Normalization, Approximation, and Semantics for Combinator Systems ⋆

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Abstract

This paper studies normalization of typeable terms and the relation between approximation semantics and filter models for Combinator Systems. It presents notions of approximants for terms, intersection type assignment, and reduction on type derivations; the last will be proved to be strongly normalizable. With this result, it is proved that every typeable term has an approximant with the same type, and a characterization of the normalization behaviour of terms using their assignable types is given. Then the two semantics are defined and compared, and it is shown that the approximants semantics is fully abstract but the filter semantics is not.

Key words: Intersection types, approximants, filter semantics, combinators

Introduction

In this paper we will focus on the relation between two approaches for semantics in the framework of Combinator Systems (CS), being the *filter semantics*, obtained by interpreting terms by the set of intersection types that can be assigned to them, and the *approximants semantics*, where terms are interpreted by the set of their approximants. Approximants are defined as rooted finite sub-trees of the (possibly infinite) normal form, based on the notion of Ω-normal forms of Huet and Lévy [16].

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The relation between the filter semantics and the approximation semantics has been studied extensively in the setting of the Lambda Calculus (LC) [6] (see [8,7,1,3]), where it has been proved that they coincide [19,3]. But, perhaps surprisingly, this has never been studied for more general notions of rewriting, such as Term Rewriting Systems (TRS) [12,17].

Within the framework of orthogonal first-order TRS, a term-like model is defined in [21], interpreting terms by the set of their approximants [18]. For these TRS it is also possible to define a semantics where types are interpreted as multi-sorted algebras [12]. Although these types are enough to describe manipulations of objects of an algebraic data-type, they do not provide an account for polymorphism, or higher order functions, which are standard in functional programming languages. A more general and expressive type system, using intersection types, has been developed in [5] for Curryfied Term Rewriting Systems (*G*TRS, first-order TRS extended with application). This type system is inspired by the Intersection Type Discipline for LC, defined in [8] (see also [7,1]): an extension of Curry's system [10,11] in that, essentially, terms are allowed to have more than one type (using the type constructor '$\cap$'). By introducing also the type constant 'ω' a type system for LC is obtained that is closed under β-equality, and interpreting terms by their assignable types gives a filter lambda model [7,3].

In this paper, based on the approach of [21], we will define a notion of approximation for CS and show the following *approximation result*: for all terms that can be assigned a type in the intersection system, there exists an approximant of that term that can be assigned the same type. For LC, such an approximation result is relatively easy to obtain, because of the presence of explicit abstraction, but in order to prove these results for abstraction-free calculi, like CS, a new technique had to be developed. This technique is that of defining *reduction on derivations* as a generalization of cut-elimination, that will be proven to be strongly normalizing. This same technique can then also be applied to other formalisms, as done for example in [4], where similar results are obtained for TRS. Strong normalization of cut-elimination has been studied in the past for several systems, but in the context of intersection types this topic had not yet been tackled.

Using the approximation result, we will show the following properties of typeable combinator terms in the intersection system:

- terms typeable without using ω are strongly normalizable,
- non-Curryfied terms that are typeable with σ from a basis B, such that ω does not occur in B and σ, are normalizable, and
- terms typeable with type $\sigma \neq \omega$ have a head-normal form.

A similar characterization of the normalization properties of terms using types in the intersection system holds for TRS, provided that the rewrite rules satisfy certain conditions [5]. In LC, these results are well-known (there is a difference though:

the characterization of normalization holds for all terms, whereas in CS it holds for non-Curryfied terms only, which are terms where each combinator of arity n is applied to at least n arguments). Perhaps less known is the fact that the notion of approximant can be useful to study the relation between typeability and normalization: in this paper we will show that the approximation result allows for a relatively easy proof of the results mentioned above for CS (a similar result for LC was shown in [3], and an abbreviated proof for more general TRS appeared in [4]).

Inspired by the approximation result, we will then focus on approximation and filter semantics of CS, as a preparation for future studies of the same semantics in the context of more general rewriting systems. There are several advantages to keeping the computational framework relatively easy at first: confluence comes for free, and a direct relation between CS and LC facilitates definitions and insight. However, note that the normalization properties of LC do not translate directly to CS, since the mappings between LC and Combinatory Logic (a particular CS defined by Curry [9]) do not preserve normal forms nor reductions (see Exp. 10).

Although TRS are very popular in the area of programming language design and their normalization properties are well-studied, there is still no thorough semantic analysis of TRS. As we have already mentioned, there exists some work in this direction, either supported by types [14] or not [21], but, for example, the relation between these models has not been studied. This paper is a first step towards filling that void, by studying two approaches to semantics for CS, the approximation semantics and the filter semantics, and comparing their expressiveness. We aim to bring these approaches to the context of TRS in future work.

Summarizing, the main contributions of this paper are:

- a strong normalization result for cut-elimination for a system with intersection types,
- a characterization of normalization properties of typeable combinator systems,
- the definition of a filter semantics for CS where terms are interpreted by their assignable types, and an approximation semantics where terms are interpreted by their approximants,
- a proof that these semantics are adequate, and
- a study of the conditions needed to obtain a full-abstraction result.

Outline

In this paper, we will, in Section 1, define Combinator Systems, for which we will develop a notion of type assignment that uses intersection types in Section 2. The intersection type assignment system we use in this paper is just the essential type assignment system for *Gi*TRS [5], restricted to CS. We will show a subject reduction

result in Section 3.

In Section 7, we will define the set of approximants of a term in CS, by introducing a special symbol $\perp$ into our systems and defining approximate normal forms. In Sections 4 to 8, we present the formal construction needed to show that any typeable term in a typeable CS has an approximant of the same type (Thm. 69). In order to prove this theorem, we will modify the type system slightly in Section 4 and introduce, in Section 5, a notion of reduction on type-derivations in this modified system. We will show that this derivation reduction is strongly normalizing (Thm. 54). A consequence of this result will be that every term typeable without using the universal type constant ω is strongly normalizable (Thm. 74).

Using the approximation result, we will then prove two normalization properties of typeable combinator systems, the first of which is a head-normalization theorem (Thm. 71). The combinatorial equivalent of the characterization of normalisation in LC no longer holds (see Section 8). However, we will prove a normalization theorem (Thm. 72) for the class of typeable non-Curryfied terms.

In Section 9, we give the definition of a filter semantics for CS, where terms are interpreted by their assignable types, and an approximation semantics, where terms are interpreted by their approximants. The approximation semantics gives a model for CS, whereas the filter semantics gives a semi-model only, except for special cases, when it gives a model.

Section 10 contains the conclusions.

1 Combinator Systems

In this section, we will give a detailed presentation of Combinator Systems (CS). CS will be defined as a special kind of applicative TRS [17], with the restriction that formal parameters of function symbols are not allowed to have structure, and right-hand sides of term rewriting rules are constructed of term-variables only. We have chosen this kind of presentation in view of a future extension of the results of this paper to full TRS, in the spirit of [5]. Notice that our treatment differs from, for example, that of [13], where only combinatory complete CS are considered.

Definition 1 (Combinator terms) *(1) An* alphabet *or* signature $\Sigma = (\mathcal{C}, \mathcal{X})$ *consists of a countable infinite set $\mathcal{X}$ of variables ranged over by $x, y, z, \ldots$, a non-empty set $\mathcal{C} = \{\mathsf{D}, \mathsf{Z}, \ldots\}$ of* combinators*, ranged over by $C, D, E, \ldots$, each equipped with an arity greater than 0, and the binary function symbol Ap (application).*

(2) The set $T(\mathcal{C}, \mathcal{X})$ of terms*, ranged over by t, is defined by:*

$$t ::= x \mid C \mid Ap\,(t_1, t_2)$$

As usual, since 'Ap' is the only function symbol, we will write $(t_1\,t_2)$ *instead of* $Ap\,(t_1, t_2)$*, and left-most, outermost brackets will be omitted, so* $t_1\,t_2\,t_3\cdots t_n$ *stands for* $Ap\,(\cdots Ap\,(Ap\,(t_1, t_2), t_3)\cdots, t_n)$.

The following is the usual notion of substitution in combinator systems.

Definition 2 (Term-substitutions) *A* term-substitution *R is a map from T*$(\mathcal{C},\mathcal{X})$ *to T*$(\mathcal{C},\mathcal{X})$*, determined by its restriction to a finite set of variables, satisfying* $\mathrm{R}\,(t_1\,t_2)$ = $\mathrm{R}\,(t_1)\mathrm{R}\,(t_2)$*. We will write* t^{R} *instead of* $\mathrm{R}\,(t)$*. If R maps* x_i *to* u_i*, for* $1 \leq i \leq n$*, we also write* $\{x_1{\mapsto}u_1, \ldots, x_n{\mapsto}u_n\}$ *for* R*, and write* $t^{\vec{u}}$ *for* t^{R}.

Combinator Systems, and the notion of rewriting on combinator terms, are defined by the following:

Definition 3 (Combinator Systems) *(1) A* combinator rule *on* $\Sigma = (\mathcal{C},\mathcal{X})$ *is a pair* (l, r) *of terms in T*$(\mathcal{C},\mathcal{X})$*, such that:*

- *(a) There are* C *and distinct* $x_1, \ldots, x_n$*, such that* $l = C\,x_1\cdots x_n$*, where* $n = arity(C)$.
- *(b) The variables occurring in* r *are contained in* l*, and* r *contains no symbols from* $\mathcal{C}$.

(2) A Combinator System *(*CS*) is a pair of an alphabet* Σ *and a set* **R** *of combinator rules on* $\Sigma = (\mathcal{C},\mathcal{X})$*, such that there is* exactly one *rule in* **R** *for each combinator* $C \in \mathcal{C}$*. This rule* (l, r) *is called* the combinator rule for C*; we will use the symbol* C *also as name for this rule and write* $l \to_C r$.

(3) A combinator rule $l \to_C r$ *determines a set of* reductions $l^{\mathrm{R}} \to_C r^{\mathrm{R}}$ *for all term-substitutions* R*. The left-hand side* l^{R} *is called a* redex*; it may be replaced by its '*contractum*'* r^{R} *inside any context* $\mathrm{C}[\ ]$*; this gives rise to* reduction steps*:* $\mathrm{C}[\,l^{\mathrm{R}}\,] \to_C \mathrm{C}[\,r^{\mathrm{R}}\,]$.

(4) We will write $t \to_{\mathbf{R}} t'$ *if there is a rule* $l \to_C r$ *in* **R** *such that* $t \to_C t'$*, and call* $\to_{\mathbf{R}}$ *the* one-step rewrite relation *generated by* **R***, and* $\to^+_{\mathbf{R}}$ *(respectively* $\to^*_{\mathbf{R}}$*) the transitive (respectively reflexive and transitive) closure of* $\to_{\mathbf{R}}$ *(the index* **R** *will be omitted when it is clear from the context). If* $t_0 \to^+ t_n$*, then* t_n *is a* reduct *of* t_0.

Example 4 (Combinatory Logic) *The standard example of a* CS *is Combinatory Logic (*CL*) – defined by Curry independently of* LC *[9] – that is, in our notation, formulated as follows:* $\mathrm{CL} = ((\mathsf{S},\mathsf{K},\mathsf{I}),\mathcal{X}),\mathbf{R})$*, where* **R** *contains the rules*

$$\begin{array}{lcl} \mathsf{S}\,x\,y\,z & \to & x\,z\,(y\,z) \\ \mathsf{K}\,x\,y & \to & x \\ \mathsf{I}\,x & \to & x \end{array}$$

The last rule was not part of the original definition, but is nowadays normally added.

We will assume that no two combinators have the same interpretation in LC (see Def. 7), so a CS like

$$\begin{aligned} \mathsf{I}x &\to x \\ \mathsf{J}x &\to x \end{aligned}$$

is excluded, since it would give an immediate counter example against any full-abstraction result with respect to the filter semantics (Section 9).

This notion of reduction on combinator terms as in Def. 3 is also known as *weak reduction* and satisfies the Church-Rosser Property (see [6]).

Proposition 5 (Church-Rosser) *If $t \to^* u$ and $t \to^* v$, then there exists a w such that $u \to^* w$ and $v \to^* w$.* ■

We now define (head-)normal forms, (head-)normalizability, strongly normalizability, unsolvable and neutral terms.

Definition 6 ((Head-)normal forms) *Let $((\mathcal{C},\mathcal{X}),\mathbf{R})$ be a* CS.

(1) A term is in normal form *with respect to* **R** *if it is irreducible.*
(2) A term t is in head-normal form *with respect to* **R** *if either*
  (a) there are a variable x and terms $t_1,\ldots,t_n$ $(n \geq 0)$ such that $t \equiv x\,t_1\cdots t_n$, or
  (b) there are a combinator $C \in \mathcal{C}$ and terms $t_1,\ldots,t_n \in T(\mathcal{C},\mathcal{X})$ such that $t \equiv C\,t_1\cdots t_n$, and $n < arity(C)$.
(3) A term is (head-)normalizable *if it can be reduced to a term in (head-)normal form. A rewrite system is* strongly normalizing *(or terminating) if all rewrite sequences are finite; it is* (head-)normalizing *if every term is (head-)normalizable.*
(4) A term is called unsolvable *if it has no head-normal form.*
(5) A term t is neutral *if there are a variable x and terms $t_1,\ldots,t_n$, such that $t \equiv x\,t_1\cdots t_n$.*

We now focus on the relation between reduction in CS and in LC.

Definition 7 *Let $\mathbf{C} = ((\mathcal{C},\mathcal{X}),\mathbf{R})$ be a* CS. *$\langle\ \rangle^{\mathbf{C}}_{\lambda} : T(\mathcal{C},\mathcal{X}) \to \Lambda$, the interpretation of combinator terms over* **C** *in* LC*, is defined by:*

$$\begin{aligned} \langle x\rangle^{\mathbf{C}}_{\lambda} &= x, && \text{for all } x \in \mathcal{X}, \\ \langle t_1\,t_2\rangle^{\mathbf{C}}_{\lambda} &= \langle t_1\rangle^{\mathbf{C}}_{\lambda}\langle t_2\rangle^{\mathbf{C}}_{\lambda}, \\ \langle C\rangle^{\mathbf{C}}_{\lambda} &= \lambda x_1\cdots x_n.\langle r\rangle^{\mathbf{C}}_{\lambda}, && \text{where } C\,x_1\cdots x_n \to r \in \mathbf{R} \end{aligned}$$

Notice that, since we assume the set of term variables for CS and LC to be the same, as well as the two notions of application on terms, $\langle r\rangle^{\mathbf{C}}_{\lambda} = r$ for every r that is the

right-hand side of a combinator rule.

Proposition 8 *Let* $\mathbf{C} = ((\mathcal{C},\mathcal{X}),\mathbf{R})$ *be a* CS*, then, for all* $t, t' \in T(\mathcal{C},\mathcal{X})$*: if* $t \to^* t'$*, then* $\langle t\rangle^{\mathbf{C}}_{\lambda} \twoheadrightarrow_{\beta} \langle t'\rangle^{\mathbf{C}}_{\lambda}$.

PROOF. By induction on the definition of $\to^*$. We only consider the case of $(C\,x_1\cdots x_n)^{\mathrm{R}} \to_C r^{\mathrm{R}}$, where $\mathrm{R} = \{x_1{\mapsto}u_1, \ldots, x_n{\mapsto}u_n\}$. Let

$$\mathrm{R}' = \{x_1{\mapsto}\langle u_1\rangle^{\mathbf{C}}_{\lambda}, \ldots, x_n{\mapsto}\langle u_n\rangle^{\mathbf{C}}_{\lambda}\}.$$

$$\begin{array}{rcl}
\text{Then } \langle (C\,x_1\cdots x_n)^{\mathrm{R}}\rangle^{\mathbf{C}}_{\lambda} & = & \langle C\,u_1\cdots u_n\rangle^{\mathbf{C}}_{\lambda} \\
& = & \langle C\rangle^{\mathbf{C}}_{\lambda}\langle u_1\rangle^{\mathbf{C}}_{\lambda}\cdots\langle u_n\rangle^{\mathbf{C}}_{\lambda} \\
& = & (\lambda x_1\cdots x_n.r)\langle u_1\rangle^{\mathbf{C}}_{\lambda}\cdots\langle u_n\rangle^{\mathbf{C}}_{\lambda} \\
& \twoheadrightarrow_{\beta} & r^{\mathrm{R}'} \\
& = & \langle r^{\mathrm{R}}\rangle^{\mathbf{C}}_{\lambda}.
\end{array}$$

The proof is completed by induction on the number of steps in $\to^*$, and by induction on the structure of contexts. ■

Although the interpretation in LC of a CS, $\langle\ \rangle^{\mathbf{C}}_{\lambda}$, respects reduction, in general, the length of the reduction sequence increases significantly. Only for particular CS it is possible to also define an interpretation of LC, $[\![\ ]\!]_{\mathbf{C}}$; the standard example is that of CL (see also [11,6,13]; in [13] also other combinatory complete CS are discussed).

Definition 9 *The mapping* $[\![\ ]\!]_{\mathrm{CL}} : \Lambda \to T(\{\mathsf{S},\mathsf{K},\mathsf{I}\},\mathcal{X})$ *is defined by:*

$$\begin{array}{rcl}
[\![x]\!]_{\mathrm{CL}} & = & x, \\
[\![\lambda x.M]\!]_{\mathrm{CL}} & = & \lambda^* x.[\![M]\!]_{\mathrm{CL}}, \\
[\![MN]\!]_{\mathrm{CL}} & = & [\![M]\!]_{\mathrm{CL}}[\![N]\!]_{\mathrm{CL}}.
\end{array}$$

where $\lambda^* x.t$*, with* $t \in T(\{\mathsf{S},\mathsf{K},\mathsf{I}\},\mathcal{X})$*, is defined by induction on the structure of* t*:*

$$\begin{array}{rcll}
\lambda^* x.x & = & \mathsf{I}, & \\
\lambda^* x.t & = & \mathsf{K}t, & \textit{if } x \textit{ not in } t, \\
\lambda^* x.t_1 t_2 & = & \mathsf{S}(\lambda^* x.t_1)(\lambda^* x.t_2). &
\end{array}$$

As for the accuracy of the above definitions, take:

$$\begin{array}{rcl}
(\lambda^* x.x)t & = & \mathsf{I}t \to t \\
(\lambda^* x.t_1)t_2 & = & \mathsf{K}t_1 t_2 = t_1, \text{if } x \text{ not in } t \\
(\lambda^* x.t_1 t_2)t_3 & = & \mathsf{S}(\lambda^* x.t_1)(\lambda^* x.t_2)t_3 = ((\lambda^* x.t_1)t_3)((\lambda^* x.t_2)t_3)
\end{array}$$

One important property of these two translations is that

$$\langle [\![M]\!]_{\mathrm{CL}} \rangle_{\lambda}^{\mathrm{CL}} \twoheadrightarrow_{\beta} M,$$

for all $M \in \Lambda$. There exists no converse of this property; moreover, the mapping $\langle\ \rangle_{\lambda}^{\mathrm{CL}}$ does not preserve normal forms or reductions:

Example 10 ([6]) *(1)* SK *is a normal form, but* $\langle \mathsf{SK} \rangle_{\lambda}^{\mathrm{CL}} \twoheadrightarrow_{\beta} \lambda xy.y$*,*
(2) $t = \mathsf{S}(\mathsf{K}(\mathsf{SII}))(\mathsf{K}(\mathsf{SII}))$ *is a normal form, but* $\langle t \rangle_{\lambda}^{\mathrm{CL}} \twoheadrightarrow_{\beta} \lambda c.(\lambda x.xx)(\lambda x.xx)$*, which does not have a* β*-normal form,*
(3) $t = \mathsf{SK}(\mathsf{SII}(\mathsf{SII}))$ *has no normal form, while* $\langle t \rangle_{\lambda}^{\mathrm{CL}} \twoheadrightarrow_{\beta} \lambda x.x$*.*

We will show in Section 8 that the combinatorial equivalent of a well-known result for intersection type assignment in the LC, i.e. the property that normalising terms can be typed with a type not containing ω, no longer holds. Take for example the CS

$$\begin{aligned} \mathsf{Z}xy &\to y, \\ \mathsf{D}x &\to xx, \end{aligned}$$

then $\mathsf{Z}(\mathsf{DD})$ is typeable with a type not containing ω (see the example before Thm. 72). Notice that, since $\mathsf{DD} \to \mathsf{DD} \to \cdots$, the term $\mathsf{Z}(\mathsf{DD})$ has no normal form. We will, however, in Section 8, prove this normalization result for 'Non-Curryfied' terms.

Definition 11 (Non-Curryfied terms) *The set* $\mathcal{T}_{NC}(\mathcal{C}, \mathcal{X})$ *of* non-Curryfied terms *is defined inductively by:*

$$t ::= xt_1 \cdots t_n\ (n \geq 0) \mid Ct_1 \cdots t_n\ (\mathit{arity}(C) \leq n)$$

Notice that $\mathcal{T}_{NC}(\mathcal{C}, \mathcal{X})$ *is a subset of* $\mathcal{T}(\mathcal{C}, \mathcal{X})$*.*

Note that $\mathsf{Z}(\mathsf{DD}) \notin \mathcal{T}_{NC}(\{\mathsf{Z}, \mathsf{D}\}, \mathcal{X})$, since Z has arity 2, and is applied here to only *one* subterm, DD. Also, none of the terms in Exp. 10 are in $\mathcal{T}_{NC}(\{\mathsf{S}, \mathsf{K}, \mathsf{I}\}, \mathcal{X})$.

As these examples show, normalization results of LC do not transfer easily to CS. In this paper, we will study the normalization properties of CS directly in the CS framework.

2 Intersection type assignment

It is well-known that in the study of normalization of reduction systems, the notion of type plays an important role, and that many of the now existing type assign-

ment systems for functional programming languages are based on (extensions of) the Curry type assignment system for LC [10,11]. The Intersection Type Discipline (ITD) presented in [8] (see also [7,1]) is an extension of Curry's system, in that, essentially, terms are allowed to have more than one type (using the type constructor '$\cap$'). By introducing also the type constant 'ω' a system is obtained that is closed under β-equality, and interpreting terms by their assignable types gives a filter lambda model [7,3].

In this section, we will develop a notion of type assignment on CS that uses intersection types. It is inspired by similar definitions presented in [13] and [5]. As in [13], we will assume that, for every combinator C, there is a basic type from which all types needed for an occurrence of C in a term can be obtained. The extension with respect to [13] is that we will not limit ourselves to basic types that are the principal type of the corresponding lambda term (see [19,2]). The differences with [5] are on the level of the language considered. In this paper, patterns are not used, i.e. rewrite rules cannot impose structure on arguments of function symbols; moreover, no function symbol is allowed to appear in the right-hand side of rewrite rules.

We will not consider general intersection types, as were defined in [7] and used in [13]. Instead, as in [5], we are going to use a restricted subset (the set of strict types, see [1]), that has the same expressive power: strict types are the representatives for equivalence classes of the types considered in the system of [7]. We will assume that ω is the same as an intersection over zero elements: if $n = 0$, then $\sigma_1 \cap \cdots \cap \sigma_n \equiv \omega$, so ω does not occur in an intersection subtype. Moreover, intersection type schemes (so also ω) occur in strict types only as subtypes at the left-hand side of an arrow type scheme.

Definition 12 (Strict types) *(1) Let Φ be a countable infinite set of type-variables, ranged over by φ. $\mathcal{T}_s$, the set of* strict types*, ranged over by $\sigma, \tau, \ldots$, is defined by:*

$$\sigma ::= \varphi \mid ((\sigma_1 \cap \cdots \cap \sigma_n) \to \sigma),\ (n \geq 0)$$

The set $\mathcal{T}$ of strict intersection types is defined by:

$$\mathcal{T} = \{(\sigma_1 \cap \cdots \cap \sigma_n) \mid n \geq 0 \ \& \ \forall\, 1 \leq i \leq n\ [\sigma_i \in \mathcal{T}_s]\}$$

As usual in the notation of types, right-most, outermost brackets will be omitted, and, as in logic, $\cap$ binds stronger than $\to$. The type ω is defined as an intersection of zero strict types.

(2) On $\mathcal{T}$, the relation $\leq$ is defined as the smallest relation satisfying:

$$\begin{array}{rl} \forall 1 \leq i \leq n\ [\sigma_1 \cap \cdots \cap \sigma_n \ \leq\ \sigma_i] & (n \geq 1) \\ \forall 1 \leq i \leq n\ [\sigma \leq \sigma_i] \ \Rightarrow\ \sigma \leq \sigma_1 \cap \cdots \cap \sigma_n & (n \geq 0) \\ \sigma \leq \tau \leq \rho \ \Rightarrow\ \sigma \leq \rho & \\ \rho \leq \sigma\ \&\ \tau \leq \mu \ \Rightarrow\ \sigma \to \tau \leq \rho \to \mu & \end{array}$$

(3) We define the relation $\sim$ by: $\sigma \sim \tau \iff \sigma \leq \tau \leq \sigma$.

We will work with types modulo $\sim$.

Lemma 13 ([3]) *For all $\sigma, \tau \in \mathcal{T}$:*

$$\begin{array}{l} \sigma \leq \tau \iff \sigma = \sigma_1 \cap \cdots \cap \sigma_n, \tau = \tau_1 \cap \cdots \cap \tau_m, \text{ for some } \sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m, \\ \qquad\qquad \text{and, for every } 1 \leq j \leq m, \text{ there is a } 1 \leq i \leq n \text{ such that } \sigma_i \leq \tau_j. \end{array}$$

Notice that, by definition, in $\sigma_1 \cap \cdots \cap \sigma_n$, all $\sigma_1, \ldots, \sigma_n$ are strict; sometimes we will deviate from this by writing $\sigma \cap \tau$ also for σ, τ not in $\mathcal{T}_s$.

Definition 14 (Bases) *(1) A* statement *is an expression of the form $t{:}\sigma$, where t is the* subject *and σ is the* predicate *of $t{:}\sigma$.*

(2) A basis *B is a set of statements with (distinct) variables as subjects, and, if $x{:}\sigma \in B$, then $\sigma \neq \omega$.*

(3) If $B_1, \ldots, B_n$ are bases, then $\bigcap\{B_1, \ldots, B_n\}$ is the basis defined as follows:

$$x{:}\sigma_1 \cap \cdots \cap \sigma_m \in \bigcap\{B_1, \ldots, B_n\}$$

if and only if $m \geq 1$ and $\{x{:}\sigma_1, \ldots, x{:}\sigma_m\}$ is the set of all statements that have x as subject that occur in $B_1 \cup \cdots \cup B_n$.

(4) The relations $\leq$ and $\sim$ are extended to bases by:

$$\begin{array}{l} B \leq B' \iff \forall x{:}\sigma' \in B'\ \exists x{:}\sigma \in B\ [\sigma \leq \sigma'] \\ B \sim B' \iff B \leq B' \leq B. \end{array}$$

We will often write $B, x{:}\sigma$ (or $B \cup \{x{:}\sigma\}$) for the basis $\bigcap\{B, \{x{:}\sigma\}\}$, when x does not occur in B. Notice that, in part 3, if $n = 0$, then $\bigcap\{B_1, \ldots, B_n\} = \emptyset$, and that $B \leq \emptyset$, for all B.

We will now recall three operations on types that are needed in the definition of type assignment and are standard in intersection systems. Substitution is the operation that instantiates a type (i.e. that replaces type-variables by types). The operation of expansion replaces a type by the intersection of a number of copies of that type. The operation of lifting replaces a type by a larger one, in the sense of $\leq$.

These three operations are of use in Def. 24, when we want to specify how, for a specific combinator, a type required by the context can be obtained from the

type provided for that combinator by the environment (Def. 23). It is possible to define type assignment with fewer or less powerful operations on types, but in order to obtain enough expressive power to prove Thm. 30:1, all three operations are needed.

Definition 15 (Type-substitution) *(1) The* type-substitution $(\alpha/\varphi) : \mathcal{T} \to \mathcal{T}$, *that replaces occurrences of* φ *by* α, *where* $\varphi \in \Phi$ *and* $\alpha \in \mathcal{T}_s \cup \{\omega\}$, *is defined by:*

$$\begin{array}{lll}
(\alpha/\varphi)\,(\varphi) & = \alpha & \\
(\alpha/\varphi)\,(\varphi') & = \varphi', & \textit{if } \varphi' \neq \varphi \\
(\alpha/\varphi)\,(\sigma{\to}\tau) & = \omega, & \textit{if } (\alpha/\varphi)\,(\tau) = \omega \\
(\alpha/\varphi)\,(\sigma{\to}\tau) & = (\alpha/\varphi)\,(\sigma) \to (\alpha/\varphi)\,(\tau), & \textit{if } (\alpha/\varphi)\,(\tau) \neq \omega \\
(\alpha/\varphi)\,(\sigma_1{\cap}\cdots{\cap}\sigma_n) & = (\alpha/\varphi)\,(\sigma'_1) \cap \cdots \cap (\alpha/\varphi)\,(\sigma'_m), & \textit{where} \\
\multicolumn{3}{r}{\{\sigma'_1,\ldots,\sigma'_m\} = \{\sigma_i \in \{\sigma_1,\ldots,\sigma_n\} \mid (\alpha/\varphi)\,(\sigma_i) \neq \omega\}}
\end{array}$$

(2) The set of type-substitutions is closed under composition: if S_1 *and* S_2 *are type-substitutions, then so is* $S_1 \circ S_2$, *where* $S_1 \circ S_2(\sigma) = S_1(S_2(\sigma))$.
(3) $S(B) = \{x{:}S(\alpha) \mid x{:}\alpha \in B \;\&\; S(\alpha) \neq \omega\}$.
(4) $S(\langle B, \sigma\rangle) = \langle S(B), S(\sigma)\rangle$.

Note that the definition of substitution in an arrow type ensures that the resulting type is still in $\mathcal{T}$.

It is possible to define a notion of type-substitution that just replaces type variables by strict types (so where $\alpha \in \mathcal{T}_s$); using such a definition, we would be forced to use the extra operation of *covering* that deals with the introduction of ω (see also [3]. To keep the set of operations small, we have decided not to follow that direction here.

Lemma 16 ([2]) *Let* S *be a type-substitution. If* $\sigma \leq \tau$, *then* $S(\sigma) \leq S(\tau)$, *and if* $B \leq B'$, *then* $S(B) \leq S(B')$. ∎

Our definition of expansion is inspired by the one given in [19] for the full intersection system in LC, we just need to make some minor changes to make sure that the type obtained is always in $\mathcal{T}$. For this, we have to check the last type-variable in arrow types (for a detailed discussion of the complexity of this operation, see [2]).

Definition 17 *The* last type-variable *of a strict type,* $last\,(\sigma)$, *is defined by:*

$$\begin{array}{ll}
last\,(\varphi) & = \varphi, \\
last\,(\sigma{\to}\tau) & = last\,(\tau).
\end{array}$$

Definition 18 (Expansion) *For every* $\mu \in \mathcal{T}$ *and* $n \geq 2$, *the pair* $\langle \mu, n\rangle$ *determines an* expansion $Ex : \mathcal{T} \to \mathcal{T}$ *which is computed with respect to* $\langle B, \sigma, E\rangle$ *as follows*

(where B is a basis, $\sigma \in \mathcal{T}$, and E is a finite set of types).

Affected variables: *The set $\mathcal{V}_\mu^{\langle B,\sigma,E\rangle}$ of type-variables is defined by:*

(1) If φ occurs in μ, then $\varphi \in \mathcal{V}_\mu^{\langle B,\sigma,E\rangle}$.

(2) If last $(\tau) \in \mathcal{V}_\mu^{\langle B,\sigma,E\rangle}$, with $\tau \in \mathcal{T}_s$ and τ (a subtype) in $\langle B,\sigma,E\rangle$, then for all type-variables φ that occur in τ: $\varphi \in \mathcal{V}_\mu^{\langle B,\sigma,E\rangle}$.

Renamings: *Let $\mathcal{V}_\mu^{\langle B,\sigma,E\rangle} = \{\varphi_1,\ldots,\varphi_m\}$. Choose $m \times n$ different type-variables $\varphi_1^1,\ldots,\varphi_n^1,\ldots,\varphi_1^m,\ldots,\varphi_n^m$, such that each φ_i^j does not occur in $\langle B,\sigma,E\rangle$, for $1 \le i \le n$ and $1 \le j \le m$. Let S_i be such that $S_i(\varphi_j) = \varphi_i^j$.*

Expansion of a type: *Ex (τ) is defined by:*

$$\begin{array}{lll} Ex(\tau_1 \cap \cdots \cap \tau_n) = Ex(\tau_1) \cap \cdots \cap Ex(\tau_n) & & \\ Ex(\tau) = S_1(\tau) \cap \cdots \cap S_n(\tau), & \text{if last}(\tau) \in \mathcal{V}_\mu^{\langle B,\sigma,E\rangle}. \\ Ex(\varphi) = \varphi, & \text{if } \varphi \notin \mathcal{V}_\mu^{\langle B,\sigma,E\rangle}. \\ Ex(\sigma \to \rho) = Ex(\sigma) \to Ex(\rho), & \text{if last}(\rho) \notin \mathcal{V}_\mu^{\langle B,\sigma,E\rangle}. \end{array}$$

Expansion of *B:* *$Ex(B) = \{x{:}Ex(\rho) \mid x{:}\rho \in B\}$.*

Expansion of *$\langle B,\sigma,E\rangle$:* *$Ex(\langle B,\sigma,E\rangle) = \langle Ex(B), Ex(\sigma), \{Ex(\rho) \mid \rho \in E\}\rangle$.*

When an expansion operation Ex is applied to a type τ without specifying $\langle B,\sigma,E\rangle$ we assume that the expansion is computed with respect to $\langle \emptyset,\tau,\emptyset\rangle$.

The proofs of the following properties are similar to those in [2].

Lemma 19 *Let Ex be the expansion determined by $\langle \mu, n\rangle$ and computed with respect to $\langle B,\sigma,E\rangle$. Then*

(1) (a) For $1 \le i \le n$, there are ρ_i and S_i such that $S_i(\rho) = \rho_i$ and $Ex(\rho) = \rho_1 \cap \cdots \cap \rho_n$, or

(b) $Ex(\rho) \in \mathcal{T}_s$.

(2) (a) For $1 \le i \le n$, there are B_i, σ_i, and S_i such that $S_i(\langle B,\sigma\rangle) = \langle B_i,\sigma_i\rangle$, and $Ex(\langle B,\sigma,E\rangle) = \langle \bigcap\{B_1,\ldots,B_n\}, \sigma_1 \cap \cdots \cap \sigma_n, E'\rangle$, or

(b) $Ex(\langle B,\sigma,E\rangle) = \langle B',\sigma',E'\rangle$, with $\sigma' \in \mathcal{T}_s$.

Lemma 20 *Let Ex be the expansion determined by $\langle \mu, n\rangle$ and computed with respect to $\langle B,\sigma,E\rangle$.*

(1) If ρ appears as (sub)-type in B, σ or E, and $\rho \le \tau$, then $Ex(\rho) \le Ex(\tau)$.

(2) If $B \le B'$, then $Ex(B) \le Ex(B')$. ∎

Definition 21 (Lifting) *A* lifting *L is an operation denoted by $\langle\langle B_0,\tau_0\rangle, \langle B_1,\tau_1\rangle\rangle$, a pair of pairs such that $\tau_0 \le \tau_1$ and $B_1 \le B_0$, and is defined by:*

$$\begin{array}{ll} L(\sigma) = \tau_1, \text{ if } \sigma = \tau_0 & L(B) = B_1, \text{ if } B = B_0 \\ L(\sigma) = \sigma, \text{ otherwise} & L(B) = B, \text{ otherwise} \end{array}$$

Definition 22 (Chains of operations on types) *A* chain *Ch* on types *is an object*

$[O_1, \ldots, O_n]$, *where each* O_i *is an operation of type-substitution, expansion or lifting, and*

$$[O_1, \ldots, O_n](\sigma) \;=\; O_n(\cdots(O_1(\sigma))\cdots)$$

We will use $*$ *to denote the operation of concatenation of chains.*

To complete the definition of type assignment, we present now the type assignment rules that are used to assign types in $\mathcal{T}$ to terms and combinator rules. In order to type the combinators, we use an environment that provides a type in $\mathcal{T}_s$ for every $C \in \mathcal{C}$, and use chains of operations to obtain the type for an occurrence of the combinator from the type provided for it by the environment.

Definition 23 (Environment) *Let* $\mathbf{C} = ((\mathcal{C}, \mathcal{X}), \mathbf{R})$ *be a* CS.

(1) An environment for $\mathbf{C}$ *is a mapping* $\mathcal{E} : \mathcal{C} \to \mathcal{T}_s$.
(2) For $C \in \mathcal{C}$, $\tau \in \mathcal{T}_s$, *and* $\mathcal{E}$ *an environment, the environment* $\mathcal{E}[C \mapsto \tau]$ *is defined by:*

$$\begin{array}{lll} \mathcal{E}[C \mapsto \tau](D) & = & \tau, \qquad \text{if } D = C, \\ \mathcal{E}[C \mapsto \tau](D) & = & \mathcal{E}(D), \text{ otherwise.} \end{array}$$

Since an environment $\mathcal{E}$ *maps all* $C \in \mathcal{C}$ *to types in* $\mathcal{T}_s$, *no combinator is mapped to* ω.

We define now type assignment on terms and combinator rules.

Definition 24 (Type assignment) *Let* $\mathbf{C} = ((\mathcal{C}, \mathcal{X}), \mathbf{R})$ *be a* CS *and* $\mathcal{E}$ *an environment for* $\mathbf{C}$.

(1) Type assignment *for terms in* $T(\mathcal{C}, \mathcal{X})$ *and* derivations *are defined by the following natural deduction system (where all types displayed are in* $\mathcal{T}_s$, *except for* τ *in rules* $(\leq)$ *and* $(\to E)$*):*

$$(\mathcal{E}) : \frac{}{B \vdash_{\mathcal{E}} C{:}\sigma}\,(\exists Ch\,[Ch(\mathcal{E}(C)) = \sigma])$$

$$(\to E) : \frac{B \vdash_{\mathcal{E}} t_1{:}\tau \to \sigma \quad B \vdash_{\mathcal{E}} t_2{:}\tau}{B \vdash_{\mathcal{E}} t_1\, t_2{:}\sigma}$$

$$(\leq) : \frac{}{B \vdash_{\mathcal{E}} x{:}\sigma}\,(x{:}\tau \in B, \tau \leq \sigma)$$

$$(\cap I) : \frac{B \vdash_{\mathcal{E}} t{:}\sigma_1 \quad \cdots \quad B \vdash_{\mathcal{E}} t{:}\sigma_n}{B \vdash_{\mathcal{E}} t{:}\sigma_1 \cap \cdots \cap \sigma_n}\,(n \geq 0)$$

If $B \vdash_{\mathcal{E}} t{:}\sigma$ is derivable using a derivation D*, we write* $\mathrm{D} :: B \vdash_{\mathcal{E}} t{:}\sigma$*. We write $B \vdash_{\mathcal{E}} t{:}\sigma$ to express that there exists a derivation* D *such that* $\mathrm{D} :: B \vdash_{\mathcal{E}} t{:}\sigma$*, and $\vdash_{\mathcal{E}} t{:}\sigma$ when $\emptyset \vdash_{\mathcal{E}} t{:}\sigma$. We will write $B \vdash^{\not\omega}_{\mathcal{E}} t{:}\sigma$ if ω is not used in the derivation.*

(2) Let $C \in \mathcal{C}$, with arity$(C) = n$. The combinator rule $C\,x_1 \cdots x_n \to r \in \mathbf{R}$ is typeable with respect to $\mathcal{E}$*, if there are $\sigma_1, \ldots, \sigma_n \in \mathcal{T}$ and $\sigma \in \mathcal{T}_s$, such that $\mathcal{E}(C) = \sigma_1 \to \cdots \to \sigma_n \to \sigma$, and $\{x_i{:}\sigma_i \mid \forall 1 \leq i \leq n\,[\sigma_i \neq \omega]\} \vdash_{\mathcal{E}} r{:}\sigma$.*

(3) **C** is typeable with respect to $\mathcal{E}$*, if every rule in* **R** *is typeable with respect to $\mathcal{E}$.*

At first sight, the formulation '*is typeable with respect to $\mathcal{E}$*' might seem a restriction on the class of systems that are considered in this paper, but it is not. Notice that an environment just maps combinators to types, without regard for the structure of their rewrite rules. The condition is added above just to ascertain that the type provided by the environment actually makes sense, and respects the structure of the rules involved.

Notice that if $B \vdash_{\mathcal{E}} t{:}\sigma$, then B can contain more statements than needed to obtain $t{:}\sigma$. Moreover, by part 2 of this definition, also $\{x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n\} \vdash_{\mathcal{E}} C\,x_1 \cdots x_n{:}\sigma$. However, just stating

'The combinator rule $l \to r$ is typeable with respect to the environment $\mathcal{E}$, if and only if there exist basis B and type σ, such that $B \vdash_{\mathcal{E}} l{:}\sigma$ and $B \vdash_{\mathcal{E}} r{:}\sigma$.'

would give a notion of type assignment that is not comparable to intersection type assignment for LC. For an example, take the combinator rule $E\,x\,y \to x\,y$. Let $\mathcal{E}(E) = \varphi_1 \to \varphi_2 \to \varphi_3$. Take $B = \{x{:}\varphi_1 \cap (\varphi_2 \to \varphi_3), y{:}\varphi_2\}$, then both $B \vdash_{\mathcal{E}} E\,x\,y{:}\varphi_3$ and $B \vdash_{\mathcal{E}} x\,y{:}\varphi_3$ are easy to derive. Notice that this combinator rule for E corresponds to the lambda term $\lambda xy.xy$, but $\varphi_1 \to \varphi_2 \to \varphi_3$ is not a correct type for this term.

The reason not to allow environments to provide types outside of $\mathcal{T}_s$ is purely practical, to obtain easier definitions. Notice that it is possible to derive an intersection type for a combinator, using rule $(\mathcal{E})$ a number of times, followed by $(\cap I)$.

The following result follows immediately.

Lemma 25 *$B \vdash_{\mathcal{E}} x{:}\sigma$ if and only if there is $x{:}\tau \in B$ such that $\tau \leq \sigma$.*

PROOF. Straightforward. ∎

Example 26 *The rules of* CL *(see Exp. 4) are typeable with respect to the environ-*

ment $\mathcal{E}_{\mathrm{CL}}$*:*

$$\begin{aligned}
\mathcal{E}_{\mathrm{CL}}(\mathsf{S}) &= (\varphi_1 \to \varphi_2 \to \varphi_3) \to (\varphi_4 \to \varphi_2) \to \varphi_1 \cap \varphi_4 \to \varphi_3, \\
\mathcal{E}_{\mathrm{CL}}(\mathsf{K}) &= \varphi_5 \to \omega \to \varphi_5, \\
\mathcal{E}_{\mathrm{CL}}(\mathsf{I}) &= \varphi_6 \to \varphi_6.
\end{aligned}$$

The term SKSI *can be typed with the type* $\alpha \to \alpha$ *with respect to* $\mathcal{E}_{\mathrm{CL}}$*: take*

$$\begin{aligned}
Ch_1 &= [(\varphi_1 \mapsto \alpha \to \alpha), (\varphi_2 \mapsto \omega), (\varphi_3 \mapsto \alpha \to \alpha), (\varphi_4 \mapsto \omega)], \\
Ch_2 &= [(\varphi_5 \mapsto \alpha \to \alpha)], \\
Ch_3 &= [(\varphi_6 \mapsto \alpha)],
\end{aligned}$$

then (notice that $Ch_1\,(\varphi_4 \to \varphi_2) = \omega$ *and* $Ch_1\,(\varphi_1 \cap \varphi_4) = \alpha \to \alpha$*)*

$$\begin{aligned}
Ch_1\,(\mathcal{E}_{\mathrm{CL}}(\mathsf{S})) &= ((\alpha \to \alpha) \to \omega \to \alpha \to \alpha) \to \omega \to (\alpha \to \alpha) \to \alpha \to \alpha \\
Ch_2\,(\mathcal{E}_{\mathrm{CL}}(\mathsf{K})) &= (\alpha \to \alpha) \to \omega \to \alpha \to \alpha \\
Ch_3\,(\mathcal{E}_{\mathrm{CL}}(\mathsf{I})) &= \alpha \to \alpha
\end{aligned}$$

and

$$\dfrac{\dfrac{\dfrac{\vdash_{\varepsilon_{\mathrm{CL}}} \mathsf{S}{:}Ch_1\,(\mathcal{E}_{\mathrm{CL}}(\mathsf{S}))\quad \vdash_{\varepsilon_{\mathrm{CL}}} \mathsf{K}{:}Ch_2\,(\mathcal{E}_{\mathrm{CL}}(\mathsf{K}))}{\vdash_{\varepsilon_{\mathrm{CL}}} \mathsf{SK}{:}\omega \to (\alpha \to \alpha) \to \alpha \to \alpha}\quad \dfrac{}{\vdash_{\varepsilon_{\mathrm{CL}}} \mathsf{S}{:}\omega}}{\vdash_{\varepsilon_{\mathrm{CL}}} \mathsf{SKS}{:}(\alpha \to \alpha) \to \alpha \to \alpha}\quad \dfrac{}{\vdash_{\varepsilon_{\mathrm{CL}}} \mathsf{I}{:}Ch_3\,(\mathcal{E}_{\mathrm{CL}}(\mathsf{I}))}}{\vdash_{\varepsilon_{\mathrm{CL}}} \mathsf{SKSI}{:}\alpha \to \alpha}$$

The definition of type assignment on CS as presented in this paper allows for the formulation of a precise relation between types assignable to terms, and those assignable to equivalent lambda terms. In fact, a result similar to part of the following property has already been proved in [13].

Definition 27 *Let* $\vdash_{\lambda\cap}$ *stand for the notion of intersection type assignment on* LC*, as defined in [3] by the following derivation rules (where all types displayed are in*

$\mathcal{T}_s$, except for τ in rules $(\to I)$, $(\to E)$ and $(\leq)$):

$$(\to I): \frac{B, x{:}\tau \vdash_{\lambda\cap} M{:}\sigma}{B \vdash_{\lambda\cap} \lambda x.M{:}\tau\to\sigma}$$

$$(\to E): \frac{B \vdash_{\lambda\cap} M{:}\tau\to\sigma \quad B \vdash_{\lambda\cap} N{:}\tau}{B \vdash_{\lambda\cap} MN{:}\sigma}$$

$$(\leq): \frac{}{B \vdash_{\lambda\cap} x{:}\sigma}\ (x{:}\tau \in B, \tau \leq \sigma)$$

$$(\cap I): \frac{B \vdash_{\lambda\cap} M{:}\sigma_1 \quad \cdots \quad B \vdash_{\lambda\cap} M{:}\sigma_n}{B \vdash_{\lambda\cap} M{:}\sigma_1\cap\cdots\cap\sigma_n}$$

The following states the relation between type assignment in CS and in LC (recall that $[\![\cdot]\!]_{\mathrm{CL}}$ is the interpretation of λ-terms in CL given in Def. 9).

Proposition 28 *If $B \vdash_{\lambda\cap} M{:}\sigma$, then $B \vdash_{\mathcal{E}_{\mathrm{CL}}} [\![M]\!]_{\mathrm{CL}}{:}\sigma$.*

PROOF. Similar to Thm. 3.7 of [13]. ∎

A more general formulation of Property 28, of course, only holds for CS that are expressive enough to encode LC. However, even for those the property is only provable if the environment assigns to the combinator symbols the principal types [19,2] of the corresponding lambda terms. For example, take $\vdash_{\lambda\cap} \lambda x.x{:}\alpha\to\alpha$ and notice that $[\![\lambda x.x]\!]_{\mathrm{CL}} = \mathsf{I}$. If $\mathcal{E}(\mathsf{I}) = (\alpha\to\alpha)\to\alpha\to\alpha$, then it is not possible to assign $\alpha\to\alpha$ to I in $\vdash_{\mathcal{E}}$.

However, we can show the following two results for CS equipped with principal environments.

Definition 29 *Let $\mathbf{C} = ((\mathcal{C},\mathcal{X}),\mathbf{R})$ be a* CS. *The environment $\mathcal{E}$ is called* principal for $\mathbf{C}$*, if for all $C \in \mathcal{C}$, $\mathcal{E}(C)$ is the principal type for $\langle C\rangle^{\mathbf{C}}_{\lambda}$ in $\vdash_{\lambda\cap}$.* [1]

Theorem 30 *Let $\mathbf{C} = ((\mathcal{C},\mathcal{X}),\mathbf{R})$ be a* CS.

(1) If $\mathcal{E}$ is principal for $\mathbf{C}$, then $B \vdash_{\lambda\cap} \langle t\rangle^{\mathbf{C}}_{\lambda}{:}\sigma$ implies $B \vdash_{\mathcal{E}} t{:}\sigma$.

[1] Since for every $l \to r \in \mathbf{R}$, r is in normal form, not containing combinators, it is possible to define the notion of principal environment directly for CS, without side-stepping to LC, but that would significantly increase the complexity of the proofs of this paper. It would not affect any of the results; in fact, the definition above would become a provable property.

(2) $B \vdash_{\mathcal{E}} t{:}\sigma$ *implies* $B \vdash_{\lambda\cap} \langle t\rangle^{\mathbf{C}}_{\lambda}{:}\sigma$.

PROOF. Assume (without loss of generality), that $\sigma \in \mathcal{T}_s$.

(1) By induction on the structure of terms in $T(\mathcal{C},\mathcal{X})$. The only case that needs attention is that of $t = C \in \mathcal{C}$, so $B \vdash_{\lambda\cap} \langle C\rangle^{\mathbf{C}}_{\lambda}{:}\sigma$. Since $\mathcal{E}$ is principal for **C**, $\mathcal{E}(C)$ is the principal type for $\langle C\rangle^{\mathbf{C}}_{\lambda}$ in $\vdash_{\lambda\cap}$ and there exists (see [2]) a chain of operations *Ch* such that $Ch(\mathcal{E}(C)) = \sigma$. But then $B \vdash_{\mathcal{E}} C{:}\sigma$ by rule $(\mathcal{E})$.

(2) By induction on the definition of $\langle\ \rangle^{\mathbf{C}}_{\lambda}$; the only alternative that needs consideration is that of $t = C \in \mathcal{C}$, and then the last rule in the derivation for $B \vdash_{\mathcal{E}} t{:}\sigma$ is $(\mathcal{E})$. Then there is a chain *Ch* such that $Ch(\mathcal{E}(C)) = \sigma$. Let $C\,x_1 \cdots x_n \to r$ be the rule for C. Then, by Def. 24:2, there are $\tau_1, \ldots, \tau_n \in \mathcal{T}$ and $\tau \in \mathcal{T}_s$, such that $\{x_1{:}\tau_1, \ldots, x_n{:}\tau_n\} \vdash_{\mathcal{E}} r{:}\tau$ and $\mathcal{E}(C) = \tau_1 \to \cdots \to \tau_n \to \tau$. Then, by induction, $\{x_1{:}\tau_1, \ldots, x_n{:}\tau_n\} \vdash_{\lambda\cap} r{:}\tau$ (notice that $\langle r\rangle^{\mathbf{C}}_{\lambda} = r$). Then, by rule $(\to I)$ of $\vdash_{\lambda\cap}$, $\vdash_{\lambda\cap} \lambda x_1 \ldots x_n.r : \tau_1 \to \cdots \to \tau_n \to \tau$; since $\vdash_{\lambda\cap}$ is closed for all three operations of substitution, expansion, and lifting (see [3]), we also have $\vdash_{\lambda\cap} \lambda x_1 \ldots x_n.r : \sigma$, so $\vdash_{\lambda\cap} \langle C\rangle^{\mathbf{C}}_{\lambda} : \sigma$. ■

3 Subject reduction

In this section we will show that the notion of type assignment defined here on CS satisfies the subject reduction property (Thm. 37). In order to achieve this, we first show that the three operations (type-substitution, expansion, and lifting) defined in the previous section can be applied to type-derivations, and are sound (the result is a well-defined derivation). We will also show that the type assignment rule $(\mathcal{E})$ is sound in the following sense: if there is an operation O such that $O(\mathcal{E}(C)) = \sigma$, then, for every type $\tau \in \mathcal{T}_s$ such that $\sigma \leq \tau$, the combinator rule for C is typeable with respect to the changed environment $\mathcal{E}[C \mapsto \tau]$.

Proposition 31 (Soundness of type-substitution) *Let S be a type-substitution.*

(1) If $B \vdash_{\mathcal{E}} t{:}\sigma$, *then* $S(B) \vdash_{\mathcal{E}} t{:}S(\sigma)$.
(2) If $C\,x_1 \cdots x_n \to r$ *is a rule typeable with respect to the environment* $\mathcal{E}$, *and* $S(\mathcal{E}(C)) \neq \omega$, *then it is typeable with respect to* $\mathcal{E}[C \mapsto S(\mathcal{E}(C))]$.

PROOF.

(1) By easy induction on the structure of derivations.
(2) By Def. 24:2, there are $\sigma_1, \ldots, \sigma_n, \sigma$, such that $\mathcal{E}(C) = \sigma_1 \to \cdots \to \sigma_n \to \sigma$, and $\{x_i{:}\sigma_i \mid \forall 1 \leq i \leq n\,[\sigma_i \neq \omega]\} \vdash_{\mathcal{E}} r{:}\sigma$. Since $S(\mathcal{E}(C)) \neq \omega$, by definition of substitution, $S(\mathcal{E}(C)) = S(\sigma_1) \to \cdots \to S(\sigma_n) \to S(\sigma)$. By 1, we have

$S(\{x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n\}) \vdash_{\mathcal{E}} r{:}S(\sigma)$, that is, $\{x_i{:}S(\sigma_i) \mid S(\sigma_i) \neq \omega\} \vdash_{\mathcal{E}} r{:}S(\sigma)$. Therefore the rule is typeable. ∎

The following essentially shows that lifting is sound:

Lemma 32 *(1) If $B \vdash_{\mathcal{E}} t{:}\sigma$ and $B' \leq B$, then $B' \vdash_{\mathcal{E}} t{:}\sigma$.*
(2) If $B \vdash_{\mathcal{E}} t{:}\sigma$ and $\sigma \leq \tau$, then $B \vdash_{\mathcal{E}} t{:}\tau$.
(3) If $B \vdash^{\omega}_{\mathcal{E}} t{:}\sigma$, $\sigma \leq \tau$, and τ is ω-free, then $B \vdash^{\omega}_{\mathcal{E}} t{:}\tau$.

PROOF. We will only give the proof for the second part; the third is similar, and the first is straightforward. We will prove it in two stages: first for σ, τ both in $\mathcal{T}_s$, then for σ, τ in $\mathcal{T}$.

$(\sigma, \tau \in \mathcal{T}_s)$ This is proven by induction on the structure of terms.
- $(t \equiv x)$ Then, by Lem. 25, there exists $x{:}\rho \in B$ such that $\rho \leq \sigma$. Since also $\rho \leq \tau$, $B \vdash_{\mathcal{E}} x{:}\tau$.
- $(t \equiv C)$ Then there is a chain *Ch* such that $Ch(\mathcal{E}(C)) = \sigma$. Since $\sigma \leq \tau$, $L = \langle\langle\emptyset, \sigma\rangle, \langle\emptyset, \tau\rangle\rangle$ is a lifting, then $Ch * [L]$ is a chain, therefore also $B \vdash_{\mathcal{E}} C{:}\tau$.
- $(t \equiv t_1\, t_2)$ So $B \vdash_{\mathcal{E}} t_1{:}\rho{\to}\sigma$, and $B \vdash_{\mathcal{E}} t_2{:}\rho$, for a certain ρ. Since $\sigma \leq \tau$, also $\rho{\to}\sigma \leq \rho{\to}\tau$; notice that both $\rho{\to}\sigma$ and $\rho{\to}\tau \in \mathcal{T}_s$. Then $B \vdash_{\mathcal{E}} t_1{:}\rho{\to}\tau$ by induction, so, by $(\to E)$, $B \vdash_{\mathcal{E}} t_1\, t_2{:}\tau$.

$(\sigma = \sigma_1 \cap \cdots \cap \sigma_m, \tau = \tau_1 \cap \cdots \cap \tau_n)$ Then, for every $1 \leq j \leq m$, $B \vdash_{\mathcal{E}} t{:}\sigma_j$. Then, by Lem. 13, for every $1 \leq i \leq n$, there is a $1 \leq j_i \leq m$ such that $\sigma_{j_i} \leq \tau_i$ and $\sigma_{j_i}, \tau_i \in \mathcal{T}_s$. Since the result has already been proved for $\mathcal{T}_s$, for every $1 \leq i \leq n$, $B \vdash_{\mathcal{E}} t{:}\tau_i$. Then by $(\cap I)$, $B \vdash_{\mathcal{E}} t{:}\tau_1 \cap \cdots \cap \tau_n$. ∎

Proposition 33 (Soundness of lifting) *Let L be a lifting.*

(1) If $B \vdash_{\mathcal{E}} t{:}\rho$, then $L(B) \vdash_{\mathcal{E}} t{:}L(\rho)$.
(2) If $C\, x_1 \cdots x_n \to r$ is a combinator rule, typeable with respect to $\mathcal{E}$, and $L(\mathcal{E}(C)) \in \mathcal{T}_s$, then it is typeable with respect to $\mathcal{E}[C \mapsto L(\mathcal{E}(C))]$.

PROOF.

(1) By Lem. 32.
(2) By Def. 24:2, there are $\sigma_1, \ldots, \sigma_n, \sigma$, such that $\mathcal{E}(C) = \sigma_1{\to}\cdots{\to}\sigma_n{\to}\sigma$ and $\{x_i{:}\sigma_i \mid \forall 1 \leq i \leq n\,[\sigma_i \neq \omega]\} \vdash_{\mathcal{E}} r{:}\sigma$. Since

$$\sigma_1{\to}\cdots{\to}\sigma_n{\to}\sigma \leq L(\sigma_1{\to}\cdots{\to}\sigma_n{\to}\sigma),$$

by Def. 12:2, there are $\tau_1, \ldots, \tau_n, \tau$, such that

$$L(\sigma_1{\to}\cdots{\to}\sigma_n{\to}\sigma) = \tau_1{\to}\cdots{\to}\tau_n{\to}\tau,$$

and for $1 \leq i \leq n$, $\tau_i \leq \sigma_i$, and $\sigma \leq \tau$. Hence $L' =$

$$\langle\langle\{x_i{:}\sigma_i \mid \forall 1 \leq i \leq n\,[\sigma_i \neq \omega]\}, \sigma\rangle, \langle\{x_i{:}\tau_i \mid \forall 1 \leq i \leq n\,[\tau_i \neq \omega]\}, \tau\rangle\rangle$$

is a lifting, and by part 1, we obtain

$$L'\left(\{x_i{:}\sigma_i \mid \forall 1 \leq i \leq n\,[\sigma_i \neq \omega]\}\right) \vdash_{\mathcal{E}} r{:}L'(\sigma),$$

so $\{x_i{:}\tau_i \mid \forall 1 \leq i \leq n\,[\tau_i \neq \omega]\} \vdash_{\mathcal{E}} r{:}\tau$. ∎

Proposition 34 (Soundness of expansion) *Let Ex be an expansion operation determined by $\langle \mu, n\rangle$, such that $Ex\langle B, \sigma, E\rangle = \langle B', \sigma', E'\rangle$.*

(1) If $B \vdash_{\mathcal{E}} t{:}\sigma$ using a set E of types for the occurrences of combinators in t, then $B' \vdash_{\mathcal{E}} t{:}\sigma'$.
(2) If $C\,x_1 \cdots x_n \to r$ is a rule, typeable with respect to $\mathcal{E}$, and $Ex(\mathcal{E}(C)) = \tau_1 \cap \cdots \cap \tau_m \in \mathcal{T}$ $(m \geq 1)$, then, for every $1 \leq j \leq m$, the rule is typeable with respect to $\mathcal{E}[C \mapsto \tau_j]$.

PROOF.

(1) By induction on $\mathcal{T}$. We will only show the part $\sigma \in \mathcal{T}_s$. Then, by Lem. 19 either:
($\sigma' = \tau_1 \cap \cdots \cap \tau_m$) Then $B' = \bigcap\{B_1, \ldots, B_m\}$, and for $1 \leq j \leq m$, there is a type-substitution S such that $S(\langle B, \sigma\rangle) = \langle B_j, \tau_j\rangle$. Then, by Thm. 31:1, for every $1 \leq j \leq m$, $B_j \vdash_{\mathcal{E}} t{:}\tau_j$. Since $B' \leq B_j$ for every $1 \leq j \leq m$, by Thm. 33, $B' \vdash_{\mathcal{E}} t{:}\tau_j$, and by ($\cap I$), $B' \vdash_{\mathcal{E}} t{:}\sigma'$.
($\sigma' \in \mathcal{T}_s$) This part is proved by induction on the structure of terms.
($t \equiv x$) Then, by ($\leq$), there is $x{:}\tau \in B$, such that $\tau \leq \sigma$. By Lem. 20:1, $Ex(\tau) \leq \sigma'$, so $B' \vdash_{\mathcal{E}} x{:}\sigma'$.
($t \equiv C$) Then, by ($\mathcal{E}$), there is a chain Ch such that $Ch(\mathcal{E}(C)) = \sigma$. Let Ex' be the expansion operation determined by $\langle \mu', n\rangle$, where μ' is the intersection of the type-variables in $\mathcal{V}_{\mu}^{\langle B,\sigma,E\rangle}$, that is, the variables affected by Ex when computing $Ex(\langle B, \sigma, E\rangle)$. Since $Ex'(\sigma) = \sigma'$, then $Ch * [Ex']$ is a chain such that $Ch * [Ex']\,(\mathcal{E}(C)) = \sigma'$. Therefore $B' \vdash_{\mathcal{E}} C{:}\sigma'$.
($t \equiv t_1\,t_2$) Then, by ($\to E$), there is τ such that $B \vdash_{\mathcal{E}} t_1{:}\tau \to \sigma$ and $B \vdash_{\mathcal{E}} t_2{:}\tau$. Let Ex' be the expansion defined by $\langle \mu', n\rangle$, where μ' is the intersection of the type-variables in $\mathcal{V}_{\mu}^{\langle B,\sigma,E\rangle}$. Note that $Ex'(B) = B'$ and $Ex'(\sigma) = \sigma'$. By induction, $B' \vdash_{\mathcal{E}} t_1{:}Ex'(\tau \to \sigma)$ and $B' \vdash_{\mathcal{E}} t_2{:}Ex'(\tau)$. Since $\sigma' \in \mathcal{T}_s$, $Ex'(\tau \to \sigma) = Ex'(\tau) \to \sigma'$, and we obtain $B' \vdash_{\mathcal{E}} t_1\,t_2{:}\sigma'$.
(2) Since $\mathcal{E}(C) \in \mathcal{T}_s$, by Lem. 19 either:
($m > 1$) By Def. 18, for every $1 \leq j \leq m$, there is a type-substitution S such that $S(\mathcal{E}(C)) = \tau_j$. The proof is completed by Thm. 31:2.

$(m = 1)$ By Def. 24:2, there are $\sigma_1, \ldots, \sigma_n, \sigma$, such that $\sigma_1 \to \cdots \to \sigma_n \to \sigma = \mathcal{E}(C)$, $\{x_i{:}\sigma_i \mid \forall 1 \leq i \leq n\,[\sigma_i \neq \omega]\} \vdash_{\mathcal{E}} r{:}\sigma$. Since $m = 1$,

$$Ex(\sigma_1 \to \cdots \to \sigma_n \to \sigma) = Ex(\sigma_1) \to \cdots \to Ex(\sigma_n) \to Ex(\sigma) \in \mathcal{T}_s.$$

By part 1, we obtain $\{x_i{:}Ex(\sigma_i) \mid \forall 1 \leq i \leq n\,[Ex(\sigma_i) \neq \omega]\} \vdash_{\mathcal{E}} r{:}Ex(\sigma)$ as required. ■

We then have:

Theorem 35 (Soundness of Chains) *(1) The set of derivations is closed under chains of operations.*

(2) Let $l \to_C r$ be a combinator rule typeable with respect to the environment $\mathcal{E}$. If $Ch(\mathcal{E}(C)) = \tau \in \mathcal{T}$, then, for every $\mu \in \mathcal{T}_s$ such that $\tau \leq \mu$, C is typeable with respect to $\mathcal{E}[C \mapsto \mu]$.

PROOF. By Propositions 31, 33, and 34. ■

Using this soundness result, we will now show that the notion of type assignment as defined in this paper satisfies the subject reduction property: if $B \vdash_{\mathcal{E}} t{:}\sigma$, and t can be rewritten to t', then $B \vdash_{\mathcal{E}} t'{:}\sigma$. Of course, this result can be obtained through the mappings $[\![\ ]\!]_{\mathrm{C}}$ and $\langle\ \rangle^{\mathrm{C}}_{\lambda}$, using the relations between the systems mentioned in the previous section, but only for combinatory complete CS and principal environments. For other CS, we must give a direct proof, for which we need the following term-substitution result.

Lemma 36 *(1) If $B \vdash_{\mathcal{E}} t{:}\sigma$, then, for every term-substitution* R *and basis B', if for every $x{:}\tau \in B$, $B' \vdash_{\mathcal{E}} x^{\mathrm{R}}{:}\tau$, then $B' \vdash_{\mathcal{E}} t^{\mathrm{R}}{:}\sigma$.*

(2) Let $C\,x_1 \cdots x_n \to r$ be a combinator rule, typeable with respect to $\mathcal{E}$. For every term-substitution R*, basis B and type μ: if $B \vdash_{\mathcal{E}} (C\,x_1 \cdots x_n)^{\mathrm{R}}{:}\mu$, then $B \vdash_{\mathcal{E}} r^{\mathrm{R}}{:}\mu$.*

PROOF.

(1) By induction on $\vdash_{\mathcal{E}}$.

$(\leq)$ Then $t = x$. Then there is $x{:}\tau \in B$, such that $\tau \leq \sigma$. Then, by Thm. 33, $B' \vdash_{\mathcal{E}} x^{\mathrm{R}}{:}\tau$ implies $B' \vdash_{\mathcal{E}} x^{\mathrm{R}}{:}\sigma$.

$(\mathcal{E})$ Then $t = C$. Immediate, since $C^{\mathrm{R}} = C$, and $C{:}\sigma$ does not depend on the basis.

$(\to E), (\cap I)$ By induction.

(2) If $C\,x_1 \cdots x_n \to r$ is a typeable combinator rule, then by Def. 24:2, there are $\sigma_1, \ldots, \sigma_n, \sigma$, such that

$$\mathcal{E}(C) = \sigma_1 \to \cdots \to \sigma_n \to \sigma \text{ and } \{x_i{:}\sigma_i \mid \forall 1 \leq i \leq n\,[\sigma_i \neq \omega]\} \vdash_{\mathcal{E}} r{:}\sigma.$$

Also, $(C\,x_1 \cdots x_n)^{\mathrm{R}} = C\,x_1{}^{\mathrm{R}} \cdots x_n{}^{\mathrm{R}}$. Since $B \vdash_{\mathcal{E}} C\,x_1{}^{\mathrm{R}} \cdots x_n{}^{\mathrm{R}}{:}\mu$, there are two cases:

($\mu \in \mathcal{T}_s$) Then there are $\mu_1, \ldots, \mu_n$, and a chain *Ch* such that $Ch(\mathcal{E}(C)) = \mu_1 \to \cdots \to \mu_n \to \mu$, and, for $1 \leq i \leq n$, $B \vdash_{\mathcal{E}} x_i{}^{\mathrm{R}}{:}\mu_i$. Since

$$\{x_i{:}\sigma_i \mid \forall 1 \leq i \leq n\,[\sigma_i \neq \omega]\} \vdash_{\mathcal{E}} r{:}\sigma,$$

we have, by Thm. 35:1, $\{x_i{:}\mu_i \mid \forall 1 \leq i \leq n\,[\mu_i \neq \omega]\} \vdash_{\mathcal{E}} r{:}\mu$. Then, by part 1, also $B \vdash_{\mathcal{E}} r^{\mathrm{R}}{:}\mu$.

($\mu = \rho_1 \cap \ldots \cap \rho_n$) Then we apply the above reasoning to each ρ_i and conclude using $(\cap I)$. ■

Using this result, the following becomes easy.

Theorem 37 (Subject reduction) *Let* $\mathbf{C} = ((\mathcal{C}, \mathcal{X}), \mathbf{R})$ *be a* CS. *For all* $t, t' \in T(\mathcal{C}, \mathcal{X})$*: if* $B \vdash_{\mathcal{E}} t{:}\sigma$ *and* $t \to^* t'$*, then* $B \vdash_{\mathcal{E}} t'{:}\sigma$.

PROOF. By induction on the length of the reduction path; the case of length 1 is proved by induction on the structure of t. Of this double induction, only the case that t itself is the term-substitution instance of a left-hand side of a combinator rule is of interest; all other cases are straightforward. Then, let $C \in \mathcal{C}$ and term-substitution R be such that $l \to_C r$, $t = l^{\mathrm{R}}$, and $t' = r^{\mathrm{R}}$. The result follows from Lem. 36:2. ■

One should remark that a subject expansion theorem, i.e. the converse of the subject reduction result:

$$\textit{If } B \vdash_{\mathcal{E}} t{:}\sigma\textit{, and } t' \to t\textit{, then } B \vdash_{\mathcal{E}} t'{:}\sigma,$$

does not hold in general. Take for example the following CS, that is typeable with respect to the given environment

$$\begin{array}{lcll} \mathsf{K}xy & \to & x & \mathcal{E}(\mathsf{K}) = \varphi_1 \to \omega \to \varphi_1 \\ \mathsf{I}x & \to & x & \mathcal{E}(\mathsf{I}) = (\varphi_2 \to \varphi_2) \to \varphi_2 \to \varphi_2 \end{array}$$

The term IK reduces to the (head-)normal form K, but can only be typed by ω with respect to $\mathcal{E}$. Of course, $(\varphi_2 \to \varphi_2) \to \varphi_2 \to \varphi_2$ is not the principal type for $\langle \mathsf{I} \rangle_\lambda^{\mathrm{CL}}$ in $\vdash_{\lambda\cap}$. In fact, we have the following result:

Theorem 38 (Subject expansion) *Let* $\mathbf{C} = ((\mathcal{C}, \mathcal{X}), \mathbf{R})$ *be a* CS*, and* $\mathcal{E}$ *be principal for* $\mathbf{C}$*, then, for all* $t, t' \in T(\mathcal{C}, \mathcal{X})$*: if* $B \vdash_{\mathcal{E}} t{:}\sigma$ *and* $t' \to t$*, then* $B \vdash_{\mathcal{E}} t'{:}\sigma$*.*

PROOF. If $B \vdash_{\mathcal{E}} t{:}\sigma$, then by Lem. 30:2, also $B \vdash_{\lambda\cap} \langle t \rangle^{\mathbf{C}}_{\lambda}{:}\sigma$. Since $t' \to t$, by Propostion 8 also $\langle t' \rangle^{\mathbf{C}}_{\lambda} \twoheadrightarrow_{\beta} \langle t \rangle^{\mathbf{C}}_{\lambda}$. Since $\vdash_{\lambda\cap}$ is closed for β-expansion, we have $B \vdash_{\lambda\cap} \langle t' \rangle^{\mathbf{C}}_{\lambda}{:}\sigma$. Then, by Thm. 30:1, we have $B \vdash_{\mathcal{E}} t'{:}\sigma$. ■

4 Restricted type assignment

Our aim is to define, in Section 5, a strongly normalizing notion of reduction on type derivations, that will be, as can be expected, a kind of Cut Elimination, guided by the occurrence of typeable redexes of $\to_{\mathbf{R}}$ in the conclusion of the type derivation. That this notion of derivation reduction is strongly normalizable will be used in Section 8 to obtain approximation and normalization results for typeable CS.

It might seem somewhat 'overkill' to define strong normalisation of derivation reduction in order to come to the usual intersection type assignment characterisations of approximation and normalisation, since, in the context of LC, these are all obtained more or less 'directly', i.e. reasoning about terms and their types: the structure of the derivations involved plays no role in proofs.

For example, the approximation result that will be proved in this paper for CS has been reached in [3] for $\vdash_{\lambda\cap}$ in LC. A problem with that result, however (or better, with the there used technique), is that it cannot be transferred to the context of CS, with its notion of weak reduction. The crucial point in the problem is that the property (for a definition of the set of approximants of a term M, $\mathcal{A}(M)$, see Def. 62):

'there is an $A \in \mathcal{A}(Mz)$ *such that* $B, z{:}\alpha \vdash_{\lambda\cap} A{:}\beta$*, and* $z \notin FV(M)$*'*
implies
'there is an $A \in \mathcal{A}(M)$ *such that* $B \vdash_{\lambda\cap} A{:}\alpha{\to}\beta$*'.*

is relatively easy to prove in LC, since the following holds:

If $A \in \mathcal{A}(Mz)$ *and* $z \notin FV(M)$*, then either:*
$A \equiv A'z$ *with* $z \notin FV(A')$ *and* $A' \in \mathcal{A}(M)$*, or* $\lambda z.A \in \mathcal{A}(M)$*.*

The first of these properties is hard to prove in arbitrary CS, because there is no known way to express abstraction adequately in CS that are not combinatory complete. Moreover, even in combinatory complete systems like CL, using the existence of a bijection through the mappings $\langle\ \rangle^{\mathrm{CL}}_{\lambda}$ and $[\![\ ]\!]_{\mathrm{CL}}$, it is not possible to prove this first property using the second. Take, for instance, the term $\mathsf{SK}y$, the environment

$\mathcal{E}_{\text{CL}}$ of Exp. 4, $B = \{z{:}\alpha\}$, and

$$\mathit{Ch} \;= [(\varphi_1 \mapsto \alpha), (\varphi_2 \mapsto \omega), (\varphi_3 \mapsto \alpha), (\varphi_4 \mapsto \alpha), (\varphi_5 \mapsto \alpha)]$$

then

$$\begin{array}{rcl} \mathit{Ch}(\mathcal{E}_{\text{CL}}(\mathsf{S})) & = & (\alpha{\to}\omega{\to}\alpha){\to}\omega{\to}\alpha{\to}\alpha \\ \mathit{Ch}(\mathcal{E}_{\text{CL}}(\mathsf{K})) & = & \alpha{\to}\omega{\to}\alpha \end{array}$$

and we can derive the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash_{\mathcal{E}_{\text{CL}}} \mathsf{S}{:}\mathit{Ch}(\mathcal{E}_{\text{CL}}(\mathsf{S}))} \quad \dfrac{}{B \vdash_{\mathcal{E}_{\text{CL}}} \mathsf{K}{:}\mathit{Ch}(\mathcal{E}_{\text{CL}}(\mathsf{K}))}}{B \vdash_{\mathcal{E}_{\text{CL}}} \mathsf{SK}{:}\omega{\to}\alpha{\to}\alpha} \quad \dfrac{}{B \vdash_{\mathcal{E}_{\text{CL}}} y{:}\omega}}{B \vdash_{\mathcal{E}_{\text{CL}}} \mathsf{SK}y{:}\alpha{\to}\alpha} \quad \dfrac{}{B \vdash_{\mathcal{E}_{\text{CL}}} z{:}\alpha}}{B \vdash_{\mathcal{E}_{\text{CL}}} \mathsf{SK}y\,z{:}\alpha}$$

Notice that $\mathcal{A}_{\text{CL}}(\mathsf{SK}yz) = \{\bot, z\}$ and that $\{z{:}\alpha\} \vdash_{\mathcal{E}_{\text{CL}}} z{:}\alpha$. Following the above property, since none of the approximants of $\mathsf{SK}yz$ is an application, we would like to obtain something like

$$[\![\lambda z.\langle z\rangle^{\mathbf{C}}_{\lambda}]\!]_{\text{CL}} \in \mathcal{A}_{\text{CL}}(\mathsf{SK}y) \text{ and } \emptyset \vdash_{\mathcal{E}_{\text{CL}}} [\![\lambda z.\langle z\rangle^{\mathbf{C}}_{\lambda}]\!]_{\text{CL}}{:}\alpha{\to}\alpha.$$

However, this fails, since

$$[\![\lambda z.\langle z\rangle^{\mathbf{C}}_{\lambda}]\!]_{\text{CL}} = \mathsf{I} \text{ and } \mathcal{A}_{\text{CL}}(\mathsf{SK}y) = \{\bot, \mathsf{S}\bot\bot, \mathsf{SK}\bot, \mathsf{S}\bot y, \mathsf{SK}y\}.$$

Therefore, a new approach to the problem of approximation and normalisation results is needed. In fact, the strong normalization result proved in Section 6 for derivation reduction deals with all these problems in *one* go: all normalisation results, as well as the approximation result, turn out to be corollaries of the main result (Thm. 54).

Since derivation reduction creates a new type derivation, some care is needed to make sure that *all* necessary sub-derivations are contracted, and no reduction is attempted where it is not possible.

In order to simplify the definition of the reduction relation, we will first define a notion of type assignment, denoted by $\vdash^{\text{r}}_{\mathcal{E}}$, that is a slight variant of the one given in Def. 24. The variation consists, essentially, of restricting bases to their relevant contents, i.e. to contain only *the types actually used* for the variables of a term. In

the next section, we will prove that derivations in this system are strongly normalizable; for this we will use the well-known method of Computability Predicates [20]. Then, in Section 8, we will show that the approximation theorem

$$\textit{If } B \vdash_{\mathcal{E}} t{:}\sigma\textit{, then there exists } a \in \mathcal{A}_{\mathcal{C}}(t) \textit{ such that } B \vdash_{\mathcal{E}} a{:}\sigma,$$

as well as the three normalization properties stated in the introduction of this paper, are consequences of this strong normalization result for $\vdash^{\mathrm{r}}_{\mathcal{E}}$.

Definition 39 (Restricted type assignment) *Let* **C** *be a* CS *and* $\mathcal{E}$ *an environment.* Restricted type assignment *and* restricted derivations *are defined by the following natural deduction system (where all types displayed are in* $\mathcal{T}_s$*, except for* τ *in rule* $(\rightarrow E)$*):*

$$(\mathcal{E}) : \frac{}{\emptyset \vdash^{\mathrm{r}}_{\mathcal{E}} C{:}\sigma}\ (\exists Ch\,[Ch(\mathcal{E}(C)) = \sigma])$$

$$(\rightarrow E) : \frac{B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t_1{:}\tau\rightarrow\sigma \quad B_2 \vdash^{\mathrm{r}}_{\mathcal{E}} t_2{:}\tau}{\bigcap\{B_1, B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} t_1 t_2{:}\sigma}$$

$$(Ax) : \frac{}{\{x{:}\sigma\} \vdash^{\mathrm{r}}_{\mathcal{E}} x{:}\sigma}$$

$$(\cap I) : \frac{B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1 \quad \cdots \quad B_n \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_n}{\bigcap\{B_1, \ldots, B_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1 \cap \cdots \cap \sigma_n}$$

We will write $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$ *if and only if there is a restricted derivation* D *that has* $B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$ *as conclusion, and* $B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$ *if there exists a* D *such that* $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$.

Notice that, in rule $(\cap I)$, if $n = 0$, then $\bigcap\{B_1, \ldots, B_n\} = \emptyset$ and $\sigma_1 \cap \cdots \cap \sigma_n = \omega$. Notice also that the main difference between $\vdash_{\mathcal{E}}$ and $\vdash^{\mathrm{r}}_{\mathcal{E}}$ lies in the fact that rule $(\leq)$ has been replaced by rule (Ax). Also, in rule $(\rightarrow E)$ for $\vdash_{\mathcal{E}}$, the bases used in left- and right-hand subderivation have to be the same, whereas for that rule in $\vdash^{\mathrm{r}}_{\mathcal{E}}$, this need not be the case: the respective bases are combined, using the operation $\bigcap\{\ \}$. We could have used this restricted system throughout this paper, without losing any important result (see also the next lemma). But since one of the objectives was to obtain at least the expressive power of the intersection type assignment system for LC (Thm. 30:1), the choice for the full system has been to allow also types in bases that are not relevant to the type assigned to the term, i.e. for derivation rule $(\leq)$ rather than (Ax). Bases are more restrictive in $\vdash^{\mathrm{r}}_{\mathcal{E}}$ because then the operation of derivation substitution (Def. 45) is easier to define.

The relation between the two notions of type assignment $\vdash^{\mathrm{r}}_{\mathcal{E}}$ and $\vdash_{\mathcal{E}}$ is strong:

Lemma 40 *(1) If $B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$, then $B \vdash_{\mathcal{E}} t{:}\sigma$.*
(2) If $B \vdash_{\mathcal{E}} t{:}\sigma$, then there is a B' such that $B \leq B'$ and $B' \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$.
(3) If $B \vdash_{\mathcal{E}} t{:}\sigma$ without using ω, then there is a B' such that $B \leq B'$ and $B' \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$ without using ω.

PROOF. By straightforward induction on the structure of derivations. ∎

Using these relations, the following lemma, that shows a subject-reduction result for restricted type assignment, becomes easy.

Theorem 41 *If $B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\tau$ and $t \to^* v$, then there exists B' such that $B \leq B'$ and $B' \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\tau$.*

PROOF. If $B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\tau$, by Lem. 40:1, also $B \vdash_{\mathcal{E}} t{:}\tau$. Since $t \to^* v$, by Thm. 37, also $B \vdash_{\mathcal{E}} v{:}\tau$. Then, by Lem. 40:2, there exists a B' such that $B \leq B'$ and $B' \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\tau$. ∎

Example 42 *Let Ch be such that $Ch(\mathcal{E}(\mathsf{K})) = \sigma{\to}\tau{\to}\sigma$, then, using Ch, we have $\{x{:}\sigma{\cap}\tau\} \vdash^{\mathrm{r}}_{\mathcal{E}} \mathsf{K}xx{:}\sigma$, $\mathsf{K}xx \to x$, and $\{x{:}\sigma\} \vdash^{\mathrm{r}}_{\mathcal{E}} x{:}\sigma$. Notice that $\{x{:}\sigma{\cap}\tau\} \leq \{x{:}\sigma\}$.*

Also the following result is a consequence of Lem 40.

Lemma 43 *If $B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$ and $\sigma \leq \tau$, then there exists B' such that $B \leq B'$ and $B' \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\tau$.*

PROOF. By Lem. 40:1, $B \vdash_{\mathcal{E}} t{:}\sigma$, and by Lem. 32, $B \vdash_{\mathcal{E}} t{:}\tau$. Hence, by Lem. 40:2, there exists B' such that $B \leq B'$ and $B' \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\tau$. ∎

We will use a short-hand notation for derivations.

Definition 44 *(1) We write $\mathrm{D} = \langle Ax\rangle$ if and only if the type derivation D consists of nothing but an application of rule (Ax), i.e. there are x and σ such that $\mathrm{D} :: \{x{:}\sigma\} \vdash^{\mathrm{r}}_{\mathcal{E}} x{:}\sigma$.*
(2) We write $\mathrm{D} = \langle\mathcal{E}\rangle$ if and only if D consists of nothing but an application of rule $(\mathcal{E})$, i.e. there are C and σ such that $\mathrm{D} :: \emptyset \vdash^{\mathrm{r}}_{\mathcal{E}} C{:}\sigma$.
(3) We write $\mathrm{D} = \langle \mathrm{D}_1, \mathrm{D}_2, {\to}E\rangle$ if and only if D is obtained from D_1 and D_2 by applying rule $(\to E)$, i.e. if there are $B_1, B_2, t_1, t_2, \sigma$, and τ such that

$$\mathrm{D}_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t_1{:}\tau{\to}\sigma,\ \mathrm{D}_2 :: B_2 \vdash^{\mathrm{r}}_{\mathcal{E}} t_2{:}\tau,\ \textit{and}\ \mathrm{D} :: \bigcap\{B_1, B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} t_1 t_2{:}\sigma.$$

(4) We write $\mathrm{D} = \langle \mathrm{D}_1, \ldots, \mathrm{D}_n, {\cap}I\rangle$ if and only if D is obtained from $\mathrm{D}_1, \ldots, \mathrm{D}_n$ by applying rule $(\cap I)$, i.e., for every $1 \leq i \leq n$, there are B_i and σ_i such that $\mathrm{D}_i :: B_i \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_i$, and $\mathrm{D} :: \bigcap\{B_1, \ldots B_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1{\cap}\cdots{\cap}\sigma_n$.

5 Derivation reduction

In this section, we will introduce a notion of reduction on derivations $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$. The effect of this reduction will be that a subderivation $\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma' \neq \omega$ for a redex t' occurring in t (due to the presence of derivation rule $(\cap I)$ there may be more than one subderivation for t') will be replaced by the derivation for its contractum, and the whole derivation for t will be updated accordingly. We will show that this notion of reduction is strongly normalizing.

Before formally defining reduction on derivations, we will define a notion of substitution on derivations, that will consist of replacing a type derivation for a variable by another derivation.

Definition 45 (Derivation substitution) Substituting $\mathrm{D}_v :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\sigma$ *for* $x{:}\sigma$ *in a derivation* $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\tau$*, denoted by*

$$\mathrm{D}' = \mathrm{D}\,[\mathrm{D}_v/x{:}\sigma] :: B'' \vdash^{\mathrm{r}}_{\mathcal{E}} t^{\{x\mapsto v\}}{:}\tau,$$

is inductively defined as follows:

(1) $\mathrm{D} = \langle Ax\rangle :: \{y{:}\tau\} \vdash^{\mathrm{r}}_{\mathcal{E}} y{:}\tau$*. If* $x = y$*, then* $\sigma = \tau$*, and* $\mathrm{D}' = \mathrm{D}_v :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\tau$*; otherwise,* $\mathrm{D}' = \mathrm{D}$*.*

(2) $\mathrm{D} = \langle \mathcal{E}\rangle :: \emptyset \vdash^{\mathrm{r}}_{\mathcal{E}} C{:}\tau$*. Then* $\mathrm{D}' = \mathrm{D}$*.*

(3) $\mathrm{D} = \langle \mathrm{D}_1, \mathrm{D}_2, {\to}E\rangle :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t_1\,t_2{:}\tau$*. We distinguish three cases:*

- *If* x *occurs in both* t_1 *and* t_2*, then* $\sigma = \sigma_1 \cap \sigma_2$*, and*

$$\begin{aligned} \mathrm{D}_1 &:: B_1, x{:}\sigma_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t_1{:}\rho{\to}\tau, \\ \mathrm{D}_2 &:: B_2, x{:}\sigma_2 \vdash^{\mathrm{r}}_{\mathcal{E}} t_2{:}\rho, \text{ and} \\ B &= \bigcap\{B_1, B_2\}, x{:}\sigma. \end{aligned}$$

  Assume (without loss of generality) that there exist $\alpha_1, \ldots, \alpha_m \in \mathcal{T}_s$ *such that* $\sigma_1 = \alpha_1 \cap \ldots \cap \alpha_j$ *and* $\sigma_2 = \alpha_{j+1} \cap \ldots \cap \alpha_m$ *and*

$$\mathrm{D}_v = \langle \mathrm{D}^1_v :: B'_1 \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\alpha_1, \ldots, \mathrm{D}^m_v :: B'_m \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\alpha_m, \cap I\rangle.$$

  Let

$$\begin{aligned} \mathrm{D}^1 &= \langle \mathrm{D}^1_v, \ldots, \mathrm{D}^j_v, \cap I\rangle :: B^1 \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\sigma_1, \text{ and} \\ \mathrm{D}^2 &= \langle \mathrm{D}^{j+1}_v, \ldots, \mathrm{D}^m_v, \cap I\rangle :: B^2 \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\sigma_2. \end{aligned}$$

  Let

$$\begin{aligned} \mathrm{D}'_1 &= \mathrm{D}_1\,[\mathrm{D}^1/x{:}\sigma_1] :: B''_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t_1^{\{x\mapsto v\}}{:}\rho{\to}\tau, \text{ and} \\ \mathrm{D}'_2 &= \mathrm{D}_2\,[\mathrm{D}^2/x{:}\sigma_2] :: B''_2 \vdash^{\mathrm{r}}_{\mathcal{E}} t_2^{\{x\mapsto v\}}{:}\rho. \end{aligned}$$

  Then $\mathrm{D}' = \langle \mathrm{D}'_1, \mathrm{D}'_2, {\to}E\rangle :: \bigcap\{B''_1, B''_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} (t_1\,t_2)^{\{x\mapsto v\}}{:}\tau$*.*

- *If x occurs just in t_1 (the case of x occurring only in t_2 is similar) then*

$$\begin{array}{rcl} \mathrm{D}_1 & :: & B_1, x{:}\sigma \vdash^{\mathrm{r}}_{\mathcal{E}} t_1{:}\rho{\to}\tau, \\ \mathrm{D}_2 & :: & B_2 \vdash^{\mathrm{r}}_{\mathcal{E}} t_2{:}\rho, \text{ and} \\ B & = & \bigcap\{B_1, B_2\}, x{:}\sigma. \end{array}$$

  Let $\mathrm{D}'_1 = \mathrm{D}_1\,[\mathrm{D}_v/x{:}\sigma]$*, then* $\mathrm{D}' = \langle \mathrm{D}'_1, \mathrm{D}_2, {\to}E\rangle$.
- *If x does not occur in $t_1\,t_2$ then* $\mathrm{D}' = \mathrm{D}$.

(4) $\mathrm{D} = \langle \mathrm{D}_1, \ldots, \mathrm{D}_n, \cap I\rangle :: \bigcap\{B_1, \ldots, B_n\}, x{:}\sigma_1{\cap}\cdots{\cap}\sigma_n \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\tau_1{\cap}\cdots{\cap}\tau_n$*, with, for* $1 \leq i \leq n$, $\mathrm{D}_i :: B_i, x{:}\sigma_i \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\tau_i$*. Since* $\mathrm{D}_v :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\sigma_1{\cap}\cdots{\cap}\sigma_n$*, reasoning as above in part 3, for* $1 \leq i \leq n$*, there are* D^i, B^i*, such that* $\mathrm{D}^i :: B^i \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\sigma_i$*. Let* $\mathrm{D}'_i = \mathrm{D}_i\,[\mathrm{D}^i/x{:}\sigma_i] :: B'_i \vdash^{\mathrm{r}}_{\mathcal{E}} t^{\{x\mapsto v\}}{:}\tau_i$*, then*

$$\mathrm{D}' = \langle \mathrm{D}'_1, \ldots, \mathrm{D}'_n, \cap I\rangle :: \bigcap\{B'_1, \ldots B'_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} t^{\{x\mapsto v\}}{:}\tau_1{\cap}\cdots{\cap}\tau_n.$$

Before coming to the definition of derivation-reduction, we need to define the concept of 'the position of a sub-derivation in a derivation'.

Definition 46 *Let* D *be a derivation, and* D′ *be a sub-derivation of* D*. The* position p of D′ in D *is defined by:*

(1) If $\mathrm{D}' = \mathrm{D}$*, then* $p = \varepsilon$.
(2) If the position of D′ *in* D_1 *is* q *and* $\mathrm{D} = \langle \mathrm{D}_1, \mathrm{D}_2, {\to}E\rangle$*, then* $p = 1q$.
(3) If the position of D′ *in* D_2 *is* q *and* $\mathrm{D} = \langle \mathrm{D}_1, \mathrm{D}_2, {\to}E\rangle$*, then* $p = 2q$.
(4) If the position of D′ *in* D_i*, for some* $1 \leq i \leq n$*, is* q*, and* $\mathrm{D} = \langle \mathrm{D}_1, \ldots, \mathrm{D}_n, \cap I\rangle$*, then* $p = q$.

Notice that if p is the position of a sub-derivation $\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma'$ in $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$, then p is also the position of an occurrence of t' in t.

Let $\langle \mathrm{D}_1, \ldots, \mathrm{D}_n, \cap I\rangle :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1{\cap}\cdots{\cap}\sigma_n$. Notice that, if $D_0 :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} u{:}\rho$ is a sub-derivation of D_j $(1 \leq j \leq n)$ at position p, then, for $1 \leq i \neq j \leq n$, either:

- there is no sub-derivation in D_i at position p, or
- D_i has a sub-derivation $\langle \cap I\rangle :: \emptyset \vdash^{\mathrm{r}}_{\mathcal{E}} u{:}\omega$ at position p, or
- D_i has a sub-derivation $\mathrm{D}'_0 :: B'' \vdash^{\mathrm{r}}_{\mathcal{E}} u{:}\rho'$ (with $\rho' \in \mathcal{T}_s$) at position p.

We can now give a definition of reduction on derivations in $\vdash^{\mathrm{r}}_{\mathcal{E}}$; this reduction corresponds to contracting a redex in the term that appears in the conclusion, and building a derivation for the contractum. The soundness of the definition is shown below.

Let $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$ be a derivation such that

- there is at least one subderivation $\mathrm{D}_p :: B_p \vdash^{\mathrm{r}}_{\mathcal{E}} t_p{:}\sigma_p$ at position p in D with $\sigma_p \in \mathcal{T}_s$, and

- $t_p = (C\,x_1 \cdots x_n)^{\mathrm{R}}$, and there is a rule $C\,x_1 \cdots x_n \to r$ such that $t \to_C t'$ at position p.

For each such subderivation, let B', B'_p be such that $B' \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma$, and $B'_p \vdash^{\mathrm{r}}_{\mathcal{E}} r^{\mathrm{R}}{:}\sigma_p$ (these exist by Thm 41). We say that D *reduces at position* p *to* $\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma$, denoted $\mathrm{D} \to_p \mathrm{D}'$, if D' is a derivation with the same tree-structure as D (that is, the same rules are applied) except for the positions p in D where a subderivation $\mathrm{D}_p :: B_p \vdash^{\mathrm{r}}_{\mathcal{E}} t_p{:}\sigma_p$ with $\sigma_p \in \mathcal{T}_s$ occurs; those subderivations are replaced by $\mathrm{D}'_p :: B'_p \vdash^{\mathrm{r}}_{\mathcal{E}} r^{\mathrm{R}}{:}\sigma_p$ in D'.

Formally:

Definition 47 (Derivation reduction) *The relation* $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma \to_p \mathrm{D}'$ *is defined by induction on* (p, σ)*:*

$(\sigma \in \mathcal{T}_s)$ *There are three cases depending on* p*:*

$(p = \varepsilon)$ *If* $t = C\,t_1 \cdots t_n$ *and there is a rule* $C\,x_1 \cdots x_n \to r$*, then*

$\mathrm{D} \to_\varepsilon \mathrm{D}' = \mathrm{D}'_0\,[\mathrm{D}'_1/x_1{:}\alpha_1, \ldots, \mathrm{D}'_n/x_n{:}\alpha_n] :: \bigcap\{B'_1, \ldots, B'_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} t' : \sigma$,

where

- $\mathrm{R} = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, $t' = r^{\mathrm{R}}$,
- *for every* $1 \le i \le n$, $\mathrm{D}'_i :: B'_i \vdash^{\mathrm{r}}_{\mathcal{E}} t_i{:}\alpha_i$*, for some* $\alpha_1, \ldots, \alpha_n$*,*
- $\mathrm{D}'_0 :: \{x_i{:}\alpha_i \mid \forall 1 \le i \le n\,[\alpha_i \neq \omega]\} \vdash^{\mathrm{r}}_{\mathcal{E}} r{:}\sigma$.

$(p = 1q)$ *Then* $\mathrm{D} = \langle \mathrm{D}_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t_1{:}\tau{\to}\sigma, \mathrm{D}_2, {\to}E\rangle :: \bigcap\{B_1, B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} t_1 t_2{:}\sigma$*, and*

$$\mathrm{D} \to_{1q} \mathrm{D}' = \langle \mathrm{D}'_1, \mathrm{D}_2, {\to}E\rangle :: \bigcap\{B', B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} t'_1 t_2{:}\sigma$$

if $\mathrm{D}_1 \to_q \mathrm{D}'_1 :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t'_1{:}\tau{\to}\sigma$.

$(p = 2q)$ *Then* $\mathrm{D} = \langle \mathrm{D}_1, \mathrm{D}_2 :: B_2 \vdash^{\mathrm{r}}_{\mathcal{E}} t_2{:}\tau, {\to}E\rangle :: \bigcap\{B_1, B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} t_1 t_2{:}\sigma$*, and*

$$\mathrm{D} \to_{2q} \mathrm{D}' = \langle \mathrm{D}_1, \mathrm{D}'_2, {\to}E\rangle :: \bigcap\{B_1, B'\} \vdash^{\mathrm{r}}_{\mathcal{E}} t_1 t'_2{:}\sigma$$

if $\mathrm{D}_2 \to_q \mathrm{D}'_2 :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t'_2{:}\tau$.

$(\sigma = \sigma_1 \cap \cdots \cap \sigma_n)$ *Then* $\mathrm{D} = \langle \mathrm{D}_1, \ldots, \mathrm{D}_n, \cap I\rangle :: \bigcap\{B_1, \ldots, B_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1 \cap \cdots \cap \sigma_n$*, where, for every* $1 \le i \le n$, $\mathrm{D}_i :: B_i \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_i$*. If there is some* $1 \le j \le n$ *such that* $\mathrm{D}_j \to_p \mathrm{D}'_j :: B'_j \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma_j$*, then*

$\mathrm{D} \to_p \mathrm{D}' = \langle \mathrm{D}'_1, \ldots, \mathrm{D}'_n, \cap I\rangle :: \bigcap\{B'_1, \ldots, B'_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma_1 \cap \cdots \cap \sigma_n$*,*

where, for $1 \le i \neq j \le n$*,*

(1) either $\mathrm{D}_i \to_p \mathrm{D}'_i :: B'_i \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma_i$*, or*

(2) there is no sub-derivation in D_i *at position* p*, or* D_i *has a sub-derivation* $\langle \cap I\rangle :: \emptyset \vdash^{\mathrm{r}}_{\mathcal{E}} u{:}\omega$ *at position* p*; then* $\mathrm{D}'_i :: B'_i \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma_i$ *is a derivation with the same structure as* D_i*, and* $B'_i = B_i$*.*

We will write $\mathrm{D} \to_{\mathcal{D}} \mathrm{D}'$ *if there is a* p *such that* D *reduces to* D' *at position* p*, and denote by* $\to^*_{\mathcal{D}}$ *its reflexive and transitive closure.*

Notice that D is reducible if and only if it contains a subderivation

$$\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} C\,u_1 \cdots u_n{:}\rho,$$

with $\rho \in \mathcal{T}_s$ and $n = \mathit{arity}(C)$. We show now that D′ is indeed a type-derivation in $\vdash^{\mathrm{r}}_{\mathcal{E}}$.

Theorem 48 (Soundness of Derivation Reduction) *Let* D *be a derivation such that* $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma \to_p \mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma$. *Then* D′ *is a well-defined derivation in* $\vdash^{\mathrm{r}}_{\mathcal{E}}$.

PROOF. We interpret a reduction step $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma \to_p \mathrm{D}'$ by the pair (p, σ), and proceed by induction on (p, σ).

$(\sigma = \sigma_1 \cap \cdots \cap \sigma_n)$ Then, for $1 \le i \le n$, there are D_i, B_i, such that $\mathrm{D}_i :: B_i \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_i$, $B = \bigcap\{B_1, \ldots, B_n\}$, and $\mathrm{D} = \langle \mathrm{D}_1, \ldots, \mathrm{D}_n, \cap I\rangle$. Moreover, since $\mathrm{D} \to_p \mathrm{D}'$, there is some $1 \le j \le n$ such that $\mathrm{D}_j \to_p \mathrm{D}'_j :: B'_j \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma_j$. Then, by the remark after Def. 46, for $1 \le i \ne j \le n$, either

(1) there is no sub-derivation in D_i at position p, or D_i has a sub-derivation $\langle \cap I\rangle :: \vdash^{\mathrm{r}}_{\mathcal{E}} u{:}\omega$ at position p; then D′ has a subderivation $\mathrm{D}'_i :: B'_i \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma_i$, with the same structure as D_i, and $B_i = B'_i$, by Def. 47. D'_i is a well-defined derivation, because t and t' coincide in the positions disjoint with p, and for p either there is no subderivation or it is a derivation for the type ω.

(2) D_i has a sub-derivation at position p such that $\mathrm{D}_i \to_p \mathrm{D}'_i :: B'_i \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma_i$, and D'_i is a well-defined derivation by induction.

So $\mathrm{D} \to_p \mathrm{D}' = \langle \mathrm{D}'_1, \ldots, \mathrm{D}'_n, \cap I\rangle :: \bigcap\{B'_1, \ldots, B'_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma_1 \cap \cdots \cap \sigma_n$ which is a well-defined derivation.

$(\sigma \in \mathcal{T}_s)$ There are three cases depending on p.

(1) If $p = \varepsilon$ then $t = (C\,x_1 \cdots x_n)^{\mathrm{R}} = C\,t_1 \cdots t_n$ and there is a combinator rule $C\,x_1 \cdots x_n \to r$. Then D has the form:

$$\dfrac{\dfrac{\dfrac{}{\emptyset \vdash^{\mathrm{r}}_{\mathcal{E}} C{:}\sigma_1 \to \cdots \to \sigma_n \to \sigma} \qquad \dfrac{\mathrm{D}_1}{B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t_1{:}\sigma_1}}{B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} C\,t_1{:}\sigma_2 \to \cdots \to \sigma_n \to \sigma} \qquad \dfrac{\mathrm{D}_n}{B_n \vdash^{\mathrm{r}}_{\mathcal{E}} t_n{:}\sigma_n}}{\bigcap\{B_1, \ldots, B_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} C\,t_1 \cdots t_n{:}\sigma}$$

Then, by Def. 24:2 and Thm. 35, $\{x_i{:}\sigma_i \mid \forall 1 \le i \le n\,[\sigma_i \ne \omega]\} \vdash_{\mathcal{E}} r{:}\sigma$, and, by Lem. 40:2, there are D'_0, and $\alpha_1, \ldots, \alpha_n$, such that, for every $1 \le i \le n$, $\sigma_i \le \alpha_i$ (α_i might be ω), and

$$\mathrm{D}'_0 :: \{x_i{:}\alpha_i \mid \forall 1 \le i \le n\,[\alpha_i \ne \omega]\} \vdash^{\mathrm{r}}_{\mathcal{E}} r{:}\sigma.$$

Then, by Lem. 43, there are $\mathrm{D}'_i, B'_i \geq B_i$ such that $\mathrm{D}'_i :: B'_i \vdash^{\mathrm{r}}_{\mathcal{E}} t_i{:}\alpha_i$, for every $1 \leq i \leq n$. Let $\mathrm{R} = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, $t' = r^{\mathrm{R}}$, then, by Def. 47, D reduces at position ε to

$$\mathrm{D}' = \mathrm{D}'_0\,[\mathrm{D}'_1/x_1{:}\alpha_1, \ldots, \mathrm{D}'_n/x_n{:}\alpha_n] :: \bigcap\{B'_1, \ldots, B'_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} t' : \sigma,$$

which is a well-defined derivation.

(2) If $p = 1q$, then $\mathrm{D} = \langle \mathrm{D}_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t_1{:}\tau{\to}\sigma, \mathrm{D}_2, {\to}E\rangle :: \bigcap\{B_1, B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} t_1 t_2{:}\sigma$, and, by induction, D_1 reduces to a well-defined $\mathrm{D}'_1 :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t'_1{:}\tau{\to}\sigma$ at position q. By Def. 47, $\mathrm{D} \to_{1q} \mathrm{D}' = \langle \mathrm{D}'_1, \mathrm{D}_2, {\to}E\rangle :: \bigcap\{B', B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} t'_1\, t_2{:}\sigma$, which is a well-defined derivation.

(3) If $p = 2q$: similar to the previous part. ∎

6 Strong normalization

In this section, we will prove that derivations in the restricted type assignment system are strongly normalizable with respect to the notion of reduction defined in the previous section. We will write *SN* (D) to indicate that D is strongly normalizable with respect to $\to_{\mathcal{D}}$.

The following properties hold:

Lemma 49 *(1) If* $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma \to_{\mathcal{D}} \mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma$, *then* $B \leq B'$, *and* $t \to t'$.

(2) Let $\mathrm{D} = \langle \mathrm{D}_1, \mathrm{D}_2, {\to}E\rangle :: \bigcap\{B_1, B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} t_1 t_2{:}\sigma$. *Then: SN* (D) *implies SN* (D_1) *and SN* (D_2).

(3) If both SN $(\mathrm{D}_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} x t_1 \cdots t_n{:}\sigma{\to}\tau)$, *and SN* $(\mathrm{D}_2 :: B_2 \vdash^{\mathrm{r}}_{\mathcal{E}} u{:}\sigma)$, *then also SN* $(\langle \mathrm{D}_1, \mathrm{D}_2, {\to}E\rangle)$.

(4) If $\mathrm{D} = \langle \mathrm{D}_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1, \mathrm{D}_2 :: B_2 \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_2, \cap I\rangle :: \bigcap\{B_1, B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1{\cap}\sigma_2$, *and* $\mathrm{D} \to_{\mathcal{D}} \mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma$ *then there are* $B'_1 \geq B_1, B'_2 \geq B_2$ *such that* $B' = \bigcap\{B'_1, B'_2\}$, *and* $\mathrm{D}_1 \to_{\mathcal{D}} \mathrm{D}'_1 :: B'_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma_1$ *or* $\mathrm{D}_2 \to_{\mathcal{D}} \mathrm{D}'_2 :: B'_2 \vdash^{\mathrm{r}}_{\mathcal{E}} t'{:}\sigma_2$.

(5) If $\mathrm{D} = \langle \mathrm{D}_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1, \mathrm{D}_2 :: B_2 \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_2, \cap I\rangle :: \bigcap\{B_1, B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1{\cap}\sigma_2$, *then SN* (D) *if and only if SN* (D_1) *and SN* (D_2).

PROOF. Direct consequence of the definition of $\to_{\mathcal{D}}$. ∎

We will use the well-known method of Computability Predicates [20].

Definition 50 (Computability predicate) *(1) Let* B *be a basis,* $t \in T(\mathcal{C}, \mathcal{X})$, *and* σ *a type. We define Comp* $(\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma)$ *recursively on* σ *by:*

(a) Comp $(\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\varphi) \iff$ *SN* (D).

(b) Comp $(\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma{\to}\tau) \iff \forall\, \mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} u{:}\sigma$
$[$*Comp* $(\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} u{:}\sigma) \Rightarrow$ *Comp* $(\langle \mathrm{D}, \mathrm{D}', {\to}E\rangle :: \bigcap\{B, B'\} \vdash^{\mathrm{r}}_{\mathcal{E}} t\,u{:}\tau)]$.

(c) $\mathit{Comp}\,(\langle \mathrm{D}_1, \ldots, \mathrm{D}_n, \cap I\rangle :: \bigcap\{B_1, \ldots, B_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1 \cap \cdots \cap \sigma_n) \iff$
$\forall 1 \leq i \leq n\,[\mathit{Comp}\,(\mathrm{D}_i :: B_i \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_i)]$.

(2) We say that a term-substitution R *is* computable in a basis B *if, for every* $x{:}\sigma \in B$*, there are* B_x *and* D_x *such that* $\mathit{Comp}\,(\mathrm{D}_x :: B_x \vdash^{\mathrm{r}}_{\mathcal{E}} x^{\mathrm{R}}{:}\sigma)$.

Notice that $\mathit{Comp}\,(\langle \cap I\rangle :: \emptyset \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\omega)$ holds for all t by part 1c when $n = 0$.

We will prove that *Comp* satisfies the standard properties of computability predicates. Recall that neutral terms are terms of the form $xt_1 \ldots t_n$ (see Def. 6).

Lemma 51 *(1)* $\mathit{Comp}\,(\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma) \Rightarrow \mathit{SN}\,(\mathrm{D})$.
(2) $\mathit{SN}\,(\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} xt_1 \cdots t_m{:}\sigma) \Rightarrow \mathit{Comp}\,(\mathrm{D})$.

PROOF. By simultaneous induction on the structure of types. The case $\sigma = \varphi$ is immediate, $\sigma = \sigma_1 \cap \cdots \cap \sigma_n$ follows from Def. 50:1c and Lem. 49:5 (note that for $\sigma = \omega$ the property $\mathit{SN}\,(\mathrm{D})$ holds trivially since D is irreducible), and for $\sigma = \alpha \to \beta$:

(1) Let x be a variable not appearing in B and t.

$$\begin{aligned}
\{x{:}\alpha\} \vdash^{\mathrm{r}}_{\mathcal{E}} x{:}\alpha \;\&\; \mathit{Comp}\,(\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\alpha\to\beta) &\Rightarrow (\mathit{IH}{:}2)\\
\mathit{Comp}\,(\mathrm{D}' :: \{x{:}\alpha\} \vdash^{\mathrm{r}}_{\mathcal{E}} x{:}\alpha) \;\&\; \mathit{Comp}\,(\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\alpha\to\beta) &\Rightarrow (50{:}1b)\\
\mathit{Comp}\,(\mathrm{D}'' = \langle \mathrm{D}, \mathrm{D}', \to E\rangle :: B, x{:}\alpha \vdash^{\mathrm{r}}_{\mathcal{E}} tx{:}\beta) &\Rightarrow (\mathit{IH}{:}1)\\
\mathit{SN}\,(\mathrm{D}'') &\Rightarrow (49{:}2)\\
\mathit{SN}\,(\mathrm{D}).&
\end{aligned}$$

(2)
$$\begin{aligned}
\mathit{SN}\,(\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} xt_1 \cdots t_m{:}\alpha\to\beta) &\Rightarrow (\mathit{IH}{:}1)\\
(\mathit{Comp}\,(\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} u{:}\alpha) \Rightarrow \mathit{SN}\,(\mathrm{D}) \;\&\; \mathit{SN}\,(\mathrm{D}')) &\Rightarrow (49{:}3)\\
(\mathit{Comp}\,(\mathrm{D}') \Rightarrow \mathit{SN}\,(\langle \mathrm{D}, \mathrm{D}', \to E\rangle :: \bigcap\{B, B'\} \vdash^{\mathrm{r}}_{\mathcal{E}} xt_1 \cdots t_m u{:}\beta)) &\Rightarrow (\mathit{IH}{:}2)\\
(\mathit{Comp}\,(\mathrm{D}') \Rightarrow \mathit{Comp}\,(\langle \mathrm{D}, \mathrm{D}', \to E\rangle)) &\Rightarrow (50{:}1b)\\
\mathit{Comp}\,(\mathrm{D}).&\quad \blacksquare
\end{aligned}$$

We will now come to the term-substitution theorem, the final construction in the proof of our strong normalization result, for which we need the following ordering:

Definition 52 *(1)* $\rhd$ *stands for the well-founded encompassment ordering:* $u \rhd v$ *if* $u \neq v$ *modulo renaming of variables, and* $v^{\mathrm{R}} = u|_p$ *for some position* p *in* u *and term-substitution* R.
(2) We define the ordering $\gg$ *on pairs – consisting of a natural number and a term – as the object* $(>_{\mathbf{N}}, \rhd)_{\mathit{lex}}$*, where* lex *denotes* lexicographic extension.
(3) Given a term t *and a term-substitution R, the* interpretation $\mathcal{I}(t^{\mathrm{R}})$ *of* t^{R} *is defined as the pair* $\langle n, t\rangle$ *where* n *is the number of combinators appearing in* t.

Note that encompassment contains the strict superterm relation.

We can now prove the term-substitution theorem.

Theorem 53 *If* $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$ *and* R *is computable in* B*, then there exists a* D' *such that Comp* $(\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t^{\mathrm{R}}{:}\sigma)$.

PROOF. We will consider the interpretation of t^{R}, and prove the theorem by Nötherian induction on $\gg$ (which is well-founded). If t is a variable, then $B = \{x{:}\sigma\}$, and since R is assumed to be computable in B, there exists a D' such that

$$\mathit{Comp}\,(\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} x^{\mathrm{R}}{:}\sigma).$$

Also, the case $\sigma = \omega$ is trivially computable. So, without loss of generality, we can assume that t is not a variable (so neither is t^{R}). Also, if $\sigma = \sigma_1 \cap \cdots \cap \sigma_n$, then the last rule applied is ($\cap I$), and we can reason on each σ_i separately, so we can focus on the case where $\sigma \in \mathcal{T}_s$.

We distinguish the following cases for t^{R}:

(t^{R} **is neutral**) Then there are $x \in \mathcal{X}, t_1, \ldots, t_n$ $(n > 0)$ such that $t^{\mathrm{R}} = x\,t_1 \cdots t_n$; also t is neutral, so there exist $z \in \mathcal{X}$ and $u_1, \ldots, u_m$ $(m > 0)$ such that $t = z\,u_1 \cdots u_m$, and $z^{\mathrm{R}} = x\,t_1 \cdots t_k$ $(k \geq 0, k + m = n)$. Since $B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$, there exist $\sigma_1, \ldots, \sigma_m, B_1, \ldots, B_m, \mathrm{D}_1, \ldots, \mathrm{D}_m$ such that

$$\mathrm{D}_0 :: \{z{:}\sigma_1 \to \cdots \to \sigma_m \to \sigma\} \vdash^{\mathrm{r}}_{\mathcal{E}} z{:}\sigma_1 \to \cdots \to \sigma_m \to \sigma, \text{ and } \mathrm{D}_j :: B_j \vdash^{\mathrm{r}}_{\mathcal{E}} u_j{:}\sigma_j,$$

for every $1 \leq j \leq m$, and $B = \bigcap\{B_1, \ldots, B_m\}$. Since $\mathcal{I}(t^{\mathrm{R}}) \gg \mathcal{I}(u_j{}^{\mathrm{R}})$, by induction, there exist D'_j such that $\mathit{Comp}\,(\mathrm{D}'_j :: B'_j \vdash^{\mathrm{r}}_{\mathcal{E}} u_j{}^{\mathrm{R}}{:}\sigma_j)$, for every $1 \leq j \leq m$. Also, since R is computable in B, there exists D'_0 such that

$$\mathit{Comp}\,(\mathrm{D}'_0 :: B'_0 \vdash^{\mathrm{r}}_{\mathcal{E}} z^{\mathrm{R}}{:}\sigma_1 \to \cdots \to \sigma_m \to \sigma).$$

Then, by Def. 50:1b,

$$\mathit{Comp}\,(\langle \cdots \langle \mathrm{D}'_0, \mathrm{D}'_1, \to E\rangle, \cdots, \mathrm{D}'_n, \to E\rangle :: \bigcap\{B'_0, B'_1, \ldots, B'_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} t^{\mathrm{R}}{:}\sigma).$$

(t^{R} **is not neutral**) Then there are $C \in \mathcal{C}, t_1, \ldots, t_n$ $(n \geq 0)$ with $t^{\mathrm{R}} = C\,t_1 \cdots t_n$. Now, three cases are possible:

(1) $t = z s_1 \ldots s_m$ $(m \leq n)$, or $t = C s_1 \cdots s_n$, and at least one of the s_i is not a variable. Since $\mathcal{I}(t^{\mathrm{R}}) \gg \mathcal{I}(s_i{}^{\mathrm{R}})$, by induction the type-derivation for $s_i{}^{\mathrm{R}}$ is computable, for every $1 \leq i \leq m$, or $1 \leq i \leq n$, respectively. Let y be a fresh variable, and $\mathrm{R}' = \mathrm{R} \cup \{y \mapsto s_i{}^{\mathrm{R}}\}$. Then $t^{\mathrm{R}} = (t[y]_i)^{\mathrm{R}'}$, and $\mathcal{I}(t^{\mathrm{R}}) \gg \mathcal{I}((t[y]_i)^{\mathrm{R}'})$. Then the type-derivation for t^{R} is computable by induction.

(2) $t = z z_1 \cdots z_m$ $(m \leq n)$. Then $z^{\mathrm{R}} = C t_1 \cdots t_k$ $(k + m = n)$. In this case we can proceed as for the case that t^{R} is neutral.

(3) $t = Cz_1 \cdots z_n$.

($n \neq 0$) Then $\mathcal{I}(t^{\mathrm{R}}) \gg \mathcal{I}(C^{\mathrm{R}})$, and $\mathrm{D}_0 :: \emptyset \vdash^{\mathrm{r}}_{\mathcal{E}} C{:}\sigma_1{\to}\cdots{\to}\sigma_n{\to}\sigma$, for certain $\sigma_1, \ldots, \sigma_n$, and *Comp* $(\mathrm{D}_0 :: \emptyset \vdash^{\mathrm{r}}_{\mathcal{E}} C{:}\sigma_1{\to}\cdots{\to}\sigma_n{\to}\sigma)$ by induction. Since R is computable in B, for every $1 \leq i \leq n$ there is D_i such that *Comp* $(\mathrm{D}_i :: B_i \vdash^{\mathrm{r}}_{\mathcal{E}} z_i{}^{\mathrm{R}}{:}\sigma_i)$, so by Def. 50:1b, also

$$\mathit{Comp}\,(\langle\cdots\langle \mathrm{D}_0, \mathrm{D}_1, {\to}E\rangle\cdots, \mathrm{D}_n, {\to}E\rangle :: \bigcap\{B_1, \ldots, B_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} t^{\mathrm{R}}{:}\sigma).$$

($n = 0$) Then $B = \emptyset$ and $\mathrm{D} = \langle\mathcal{E}\rangle$. Let $\sigma = \sigma_1{\to}\cdots{\to}\sigma_n{\to}\varphi$; in order to prove that there exists a D' such that *Comp* $(\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} C{:}\sigma)$ it is sufficient to prove

$$\begin{array}{l}\forall 1 \leq i \leq n\ \exists \mathrm{D}_i\ [\mathit{Comp}\,(\mathrm{D}_i :: B_i \vdash^{\mathrm{r}}_{\mathcal{E}} u_i{:}\sigma_i)] \Rightarrow \\ \quad \mathit{Comp}\,(\langle\cdots\langle \mathrm{D}, \mathrm{D}_1, {\to}E\rangle, \cdots, \mathrm{D}_n, {\to}E\rangle :: \underline{B} \vdash^{\mathrm{r}}_{\mathcal{E}} C\,u_1\cdots u_n{:}\varphi).\end{array}$$

with $\underline{B} = \bigcap\{B_1, \ldots, B_n\}$. Take $\mathrm{D}_0 = \langle\cdots\langle \mathrm{D}, \mathrm{D}_1, {\to}E\rangle\cdots, \mathrm{D}_n, {\to}E\rangle$, then by Def. 50:1a it suffices to prove

$$\forall 1 \leq i \leq n\ \exists \mathrm{D}_i\ [\mathit{Comp}\,(\mathrm{D}_i :: B_i \vdash^{\mathrm{r}}_{\mathcal{E}} u_i{:}\sigma_i)] \Rightarrow \mathit{SN}\,(\mathrm{D}_0).$$

We will proceed by induction on the sum of the maximal lengths of the reduction paths on the derivations $\mathrm{D}_i :: B \vdash^{\mathrm{r}}_{\mathcal{E}} u_i{:}\sigma_i$ (notice that these derivations are strongly normalizable by Lem. 51:1, since they are computable).

Consider all possible rewrite steps out of D_0.

(a) $\mathrm{D}_0 \to_{\mathcal{D}} \mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} Cu_1 \ldots u_{i-1}u_i'u_{i+1} \ldots u_n{:}\varphi$, where the reduction took place in u_i. Then by the inner induction *SN* (D').

(b) $\mathrm{D}_0 \to_{\mathcal{D}} \mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\varphi$ at the outermost level. Then there are a rule $C\,x_1\cdots x_i \to r$ where $i = \mathit{arity}(C)$, term-variables $x_{i+1}, \ldots, x_n$, and term-substitution $\mathrm{R}_1 = \{x_1 \mapsto u_1, \ldots, x_n \mapsto u_n\}$ such that $v = (r\,x_{i+1}\cdots x_n)^{\mathrm{R}_1}$. Since *Comp* $(\mathrm{D} :: B_i \vdash^{\mathrm{r}}_{\mathcal{E}} u_i{:}\sigma_i)$ for all $1 \leq i \leq n$, R_1 is computable in $\{x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n\}$. Then

$$\mathcal{I}(C^{\mathrm{R}}) \gg \mathcal{I}((r\,x_{i+1}\cdots x_n)^{\mathrm{R}_1}),$$

so by the external induction *Comp* $(\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\varphi)$, and *SN* (D') by Def. 50:1a.

Since for all D' such that $\mathrm{D}_0 \to_{\mathcal{D}} \mathrm{D}'$ we have proved *SN* (D'), we deduce *SN* (D_0) as required. ∎

The main result of this section is then the strong normalization theorem for derivation reduction in $\vdash^{\mathrm{r}}_{\mathcal{E}}$.

Theorem 54 (Strong normalisation) *If* $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$*, then* *SN* (D).

PROOF. Let $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$. Take R such that $x^{\mathrm{R}} = x$, then R is computable in B by

Lem. 51:2. Then *Comp* $(\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma)$ follows from Thm. 53, and, by Lem. 51:1, *SN* (D). ■

7 Approximants

Now we will develop, essentially following [22] (see also [6]), a notion of approximant for combinator terms. This will be done by introducing a special symbol $\bot$ into the definition of terms.

Definition 55 (Combinator terms with $\bot$) *Let* $\mathbf{C} = ((\mathcal{C}, \mathcal{X}), \mathbf{R})$ *be a* CS.

(1) The set $T(\mathcal{C}, \mathcal{X}, \bot)$ *is defined by:*

$$t \ ::= \ \bot \mid x \mid C \mid Ap\,(t_1, t_2)$$

(2) The notion of rewriting of Def. 3 extends naturally to terms in $T(\mathcal{C}, \mathcal{X}, \bot)$*, and we will use the same symbol '* $\to_{\mathbf{R}}$ *' to denote the rewriting relation induced by* $\mathbf{C}$ *on* $T(\mathcal{C}, \mathcal{X}, \bot)$.

The relation $\sqsubseteq$ on terms, as given in the following definition, takes $\bot$ to be the smallest term.

Definition 56 *(1) We define the relation* $\sqsubseteq$ *on* $T(\mathcal{C}, \mathcal{X}, \bot)$ *inductively by:*

$$\begin{aligned} \bot \ &\sqsubseteq \ t, \\ t \ &\sqsubseteq \ t, \\ t_1 \sqsubseteq u_1 \ \& \ t_2 \sqsubseteq u_2 \ &\Longleftrightarrow \ t_1 t_2 \sqsubseteq u_1 u_2. \end{aligned}$$

(2) t *and* u *are called* compatible *if there exists a* v *such that* $t \sqsubseteq v$ *and* $u \sqsubseteq v$.

We will now come to the definition of approximate normal forms and of direct approximants. The general idea is that a direct approximant of a term t is constructed out of t by replacing all redexes and potential redexes in t by $\bot$ (a potential redex is a subterm that *could* be a redex if $\bot$ were to be replaced by an appropriate term).

Definition 57 (Approximate normal forms) *Let* $\mathbf{C} = ((\mathcal{C}, \mathcal{X}), \mathbf{R})$ *be a* CS.

(1) $\mathcal{A}_{\mathcal{C}}$*, the set of* approximate normal forms *of* $T(\mathcal{C}, \mathcal{X}, \bot)$*, ranged over by* a*, is defined by:*

$$a \ ::= \ \bot \mid x a_1 \cdots a_n \ (n \geq 0) \mid C a_1 \cdots a_n \ (n < \mathit{arity}(C)).$$

(2) $\mathcal{DA}(t)$, the direct approximant of t *with respect to* **C** *is defined by:*

$$\begin{aligned}
\mathcal{DA}(x) &= x \\
\mathcal{DA}(C) &= C \\
\mathcal{DA}(t_1\,t_2) &= \bot, \textit{ if } \mathcal{DA}(t_1) = \bot \qquad \textit{or} \\
&\qquad\qquad \mathcal{DA}(t_1) = C\,a_1 \cdots a_n, \textit{ and arity}(C) = n{+}1 \\
&= \mathcal{DA}(t_1)\,\mathcal{DA}(t_2), \textit{otherwise}
\end{aligned}$$

Notice that every normal form in $T(\mathcal{C},\mathcal{X})$ *is also an approximate normal form.*

For $\sqsubseteq$, the following properties hold:

Lemma 58 *(1)* $t \sqsubseteq u \sqsubseteq v \;\Rightarrow\; t \sqsubseteq v$.
(2) t *is a head-normal form* $\iff \exists a \in \mathcal{A}_{\mathcal{C}}\,[a \sqsubseteq t \;\&\; a \neq \bot]$.
(3) If $a \in \mathcal{A}_{\mathcal{C}}$ *and* $a \sqsubseteq t$, *then* $a \sqsubseteq \mathcal{DA}(t)$.

PROOF. By induction on the definition of $\sqsubseteq$. ∎

The relation between reduction and $\sqsubseteq$ is expressed by:

Lemma 59 *(1)* $a \in \mathcal{A}_{\mathcal{C}} \;\&\; v \to^* w \;\&\; a \sqsubseteq v \;\Rightarrow\; a \sqsubseteq w$.
(2) $t_0 \sqsubseteq t \;\&\; t_0 \to t_1 \;\Rightarrow\; \exists t'[t \to t' \;\&\; t_1 \sqsubseteq t']$.

PROOF. By induction on the structure of terms. ∎

We will now introduce a notion of 'join' on terms containing $\bot$, that is of use in the proof of Lem. 68.

Definition 60 *On* $T(\mathcal{C},\mathcal{X},\bot)$, *the partial mapping* $\sqcup : T(\mathcal{C},\mathcal{X},\bot) \times T(\mathcal{C},\mathcal{X},\bot) \to T(\mathcal{C},\mathcal{X},\bot)$ *is defined by:*

$$\begin{aligned}
\bot \sqcup t = t \sqcup \bot &= t \\
t \sqcup t &= t \\
(t_1\,t_2) \sqcup (u_1\,u_2) &= (t_1 \sqcup u_1)(t_2 \sqcup u_2)
\end{aligned}$$

The last alternative defines the join on applications in a more general way than that of [15], which would state that $(t_1\,t_2) \sqcup (u_1\,u_2) \sqsubseteq (t_1 \sqcup u_1)(t_2 \sqcup u_2)$, since it is not always sure that a join of two arbitrary terms exists. However, we will use our more general definition only on terms that are compatible, so the conflict is only apparent. So, when we write a term as $v \sqcup u$, we assume v and u to be compatible.

The following lemma shows that $\sqcup$ acts as least upper bound for compatible terms.

Lemma 61 *If $t_1 \sqsubseteq t$ and $t_2 \sqsubseteq t$, then $t_1 \sqcup t_2$ is defined, and: $t_1 \sqsubseteq t_1 \sqcup t_2$, $t_2 \sqsubseteq t_1 \sqcup t_2$, and $t_1 \sqcup t_2 \sqsubseteq t$.*

PROOF. By induction on the structure of terms. ∎

Approximants of terms are defined by:

Definition 62 (Approximants) $\mathcal{A}_{\mathcal{C}}(t) = \{a \in \mathcal{A}_{\mathcal{C}} \mid \exists u\,[t \to^* u \;\&\; a \sqsubseteq u]\}$ *is the* set of approximants of t.

In Section 9, using this definition, we will define a semantics for CS, and we will need the following properties relating approximants and reduction.

Lemma 63 *(1)* $t \to^* t' \;\Rightarrow\; \mathcal{A}_{\mathcal{C}}(t) = \mathcal{A}_{\mathcal{C}}(t')$.
(2) $a, a' \in \mathcal{A}_{\mathcal{C}}(t) \;\Rightarrow\; a \sqcup a' \in \mathcal{A}_{\mathcal{C}}(t)$.

PROOF.

(1) $(\subseteq)$

$$\begin{aligned}
t \to^* t' \;\&\; a \in \mathcal{A}_{\mathcal{C}}(t) \;&\Rightarrow \\
t \to^* t' \;\&\; \exists v\,[t \to^* v \;\&\; a \sqsubseteq v] \;&\Rightarrow\; (\textit{Prop.}\ 5) \\
\exists v, w\,[t \to^* v \;\&\; v \to^* w \;\&\; t' \to^* w \;\&\; a \sqsubseteq v] \;&\Rightarrow\; (\textit{Lem.}\ 59{:}1) \\
\exists w\,[t' \to^* w \;\&\; a \sqsubseteq w] \;&\Rightarrow\; a \in \mathcal{A}_{\mathcal{C}}(t').
\end{aligned}$$

$(\supseteq)$

$$\begin{aligned}
t \to^* t' \;\&\; a \in \mathcal{A}_{\mathcal{C}}(t') \;&\Rightarrow \\
t \to^* t' \;\&\; \exists v\,[t' \to^* v \;\&\; a \sqsubseteq v] \;&\Rightarrow \\
\exists v\,[t \to^* v \;\&\; a \sqsubseteq v] \;&\Rightarrow\; a \in \mathcal{A}_{\mathcal{C}}(t).
\end{aligned}$$

(2)

$$\begin{aligned}
a \in \mathcal{A}_{\mathcal{C}}(t) \;\&\; a' \in \mathcal{A}_{\mathcal{C}}(t) \;&\Rightarrow\; (\textit{Def.}\ 62) \\
\exists u, u'\,[t \to^* u \;\&\; a \sqsubseteq u \;\&\; t \to^* u' \;\&\; a' \sqsubseteq u'] \;&\Rightarrow\; (\textit{Prop.}\ 5 \;\&\; 59{:}1) \\
\exists u, u', v\,[t \to^* u \to^* v \;\&\; t \to^* u' \to^* v \;\&\; a \sqsubseteq v \;\&\; a' \sqsubseteq v] \;&\Rightarrow\; (\textit{Lem.}\ 61) \\
\exists v\,[t \to^* v \;\&\; a \sqcup a' \sqsubseteq v] \;&\Rightarrow\; a \sqcup a' \in \mathcal{A}_{\mathcal{C}}(t).
\end{aligned}$$

Lemma 64 *If $\mathcal{A}_{\mathcal{C}}(t) = \{\bot\}$, then t is unsolvable.*

PROOF. If $\mathcal{A}_{\mathcal{C}}(t) = \{\bot\}$, then, for all v such that $t \to^* v$, and $a \in \mathcal{A}_{\mathcal{C}}$, if $a \sqsubseteq v$, then $a = \bot$. So, in particular, there is no v such that $t \to^* v$ and v is of the shape $x a_1 \cdots a_n$, with $(n \geq 0)$ or $C a_1 \cdots a_n$ with $(n < \textit{arity}(C))$, since otherwise $x \bot \cdots \bot \sqsubseteq v$ or $C \bot \cdots \bot \sqsubseteq v$. Therefore, t does not reduce to a term in head normal form: it is unsolvable. ∎

The following result is crucial for the proof of Lem. 78:

Lemma 65 *Let $t_1, t_2 \in \mathcal{T}(\mathcal{C}, \mathcal{X})$, $a \in \mathcal{A}_{\mathcal{C}}(t_1\, t_2)$, then there exist $a_1 \in \mathcal{A}_{\mathcal{C}}(t_1)$, $a_2 \in \mathcal{A}_{\mathcal{C}}(t_2)$ and u' such that $a_1\, a_2 \to^* u'$ and $a \sqsubseteq u'$.*

PROOF. The case $a = \bot$ is trivial. For $a \neq \bot$: assume $t_1\, t_2 \to^* u$ and $a \sqsubseteq u$, then either:

(1) $u = u_1\, u_2$, and $t_j \to^* u_j$, for $j = 1, 2$. Since $a \sqsubseteq u_1\, u_2$ and $a \neq \bot$, there are a_1, a_2 such that $a = a_1\, a_2$, and $a_j \sqsubseteq u_j$, for $j = 1, 2$. Notice that $a_1\, a_2 \in \mathcal{A}_{\mathcal{C}}$, and take $u' = a$.

(2) There exist $C, p_1, \ldots, p_n$ such that $C\, x_1 \cdots x_n \to r$,

$$t_1\, t_2 \to^* C\, p_1 \cdots p_n \to r^{\vec{p}} \to^* u,$$

and none of the reductions in the first part of this sequence take place at the root position. Since some of the reductions that take place after contracting the redex $C\, p_1 \cdots p_n$ are in fact residuals of redexes already occurring in $p_1, \ldots, p_n$, we can take the reduction sequence that first contracts all redexes (and their residuals) that already occur in $p_1, \ldots, p_n$. Then, since the rewrite system is orthogonal (i.e. rules are left linear and without superpositions), there exists $p'_1, \ldots, p'_n$ and v such that

$$t_1\, t_2 \to^* C\, p_1 \cdots p_n \to^* C\, p'_1 \cdots p'_n \to r^{\vec{p'}} \to^* v \text{ and } u \to^* v$$

and in the reduction sequence $r^{\vec{p'}} \to^* v$ we mimic $r^{\vec{p}} \to^* u$, but only contract redexes that are created *after* the redex $C\, p'_1 \cdots p'_n$ was contracted. Take $a_i = \mathcal{DA}(p'_i)$, for $1 \leq i \leq n$, then the redexes that are erased have no relevance to the sequence $r^{\vec{p'}} \to^* v$; moreover, there is only one redex in $C\, a_1 \cdots a_n$, being that term itself, and both $C\, a_1 \cdots a_{n-1}$ and a_n are in $\mathcal{A}_{\mathcal{C}}$. Notice that $t_1 \to^* C\, p'_1 \cdots p'_{n-1}$, $C\, a_1 \cdots a_{n-1} \sqsubseteq C\, p'_1 \cdots p'_{n-1}$, and $t_2 \to^* p'_n$, $a_n \sqsubseteq p'_n$.

We now focus on the reduction sequence

$$C\, p'_1 \cdots p'_n \to r^{\vec{p'}} \to^* v$$

Notice that, by the construction sketched above, only redexes that are newly created are contracted, and that any redex created in this sequence corresponds to a redex being created for a sequence starting with $C\, a_1 \cdots a_n$, therefore

$$C\, a_1 \cdots a_n \to r^{\vec{a}} \to^* u',$$

and each term created in this reduction is smaller than (in the sense of $\sqsubseteq$) the corresponding term in the reduction sequence above (hence $u' \sqsubseteq v$), and each redex in u' corresponds to a redex in v. Take $a' = \mathcal{DA}(v)$, then $a' \sqsubseteq v$, and all redexes are masked by $\bot$. Since $u' \sqsubseteq v$ by masking all the 'old' redexes, we also have that $a' = \mathcal{DA}(u')$. Since $a \sqsubseteq u$, also $a \sqsubseteq v$ (by Lem. 59:1), and therefore $a \sqsubseteq a'$ (by Lem. 58:3). We then deduce $a \sqsubseteq u'$. ∎

To come to a notion of type assignment on $T(\mathcal{C}, \mathcal{X}, \bot)$, the definition of type assignment as given in Def. 24 and 39 need *not* be changed, it suffices that the terms are allowed to be in $T(\mathcal{C}, \mathcal{X}, \bot)$. In particular, $\mathcal{E}$ does not produce a type for $\bot$; since $\bot \notin \mathcal{C}$, and because of Def. 24, this implies that $\bot$ can only appear in (sub)terms that are typed with ω.

The following property is needed in the proof of Thm. 72:

Lemma 66 *If $B \vdash_{\mathcal{E}} t{:}\sigma$, where B, σ are ω-free, and t is a combinator-free normal form, then t is $\bot$-free.*

PROOF. By induction on t.

$(t = \bot\, t_1 \cdots t_n,\ n \geq 0)$ Impossible, since $\sigma \neq \omega$.
$(t = x\, t_1 \cdots t_n,\ n \geq 0)$ Without loss of generality, we can assume $\sigma \in \mathcal{T}_s$. Then there are $\sigma_1, \ldots, \sigma_n$ such that $B \vdash_{\mathcal{E}} x{:}\sigma_1 \to \cdots \to \sigma_n \to \sigma$, and $B \vdash_{\mathcal{E}} t_i{:}\sigma_i$, for every $1 \leq i \leq n$. Therefore, there are $\sigma'_1, \ldots, \sigma'_{n+1}$ with $x{:}\sigma'_1 \to \ldots \to \sigma'_n \to \sigma'_{n+1} \in B$, all $\sigma'_1, \ldots, \sigma'_{n+1}$ are ω-free, $\sigma_i \leq \sigma'_i$ for $1 \leq i \leq n$, and $\sigma'_{n+1} \leq \sigma$. Then, by Lem. 32:2, $B \vdash_{\mathcal{E}} t_i{:}\sigma'_i$, for $1 \leq i \leq n$. Then, by induction, t_i does not contain $\bot$, for $1 \leq i \leq n$.
$(t = C\, t_1 \cdots t_n)$ Impossible, since t is combinator-free. ∎

In Lem. 68, we will need the following result.

Lemma 67 *(1) If* $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$, $t \sqsubseteq v$, *then there exists* $\mathrm{D}' :: B \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\sigma$, *where the type-derivation* D' *has the same tree-structure as* D *(that is, the same rules are applied).*
(2) If $\mathrm{D} :: B \vdash_{\mathcal{E}} t{:}\sigma$, *and* $t \sqsubseteq v$, *then there exists* $\mathrm{D}' :: B \vdash_{\mathcal{E}} v{:}\sigma$.

PROOF.

(1) By induction on the structure of derivations.
$(\to E)$ $\mathrm{D} = \langle \mathrm{D}_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t_1{:}\rho \to \tau, \mathrm{D}_2 :: B_2 \vdash^{\mathrm{r}}_{\mathcal{E}} t_2{:}\rho, \to E\rangle :: \underline{B} \vdash^{\mathrm{r}}_{\mathcal{E}} t_1\, t_2{:}\tau$, with $\underline{B} = \bigcap\{B_1, B_2\}$. Then there are $v_1 \sqsupseteq t_1, v_2 \sqsupseteq t_2$ such that $v = v_1\, v_2$, and $\mathrm{D}'_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} v_1{:}\rho \to \tau$ and $\mathrm{D}'_2 :: B_2 \vdash^{\mathrm{r}}_{\mathcal{E}} v_2{:}\rho$ by induction. Therefore there exists

$$\langle \mathrm{D}'_1, \mathrm{D}'_2, \to E\rangle :: \bigcap\{B_1, B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} v_1\, v_2{:}\tau,$$

which has the same structure as D.
$(\cap I)$ $\mathrm{D} = \langle \mathrm{D}_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1, \ldots, \mathrm{D}_n :: B_n \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_n, \cap I\rangle :: \underline{B} \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1 \cap \cdots \cap \sigma_n$, with $n \geq 0$, and $\underline{B} = \bigcap\{B_1, \ldots, B_n\}$. Then, by induction, for $1 \leq i \leq n$, $\mathrm{D}_i :: B_i \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\sigma_i$, so also

$$\langle \mathrm{D}_1, \ldots, \mathrm{D}_n, \cap I\rangle :: \bigcap\{B_1, \ldots, B_n\} \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\sigma_1 \cap \cdots \cap \sigma_n.$$

(Ax), $(\mathcal{E})$ Immediate.

Notice that the only interesting case is hidden in the last part: $n = 0$. Then, in particular, t can be $\bot$, so $B = \emptyset$, and v can be any term; remember that $\emptyset \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\omega$ for all v.

(2) If $\mathrm{D} :: B \vdash_{\mathcal{E}} t{:}\sigma$, then, by Lem. 40:2, there is $B' \geq B$ such that $\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$. Since $t \sqsubseteq v$, by the first part also $\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\sigma$. Then also $\mathrm{D}' :: B \vdash_{\mathcal{E}} v{:}\sigma$. ■

8 Approximation and normalization

In this section we will give the proofs for the approximation and normalisation results.

We will need the following intermediate result.

Lemma 68 *Let* $\mathbf{C} = ((\mathcal{C},\mathcal{X}),\mathbf{R})$ *be a* CS*, then, for all* $t \in T(\mathcal{C},\mathcal{X})$*: if* $\mathrm{D} :: B \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$ *is in normal form with respect to* $\to_{\mathcal{D}}$*, then there exists an* $a \in \mathcal{A}_{\mathcal{C}}$ *and* D' *such that* $a \sqsubseteq t$ *and* $\mathrm{D}' :: B \vdash^{\mathrm{r}}_{\mathcal{E}} a{:}\sigma$.

PROOF. By induction on the structure of derivations.

$(\to E)$ Then $\mathrm{D} =$

$$\langle \mathrm{D}_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t_1{:}\tau\to\sigma, \mathrm{D}_2 :: B_2 \vdash^{\mathrm{r}}_{\mathcal{E}} t_2{:}\tau, \to E\rangle :: \bigcap\{B_1, B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} t_1 t_2{:}\sigma.$$

Then, by induction, there are $a_1 \sqsubseteq t_1, a_2 \sqsubseteq t_2$ such that $\mathrm{D}'_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} a_1{:}\tau\to\sigma$, and $\mathrm{D}'_2 :: B_2 \vdash^{\mathrm{r}}_{\mathcal{E}} a_2{:}\tau$, and

$$\langle \mathrm{D}'_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} a_1{:}\tau\to\sigma, \mathrm{D}'_2 :: B_2 \vdash^{\mathrm{r}}_{\mathcal{E}} a_2{:}\tau, \to E\rangle :: \bigcap\{B_1, B_2\} \vdash^{\mathrm{r}}_{\mathcal{E}} a_1 a_2{:}\sigma.$$

By Def. 56 we know that $a_1 a_2 \sqsubseteq t_1 t_2$.

Now $a_1 a_2 \notin \mathcal{A}_{\mathcal{C}}$ if there is a $C \in \mathcal{C}$ such that $a_1 = C\, a_1^1 \cdots a_1^{n-1}$ and $\mathit{arity}(C) = n$. But then there are $t_1^1, \ldots, t_1^{n-1}$ with $t_1 = C\, t_1^1 \cdots t_1^{n-1}$, and $t = C\, t_1^1 \cdots t_1^{n-1} t_2$. In particular, by the remark after Def. 47, D is reducible, which is impossible. So $a_1 a_2 \in \mathcal{A}_{\mathcal{C}}$.

$(\cap I)$ $\mathrm{D} = \langle \mathrm{D}_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1, \ldots, \mathrm{D}_n :: B_n \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_n, \cap I\rangle :: \underline{B} \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma_1 \cap \cdots \cap \sigma_n$, with $\underline{B} = \bigcap\{B_1, \ldots, B_n\}$. By induction, for $1 \leq i \leq n$, there is an $a_i \sqsubseteq t$ in $\mathcal{A}_{\mathcal{C}}$ such that $\mathrm{D}'_i :: B_i \vdash^{\mathrm{r}}_{\mathcal{E}} a_i{:}\sigma_i$. Take $a = a_1 \sqcup \cdots \sqcup a_n$. Since, for $1 \leq i \leq n$, $a_i \sqsubseteq a$, by Lem. 67 also $\mathrm{D}''_i :: B_i \vdash^{\mathrm{r}}_{\mathcal{E}} a{:}\sigma_i$, so we get

$$\langle \mathrm{D}''_1 :: B_1 \vdash^{\mathrm{r}}_{\mathcal{E}} a{:}\sigma_1, \ldots, \mathrm{D}''_n :: B_n \vdash^{\mathrm{r}}_{\mathcal{E}} a{:}\sigma_n, \cap I\rangle :: \underline{B} \vdash^{\mathrm{r}}_{\mathcal{E}} a{:}\sigma_1 \cap \cdots \cap \sigma_n.$$

Since $a_i \sqsubseteq t$ for all $1 \leq i \leq n$, by Lem. 61 also $a \sqsubseteq t$. Notice that if $n = 0$, then $a = \bot$.

The cases $(\mathcal{E})$ and (*Ax*) are immediate. ∎

Theorem 69 (Approximation) *Let* $\mathbf{C} = ((\mathcal{C},\mathcal{X}),\mathbf{R})$ *be a* CS*, then: if* $B \vdash_{\mathcal{E}} t{:}\sigma$*, then there exists an* $a \in \mathcal{A}_{\mathcal{C}}(t)$ *such that* $B \vdash_{\mathcal{E}} a{:}\sigma$*, for all* $t \in T(\mathcal{C},\mathcal{X})$*.*

PROOF. By Lem. 40:2, for D such that $\mathsf{D} :: B \vdash_{\mathcal{E}} t{:}\sigma$, there are D' and B' such that $\mathsf{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$, and $B \leq B'$. Then, by Thm. 54, $SN(\mathsf{D}')$. Let $\mathsf{D}'' :: B'' \vdash^{\mathrm{r}}_{\mathcal{E}} v{:}\sigma$ be a normal form of D' with respect to $\rightarrow_{\mathcal{D}}$. Then by Lem. 68, there is an $a \in \mathcal{A}_{\mathcal{C}}$ such that $a \sqsubseteq v$ and $\mathsf{D}''' :: B'' \vdash^{\mathrm{r}}_{\mathcal{E}} a{:}\sigma$. Then, by Lem. 49, $B' \leq B''$, and $t \rightarrow^* v$, therefore $a \in \mathcal{A}_{\mathcal{C}}(t)$. Also, by Lem. 40:1 and Lem. 32:1, $B \vdash_{\mathcal{E}} a{:}\sigma$. ∎

For principal environments we can show that the converse of this result also holds.

Theorem 70 *Let* $\mathbf{C} = ((\mathcal{C},\mathcal{X}),\mathbf{R})$ *be a* CS*, and* $\mathcal{E}$ *be principal for* $\mathbf{C}$*, then, for all* $t \in T(\mathcal{C},\mathcal{X})$*: if there is an* $a \in \mathcal{A}_{\mathcal{C}}(t)$ *such that* $B \vdash_{\mathcal{E}} a{:}\sigma$*, then* $B \vdash_{\mathcal{E}} t{:}\sigma$*.*

PROOF. If $a \in \mathcal{A}_{\mathcal{C}}(t)$ such that $B \vdash_{\mathcal{E}} a{:}\sigma$, then there exists a v such that $t \rightarrow^* v$ and $a \sqsubseteq v$. But then, by Lem. 67, also $B \vdash_{\mathcal{E}} v{:}\sigma$. Since $\mathcal{E}$ is principal for $\mathbf{C}$, by Thm. 38, also $B \vdash_{\mathcal{E}} t{:}\sigma$. ∎

Theorem 71 (Head-normalisation) *Let* $t \in T(\mathcal{C},\mathcal{X})$*. If* $B \vdash_{\mathcal{E}} t{:}\sigma$*, and* $\sigma \neq \omega$*, then* t *has a head-normal form.*

PROOF. If $B \vdash_{\mathcal{E}} t{:}\sigma$, then by Thm. 69, there is an $a \in \mathcal{A}_{\mathcal{C}}(t)$ such that $B \vdash_{\mathcal{E}} a{:}\sigma$. Since $\sigma \neq \omega$, $a \neq \bot$, and since $a \in \mathcal{A}_{\mathcal{C}}$, there are x or C, and terms $a_1, \ldots, a_n$ such that $a = x a_1 \cdots a_n$, or $a = C a_1 \cdots a_n$ with $arity(C) < n$. Also, since $a \in \mathcal{A}_{\mathcal{C}}(t)$, there is a v such that $t \rightarrow^* v$ and $a \sqsubseteq v$. Since $a \sqsubseteq v$, there are $t_1, \ldots, t_n$ such that either $v = x t_1 \cdots t_n$, or $v = C t_1 \cdots t_n$, with $arity(C) < n$. But then v *is in* head-normal form, so t *has* a head-normal form. ∎

The combinatorial equivalent of another well-known result for intersection type assignment in the LC, i.e. the property

If $B \vdash_{\mathcal{E}} t{:}\sigma$*, and* B, σ *are* ω*-free, then* t *has a normal form*

no longer holds. Take for example the CS

$$\begin{array}{lcll} \mathsf{Z}xy & \rightarrow & y & \mathcal{E}(\mathsf{Z}) = \omega{\rightarrow}\varphi_1{\rightarrow}\varphi_1, \\ \mathsf{D}x & \rightarrow & xx & \mathcal{E}(\mathsf{D}) = ((\varphi_2{\rightarrow}\varphi_3){\cap}\varphi_2){\rightarrow}\varphi_3 \end{array}$$

then $\mathsf{Z}(\mathsf{DD})$ is typeable with a type not containing ω, but the term $\mathsf{Z}(\mathsf{DD})$ has no normal form. However, we can prove this result for the class of typeable non-Curryfied terms.

Theorem 72 (Normalisation) *Let $t \in T_{NC}(\mathcal{C},\mathcal{X})$. If $B \vdash_{\mathcal{E}} t{:}\sigma$, and B, σ are ω-free, then t has a normal form.*

PROOF. By Thm. 69, there is an $a \in \mathcal{A}_{\mathcal{C}}(t)$ such that $B \vdash_{\mathcal{E}} a{:}\sigma$. Notice that if $t \in T_{NC}(\mathcal{C},\mathcal{X})$, and t' is a reduct of t then also $t' \in T_{NC}(\mathcal{C},\mathcal{X})$. Therefore, a cannot contain any $C \in \mathcal{C}$. Then $a = x\,a_1 \cdots a_n$, where each a_i contains only variables and possibly $\bot$. But, by Lem. 66, a does not contain $\bot$. Now, since $a \in \mathcal{A}_{\mathcal{C}}(t)$, there exists $v \in T(\mathcal{C},\mathcal{X})$ such that $t \to^* v$ and $a \sqsubseteq v$. Since a does not contain $\bot$, $v = a$, and since a is in normal form, t has a normal form. ■

We will now show that, using Thm. 54, all terms typeable in the subsystem of $\vdash_{\mathcal{E}}$ that does not use ω ($\vdash^{\omega}_{\mathcal{E}}$), are strongly normalizable.

Lemma 73 *(1) If* D *is a derivation in* $\vdash^{\omega}_{\mathcal{E}}$*, and* $\mathrm{D} \to_{\mathcal{D}} \mathrm{D}'$*, then also* D' *is a derivation in* $\vdash^{\omega}_{\mathcal{E}}$*.*
(2) $\mathrm{D} :: B \vdash^{\omega}_{\mathcal{E}} t{:}\sigma \to_{\mathcal{D}} \mathrm{D}' :: B' \vdash^{\omega}_{\mathcal{E}} t'{:}\sigma$*, if and only if* $t \to t'$*.*

PROOF. By Def. 47, and Lem. 40:3. ■

Thus, in the type system $\vdash^{\omega}_{\mathcal{E}}$, $\to_{\mathcal{D}}$ mimics $\to$ and vice-versa. This observation immediately leads to the following result.

Theorem 74 *Let $t \in T(\mathcal{C},\mathcal{X})$. If $B \vdash^{\omega}_{\mathcal{E}} t{:}\sigma$, then t is strongly normalizable.*

PROOF. Let D be such that $\mathrm{D} :: B \vdash^{\omega}_{\mathcal{E}} t{:}\sigma$. Since also $\mathrm{D} :: B \vdash_{\mathcal{E}} t{:}\sigma$, by Lem. 40:3, there are D', B' such that $B \leq B'$, and $\mathrm{D}' :: B' \vdash^{\mathrm{r}}_{\mathcal{E}} t{:}\sigma$ without using ω. By Thm. 54, D' is strongly normalizable with respect to $\to_{\mathcal{D}}$. By Lem. 73:2, all derivation redexes in D' correspond to redexes in t and vice-versa, a property that is preserved under reduction. So also t is strongly normalizable. ■

It is worthwhile to notice that, unlike for LC with $\vdash_{\lambda\cap}$, the reverse implication of the three theorems does not hold in general. For this, it is sufficient to note that a subject expansion theorem does not hold (see also the last remark of Section 3).

Another aspect worth noting is that, unlike in LC, no longer every term in normal form is typeable without ω in basis and type. Take for example

$$t = \mathsf{S}(\mathsf{K}(\mathsf{SII}))(\mathsf{K}(\mathsf{SII})),$$

and note that, by Property 30 every type assignable to t (regardless of the environment used) is a type assignable to $\lambda y.(\lambda x.xx)(\lambda x.xx)$ in $\vdash_{\lambda\cap}$. Since this last term has no head-normal form, only ω can be assigned to it.

9 Semantics

In this section, we will define two semantics for CS. The first is a filter model, where terms will be interpreted by the set of their assignable types; the second an approximation model, where terms will be interpreted by the set of their approximants.

Definition 75 (Filters) *(1) A subset d of $\mathcal{T}$ is a* filter *if and only if:*
(a) If $\sigma_1, \ldots, \sigma_n \in d$ $(n \geq 0)$, then $\sigma_1 \cap \cdots \cap \sigma_n \in d$.
(b) If $\sigma \in d$ and $\sigma \leq \tau$, then $\tau \in d$.
(2) If V is a subset of $\mathcal{T}$, then $\uparrow V$ is the smallest filter that contains V, and $\uparrow\sigma = \uparrow\{\sigma\}$.
(3) $\mathcal{F} = \{d \subseteq \mathcal{T} \mid d$ is a filter$\}$.

Notice that a filter is never empty, since by part 1a, for all d, $\omega \in d$. $\langle \mathcal{F}, \subseteq \rangle$ is a cpo and henceforward it will be considered with the corresponding Scott topology.

Definition 76 *(1) Application on $\wp\mathcal{A}_\mathcal{C}$, $\cdot : \wp\mathcal{A}_\mathcal{C} \times \wp\mathcal{A}_\mathcal{C} \to \wp\mathcal{A}_\mathcal{C}$, is defined by:*

$$A_1 \cdot A_2 \;=\; \{a \in \mathcal{A}_\mathcal{C} \mid \exists a_1 \in A_1, a_2 \in A_2, u\, [a_1 a_2 \to^* u \;\&\; a \sqsubseteq u]\}.$$

(2) Application on $\mathcal{F}$, $\cdot : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, is defined by:

$$d \cdot e \;=\; \uparrow\{\sigma \mid \exists\, \tau \in e\, [\tau \to \sigma \in d]\}.$$

We will define two interpretations of terms:

Definition 77 *(1) The interpretation of terms in the domain of approximants over $\mathcal{C}$ is defined as: $[\![t]\!]^{\mathcal{A}}_{\mathcal{C}} = \mathcal{A}_\mathcal{C}(t) = \{a \in \mathcal{A}_\mathcal{C} \mid \exists u\, [t \to^* u \;\&\; a \sqsubseteq u]\}$.*
(2) Let ξ be a valuation of term variables in $\mathcal{F}$; we write $\xi \models B$ if and only if, for all $x{:}\sigma \in B$, $\sigma \in \xi(x)$. $[\![t]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$, the interpretation of terms in $\mathcal{F}$ via ξ and $\mathcal{E}$ is defined by: $[\![t]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} = \{\sigma \mid \exists B\, [\xi \models B \;\&\; B \vdash_\mathcal{E} t{:}\sigma]\}$.

Notice that, by rule $(\cap I)$ and Thm. 33, $\{\sigma \mid \exists B\, [B \vdash_\mathcal{E} t{:}\sigma]\} \in \mathcal{F}$.

Both applications are well-defined, in the sense that they respect application on terms.

Lemma 78 *(1)* $[\![t_1]\!]^{\mathcal{A}}_{\mathcal{C}} \cdot [\![t_2]\!]^{\mathcal{A}}_{\mathcal{C}} = [\![t_1\, t_2]\!]^{\mathcal{A}}_{\mathcal{C}}$.
(2) $[\![t_1]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} \cdot [\![t_2]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} = [\![t_1\, t_2]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$.

PROOF.

$$
\begin{array}{lll}
(1)\ (\subseteq) & [\![t_1]\!]^{\mathcal{A}}_{\mathcal{C}} \cdot [\![t_2]\!]^{\mathcal{A}}_{\mathcal{C}} & = \ (\textit{Def.}\ 76{:}1) \\
\{a \in \mathcal{A}_{\mathcal{C}} \mid \exists a_1 \in [\![t_1]\!]^{\mathcal{A}}_{\mathcal{C}}, a_2 \in [\![t_2]\!]^{\mathcal{A}}_{\mathcal{C}}, u[a_1\, a_2 \to^* u \ \&\ a \sqsubseteq u]\} & & = \\
\{a \in \mathcal{A}_{\mathcal{C}} \mid \exists a_1, a_2 \in \mathcal{A}_{\mathcal{C}}, u\ [\exists u_1\ [t_1 \to^* u_1 \ \&\ a_1 \sqsubseteq u_1]\ \& & & \\
\qquad \exists u_2\ [t_2 \to^* u_2 \ \&\ a_2 \sqsubseteq u_2]\ \&\ a_1\, a_2 \to^* u \ \&\ a \sqsubseteq u]\} & & \subseteq \ (\textit{Lem.}\ 59{:}2) \\
\{a \in \mathcal{A}_{\mathcal{C}} \mid \exists u\ [t_1\, t_2 \to^* u \ \&\ a \sqsubseteq u]\} & & = \ [\![t_1\, t_2]\!]^{\mathcal{A}}_{\mathcal{C}} \\
(\supseteq) \quad [\![t_1\, t_2]\!]^{\mathcal{A}}_{\mathcal{C}} = \{a \in \mathcal{A}_{\mathcal{C}} \mid \exists u\ [t_1\, t_2 \to^* u \ \&\ a \sqsubseteq u]\} & & \subseteq \ (\textit{Lem.}\ 65) \\
\{a \in \mathcal{A}_{\mathcal{C}} \mid \exists a_1 \in [\![t_1]\!]^{\mathcal{A}}_{\mathcal{C}}, a_2 \in [\![t_2]\!]^{\mathcal{A}}_{\mathcal{C}}, u[a_1\, a_2 \to^* u \ \&\ a \sqsubseteq u]\} & & = \ [\![t_1]\!]^{\mathcal{A}}_{\mathcal{C}} \cdot [\![t_2]\!]^{\mathcal{A}}_{\mathcal{C}} \\
(2) & [\![t_1]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} \cdot [\![t_2]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} & = \\
& \uparrow\{\sigma \mid \exists\ \tau \in [\![t_2]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}\ [\tau{\to}\sigma \in [\![t_1]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}]\} & = \\
\uparrow\{\sigma \mid \exists \tau\ [\exists B_1\ [B_1 \vdash_{\mathcal{E}} t_1{:}\tau{\to}\sigma] \ \&\ \exists B_2\ [B_2 \vdash_{\mathcal{E}} t_2{:}\tau]]\} & & = \ (\subseteq\colon B = \bigcap\{B_1, B_2\}) \\
& \uparrow\{\sigma \mid \exists \tau, B\ [B \vdash_{\mathcal{E}} t_1{:}\tau{\to}\sigma \ \&\ B \vdash_{\mathcal{E}} t_2{:}\tau]\} & = \\
& \uparrow\{\sigma \mid \exists B\ [B \vdash_{\mathcal{E}} t_1\, t_2{:}\sigma]\} & = \\
& \{\sigma \mid \exists B\ [B \vdash_{\mathcal{E}} t_1\, t_2{:}\sigma]\} & = \ [\![t_1\, t_2]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} \qquad \blacksquare
\end{array}
$$

As seen above in Lem. 63:1, if $t \to^* t'$, then $\mathcal{A}_{\mathcal{C}}(t) = \mathcal{A}_{\mathcal{C}}(t')$, which implies that, at least, if $t \to^* t'$, then $[\![t]\!]^{\mathcal{A}}_{\mathcal{C}} = [\![t']\!]^{\mathcal{A}}_{\mathcal{C}}$. The converse does not hold, since unsolvable terms that are not in $\to^*$, still have the same image under $[\![\ ]\!]^{\mathcal{A}}_{\mathcal{C}}$, namely $\bot$. We now formalize these properties.

The relation $=_{\mathbf{R}}$ is the reflexive, symmetric and transitive closure of $\to_{\mathbf{R}}$:

Definition 79 *Let* $((\mathcal{C},\mathcal{X}),\mathbf{R})$ *be a* CS. *We define the equivalence relation* $=_{\mathbf{R}} \subseteq T(\mathcal{C},\mathcal{X}) \times T(\mathcal{C},\mathcal{X})$ *by:*

$$
\begin{array}{rcl}
t \to^*_{\mathbf{R}} v & \Rightarrow & t =_{\mathbf{R}} v \\
t =_{\mathbf{R}} v & \Rightarrow & v =_{\mathbf{R}} t \\
t =_{\mathbf{R}} v \ \&\ v =_{\mathbf{R}} w & \Rightarrow & t =_{\mathbf{R}} w
\end{array}
$$

Lemma 80 *If* $t =_{\mathbf{R}} v$, *then there exists* u *such that* $t \to^*_{\mathbf{R}} u$ *and* $v \to^*_{\mathbf{R}} u$.

PROOF. By induction on the definition of $=_{\mathbf{R}}$. If $t =_{\mathbf{R}} v \ \&\ v =_{\mathbf{R}} w \ \Rightarrow\ t =_{\mathbf{R}} w$, then, by induction there are u_1 and u_2 such that $t \to^*_{\mathbf{R}} u_1$ and $v \to^*_{\mathbf{R}} u_1$, and $v \to^*_{\mathbf{R}} u_2$ and $w \to^*_{\mathbf{R}} u_2$. Since $v \to^*_{\mathbf{R}} u_1$ and $v \to^*_{\mathbf{R}} u_2$, by Property 5, there exist a u_3 such

that $u_1 \to^*_{\mathbf{R}} u_3$ and $u_2 \to^*_{\mathbf{R}} u_3$. But then, in particular, $t \to^*_{\mathbf{R}} u_3$ and $w \to^*_{\mathbf{R}} u_3$. The other cases are straightforward. ■

The approximant semantics is adequate, in that it equates terms that are equal in the theory **R**.

Theorem 81 (Adequacy of the Approximation Model) *If* $t =_{\mathbf{R}} v$*, then* $[\![t]\!]^{\mathcal{A}}_{\mathcal{C}} = [\![v]\!]^{\mathcal{A}}_{\mathcal{C}}$.

PROOF. Consequence of Lem. 80 and 63:1. ■

The converse of this result, '*If* $[\![t]\!]^{\mathcal{A}}_{\mathcal{C}} = [\![v]\!]^{\mathcal{A}}_{\mathcal{C}}$*, then* $t =_{\mathbf{R}} v$' does not hold.

Example 82 *Take the* CS

$$\begin{array}{lcl} \mathsf{S}xyz & \to & xz(yz) \\ \mathsf{K}xy & \to & x \\ \mathsf{D}x & \to & xx \\ \mathsf{W}x & \to & xxx \end{array}$$

Notice that $\mathsf{SK(DD)}$ *and* $\mathsf{SK(WW)}$ *both have only one redex, and that this property is preserved under reduction. Then*

$$\mathsf{SK(DD)} \to \mathsf{SK(DD)} \to \mathsf{SK(DD)} \to \cdots$$

and

$$\mathsf{SK(WW)} \to \mathsf{SK(WWW)} \to \mathsf{SK(WWWW)} \to \cdots,$$

so

$$[\![\mathsf{SK(DD)}]\!]^{\mathcal{A}}_{\mathcal{C}} = \{\bot, \mathsf{S}\bot\bot, \mathsf{SK}\bot\} = [\![\mathsf{SK(WW)}]\!]^{\mathcal{A}}_{\mathcal{C}},$$

but there is no u *such that both* $\mathsf{SK(DD)} \to^* u$ *and* $\mathsf{SK(WW)} \to^* u$.

We could modify the relation $=_{\mathbf{R}}$ to identify all unsolvable terms, so as to obtain $\mathsf{SK(DD)} \approx_{\mathbf{R}} \mathsf{SK(WW)}$ (this is used also for LC).

Definition 83 *Let* $((\mathcal{C}, \mathcal{X}), \mathbf{R})$ *be a* CS*. We define the equivalence relation* $\approx_{\mathbf{R}} \subseteq$

$T(\mathcal{C},\mathcal{X}) \times T(\mathcal{C},\mathcal{X})$ *by:*

$$\begin{array}{rcl} t \to_{\mathbf{R}}^{*} v & \Rightarrow & t \approx_{\mathbf{R}} v \\ t, v \text{ are unsolvable} & \Rightarrow & t \approx_{\mathbf{R}} v \\ t \approx_{\mathbf{R}} v & \Rightarrow & v \approx_{\mathbf{R}} t \\ t \approx_{\mathbf{R}} v \;\&\; v \approx_{\mathbf{R}} w & \Rightarrow & t \approx_{\mathbf{R}} w \\ t \approx_{\mathbf{R}} v & \Rightarrow & wt \approx_{\mathbf{R}} wv \;\&\; tw \approx_{\mathbf{R}} vw \end{array}$$

Notice that $\mathsf{SK}(\mathsf{DD}) \approx_{\mathbf{R}} \mathsf{SK}(\mathsf{WW})$.

Theorem 84 *If* $t \approx_{\mathbf{R}} v$*, then* $[\![t]\!]_{\mathcal{C}}^{\mathcal{A}} = [\![v]\!]_{\mathcal{C}}^{\mathcal{A}}$.

PROOF. By induction on the definition of $\approx_{\mathbf{R}}$. The case $t \to_{\mathbf{R}}^{*} v$ follows from Lem. 63:1. If t, v are unsolvable, then $[\![t]\!]_{\mathcal{C}}^{\mathcal{A}} = \{\bot\} = [\![v]\!]_{\mathcal{C}}^{\mathcal{A}}$. The last case is a consequence of Lem. 78. The other two cases follow by induction. ■

Although, by $\approx_{\mathbf{R}}$, terms are equated that are unsolvable, still we do not get a full-abstraction result, since it can be that solvable terms have the same infinite set of approximants, whilst sharing no terms during reduction.

Example 85 *Take*

$$\begin{array}{rcl} \mathsf{T}xy & \to & y(xxy) \\ \mathsf{Y}xy & \to & y(xy(xy)) \\ \mathsf{X}xy & \to & x(yy) \end{array}$$

Then we have the following reduction sequences:

$$\begin{array}{rcl} \mathsf{YX}z & \to & z(\mathsf{X}z(\mathsf{X}z)) \\ & \to & z(z(\mathsf{X}z(\mathsf{X}z))) \\ & \to & z(z(z(\mathsf{X}z(\mathsf{X}z)))) \\ & \cdots & \\ & \to & z(z(z(z(z(z\cdots))))) \end{array} \qquad \begin{array}{rcl} \mathsf{TT}z & \to & z(\mathsf{TT}z) \\ & \to & z(z(\mathsf{TT}z)) \\ & \to & z(z(z(\mathsf{TT}z))) \\ & \cdots & \\ & \to & z(z(z(z(z(z\cdots))))) \end{array}$$

In particular,

$$[\![\mathsf{YX}z]\!]_{\mathcal{C}}^{\mathcal{A}} = \{\bot, z\bot, z(z\bot), z(z(z\bot)), \ldots\} = [\![\mathsf{TT}z]\!]_{\mathcal{C}}^{\mathcal{A}},$$

but not $\mathsf{YX}z \approx_{\mathbf{R}} \mathsf{TT}z$.

We can obtain a full-abstraction result for the approximation semantics using the following relation:

Definition 86 *Let $((\mathcal{C},\mathcal{X}),\mathbf{R})$ be a* CS. *The relation $\approx_{\mathbf{R}}^{hnf}$ is defined coinductively as follows: $t \approx_{\mathbf{R}}^{hnf} u$ if and only if either*

(1) t and u are both unsolvable, or
(2) if $C\,t_1 \cdots t_n$ is a head normal form of t (resp. u), then there is a head normal form $C\,u_1 \cdots u_n$ of u (resp. t) such that, for $1 \leq i \leq n$, $t_i \approx_{\mathbf{R}}^{hnf} u_i$, or
(3) if $x\,t_1 \cdots t_n$ is a head normal form of t (resp. u), then there is a head normal form $x\,u_1 \cdots u_n$ of u (resp. t) such that, for $1 \leq i \leq n$, $t_i \approx_{\mathbf{R}}^{hnf} u_i$.

Theorem 87 (Full Abstraction of the Approximation Model)
$t \approx_{\mathbf{R}}^{hnf} u$ if and only if $[\![t]\!]_{\mathcal{C}}^{\mathcal{A}} = [\![u]\!]_{\mathcal{C}}^{\mathcal{A}}$.

PROOF.

(if) By coinduction. It is sufficient to show that if $[\![t]\!]_{\mathcal{C}}^{\mathcal{A}} = [\![u]\!]_{\mathcal{C}}^{\mathcal{A}}$ then either
(1) t, u are unsolvable, or
(2) if $C\,t_1 \cdots t_n$ is a head normal form of t (resp. u), then $C\,u_1 \cdots u_n$ is a head normal form of u (resp. t), and $[\![t_i]\!]_{\mathcal{C}}^{\mathcal{A}} = [\![u_i]\!]_{\mathcal{C}}^{\mathcal{A}}$ for $1 \leq i \leq n$, or
(3) if $x\,t_1 \cdots t_n$ is a head normal form of t (resp. u), then $x\,u_1 \cdots u_n$ is a head normal form of u (resp. t), and $[\![t_i]\!]_{\mathcal{C}}^{\mathcal{A}} = [\![u_i]\!]_{\mathcal{C}}^{\mathcal{A}}$ for $1 \leq i \leq n$.

This is a straightforward consequence of the fact that u and t have the same set of approximants.

(only if) We take $a \in [\![t]\!]_{\mathcal{C}}^{\mathcal{A}}$ and show $a \in [\![u]\!]_{\mathcal{C}}^{\mathcal{A}}$ by induction on the depth of a.
$(a = \bot)$ Trivial.
$(a = Ca_1 \ldots a_n)$ Then t has a head normal form $C\,t_1 \cdots t_n$, and therefore u has a head normal form $C\,u_1 \cdots u_n$ such that $t_i \approx_{\mathbf{R}}^{hnf} u_i$ for $1 \leq i \leq n$. Since $a_i \in [\![t_i]\!]_{\mathcal{C}}^{\mathcal{A}}$ and its depth is smaller than that of a, by induction we conclude that $a_i \in [\![u_i]\!]_{\mathcal{C}}^{\mathcal{A}}$. Therefore $a \in [\![u]\!]_{\mathcal{C}}^{\mathcal{A}}$.
$(a = xa_1 \ldots a_n)$ Similar. ∎

The filter semantics gives a semi-model with respect to $\rightarrow_{\mathbf{R}}$, as the following theorem shows.

Theorem 88 *If $t \rightarrow_{\mathbf{R}}^{*} v$, then $[\![t]\!]_{\xi,\mathcal{E}}^{\mathcal{F}} \subseteq [\![v]\!]_{\xi,\mathcal{E}}^{\mathcal{F}}$.*

PROOF. Take $\sigma \in [\![t]\!]_{\xi,\mathcal{E}}^{\mathcal{F}}$. Then $\exists B\ [B \vdash_{\mathcal{E}} t{:}\sigma]$, and, since $t \rightarrow_{\mathbf{R}}^{*} v$, by Thm. 37, also $\exists B\ [B \vdash_{\mathcal{E}} v{:}\sigma]$, so $\sigma \in [\![v]\!]_{\xi,\mathcal{E}}^{\mathcal{F}}$. ∎

In view of the fact that type assignment in $\vdash_{\mathcal{E}}$ is not closed for subject-expansion (see the remark at the end of Section 3), it is, in general, not possible to show a stronger result like '*If* $t =_{\mathbf{R}} v$*, then* $[\![t]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} = [\![v]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$'. However, when using a principal environment, this result holds.

Theorem 89 (Adequacy of the Filter Model) *Let* $\mathbf{C} = ((\mathcal{C},\mathcal{X}),\mathbf{R})$ *be a* CS*, and* $\mathcal{E}$ *be principal for* $\mathbf{C}$*, then* $t =_{\mathbf{R}} v$ *implies* $[\![t]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} = [\![v]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$.

PROOF. By Thm. 37 and 38. ∎

We even have the following result easily.

Theorem 90 *Let* $\mathbf{C} = ((\mathcal{C},\mathcal{X}),\mathbf{R})$ *be a* CS*, and* $\mathcal{E}$ *be principal for* $\mathbf{C}$*, then, for all* $t, v \in T(\mathcal{C},\mathcal{X})$*:* $t \approx_{\mathbf{R}} v$ *implies* $[\![t]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} = [\![v]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$.

PROOF. By induction on the definition of $\approx_{\mathbf{R}}$. The case $t \to^*_{\mathbf{R}} v$ is covered by Thm. 37 and 38. If t, v are unsolvable, then by Thm. 71, $[\![t]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} = \{\omega\} = [\![v]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$. The last case is a consequence of Lem. 78. The other two cases follow by straightforward induction. ∎

The converse of these results do not hold.

Example 91 *Take* $\mathsf{T}, \mathsf{Y}, \mathsf{X}$ *as in Exp. 85, and let*

$$\begin{aligned}
\mathcal{E}(\mathsf{T}) &= ((\varphi_1\to\varphi_2\to\varphi_3)\cap\varphi_1)\to((\varphi_3\to\varphi_4)\cap\varphi_2)\to\varphi_4,\\
\mathcal{E}(\mathsf{Y}) &= ((\varphi_3\to\varphi_5\to\varphi_1)\cap(\varphi_4\to\varphi_5))\to((\varphi_1\to\varphi_2)\cap\varphi_3\cap\varphi_4)\to\varphi_2,\\
\mathcal{E}(\mathsf{X}) &= (\varphi_1\to\varphi_2)\to((\varphi_3\to\varphi_1)\cap\varphi_3)\to\varphi_2,
\end{aligned}$$

then

$$\begin{aligned}
[\![\mathsf{YX}]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} = \{&\omega, (\omega\to\varphi_1)\to\varphi_1, ((\omega\to\varphi_1)\cap(\varphi_1\to\varphi_2))\to\varphi_2,\\
&((\omega\to\varphi_1)\cap(\varphi_1\to\varphi_2)\cap(\varphi_2\to\varphi_3))\to\varphi_3,\ldots\} = [\![\mathsf{TT}]\!]^{\mathcal{F}},
\end{aligned}$$

(notice that these types correspond directly to the approximants of Exp. 85) but neither $\mathsf{YX} =_{\mathbf{R}} \mathsf{TT}$*, nor* $\mathsf{YX} \approx_{\mathbf{R}} \mathsf{TT}$.

For the filter semantics, we have, as can be expected:

Theorem 92 *Let* $\mathbf{C} = ((\mathcal{C},\mathcal{X}),\mathbf{R})$ *be a* CS*, and* $\mathcal{E}$ *be principal for* $\mathbf{C}$*, then, for all* $t, u \in T(\mathcal{C},\mathcal{X})$*:* $t \approx^{hnf}_{\mathbf{R}} u$ *implies* $[\![t]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} = [\![u]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$.

PROOF. If $t \approx^{hnf}_{\mathbf{R}} u$, then, by Thm. 87, $[\![t]\!]^{\mathcal{A}}_{\mathcal{C}} = [\![u]\!]^{\mathcal{A}}_{\mathcal{C}}$. Let $\sigma \in [\![t]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$ (the other case is similar), then there exists a B such that $B \vdash_{\mathcal{E}} t{:}\sigma$. Then, by Thm. 69, there exists an $a \in \mathcal{A}_{\mathcal{C}}(t)$ such that $B \vdash_{\mathcal{E}} a{:}\sigma$. Since $\mathcal{A}_{\mathcal{C}}(t) = [\![t]\!]^{\mathcal{A}}_{\mathcal{C}} = [\![u]\!]^{\mathcal{A}}_{\mathcal{C}} = \mathcal{A}_{\mathcal{C}}(u)$, $a \in \mathcal{A}_{\mathcal{C}}(u)$, and by Thm. 70, $B \vdash_{\mathcal{E}} u{:}\sigma$, so $\sigma \in [\![u]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$. ■

Perhaps surprisingly (at least for LC, the approximation and the filter semantics coincide [19,3]), we do not have a full-abstraction result with respect to filter semantics.

Example 93 *Take*

$$\begin{array}{lcl} \mathsf{K}xy & \to & x \\ \mathsf{I}x & \to & x \end{array} \qquad \text{and} \qquad \begin{array}{lcl} \mathcal{E}(\mathsf{K}) & = & (\varphi_1 \to \varphi_2) \to \varphi_1 \to \varphi_2 \\ \mathcal{E}(\mathsf{I}) & = & \varphi_1 \to \varphi_1 \end{array}$$

Then

$$[\![\mathsf{KI}]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} = [\![\mathsf{I}]\!]^{\mathcal{F}}_{\xi,\mathcal{E}},$$

but neither $\mathsf{KI} =_{\mathbf{R}} \mathsf{I}$*, nor* $\mathsf{KI} \approx_{\mathbf{R}} \mathsf{I}$*, nor* $\mathsf{KI} \approx^{hnf}_{\mathbf{R}} \mathsf{I}$.

The relation between the two semantics is formulated by:

Theorem 94 $[\![t]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} \subseteq \bigcup_{a \in [\![t]\!]^{\mathcal{A}}_{C}} [\![a]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$.

PROOF. If $\sigma \in [\![t]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$, then there is a B such that $B \vdash_{\mathcal{E}} t{:}\sigma$. Then, by Thm. 69, there is an $a \in \mathcal{A}_{\mathcal{C}}(t)$ such that $B \vdash_{\mathcal{E}} a{:}\sigma$. ■

Note that the inclusion is strict, since the Subject Expansion property does not hold in general. Also, as can be expected:

Theorem 95 *Let* $\mathbf{C} = ((\mathcal{C},\mathcal{X}),\mathbf{R})$ *be a* CS*,* $\mathcal{E}$ *principal for* $\mathbf{C}$*. For all* $t \in T(\mathcal{C},\mathcal{X})$*,* $\bigcup_{a \in [\![t]\!]^{\mathcal{A}}_{C}} [\![a]\!]^{\mathcal{F}}_{\xi,\mathcal{E}} \subseteq [\![t]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$.

PROOF. If $\sigma \in \bigcup_{a \in [\![t]\!]^{\mathcal{A}}_{C}} [\![a]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$, then there exists $a \in [\![t]\!]^{\mathcal{A}}_{\mathcal{C}}$, B such that $B \vdash_{\mathcal{E}} a{:}\sigma$. Then, by Thm. 70, also $B \vdash_{\mathcal{E}} t{:}\sigma$, so $\sigma \in [\![t]\!]^{\mathcal{F}}_{\xi,\mathcal{E}}$. ■

10 Conclusions

The approximation result has important consequences both from a computational point of view, since it allows us to characterize the normalization properties of typeable terms, and from a semantic point of view, since it allows us to study the relations between filter models and approximantion models. This is true both for LC and for CS, but the characterizations of normalization and the relations between the models are different in each case. The most striking difference is probably the fact that the models do not coincide in general in the case of CS (the filter model is only a semi-model in general) whereas they do coincide for the LC. Of course, the lack of Subject Expansion in CS explains the fact that we only have a semi-model. However, the fact that for CS the approximation model is fully abstract, but the filter model is not, is related to the fact that we have a "weak" form of reduction in CS, compared with the reduction in LC.

The proof of the approximation result uses a notion of Cut Elimination (Derivation Reduction) which is new in the context of intersection types. It could be adapted to other rewriting systems (in particular, the LC and TRS), where it also helps to obtain easier proofs of the characterisation of normalisation properties of typeable terms. In the case of LC this remains an open issue, whereas for TRS the proof was sketched in [4]. In the future we hope to be able to extend the semantic study presented in this paper to the more general TRS studied in [4].

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