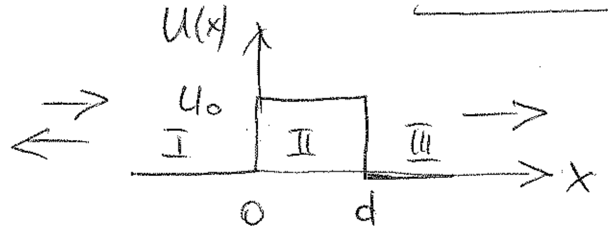
Symmetric potential barrier



Transmission to the RIGHT!

(I) $-\frac{\hbar^2}{2m}\psi'' = E\psi$, $\psi_I = e^{ikx} + Be^{-ikx}$, $k = \frac{1}{\hbar}\sqrt{2mE}$

(II) $\psi_{II} = Ce^{iqx} + De^{-iqx}$, $q = \frac{1}{\hbar}\sqrt{2m(E-U_0)}$

(III) $\psi_{III} = Ee^{ikx}$

Continuity of ψ and ψ' at $\underline{x=0}$:

$$\begin{cases} 1+B = C+D \\ ik(1-B) = iq(C-D) \end{cases} \qquad (1)$$

Continuity of ψ, ψ' at $\underline{x=d}$:

$$\begin{cases} Ce^{iqd} + De^{-iqd} = Ee^{ikd} \\ iq(Ce^{iqd} - De^{-iqd}) = ikEe^{ikd} \end{cases} \qquad (2)$$

From (2):

$$\frac{q}{k}(Ce^{iqd} - De^{-iqd}) = Ce^{iqd} + De^{-iqd}$$

$$Ce^{iqd}\left(\frac{q}{k} - 1\right) = De^{-iqd}\left(1 + \frac{q}{k}\right)$$

$$C = De^{-2iqd}\frac{k+q}{q-k} = \frac{q+k}{q-k}e^{-2iqd}D \equiv \varkappa D$$

Then, $\cancel{1+B = D\left[1 + \frac{q+k}{q-k}e^{-2iqd}\right]}$ from (1):

$B = C + D - 1$ ~~= 1~~ and $1 - B = \frac{q}{k}(C-D)$, and

$$C + D - 1 = 1 - \frac{q}{k}(C - D)$$

$$D(1+\varkappa) - 1 = 1 - \frac{q}{k} D(\varkappa - 1)$$

$$D\left[1 + \varkappa + \frac{q}{k}(\varkappa - 1)\right] = 2, \quad D = \frac{2}{1 + \varkappa - \frac{q}{k}(1-\varkappa)}$$

$$= \frac{2}{1 - \frac{q}{k} + \varkappa\left(1 + \frac{q}{k}\right)} = \frac{2k}{(k-q) + (k+q)\varkappa}$$

$$= \frac{2k}{(k-q) + (k+q)\frac{q+k}{q-k} e^{-2iqd}} = \frac{2k(q-k)}{-(k-q)^2 + (k+q)^2 e^{-2iqd}}$$

$$\boxed{D = \frac{2k(k-q)}{(k-q)^2 - (k+q)^2 e^{-2iqd}}}$$

$$\boxed{C = \frac{q+k}{q-k} e^{-2iqd} D = \frac{-2k(q+k)\, e^{-2iqd}}{(k-q)^2 - (k+q)^2 e^{-2iqd}}}$$

$$E = e^{i(q-k)d} C + D e^{-i(q+k)d}$$

$$= \left[\frac{q+k}{q-k} e^{-2iqd} e^{i(q-k)d} + e^{-i(q+k)d}\right] D$$

$$= D \cdot \left[\frac{q+k}{q-k} e^{-i(\cancel{3}q+k)d} + e^{-i(q+k)d}\right] = D\left[\frac{q+k}{q-k} + 1\right] e^{-i(q+k)d}$$

$$\cancel{E = 2k \frac{-(q+k) e^{i(3q-k)d} + e^{-i(q+k)d}(q-k)}{(k-q)^2 - (k+q)^2 e^{-2iqd}}} = D \frac{2q}{q-k} e^{-i(q+k)d}$$

$$B = C + D - 1 = \frac{2k(k-q)}{(k-q)^2-(k+q)^2e^{-2iqd}} + \frac{-2k(q+k)e^{-2iqd}}{(k-q)^2-(k+q)^2e^{-2iqd}} - 1$$

$$= \frac{2k(k-q) - 2k(q+k)e^{-2iqd} - (k-q)^2 + (k+q)^2e^{-2iqd}}{(k-q)^2-(k+q)^2e^{-2iqd}}$$

$$= \frac{1}{\dots}\left\{(k-q)(2k-k+q) + (k+q)(k+q-2k)\,e^{-2iqd}\right\}$$

$$= \frac{1}{\dots}\left\{(k-q)(k+q) + (k+q)(q-k)\,e^{-2iqd}\right\}$$

$$= \frac{1}{\dots}(k-q)(k+q)\left(1 - e^{-2iqd}\right),$$

$$\boxed{B = \frac{(k^2-q^2)(1-e^{-2iqd})}{(k-q)^2-(k+q)^2e^{-2iqd}}} = \frac{(k^2-q^2)(e^{iqd}-e^{-iqd})}{(k-q)^2e^{iqd}-(k+q)^2e^{-iqd}}$$

~~The reflection coefficient is given by:~~

$$R = |B|^2 = \cancel{(k^2-q^2)^2 \frac{(1-e^{-2iqd})(1-e^{+2iq^*d})}{\quad}}$$

(a) $\underline{\underline{E<U_0}}$ $\quad q = i\lambda,\ \lambda = \frac{i}{\hbar}\sqrt{2m(U_0-E)},\ q^2=-\lambda^2$

$$B = \frac{(k^2+\lambda^2)(1-e^{2\lambda d})}{(k-i\lambda)^2-(k+i\lambda)^2e^{2\lambda d}}$$

$$R = |B|^2 = \frac{(k^2+\lambda^2)^2(1-e^{2\lambda d})^2}{F},$$

$$F = \left[(k-i\lambda)^2-(k+i\lambda)^2e^{2\lambda d}\right]\left[(k+i\lambda)^2-(k-i\lambda)^2e^{2\lambda d}\right]$$

$$= (k^2+\lambda^2)^2 + (k^2+\lambda^2)^2e^{4\lambda d} - (k-i\lambda)^4e^{2\lambda d} - (k+i\lambda)^4e^{2\lambda d}$$

$$= (k^2+\lambda^2)^2\left[1+e^{4\lambda d}\right] - e^{2\lambda d}\left[(k-i\lambda)^4+(k+i\lambda)^4\right]$$

$$= (k^2+\lambda^2)^2(1+e^{4\lambda d}) - e^{2\lambda d}\Big[k^4 + \cancel{4k^3(-i\lambda)} + 6k^2(-i\lambda)^2 + \cancel{4k(-i\lambda)^3} + (-i\lambda)^4 +$$

$$+ k^4 + \cancel{4k^3(+i\lambda)} + 6k^2(i\lambda)^2 + \cancel{4k(i\lambda)^3} + (i\lambda)^4\Big]$$

All the coefficients are:

$$B = \frac{k^2 - q^2}{(k^2+q^2) + 2ikq\,\text{ctan}(qd)}$$

$$E = \frac{-2kq\, e^{-ikd}}{i(k^2+q^2)\sin qd - 2kq\cos qd}\,, \quad D = \frac{k(k-q)\, e^{iqd}}{i(k^2+q^2)\sin qd - 2kq\cos qd}$$

$$C = \frac{-k(q+k)\, e^{-iqd}}{i(k^2+q^2)\sin qd - 2kq\cos qd}$$

(i) $\underline{E > U_0}$ $\to$ q - real, the above eqs. can be used without change.

The transmission $T = j_{\to}^{III} / j_{\to}^{I}$

$$j_{\to} = \frac{i\hbar}{2m}(\psi_{\to}\psi_{\to}^{*\prime} - \psi_{\to}^{*}\psi_{\to}^{\prime})\,, \quad \psi_{\to} = A e^{ikx},$$

then
$$j_{\to} \frac{i\hbar}{2m}\left[A e^{ikx} A^*(-ik) e^{-ikx} - A^* e^{-ikx} A(ik) e^{ikx}\right]$$
$$= \frac{i\hbar}{2m}(+ik)\left[-|A|^2 - |A|^2\right] = \frac{\hbar k}{m}|A|^2$$

so that each exponent contributes this to the current.

$$T = \frac{\hbar k/m\, |E|^2}{\hbar k/m\, 1} = |E|^2 = \frac{4k^2q^2 e^{-ikd} e^{ikd}}{\left(i(k^2+q^2)\sin qd - 2kq\cos qd\right)\left(-i(k^2+q^2)S - 2kqC\right)}$$

$$= \frac{4k^2q^2}{(k^2+q^2)^2 \underbrace{\sin^2 qd}_{S^2} + 4k^2q^2 \underbrace{\cos^2 qd}_{C^2 = 1-S^2}} = \frac{1}{\frac{(k^2+q^2)^2}{4k^2q^2}S^2 + 1 - S^2} =$$

$$= \left[1 - \left(1 - \frac{(k^2+q^2)^2}{4k^2q^2}\right)S^2\right]^{-1} = \left[1 - \frac{4k^2q^2 - (k^2+q^2)^2}{4k^2q^2} S^2\right]^{-1}$$

$$\boxed{T = \left[1 + \frac{(k^2-q^2)^2}{4k^2q^2}\sin^2 qd\right]^{-1}}$$

Reflection:

$$R = \frac{j^{I}_{\leftarrow}}{j^{I}_{\rightarrow}} = |B|^2 = \frac{(k^2-q^2)^2}{[(k^2+q^2)+i\,2kq\,\mathrm{ctan}\,qd][(k^2+q^2)-i\,2kq\,\mathrm{ctan}\,qd]}$$

where:

$$[\ldots][\ldots] = (k^2+q^2)^2 + 4k^2q^2\,\mathrm{ctan}^2(qd)$$

$$\boxed{R = \frac{(k^2-q^2)^2}{(k^2+q^2)^2+4k^2q^2\,\mathrm{ctan}^2(qd)}}$$

It is also checked that

$$\boxed{T+R=1}$$

or $\boxed{|E|^2+|B|^2=1}$

(ii) $\underline{E<U_0}$ $\quad q = i\lambda$, $\quad \lambda = \frac{1}{\hbar}\sqrt{2m(U_0-E)}$ is real.

In this case

$$E = \frac{-2ki\lambda\, e^{-ikd}}{i(k^2-\lambda^2)\sin(i\lambda d) - 2ik\lambda\cos(i\lambda d)} = \cancel{\frac{2ki\lambda\, e^{-ikd}}{-(k^2-\lambda^2)\sinh(\lambda d)-2ik\lambda\cosh}}$$

$$= \frac{-2ki\lambda\, e^{-ikd}}{-(k^2-\lambda^2)\sinh(\lambda d) - 2ik\lambda\cosh(\lambda d)} = \frac{i2k\lambda\, e^{-ikd}}{(k^2-\lambda^2)\sinh(\lambda d)+2ik\lambda\cosh(\lambda d)}$$

so that in this case

$$T = |E|^2 = \frac{i2k\lambda e^{-ikd}\,(-i2k\lambda)\,e^{ikd}}{[(k^2-\lambda^2)\sinh(\lambda d)+2ik\lambda\cosh(\lambda d)][(k^2-\lambda^2)\sinh(\lambda d)-2ik\lambda\cosh(\lambda d)]}$$

$$= \frac{4k^2\lambda^2}{(k^2-\lambda^2)^2\sinh^2(\lambda d)+4k^2\lambda^2\cosh^2(\lambda d)}, \quad \boxed{\cosh^2(\lambda d) = 1+\sinh^2(\lambda d)}$$

$$\hookrightarrow = \frac{4k^2\lambda^2}{(k^2-\lambda^2)^2 s^2 + 4k^2\lambda^2(1+s^2)} = \frac{4k^2\lambda^2}{4k^2\lambda^2 + s^2\underbrace{[(k^2-\lambda^2)^2+4k^2\lambda^2]}_{(k^2+\lambda^2)^2}}$$

$$\boxed{T = \left[1+\frac{(k^2+\lambda^2)^2}{4k^2\lambda^2}\sinh^2(\lambda d)\right]^{-1}}$$

and

$$B = \frac{k^2+\lambda^2}{(k^2-\lambda^2) - 2k\lambda\,\mathrm{ctan}(i\lambda d)}, \qquad \mathrm{ctan}(i\lambda d) = \frac{\cos(i\lambda d)}{\sin(i\lambda d)} =$$

$$= \frac{\cosh(\lambda d)}{i\sinh(\lambda d)} = -i\coth(\lambda d)$$

$$B = \frac{k^2+\lambda^2}{(k^2-\lambda^2) + 2k\lambda i\coth(\lambda d)}$$

and hence

$$R = |B|^2 = \frac{(k^2+\lambda^2)^2}{(k^2-\lambda^2)^2 + 4k^2\lambda^2\coth^2(\lambda d)} = \frac{(k^2+\lambda^2)^2\sinh^2(\lambda d)}{(k^2-\lambda^2)^2\sinh^2(\lambda d) + 4k^2\lambda^2\cosh^2(\lambda d)}$$

$$= \frac{(k^2+\lambda^2)\,s^2}{(k^2-\lambda^2)^2 s^2 + 4k^2\lambda^2(1+s^2)} = \frac{(k^2+\lambda^2)\,s^2}{s^2\big[\underbrace{(k^2-\lambda^2)^2+4k^2\lambda^2}_{(k^2+\lambda^2)^2}\big] + 4k^2\lambda^2}$$

$$= \frac{(k^2+\lambda^2)\,s^2}{(k^2+\lambda^2)^2 s^2 + 4k^2\lambda^2} = \left[1 + \frac{4k^2\lambda^2}{(k^2+\lambda^2)^2\sinh^2(\lambda d)}\right]^{-1}$$

It is checked that

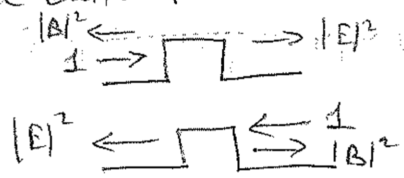
$$T+R = \frac{4k^2\lambda^2}{(k^2-\lambda^2)^2 s^2 + 4k^2\lambda^2 c^2} + \frac{(k^2+\lambda^2)^2 s^2}{(k^2-\lambda^2)^2 s^2 + 4k^2\lambda^2 c^2}$$

$$= \frac{4k^2\lambda^2 + (k^2+\lambda^2)^2 s^2}{(k^2-\lambda^2)^2 s^2 + 4k^2\lambda^2 c^2} = \frac{4k^2\lambda^2 + (k^2+\lambda^2)^2 s^2}{(k^2-\lambda^2)^2 s^2 + 4k^2\lambda^2(1+s^2)}$$

$$= \frac{4k^2\lambda^2 + (k^2+\lambda^2)^2 s^2}{(k^2+\lambda^2)^2 s^2 + 4k^2\lambda^2} = 1 \qquad \boxed{T+R=1}$$

or $\boxed{|E|^2+|B|^2=1}$

The overall current is zero due to both directions:



The current on the left is

$$\frac{\hbar k}{m}1 - \frac{\hbar k}{m}|B|^2 - \frac{\hbar k}{m}|E|^2 = 0$$

The same on the right $= 0$.