one $\pi \in \mathcal{A}_\phi$ is square integrable $\iff$ all $\pi \in \mathcal{A}_\phi$ are square integrable $\iff$ $\phi(W_F')$ is not contained in a proper Levi
- one $\pi \in \mathcal{A}_\phi$ is tempered $\iff$ all $\pi \in \mathcal{A}_\phi$ are tempered $\iff$ $\phi(W_F)$ is bounded
- compatibility for certain maps $H(F) \to G(F)$ (abelian kernel and cokernel).

The fibres $\mathcal{A}_\phi$ are called L-packets and ϕ is its Langlands parameter. This really would be an arithmetic parametrization of the representations of $G(F)$.

What do we know (or did Langlands know)?

- If F is non-archimedean and π is spherical (unramified) then the parameter ϕ is unramified (trivial on I_F and $\mathbb{G}_a$), $\mathcal{A}_\phi$ is a singleton, and $\phi(\Phi_F) = A_\pi \in \widehat{T}$ is the Satake parameter of π.
- If $G = T$ is a torus, this was established by Langlands in 1968.
- if $F = \mathbb{R}$ or $\mathbb{C}$ this was proved in complete generality by Langlands in 1973. This is *the* Langlands classification.
- if $G = GL_n$ this is subsumed by the L-function version.
- Sporadic special cases (GSp_4, Sp_4, generic representations of SO_{2n+1}).

k **a global field.** If we had a global Weil-Deligne group W_k' we could give an analogous global statement. Without that we can essentially do what we did for GL_n, that is

A. Restrict to parameters of Galois type, i.e., of the form $\phi : Gal(\bar{k}/k) \to {}^LG$ or $\phi : W_k \to {}^LG$, and ask what type of automorphic representations are parametrized by these.

B. Local/global compatibility. If $\pi = \otimes' \pi_v$ is a cuspidal automorphic representation of $G(\mathbb{A})$ then assuming the LLC for $G(k_v)$ each component π_v determines a Langlands parameter $\phi_v \in \Phi(G(k_v)$ and the collection $\phi = \{\phi_v\}$ should fit together to give a "global parameter" for π and so have an inferred compatibility.

4. Langlands L-functions

Remember, Langlands was trying to understand certain Euler products in terms of L-functions. One goal of this parametrization was to define suitable L-functions. This was done as follows.

We need one extra piece of data: a representation

$$r : {}^LG \to GL_n(\mathbb{C})$$

which is continuous and such that the restriction of r to $\widehat{G}$ is complex analytic.

Local Langlands L-functions. If π is an irreducible admissible representation of $G(F)$ with Langlands parameter $\phi \in \Phi(G)$, i.e., $\pi \in \mathcal{A}_\phi$, then if we compose the parameter ϕ with our representation r we obtain a Weil-Deligne (or simply Artin) representation $r \circ \phi : W_F' \to$

$GL_n(\mathbb{C})$. We use this to then define the local L-function attached to π and r:

$$L(s,\pi,r) = L(s, r\circ\phi)$$
$$\varepsilon(s,\pi,r,\psi) = \varepsilon(s, r\circ\phi,\psi)$$

Note that the elements of $\mathcal{A}_\phi$ cannot be distinguished by this class of (Langlands) L-functions, hence the name L-packet.

Global Langlands L-functions. Even though we do not have a (even hypothetical) global Langlands correspondence, we can none the less define global L-functions by local/global compatibility and Euler products. Let k be a global field. If $r : {}^LG \to GL_n(\mathbb{C})$ then by the compatibility of the local and global L-groups, r will determine a complex representation of ${}^LG_v = {}^LG(k_v)$, which we still denote by r:

$$r : {}^LG_v \to {}^LG \to GL_n(\mathbb{C}).$$

If $\pi \simeq \otimes'\pi_v$ is an irreducible automorphic representation of $G(\mathbb{A})$, then for each local component π_v we (hope to) have a local Langlands parameter $\phi_v \in \Phi(G_v)$ And we set

$$L(s,\pi,r) = \prod_v L(s,\pi_v,r) = \prod_v L(s, r\circ\phi_v) = L(s, r\circ\{\phi_v\})$$
$$\varepsilon(s,\pi,r) = \prod_v \varepsilon(s,\pi_v,r,\psi_v) = \prod_v \varepsilon(s, r\circ\phi_v,\psi_v) = \varepsilon(s, r\circ\{\phi_v\})$$

• Even though we don't know the LLC in general, we do know how to parametrize all unramified local representations and all representations for $v|\infty$. So there is a finite set of places S outside of which the local component π_v is parametrized. Se we have a well defined partial L-function

$$L^S(s,\pi,r) = \prod_{v\notin S} L(s,\pi_v,r).$$

This is very similar to the situation in Artin's first paper on his L-functions where he only had a definition of the local factors at the unramified places. Langlands proved that these partial L-functions converged for $Re(s) \gg 0$.

It is an open problem in general to determine the L-factors for $v \in S$ such that the resulting Euler Product has meromorphic continuation and FE. This can be thought of as an analytic/arithmetic problem independent of the local classification. There are currently two methods for this.

(a) The Langlands-Shahidi method. Here one analyzes the $L(s,\pi,r)$ for those π and r that occur in the Fourier coefficients of Eisenstein series. These are essentially Langlands' original Euler products. This is essentially complete by the work of Shahidi.

(b) Integral representations. This method is active, but somewhat stuck at present. The L-functions defined this way figured in the LLC and GLC for GL_n. Often in the theory of integral representations one identifies the L-functions represented in terms of Langlands L-functions by doing a local unramified calculation. Very seldom in the local ramified analysis completed (except for GL_n).

Exercises

1. Take the following two low rank algebraic groups and work out the roots, positive roots, and simple roots and simple coroots.

(a) Sp_4. Take the symplectic form to be represented by $J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$ and define

$$Sp_4(K) = \{g \in GL_4(K) \mid {}^t gJg = J\}$$

With this form, one can obtain a Borel subgroup by intersecting with the Borel of GL_4.

(b) SO_5. Now take the symmetric form to be represented by $S = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$ and then

$$SO_5(K) = \{g \in GL_5(K) \mid {}^t gSg = S\}$$

Again, with this form, one can obtain a Borel subgroup by intersecting with the Borel of GL_5.

2. Can you see any hints of the Langlands duality between Sp_4 and SO_5.

References

• Langlands computation of the constant term of Eisenstein series:

R. P. Langlands, *Euler Products.* Yale University Press, 1971.

• The Satake classification:

I. Satake, Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields. *Publ. Math. IHES* **18** (1963), 5–70.

see also

P. Cartier, Representations of p-adic groups: a survey. *Automorphic forms, representations and L-functions* (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, *Proc. Sympos. Pure Math.*, XXXIII, Amer. Math. Soc., Providence, R.I., 1979, 111-155.

• Structure theory of algebraic groups:

J. Tits, Reductive groups over local fields. *Automorphic forms, representations and L-functions* (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, 29–69.

as well as

T. Springer, *Linear Algebraic Groups*, second edition. Progress in Math. Vol 9, Birkhäuser, Boston–Basel–Berlin, 1998.

particularly Chapter 16.

• Dual groups, L-groups, Langlands Conjecture, and L-functions.

Some of Langlands original papers: (Note: The references below are to the published versions, some of which appeared long after the results were available in preprint form. All papers of Langlands are available at *http://publications.ias.edu/rpl*)

R. P. Langlands, Letter to A. Weil, 1967, available at *http://publications.ias.edu/rpl*

R. P. Langlands, The representation theory of abelian algebraic groups, *Pacific J. Math.* Olga Taussky-Todd Memorial Volume (1988), 231–250.

R. P. Langlands, Problems in the theory of automorphic forms. *Lectures in modern analysis and applications, III.* Lecture Notes in Math., Vol. 170, Springer, Berlin, 1970, 18–61.

R. P. Langlands, On the classification of irreducible representations of real algebraic groups. *Representation theory and harmonic analysis on semisimple Lie groups*, Math. Surveys Monogr., 31, Amer. Math. Soc., Providence, RI, 1989, 101-170.

Surveys:

A. Borel, Automorphic L-functions. *Automorphic forms, representations and L-functions* (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, 27–61.

J. Cogdell, Dual groups and Langlands functoriality. *An Introduction to the Langlands Program* (J. Bernstein and S. Gelbart, eds.). Birkhäuser, Boston, 2003, 251–268.