LECTURE 1: LANGLANDS CORRESPONDENCES FOR GL_n

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1. Converse Theorems

We begin with variations on the theme of *Converse Theorems*. This is a coda to SF's lectures.

In 1921–1922 Hamburger had characterized the Riemann zeta function by its analytic properties: convergence, continuation, and functional equation (FE). Hecke sought to do something similar for number fields (quadratic imaginary). Towards this goal he stated and proved what we think of as the first Converse Theorem (CT):

Theorem. [Hecke, 1936] *Let $D(s) = \sum \frac{a_n}{n^s}$ be a Dirichlet series which converges for $Re(s) \gg 0$, has an entire continuation to $\mathbb{C}$ and satisfies a functional equation*

$$\Lambda_D(s) = (2\pi)^{-s}\Gamma(s)D(s) = \lambda_D(k-s).$$

Then

$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau} \in S_k(SL_2(\mathbb{Z})).$$

Maaß proved a similar result for Maaß forms in 1949.

In 1967 Weil extended this Dirichlet series associated to cusp forms for $\Gamma_0(N)$. However, to do this he requires analytic control (convergence, continuation, functional equation) for all twists $D_\chi(s) = \sum \frac{\chi(n)a_n}{n^s}$ for all Dirichlet characters χ of conductor relatively prime to N.

The GL_n representation theoretic version of this was written down in 1994. It takes a slightly different form.

Theorem. *Let $\pi = \otimes'\pi_v$ be an irreducible admissible representation of $GL_n(\mathbb{A})$ such that*

bullet the central character ω_π is automorphic for $GL_1(\mathbb{A})$
- *$L(s,\pi) = \prod_v L(s,\pi_v)$ converges for $Re(s) \gg 0$*

Furthermore suppose that for all cuspidal automorphic π' of $GL_m(\mathbb{A})$, $1 \le m \le n-1$ we have that $L(s,\pi\times\pi')$

- *extends to an entire function of s*
- *is bounded in vertical strips*

- *FE:* $L(s,\pi\times\pi') = \varepsilon(s,\pi\times\pi')L(1-s.\tilde{\pi}\times\tilde{\pi'})$

Then $V_\pi \hookrightarrow \mathcal{A}_0(GL_n(\mathbb{A}))$*, i.e.,* π *is cuspidal automorphic.*

Note that this is based on an Euler product rather than a Dirichlet series.

Meta-Theorem. *Let* $\Lambda(s) = \prod_v \Lambda_v(s)$ *be an Euler product of degree* n *(over* k*) for which we expect entire continuation and a sufficiently rich family of FE. Then there should be a cuspidal automorphic representation* π *of* $GL_n(\mathbb{A})$ *such that*

$$\Lambda(s) = L(s,\pi).$$

Finding such a π is the question of *modularity* or *automorphiy.*

2. Artin L-functions

The Artin L-functions (and their generalizations) give us a rich source of degree n Euler products that we expect to be nice.

Artin defined his L-functions is a series of papers in 1923, 1930, and 1931. Abelian global class field theory (CFT) was new and exciting. I think Artin was already looking towards a non-abelian CFT. He took as his starting point a factorization theorem due to Weber and Hecke.

Theorem. *Let* L/K *be an abelian extension with* L *class field to* $H \subset I_K(\mathfrak{m})$*. Then*

$$\zeta_L(s) = \prod_{\chi\in(I_K(\mathfrak{m})/H)^\wedge} L(s,\chi) = \zeta_K \prod_{\chi\neq\chi_0} L(s,\chi).$$

During the same period we find the beginning of the representation theory of finite groups due to Frobenius. Artin seemed particularly inspired by two crucial ideas of Frobenius.

(a) non-abelian characters: If $\rho : G = Gal(L/K) \to GL_n(\mathbb{C})$ is a representation of $Gal(L/K)$ then

$$\chi_\rho(g) = Tr(\rho(g))$$

is a class function on G and the χ_ρ for ρ irreducible span all class functions.

(b) the Frobenius substitution: Suppose L/K is an extension of number fields. Let $\mathfrak{p} \subset \mathfrak{o}_K$ be a prime ideal of K and $\mathfrak{P} \subset \mathfrak{o}_L$ a prime of L above $\mathfrak{p}$. Let

$$D_\mathfrak{P} = \{\sigma \in \mathrm{Gal}(L/K) \mid \sigma\mathfrak{P} = \mathfrak{P}\}$$

be the decomposition group for $\mathfrak{P}$. Then we have a short exact sequence

$$1 \longrightarrow I_\mathfrak{P} \longrightarrow D_\mathfrak{P} \longrightarrow \mathrm{Gal}(\mathbb{L}_\mathfrak{P}/\mathbb{K}_\mathfrak{p}) \longrightarrow 1$$

where we use $\mathbb{L}_{\mathfrak{P}}$ and $\mathbb{K}_{\mathfrak{p}}$ for the residue fields and $I_{\mathfrak{P}}$ is the inertia subgroup. Then $\mathrm{Gal}(\mathbb{L}_{\mathfrak{P}}/\mathbb{K}_{\mathfrak{p}})$ is a cyclic group generated by $\varphi_{\mathfrak{P}}$ defined by $\varphi_{\mathfrak{P}}(x) = x^{N(\mathfrak{p})}$. Then a *Frobenius substitution* $\sigma_{\mathfrak{P}}$ for $\mathfrak{P}$ is any inverse image of $\varphi_{\mathfrak{P}}$ in $D_{\mathfrak{P}}$. Note that

(i) $\sigma_{\mathfrak{P}}$ is well defined up to the inertia subgroup $I_{\mathfrak{P}}$, and if $\mathfrak{p} \not| \mathfrak{D}_{L/K}$ then $I_{\mathfrak{P}} = \{e\}$.
(ii) If $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are any two primes over $\mathfrak{p}$, say for $\mathfrak{p} \not| \mathfrak{D}_{L/K}$, then $\sigma_{\mathfrak{P}_1}$ and $\sigma_{\mathfrak{P}_2}$ are conjugate in $\mathrm{Gal}(L/K)$, so for any class function χ of $\mathrm{Gal}(L/K)$ we have

$$\chi(\sigma_{\mathfrak{P}_1}) = \chi(\sigma_{\mathfrak{P}_2}) = \chi(\sigma_{\mathfrak{p}})$$

where we can let $\sigma_{\mathfrak{p}}$ denote the conjugacy class of the $\sigma_{\mathfrak{P}}$ in $\mathrm{Gal}(L/K)$.

Conceptually, Artin viewed the formula of Weber-Hecke as analogous to the decomposition of the regular representation

$$r_G = \mathrm{Ind}_{\{e\}}^G(\mathbf{1}) = \bigoplus_{\rho \text{ irred.}} \dim(\rho)\rho \quad \text{or} \quad \chi_{r_G} = Ind(\chi_0) = \sum_{\chi \in \widehat{G}} \chi(1)\chi.$$

For these to actually to coincide he would need a theory of L-functions $L(s,\rho) = L(s,\chi,L/K)$ such that

- $L(s,\chi_0,K/K) = \zeta_K(s)$
- additive: $L(s,\chi_1+\chi_2,L/K) = L(s,\chi_1,L/K)L(s,\chi_2,L/K)$

and if we have a tower $L \supset M \supset K$ and ρ a representation of $Gal(M/K)$ and ρ_0 a representation of $Gal(L/M)$ the the theory behaves well with respect to

- inflation: $L(s,\mathrm{Infl}(\rho),L/K) = L(s,\rho,M/K)$
- induction: $L(s,\mathrm{Ind}(\rho_0),L/K) = L(s,\rho_0,L/M)$

How did Artin define his L-functions? In the Weber situation

$$L(s,\chi) = \prod_{\mathfrak{p}} (1-\chi(\mathfrak{p})N(\mathfrak{p})^{-s})^{-1} \quad \text{so} \quad \mathrm{Log}(L(s,\chi)) = \sum_{\mathfrak{p}} \left(-\sum_{\ell} \frac{\chi(\mathfrak{p}^\ell)}{\ell N(\mathfrak{p})^\ell} \right).$$

At the unramified places, so the Frobenius class was well defined, Artin replaced $\chi \to \chi_\rho$ and $\mathfrak{p} \to \sigma_{\mathfrak{p}}$. When this gets folded back up he had

$$L(s,\rho) = L(s,\chi_\rho,L/K) = \prod_{\mathfrak{p} \text{ unram.}} \det(I_n - \rho(\sigma_{\mathfrak{p}})N(\mathfrak{p})^{-s})^{-1}.$$

In his 1930 paper he defined the Euler factors for ramified primes:

$$L_{\mathfrak{p}}(s,\chi_\rho,L/K) = \det(I_n - N(\mathfrak{p})^{-s}\rho(\sigma_{\mathfrak{p}})|V_\rho^{I_{\mathfrak{p}}})^{-1}$$

and the archimedean Gamma-factors $L_\infty(s,\chi_\rho,L/K)$. He then proved

Theorem. *$L(s,\rho) = L(s,\chi_\rho,L/K)$ converges for $Re(s) \gg 0$ and is additive, inductive, and behaves well under inflation.*

Moreover indeed from the decomposition of the regular representation of $G = Gal(L/K)$ he obtained his factorization.

$$\zeta_L(s) = \zeta_K(s) \prod_{\substack{\chi \in \hat{G} \\ \chi \neq \chi_0}} L(s, \chi, L/K)^{\chi(1)}$$

even for non-abelian L/K.

Comparing this factorization to that of Hecke-Weber in the abelian case, Artin conjectured his reciprocity law, namely if L/K is a class field for $H \subset I_K(\mathfrak{m})$ then the isomorphism

$$I_K(\mathfrak{m})/H \simeq G \quad \text{is given by} \quad \mathfrak{p} \mapsto \sigma_{\mathfrak{p}}$$

which he proved in 1927 without L-functions.

Artin also proved a beautiful fact in finite group theory, later improved by Brauer, that gave him continuation and functional equation for his L-functions. Brauer's version is

Theorem. *Let G be a finite group, χ a rational character (so a sum of irreducible characters with rational coefficients). Then*

$$\chi = \sum_i n_i \mathrm{Ind}_{H_i}^G(\psi_i)$$

with the H_i "elementary" subgroups (abelian) and $n_i \in \mathbb{Z}$.

Applying this to a single χ_ρ we obtain

Corollary. *Given $\rho : Gal(L/K) \to GL_n(\mathbb{C})$ there are elementary subgroups $H_i \subset Gal(L/K)$ and abelian characters ψ_i of H_i such that*

$$L(s, \chi_\rho, L/K) = \prod_i L(s, \psi_i, L/L^{H_i})^{n_i}.$$

Since the $L(s, \psi_i, L/L^{H_i})$ are abelian Hecke L-functions Artin obtained

Theorem. *$L(s, \rho) = L(s, \chi_\rho, L/K)$ extends to a meromorphic function of s and satisfies a functional equation*

$$L(s, \rho) = \varepsilon(s, \rho) L(1 - s, \rho^\vee).$$

Artin Conjecture. *If ρ is irreducible and non-trivial, $L(s, \rho)$ is entire.*

3. Langlands Correspondence, I

If we combine the theorem of Artin/Brauer and Artin's Conjecture with our meta-theorem from the converse theorem, we obtain a first version of the Langlands conjectures/correspondences. (This is **not** the way Langlands was led to it!)

Global Langlands Correspondence/Conjecture, I. *Let k be a global field. There is a compatible family of injections*

$$\left\{\begin{matrix}\rho : Gal(\bar{k}/k) \to GL_n(\mathbb{C}) \\ \textit{irreducible, continuous}\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\pi \textit{ cuspidal automorphic} \\ \textit{representations of } GL_n(\mathbb{A})\end{matrix}\right\}$$

denoted $\rho \mapsto \pi(\rho)$, such that

$$L(s, \rho_1 \otimes \rho_2) = L(s, \pi(\rho_1) \times \pi(\rho_2))$$
$$\varepsilon(s, \rho_1 \otimes \rho_2) = \varepsilon(s, \pi(\rho_1) \times \pi(\rho_2)).$$

For $n = 1$ this is (adelic) CFT.

Since the L-functions on both sides are defined by Euler products, we can formulate local versions as well.

Local Langlands Correspondence/Conjecture, I. *Let F be a local field. There is a compatible family of injections*

$$\left\{\begin{matrix}\rho : Gal(\bar{F}/F) \to GL_n(\mathbb{C}) \\ \textit{continuous}\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\pi \textit{ irreducible admissible} \\ \textit{representations of } GL_n(F)\end{matrix}\right\}$$

denoted $\rho \mapsto \pi(\rho)$, such that

$$L(s, \rho_1 \otimes \rho_2) = L(s, \pi(\rho_1) \times \pi(\rho_2)$$
$$\varepsilon(s, \rho_1 \otimes \rho_2, \psi) = \varepsilon(s, \pi(\rho_1) \times \pi(\rho_2), \psi).$$

For $n = 1$ this is LCFT.

4. Langlands Correspondence, II

One of Langlands' insights is that it should be possible to replace the *injections* by *bijections.*

Already in the case $n = 1$ Weil realized that the use of the Galois group is not rich enough to give LCFT at the archimedean places. On the other hand, Artin's analysis of his L-functions at the finite places only used the Frobenius $\sigma_{\mathfrak{P}} = \sigma_v$ and the inertia subgroup $I_{\mathfrak{P}} = I_v$. To put the archimedean and non-archimedean places on an even footing, Weil introduces what we now call the *Weil group.*

• F **archimedean.** If $F = \mathbb{C}$ then $W_{\mathbb{C}} = \mathbb{C}^\times$. If $F = \mathbb{R}$ we set $W_{\mathbb{R}} = \mathbb{C}^\times \bigcup j\mathbb{C}^\times$ with the relations $jzj^{-1} = \bar{z}$ and $j^2 = -1 \in \mathbb{C}^\times$. In either case we have

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_F \longrightarrow \mathrm{Gal}(\bar{F}/F) \longrightarrow 1$$

and $W_F^{ab} \simeq F^\times$ is LCFT.

F **non-archimedean.** Weil defined W_F as the semidirect product $W_F = \langle I_F, \sigma_F\rangle = I_F \rtimes \sigma_F^{\mathbb{Z}}$ where $I_F \subset Gal(\bar{F}/F)$ is the inertia subgroup, endowed with its usual profinite topology, and σ_F is a choice of Frobenius element. $I_F \subset W_F$ is open and multiplication by σ_F is continuous. In this case we have

$$W_F \hookrightarrow Gal(\bar{F}/F) \quad \text{with dense image}$$

and again $W_F^{ab} \simeq F^\times$ is LCFT.

• k **global.** There is only a cohomological definition of the global Weil group W_k. At a finite level $W_{L/k}$ is the extension of the Galois group $Gal(L/k)$ associated to the fundamental class $u \in H^2(Gal(L/K), C_L)$ where $C_L = \mathbb{A}_L^\times / L^\times$ is the idele class group, so that

$$1 \longrightarrow C_L \longrightarrow W_{L/k} \longrightarrow \mathrm{Gal}(L/k) \longrightarrow 1$$

then one takes an inverse limit. It is compatible with the local Weil groups and CFT says $W_k^{ab} \simeq k^\times \backslash \mathbb{A}_k^\times$.

Global & Local Langlands Correspondence/Conjecture, II. Replace $Gal(\bar{k}/k) \to W_k$ and $Gal(\bar{F}/F) \to W_F$. Then, at least for $n = 1$ we have bijections between one dimensional representations of the Weil groups, so factoring through the abelianization, and automorphic representations of $GL_1(\mathbb{A})$, that is, idele class characters. This is Artin's reciprocity law at the Weil group level. Also, for F local archimedean, we had bijections for all n by the Langlands classification of representations of real algebraic groups.

5. Langlands Correspondence, III.

Deligne notices that for F non-archimedean local field, all irreducible representations of $GL_2(F)$ were not accounted for by two dimensional representations of W_F. So there was no bijection for $n = 2$. The problem was the disconnect between the profinite topology of $I_F \subset W_F$ and the archimedean topology of $\mathbb{C}$.

Deligne had also studied the ℓ-adic representations of W_F

$$\rho : W_F \to GL_n(\bar{\mathbb{Q}}_\ell) = GL(V_\rho).$$

Here there is an additional structure that one gets "for free" (actually a Theorem of Grothendieck): there is a *nilpotent monodromy operator* on V_ρ that is normalized by $\rho(W_F)$. Deligne defined what we now call the *Weil-Deligne group*, W_F', to include this monodromy operator as part of its intrinsic structure.

$$W_F' = W_F \ltimes \mathbb{G}_a \quad \text{with} \quad wxw^{-1} = ||w||x$$

where $|| \cdot ||$ is the norm on W_F coming from the LCFT isomorphism

$$W_F \longrightarrow W_F^{ab} \xrightarrow{\sim} F^\times \xrightarrow{||\cdot||} q^{\mathbb{Z}}$$

normalized so that

$$||w|| = 1 \text{ if } w \in I_F, \quad ||\sigma_F|| = q, \quad ||\Phi_F|| = q^{-1}$$

so that the geometric Frobenius $\Phi_F = \sigma_F^{-1}$ corresponds to a uniformizer ϖ_F for F.

A Weil-Deligne representation $\rho' : W_F' \to GL(V)$ then corresponds to a pair $\rho' = (\rho, N)$ where

$$\rho : W_F \to GL(V), \quad N \in \mathrm{End}(V) \text{ nilpotnet}, \quad \text{with } \rho(w)N\rho(w)^{-1} = ||w||N$$

where ρ is continuous with respect to the discrete topology on $GL(V)$. The local L-function for a Weil-Deligne representation is

$$L(s, \rho') = \det(I_n - q^{-s}\rho(\Phi_F)|\ker(N)^{I_F})^{-1}.$$

For archimedean fields, we already had local bijections, so we just take $W_F' = W_F$.

For global fields k *there is no "natural" global Weil-Deligne group* that we know of. (Arthur refers to W_F' as a local Langlands group. His global Langlands group W_k' is a *hypothetical* global Weil-Deligne group.)

We are left with:

Local Langlands Correspondence. *Let F be a non-archimedean local field. Then there is a unique compatible system of bijections*

$$\begin{Bmatrix} \rho : W_F' \to GL_n(\mathbb{C}) \\ \Phi_F \textit{ semisimple} \end{Bmatrix} \leftrightarrow \begin{Bmatrix} \pi \textit{ irreducible admissible} \\ \textit{representations of } GL_n(F) \end{Bmatrix}$$

for all $n \geq 1$, denoted $\rho \mapsto \pi(\rho)$, such that

(i) *For $n = 1$ this is LCFT (suitable normalized)*
(ii) *We have equality of twisted L- and ε-factors*

$$L(s, \rho_1 \otimes \rho_2) = L(s, \pi(\rho_1) \times \pi(\rho_2))$$
$$\varepsilon(s, \rho_1 \otimes \rho_2, \psi) = \varepsilon(s, \pi(\rho_1) \times \pi(\rho_2), \psi).$$

(iii) $\pi(\det(\rho)) = \omega_{\pi(\rho)}$
(iv) $\pi(\rho^\vee) = \widetilde{\pi(\rho)}$
(v) *For all characters χ of $W_F^{ab} \simeq F^\times$ we have $\pi(\rho \otimes \chi) = \pi(\rho) \otimes \chi$.*

• For $F = \mathbb{R}$ or $\mathbb{C}$ the analogous result was known for general G by Langlands (1973).

• If $Char(F) = p > 0$ this was established by Laumon–Rapoport–Stuhler in 1993.

• For $Char(F) = 0$ this was established by Harris–Taylor, then followed by Henniart, in 2000. Both proofs used local/global techniques and the cohomology of Shimura varieties (hence the geometric normalization). Recently Scholze has given a simplified version of the Harris–Taylor proof.

• This is *local non-abelian class field theory.*

Global Langlands Correspondence? What can we do globally without a global Weil–Deligne group W_k'? There are a few options.

A. Stick to Galois Representations.

Conjecture. *Let k be a global field. There is a compatible family of bijections*

$$\left\{\begin{matrix}\rho : Gal(\bar{k}/k) \to GL_n(\mathbb{C}) \\ \textit{irreducible, continuous}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\pi \textit{ cuspidal automorphic} \\ \textit{representations of } GL_n(\mathbb{A}) \\ \textit{of Galois type}\end{matrix}\right\}$$

denoted $\rho \mapsto \pi(\rho)$, satisfying conditions as above.

Being of Galois type is a condition on the archimedean component π_∞ of π.

A version of this was established over global function fields (global characteristic p) by Drinfeld (1988) and L. Lafforgue (2002).

B. Local/Global compatibility.

If we begin with say a representation $\rho : W_k \to GL_n(\mathbb{C})$, then for each place v of k we get a local representation $\rho_v : W_{k_v} \to W_k \to GL_n(\mathbb{C})$ and so by the LLC a family of local representations $\pi(\rho_v)$ of $GL_n(k_v)$. Then conjecturally $\pi(\rho) = \otimes' \pi(\rho_v)$ should be an automorphic representation of $GL_n(\mathbb{A})$. This should be compatible with A.

Exercises

1. Derive Artin's factorization of $\zeta_L(s)$ from the formal properties of the Artin L-function. (Use the decomposition of the regular representation.)

2. For F a local field, show $W_F^{ab} =\simeq F^\times$. For $F = \mathbb{R}$, $\mathbb{C}$ you can do this by hand. For F non-archimedean, you can only do this if you know LCFT.

References

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A. Weil, Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, *Math. Ann.* **168** (1967), 149–156.

J. Cogdell and I.I. Piaterski-Shapiro, Converse Theorems for GL_n, *Publ. Math. Inst. Hautes Études Sci.* **79** (1994), 157–214.

• The basic papers on Artin L-functions are:

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E. Artin, Die gruppentheoretische Struktur der Diskriminante algebraischer Zahlkörper, *J. Reine Angew. Math.* **164** (1931), 1–11.

and one can also look at:

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• References on the Langlands Correspondence for GL_n:

P. Deligne, first reference above.

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J. Cogdell, Langlands conjectures for GL_n. *An Introduction to the Langlands Program* (J. Bernstein and S. Gelbart, eds.). Birkhäuser, Boston, 2003, 227–249.

More references (eg, to Drinfeld and Lafforgue) can be found in the bibliography of this survey.