Abstract for Iwasawa 2015 Poster Session
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Semisimplicity of Classical Iwasawa Modules

Let p be an odd prime and K a number field. Let K_∞/K be a $\mathbb{Z}_p$-extension of K, let $\Gamma = \mathrm{Gal}(K_\infty/K)$ and let $\Lambda = \mathbb{Z}_p[[\Gamma]]$. Finally, let L_∞ denote the pro-p Hilbert class field of K_∞ and $X = \mathrm{Gal}(L_\infty/K_\infty)$, which was shown by Iwasawa to be a finitely generated torsion Λ-module.

Given a finitely generated torsion Λ-module Y, one can show that the submodule Y^Γ is pseudo-isomorphic to the quotient Y_Γ. Furthermore, there is an obvious Λ-module homomorphism $Y^\Gamma \to Y_\Gamma$, namely the composition of the inclusion $\iota : Y^\Gamma \to Y$ with the projection $\pi : Y \to Y_\Gamma$. However, for an arbitrary Y, this composition need not be a pseudo-isomorphism.

We say that a torsion Λ-module Y is *T-semisimple* if the map $\pi \circ \iota : Y^\Gamma \to Y_\Gamma$ is a pseudo-isomorphism and we say that a $\mathbb{Z}_p$-extension K_∞/K is T-semisimple if the associated Λ-module $X = \mathrm{Gal}(L_\infty/K_\infty)$ is T-semisimple.

In [2], Greenberg proved that if $K/\mathbb{Q}$ is abelian and K_∞/K is the cyclotomic $\mathbb{Z}_p$-extension, then K_∞ is T-semisimple. In [1], Carroll and Kisilevsky extended Greenberg's result to show that certain other $\mathbb{Z}_p$-extensions of an abelian number field which are Galois over $\mathbb{Q}$ are T-semisimple. However, Kisilevsky and Jaulent independently showed that a $\mathbb{Z}_p$-extension K_∞ of an abelian number field K need not be T-semisimple, even if $K_\infty/\mathbb{Q}$ is Galois (in [4] and [3], respectively).

I will review these results and show how Greenberg's result can be used to prove that T-semisimplicity is in some sense a generic condition when $K/\mathbb{Q}$ is abelian. More precisely, suppose that $K/\mathbb{Q}$ is a complex abelian extension. Let $\tilde{K}_\infty$ denote the compositum of all $\mathbb{Z}_p$-extensions of K and $\tilde{\Gamma} = \mathrm{Gal}(\tilde{K}_\infty/K)$. From Leopoldt's conjecture, we know that $\tilde{\Gamma} \simeq \mathbb{Z}_p^{[K:\mathbb{Q}]/2+1}$ and $\tilde{\Gamma}^+ \simeq \mathbb{Z}_p$. Let us call a $\mathbb{Z}_p$-extension K_∞/K anti-cyclotomic if complex conjugation (in $\mathrm{Gal}(K/\mathbb{Q})$) acts on $\Gamma = \mathrm{Gal}(K_\infty/K)$ nontrivially (i.e., if $K_\infty \subseteq \tilde{K}_\infty^{\tilde{\Gamma}^+}$). By studying decomposition subgroups, I will show that T-semisimplicity for the cyclotomic $\mathbb{Z}_p$-extension of K implies that every other $\mathbb{Z}_p$ extension of K in which all primes above p ramify and which is not anti-cyclotomic is also T-semisimple.

References

[1] J. Carroll and H. Kisilevsky. On the Iwasawa invariants of certain $\mathbb{Z}_p$-extensions. *Compositio Mathematica*, 49(2):217–229, 1983.

[2] Ralph Greenberg. On a certain l-adic representation. *Inventiones mathematicae*, 21(1-2):117–124, 1973.

[3] Jean-François Jaulent. Sur la théorie des genres dans les tours métabéliennes. *Seminaire de Théorie des Nombres de Bordeaux*, 11:1–18, 1981-1982.

[4] H. Kisilevsky. Some non-semi-simple Iwasawa modules. *Compositio Mathematica*, 49(3):399–404, 1983.